2021

Elementary College Geometry (2021 ed.)

Henry Africk
CUNY New York City College of Technology

Elementary
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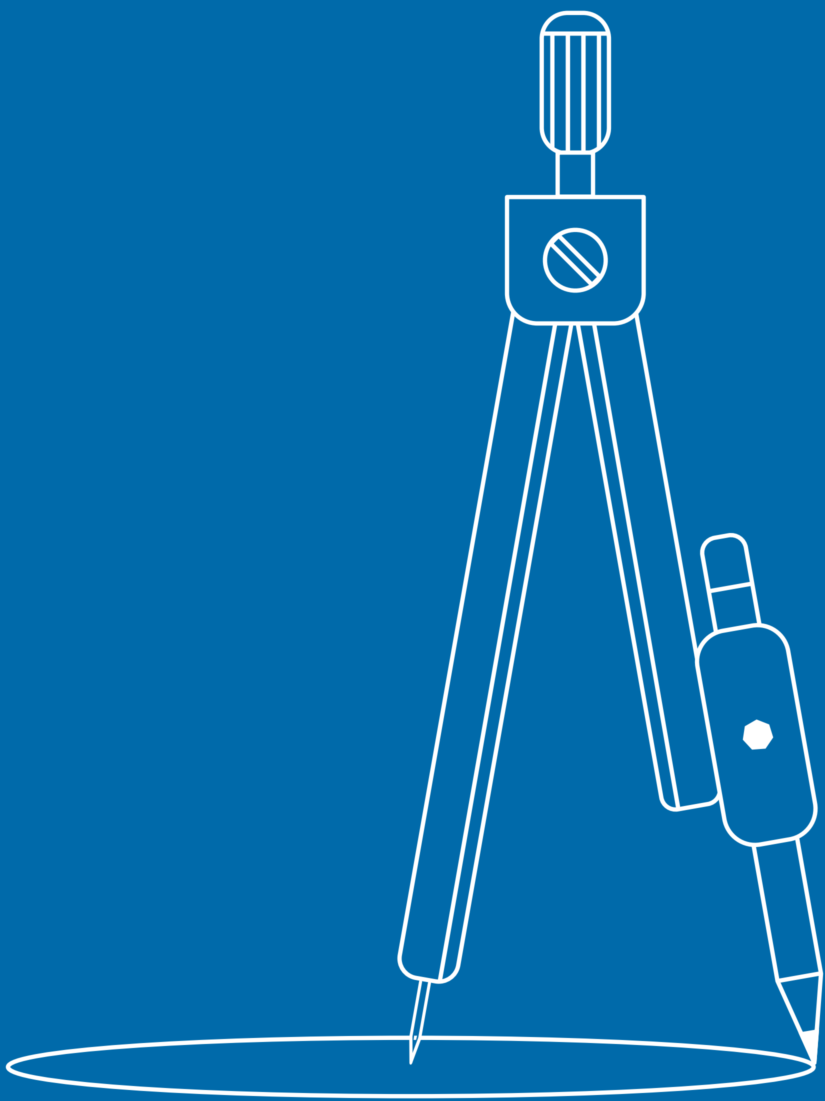

CUNY Academic Works

Elementary College Geometry

Henry Africk

NEW YORK CITY COLLEGE OF TECHNOLOGY

2021 edition with corrections and new diagrams

PREFACE

This text is intended for a brief introductory course in plane geometry. It covers the topics from elementary geometry that are most likely to be required for more advanced mathematics courses. The only prerequisite is a semester of algebra.

The emphasis is on applying basic geometric principles to the numerical solution of problems. For this purpose the number of theorems and definitions is kept small. Proofs are short and intuitive, mostly in the style of those found in a typical trigonometry or precalculus text. There is little attempt to teach theorem-proving or formal methods of reasoning. However the topics are ordered so that they may be taught deductively.

The problems are arranged in pairs so that just the odd-numbered or just the even-numbered can be assigned. For assistance, the student may refer to a large number of completely worked-out examples. Most problems are presented in diagram form so that the difficulty of translating words into pictures is avoided. Many problems require the solution of algebraic equations in a geometric context, These are included to reinforce the student's algebraic and numerical skills. A few of the exercises involve the application of geometry to simple practical problems, These serve primarily to convince the student that what he or she is studying is useful. Historical notes are added where appropriate to give the student a greater appreciation of the subject.

This book is suitable for a course of about 45 semester hours. A shorter course may be devised by skipping proofs, avoiding the more complicated problems and omitting less crucial topics.

I would like to thank my colleagues at New York City Technical College who have contributed, directly or indirectly, to the development to this work. In particular, I would like to acknowledge the influence of L. Chosid, M. Graber, S, Katoni, F. Parisi and E. Stern.

Henry Africk
New York City Technical College
City University of New York

ACKNOWLEDGEMENT

I would especially like to thank Ümit Kaya for the typesetting and graphics which have greatly enhanced the appearance of the original work.

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CHAPTER 1

LINES, ANGLES and TRIANGLES

1.1 LINES

Geometry (from Greek words meaning earth-measure) originally developed as a means of surveying land areas. In its simplest form, it is a study of figures that can be drawn on a perfectly smooth flat surface, or **plane**. It is this **plane geometry** which we will study in this book and which serves as a foundation for trigonometry, solid and analytic geometry and calculus.

The simplest figures that can be drawn on a plane are the point and the line. By a **line** we will always mean a **straight line**. Through two distinct points one and only one (straight) line can be drawn. The line through points A and B will be denoted by $\overleftrightarrow{AB}$ (Figure 1).

The arrows indicate that the line extends indefinitely in each direction. The **line segment** from A to B consists of A, B and that part of $\overleftrightarrow{AB}$ between A and B. It is denoted by AB.* The **ray** $\overrightarrow{AB}$ is the part of $\overleftrightarrow{AB}$ which begins at A and extends indefinitely in the direction of B.

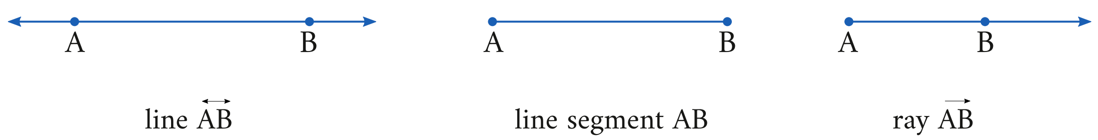


Figure 1. Line $\overleftrightarrow{AB}$, line segment AB, and ray $\overrightarrow{AB}$.

We assume everyone is familiar with the notion of **length** of a line segment and how it can be measured inches, or feet, or meters etc. The **distance** between two points A and B is the same as the length of AB.

Two line segments are **equal** if they have the same length.

In Figure 2, AB = CD.

Figure 2. AB = CD.

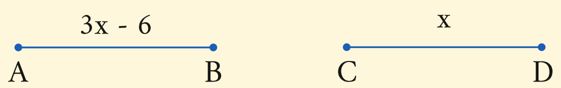
* Some textbooks use the notation $\overline{AB}$ for line segment

We often indicate two line segments are equal by marking them in the same way. In Figure 3, $AB = CD$ and $EF = GH$.

Figure 3. AB = CD *and* EF = GH.

? EXAMPLE A

Find x if AB = CD:



SOLUTION

$$\begin{aligned} AB &= CD \\ 3x - 6 &= x \\ 3x - x &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

CHECK

$$\begin{array}{r|l} AB & CD \\ 3x - 6 & x \\ 3(3) - 6 & 3 \\ 9 - 6 & \\ 3 & \end{array}$$

ANSWER: $x = 3$.

Notice that in Example A we have not indicated the unit of measurement. Strictly speaking, we should specify that $AB = 3x - 6$ inches (or feet or meters) and that $CD = x$ inches. However since the answer would still be $x = 3$ we will usually omit this information to save space.

We say that B is the **midpoint** of AC if B is A point on AC and AB = BC (see Figure 4).

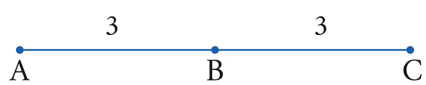


Figure 4. B is the midpoint of AC.

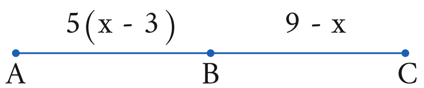
? EXAMPLE B

Find x and AC if B is the midpoint of AC and $AB = 5(x - 3)$ and $BC = 9 - x$.

SOLUTION

We first draw a picture to help visualize the given information:

Since 3 is a midpoint,



$$\begin{aligned} AB &= BC \\ 5(x-3) &= 9-x \\ 5x - 15 &= 9 - x \\ 5x + x &= 9 + 15 \\ 6x &= 24 \\ x &= 4 \end{aligned}$$

We obtain $AC = AB + BC = 5 + 5 = 10$.

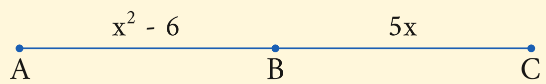
CHECK

$$\begin{array}{r|l} AB = & BC \\ 5(x-3) & 9-x \\ 5(4-3) & 9-4 \\ 5(1) & 5 \\ 5 & \end{array}$$

ANSWER : $x = 4$, $AC = 10$.

? EXAMPLE C

Find AB if B is the midpoint of AC:



SOLUTION

$$\begin{aligned} AB &= BC \\ x^2 - 6 &= 5x \\ x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \end{aligned}$$

$$\begin{array}{r|r} x - 6 = 0 & x + 1 = 0 \\ x = 6 & x = -1 \end{array}$$

CHECK

$$\begin{array}{r|l} AB = & BC \\ x^2 - 6 & 5x \\ 6^2 - 6 & 5(6) \\ 36 - 6 & 30 \\ 30 & \end{array}$$

If x =6 then
AB = $x^2 - 6 = 6^2 - 6 = 36 - 6 = 30$.

If x = −1 then
AB = $(-1)^2 - 6 = 1 - 6 = -5$.

We reject the answer x = −1 and AB = −5 because the length of a line segment is always positive. Therefore x = 6 and AB = 30.

ANSWER : AB = 30.

Three points are **collinear** if they lie on the same line.

Figure 5. A, B and C are collinear
AB = 5, BC = 3 and AC =8.

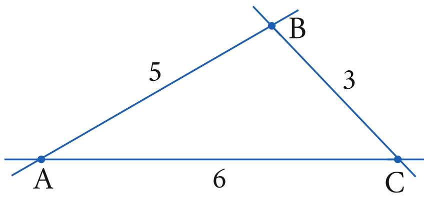


Figure 6. A, B and C are not collinear
AB = 5, BC = 3 and AC = 6.

A, B, and C are collinear if and only if AB + BC = AC.

? EXAMPLE D

If A, B, and C are collinear and AC = 7, find x:

SOLUTION

$$\begin{aligned} AB + BC &= AC \\ 8 - 2x + x + 1 &= 7 \\ 9 - x &= 7 \\ 2 &= x \end{aligned}$$

ANSWER : x = 2.

CHECK

AB + BC	= AC
8 - 2x + x + 1	7
8 - 2(2) + 2 + 1	
8 - 4 + 3	
4 + 3	
7	

HISTORICAL NOTE

Geometry originated in the solution of practical problems. The architectural remains of Babylon, Egypt, and other ancient civilizations show a knowledge of simple geometric relationships. The digging of canals, erection of buildings, and the laying out of cities required computations of lengths, areas, and volumes. Surveying is said to have developed in Egypt so that tracts of land could be relocated after the annual overflow of the Nile. Geometry was also utilized by ancient civilizations in their astronomical observations and the construction of their calendars.

The Greeks transformed the practical geometry of the Babylonians and Egyptians into an organized body of knowledge. Thales (c, 636 - c. 546 B,C.), one of the "seven wise men" of antiquity, is credited with being the first to obtain geometrical results by logical reasoning, instead of just by intuition and experiment. Pythagoras (c. 582 - c. 507 B,C.) continued the work of Thales. He founded the Pythagorean school, a mystical society devoted to the unified study of philosophy, mathematics, and science. About 300 B,C., Euclid, a Greek teacher of mathematics at the university at Alexandria, wrote a systematic exposition of elementary geometry called the *Elements*. In his *Elements*, Euclid used a few simple principles, called **axioms** or **postulates**, to derive most of the mathematics known at the time. For over 2000 years, Euclid's *Elements* has been accepted as the standard textbook of geometry and is the basis for most other elementary texts, including this one.

PROBLEMS

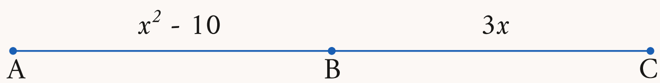
1. Find x if AB = CD :

2. Find x if AB = CD :

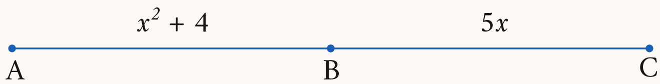
3. Find x and AC if B is the midpoint of AC and AB = 3(x−5) and BC = x + 3.

4. Find x and AC if B is the midpoint of AC and AB = 2x + 9 and BC = 5(x − 9).

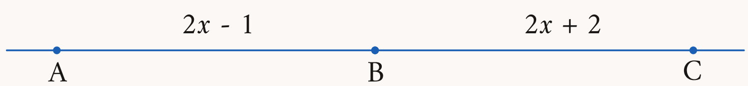
5. Find AB if B is the midpoint of AC :



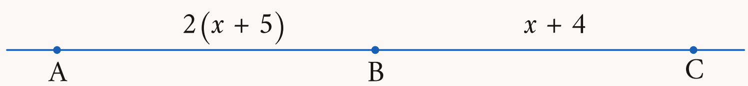
6. Find AB if B is the midpoint of AC :



7. If A, B and C are collinear and AC = 13 find x.



8. If A, B and C are collinear and AC = 26 find x.



1.2 ANGLES

An **angle** is the figure formed by two rays with a common end point. The two rays are called the **sides** of the angle and the common end point is called the **vertex** of the angle. The symbol for angle is $\angle$.

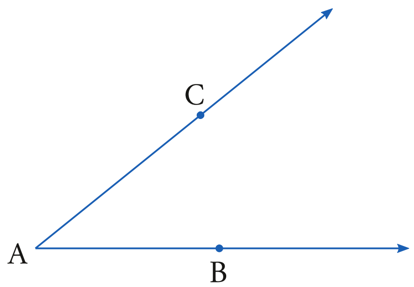


Figure 1. Angle BAC has vertex A and sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

The angle in Figure 1 has vertex A and sides AB and AC. It is denoted by $\angle BAC$ or $\angle CAB$ or simply A . When three letters are used, the middle letter is always the vertex. In Figure 2 we would not use the notation $\angle A$ as an abbreviation for $\angle BAC$ because it could also mean $\angle CAD$ or $\angle BAD$. We could however use the simplier name $\angle x$ for $\angle BAC$ if "x" is marked in as shown.

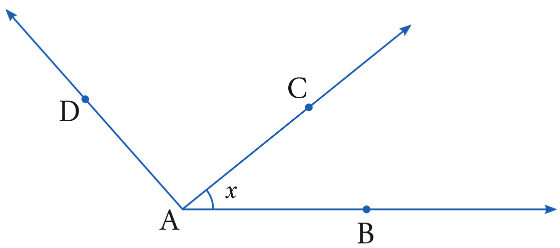


Figure 2. $\angle BAC$ may also be denoted by $\angle x$.

Angles can be measured with an instrument called a **protractor.** The unit of measurement is called a **degree** and the symbol for degree is °.

To measure an angle, place the center of the protractor (often marked with a cross or a small circle) on the vertex of the angle. Position the protractor so that one side of the angle cuts across 0, at the beginning of the scale, and so that the other side cuts across a point further up on the scale. We use either the upper scale or the lower scale, whichever is more convenient. For example, in Figure 3, one side of $\angle BAC$ crosses 0 on the lower scale and the other side crosses 50 on the lower scale. The measure of $\angle BAC$ is therefore 50° and we write $\angle BAC = 50°$.

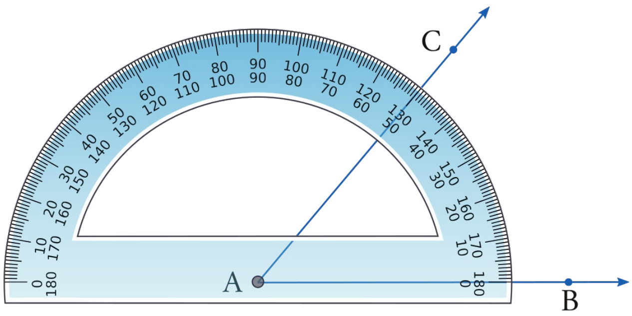


Figure 3. The protractor shows $\angle BAC = 50°$.

In Figure 4, side $\overrightarrow{AD}$ of $\angle DAC$ crosses 130 on the upper scale. Therefore we look on the upper scale for the point at which $\overrightarrow{AC}$ crosses and conclude that $\angle DAC = 130°$.

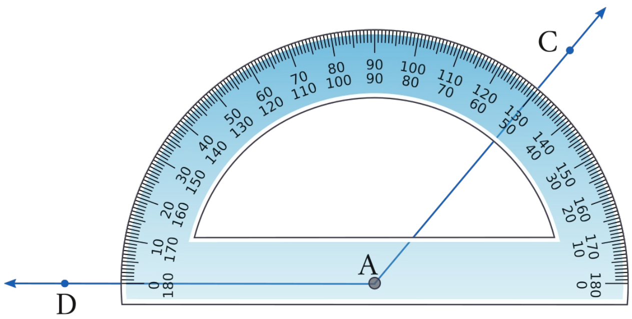


Figure 4. $\angle DAC = 130°$.

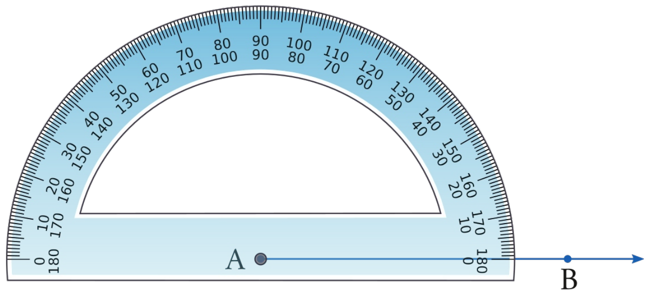
? EXAMPLE A

Draw an angle of 40° and label it $\angle BAC$.

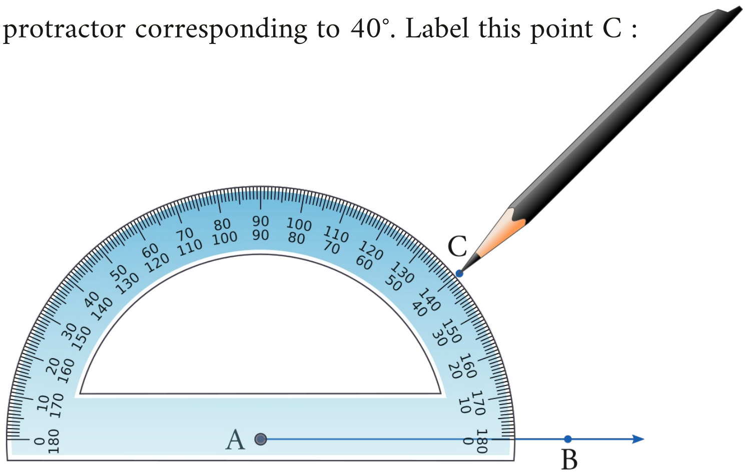
SOLUTION

Draw ray $\overrightarrow{AB}$ using a straight edge:

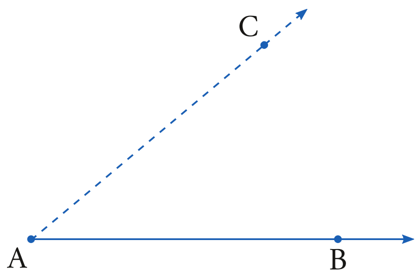
Place the protractor so that its center coincides with A and $\overrightarrow{AB}$ crosses the scale at 0:



Mark the place on the protractor corresponding to 40°. Label this point C :



Connect A with C :



Two angles are said to be **equal** if they have the same measure in degrees. We often indicate two angles are equal by marking them in the same way. In Figure 5, $\angle A = \angle B$.

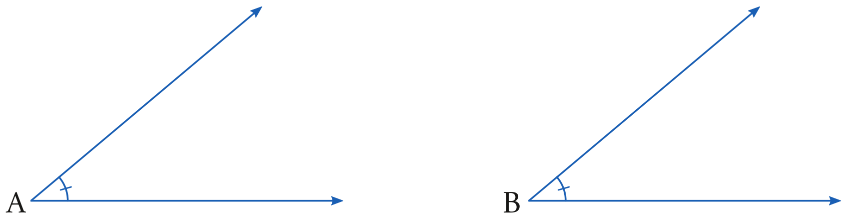


Figure 5. Equal angles.

An **angle bisector** is a ray which divides an angle into two equal angles. In Figure 6, $\overrightarrow{AC}$ is an angle bisector of $\angle BAD$. We also say $\overrightarrow{AC}$ **bisects** $\angle BAD$.

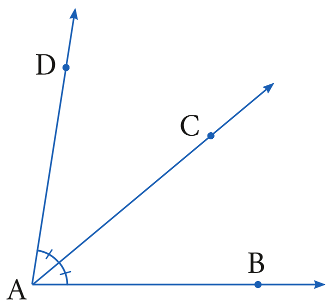
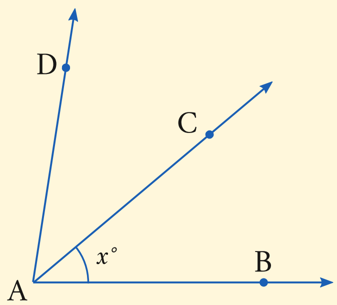


Figure 6. $\overrightarrow{AC}$ bisects $\angle BAD$.

? EXAMPLE B

Find x if $\overrightarrow{AC}$ bisects $\angle BAD$ and $\angle BAD = 80^\circ$:



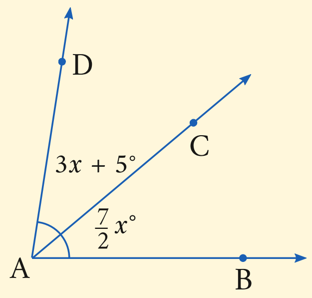
SOLUTION

$$x^\circ = \frac{1}{2}(\angle BAD) = \frac{1}{2}(80^\circ) = 40^\circ$$

ANSWER : $x = 40$.

? EXAMPLE C

Find x if $\overrightarrow{AC}$ bisects $\angle BAD$:



SOLUTION

$$\angle BAC = \angle CAD$$
$$\frac{7}{2}x = 3x + 5$$
$$(2)\frac{7}{2}x = (2)(3x + 5)$$
$$7x = 6x + 10$$
$$7x - 6x = 10$$
$$x = 10$$

ANSWER : x = 10.

CHECK

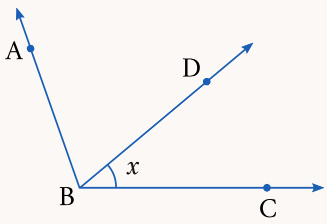
$$\angle BAC = \angle CAD$$

$\frac{7}{2}x$	$3x + 5$
$\frac{7}{2}(10)^\circ$	$3(10)^\circ + 5^\circ$
35°	$30^\circ + 5^\circ$
	35°

PROBLEMS

1 - 6. For each figure, give another name for ∠x:

1.



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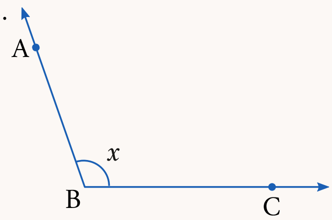

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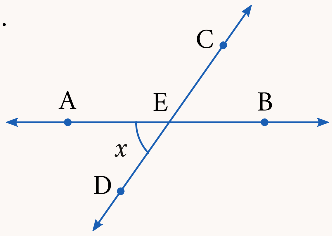

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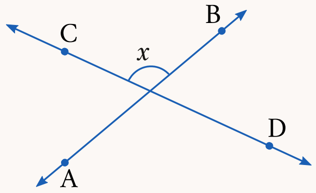

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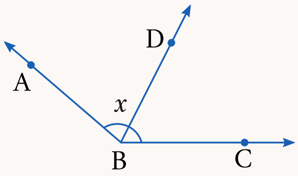

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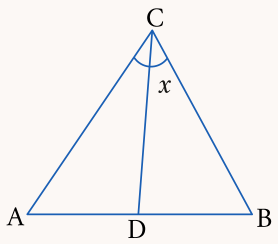
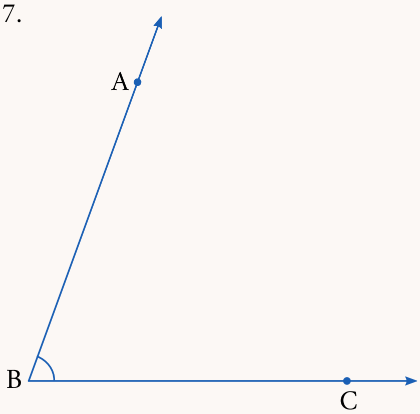


7 - 16. Measure each of the indicated angles:

7.



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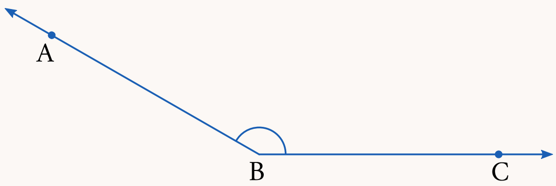

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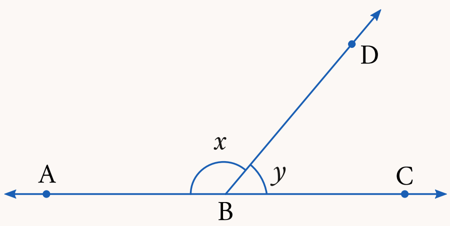

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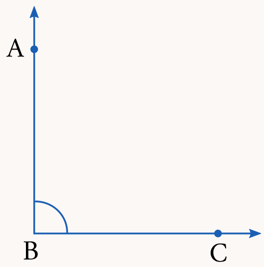

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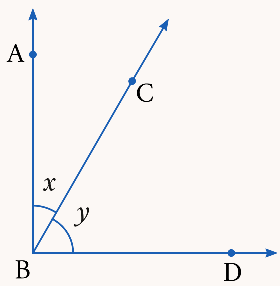

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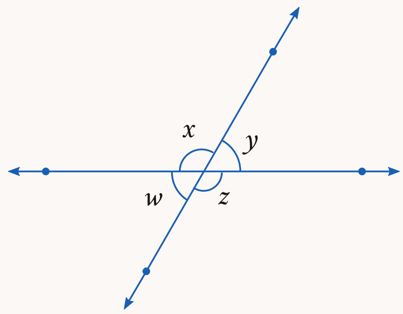

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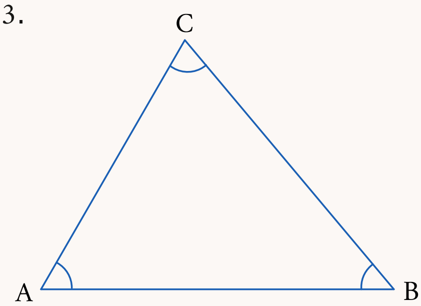

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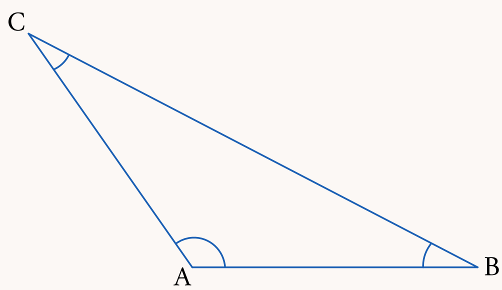

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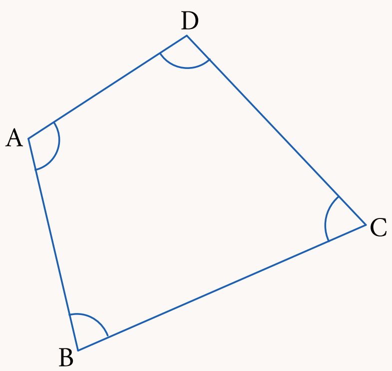

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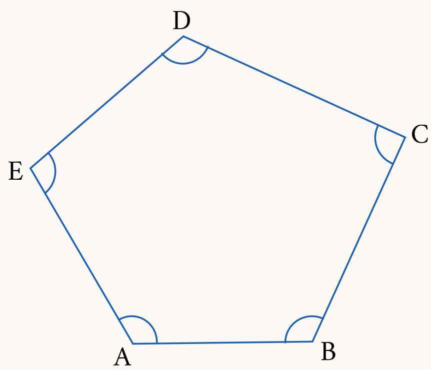
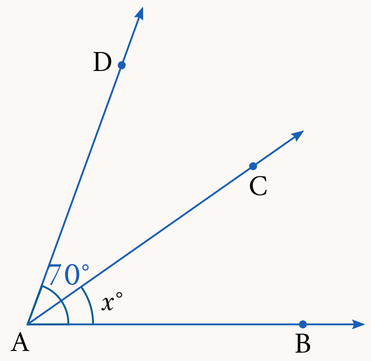


17 - 24. Draw and label each angle :

17. $\angle BAC = 30°$
18. $\angle BAC = 40°$
19. $\angle ABC = 45°$
20. $\angle EFG = 60°$
21. $\angle RST = 72°$
22. $\angle XYZ = 90°$
23. $\angle PQR = 135°$
24. $\angle JKL = 164°$

25 - 28. Find x if $\overrightarrow{AC}$ bisects $\angle BAD$:

25.



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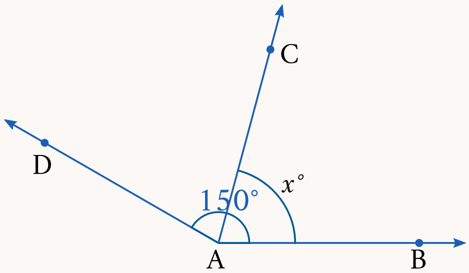

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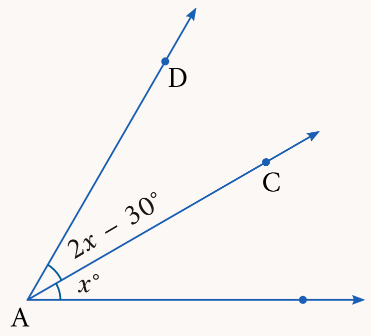

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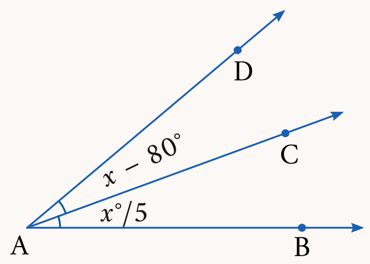


1.3 ANGLE CLASSIFICATIONS

Angles are classified according to their measures as follow:

- An **acute angle** is an angle whose measure is between 0° and 90°.
- A **right angle** is an angle whose measure is 90°. We often use a little square to indicate a right angle.
- An **obtuse angle** is an angle whose measure is between 90° and 180°.
- A **straight angle** is an angle whose measure is 180°. A straight angle is just a straight line with one of its points designated as the vertex.
- A **reflex angle** is an angle whose measure is greater than 180°.

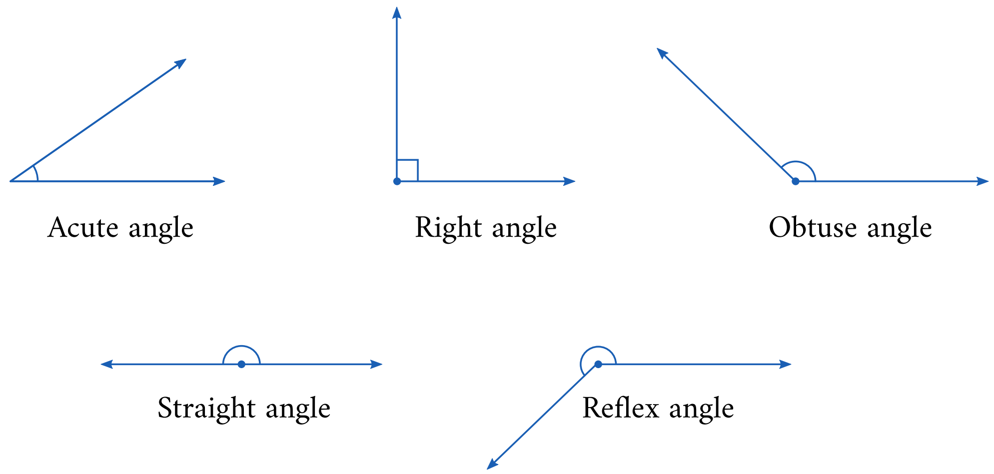


Figure 1. Angles classified according to their measures.

Notice that an angle can be measured in two ways. In Figure 2, ∠ABC is a reflex of 240° or an obtuse angle of 120° depending on how it is measured. Unless otherwise indicated, we will always assume the angle has measure less than 180°.

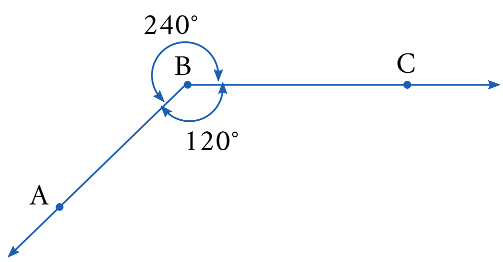


Figure 2. ∠ABC can be measured in two different ways.

Two lines are **perpendicular** if they meet to form right angles. In Figure 3, $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$. The symbol for perpendicular is $\perp$ and we write $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

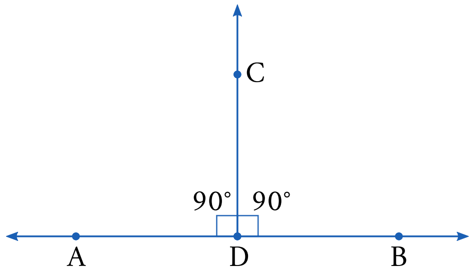


Figure 3. $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$.

The **perpendicular bisector** of a line segment is a line perpendicular to the line segment at its midpoint. In Figure 4, $\overleftrightarrow{CD}$ is a perpendicular bisector of AB .

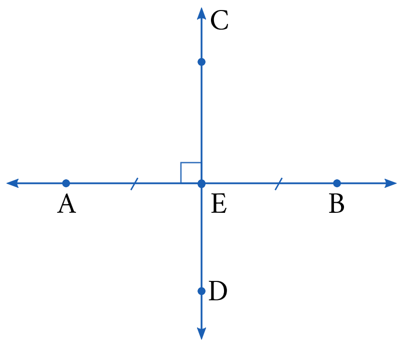


Figure 4. $\overleftrightarrow{CD}$ is a perpendicular bisector of AB.

Two angles are called **complementary** if the sum of their measures is 90°. Each angle is called the **complement** of the other. For example, angles of 60° and 30° are complementary.

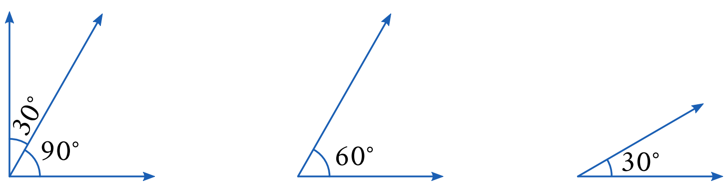


Figure 5. Complementary angles.

? EXAMPLE A

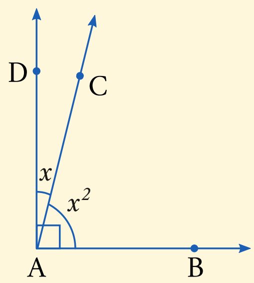
Find the complement of a 40° angle.

SOLUTION $90° - 40° = 50°$

ANSWER : $x = 50°$

? EXAMPLE B

Find x and the complementary angles.



SOLUTION

Since $\angle BAD = 90°$,

$$\begin{aligned} x^2 + x &= 90° \\ x^2 + x - 90 &= 0 \\ (x-9)(x+10) &= 0 \end{aligned}$$

$$x - 9 = 0 \quad \text{or} \quad x + 10 = 0$$
$$x = 9 \qquad\qquad x = -10$$

$$\begin{aligned} \angle CAD &= x = 9° \\ \angle BAC &= x^2 = 9^2 = 81° \\ \angle BAC + \angle CAD &= 81° + 9° = 90° \end{aligned}$$

$\angle CAD = x = -10°$

We reject this answer because the measure of an angle is always positive.*

CHECK

$x = 9°$,

$$\begin{array}{l|l} x^2 + x & = 90° \\ 9^2 + 9 & \\ 81 + 9 & \\ 90° & \end{array}$$

ANSWER : $x = 9$, $\angle CAD = 9°$, $\angle BAC = 81°$

*In trigonometry, when directed angles are introduced, angles can have negative measure. In this book, however, all angles will be thought of as having positive measure.

Two angles are called **supplementary** if the sum of their measures is 180°. Each angle is called the **supplement** of the other. For example, angle of 150° and 30° are supplementary.

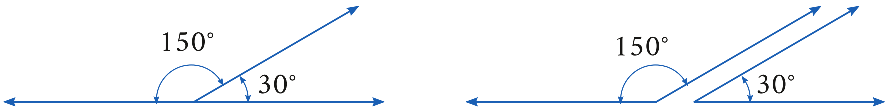


Figure 6. Supplementary angles.

? EXAMPLE C

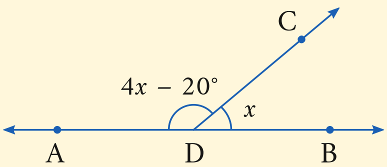
Find the supplementary of an angle 40°.

SOLUTION $180° - 40° = 140°$

ANSWER : $x = 140°$

? EXAMPLE D

Find x and the supplementary angles.



SOLUTION

Since $\angle ADB = 180°$,

$$\begin{aligned} 4x - 20 + x &= 180 \\ 5x &= 180 + 20 \\ 5x &= 200 \\ x &= 40 \end{aligned}$$

$\angle ADC = 4x - 20° = 4(40) - 20 = 160 - 20 = 140°$
$\angle BDC = x = 40°$
$\angle ADC + \angle BDC = 140° + 40° = 180°.$

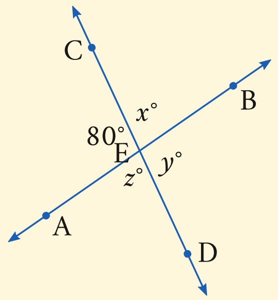
CHECK

$$\begin{array}{r|l} 4x - 20 + x & = \ 180° \\ 4(40) - 20 + 40 & \\ 160 - 20 + 40 & \\ 140 + 40 & \\ 180° & \end{array}$$

ANSWER : $x = 40°$, $\angle ADC = 140°$, $\angle BDC = 40°$

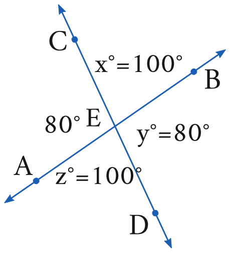
? EXAMPLE E

Find x, y, z:



SOLUTION

x° = 180° - 80° = 100° because x° and 80° are the measures of supplementary angles.
y° = 180° - x° = 180° - 100° = 80°
z° = 180° - 80° = 100°



ANSWER : x = 100°, y = 80°, z = 100°

When two lines intersect as in *Example E*, they form two pairs of angles that are opposite to each other called **vertical** angles. In Figure 7, $\angle x$ and $\angle x'$ are one pair of vertical angles. y and $\angle y'$ are the other pair of vertical angles. As suggested by *Example E*, $\angle x = \angle x'$ and $\angle y = \angle y'$. To see this in general, we can reason as follows: $\angle x$ is the supplement of $\angle y$ so $\angle x = 180° - \angle y$. $\angle x'$ is also the supplement of $\angle y$ so $\angle x' = 180° - \angle y$.
Therefore $\angle x = \angle x'$. Similarly, we can show $\angle y = \angle y'$. Therefore vertical angles are always equal.

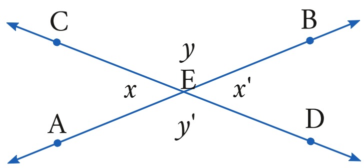
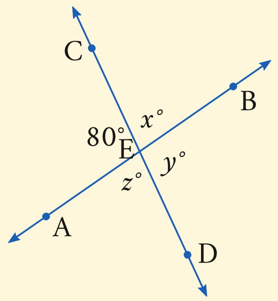


***Figure** 7.* $\angle x$, $\angle x'$ and $\angle y$, and $\angle y'$ are pairs of vertical angles.

We can now use "vertical angles are equal" in solving problems:

? EXAMPLE E (REPEATED)

Find x, y, z:



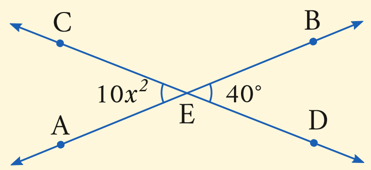
SOLUTION

$\angle x = 180° - 80° = 100°$ because $\angle x$ is the supplement of 80°.
$\angle y = 80°$ because vertical angles are equal.
$\angle z = \angle x = 100°$ because vertical angles are equal.

ANSWER : $x = 100°, y = 80°, z = 100°$

? EXAMPLE F

Find x:



SOLUTION

Since the vertical angles are equal, $10x^2 = 40°$.

Method 1

$$\begin{aligned} 10x^2 &= 40 \\ 10x^2 - 40 &= 0 \\ 10(x^2 - 4) &= 0 \\ 10(x-2)(x+2) &= 0 \end{aligned}$$

$$x + 2 = 0 \qquad x - 2 = 0$$
$$x = -2 \qquad x = 2$$

Method 2

$$\begin{aligned} 10x^2 &= 40 \\ \frac{10x^2}{10} &= \frac{40}{10} \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

If $x = 2$ then $\angle AEC = 10x^2 = 10(2)^2 = 10(4) = 40°$.
If $x = -2$ then $\angle AEC = 10x^2 = 10(-2)^2 = 10(4) = 40°$.

We accept the solution $x = -2$ even though x is negative because the value of the angle $10x^2$ is still positive.

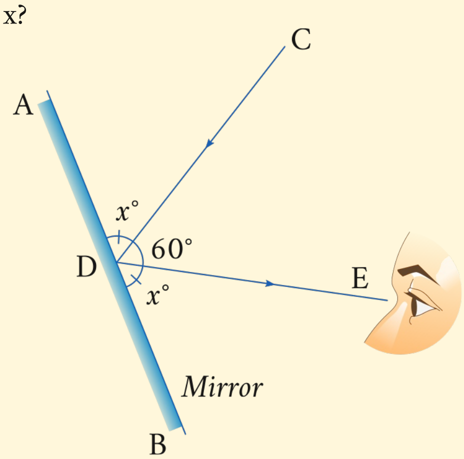
CHECK

$$\begin{array}{rcl} x & = & 2 \\ 10x^2 & = & 40° \\ 10(2)^2 & | & \\ 40 & | & \end{array} \qquad \begin{array}{rcl} x & = & -2 \\ 10x^2 & = & 40° \\ 10(-2)^2 & | & \\ 40 & | & \end{array}$$

ANSWER : $x = 2$ or $x = -2$

? EXAMPLE G

In the diagram, AB represents a mirror, CD represents a ray of light approaching the mirror from C and E represents the eye of a person observing the ray as it is reflected from the mirror at D. According to a law of physics, ∠CDA equals ∠EDB. If ∠CDE = 60°, how much is x?



SOLUTION

Let x° = ∠CDA = ∠EDB

$$\begin{aligned} x + x + 60 &= 180 \\ 2x + 60 &= 180 \\ 2x &= 120 \\ x &= 60° \end{aligned}$$

ANSWER : $x = 60°$

THEOREMS AND POSTULATES

The statement "vertical angles are always equal" is an example of a theorem. A **theorem** is a statement which we can prove to be true. A **proof** is a process of reasoning which uses statements already known to be true to show the truth of a new statement. An example of a proof is the discussion preceding the statement "Vertical angles are always equal." We used facts about supplementary angles that were already known to establish the new statement, that "Vertical angles are always equal."

Ideally we would like to prove all statements in mathematics which we think are true. However before we can begin proving anything we need some true statements with which to start. Such statements should be so self-evident as not to require proofs themselves. A statement of this kind, which we assume to be true without proof, is called a **postulate** or an **axiom**. An example of a postulate is the assumption that all angles can be measured in degrees. This was used without actually being stated in our proof that "Vertical angles are always equal."

Theorems, proofs, and postulates constitute the heart of mathematics and we will encounter many more of them as we continue our study of geometry.

PROBLEMS

1. Find the complement of an angle of

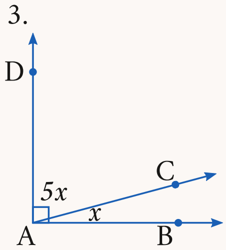
   a. 37° b. 45° c. 53° d. 60°

2. Find the complement of an angle of

   a. 30° b. 40° c. 50° d. 81°

3 - 6. Find x and the complementary angles:

3.



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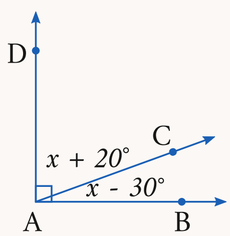

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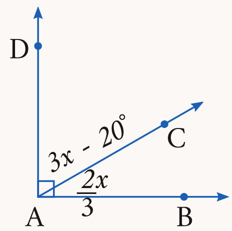

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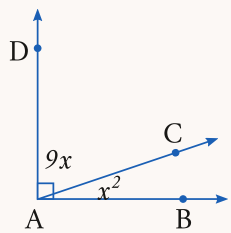


7. Find the supplement of an angle of

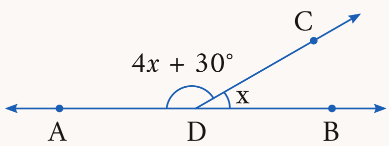
a. 30° b. 37° c. 90° d. 20°

8. Find the supplement of an angle of

a. 45° b. 52° c. 85° d. 135°

9 - 14. Find x and the supplementary angles:

9.



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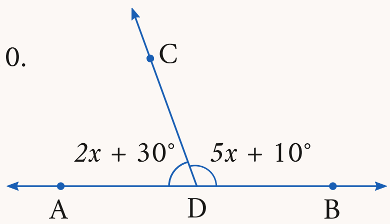

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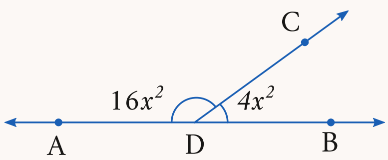

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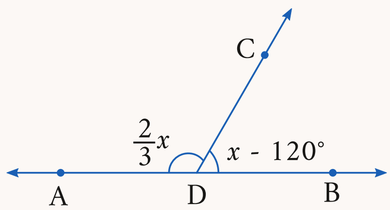

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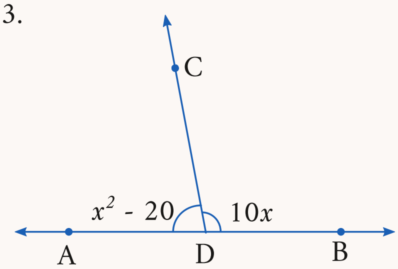

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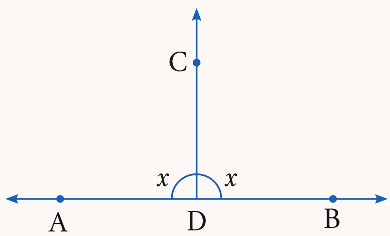
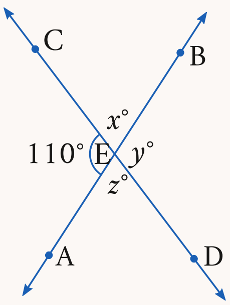


15 - 22. Find x, y and z :

15.



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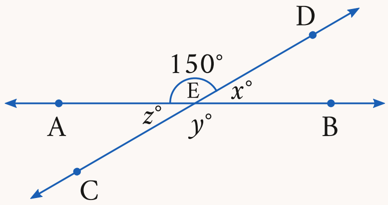

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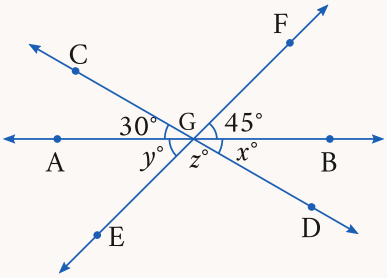

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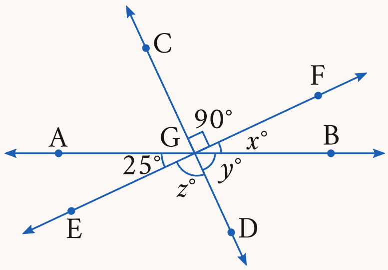

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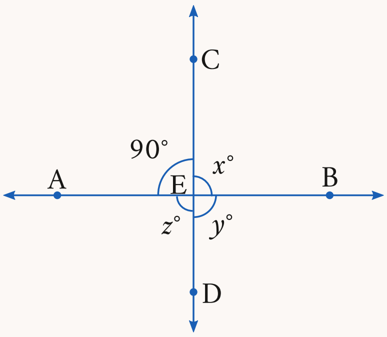

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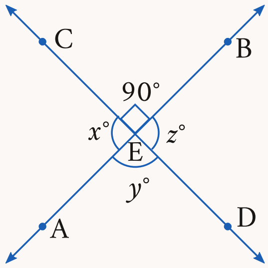

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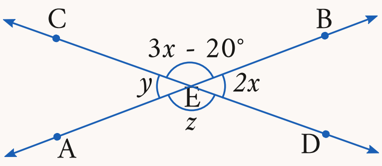

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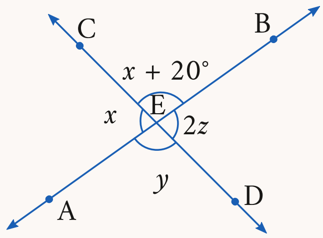
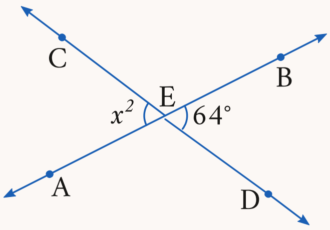


23 - 26. Find x :

23.



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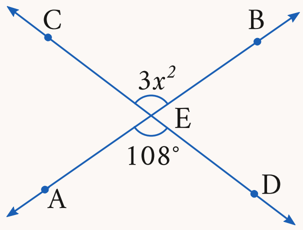

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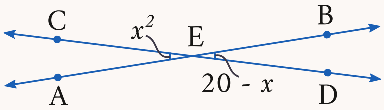

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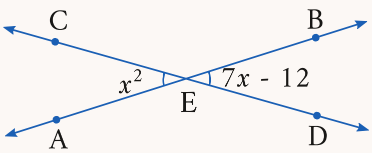
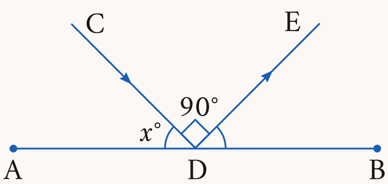
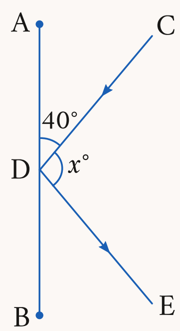


27. Find x if ∠CDA = ∠EDB :



28. Find x if ∠CDA = ∠EDB = 40°:



1.4 PARALLEL LINES

Two lines are **parallel** if they do not meet, no matter how far they are extended. The symbol for parallel is $\parallel$. In Figure 1, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The arrow marks are used to indicate the lines are parallel.

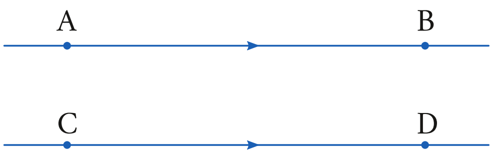


Figure 1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel. They do not meet no matter how far they are extended.

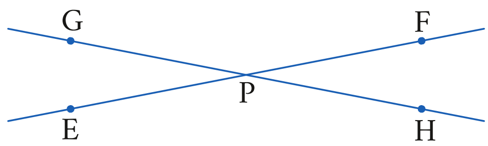


Figure 2. $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are not parallel. They meet at point P.

We make the following assumption about parallel lines, called the **parallel postulate**: Through a point not on a given line one and only one line can be drawn parallel to the given line.

So in Figure 3, there is exactly one line that can be drawn through C that is parallel to $\overleftrightarrow{AB}$.

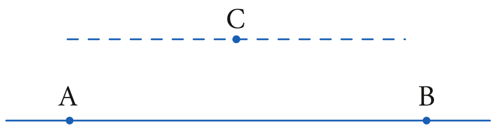


Figure 3. There is exactly one line that can be drawn through C parallel to $\overleftrightarrow{AB}$.

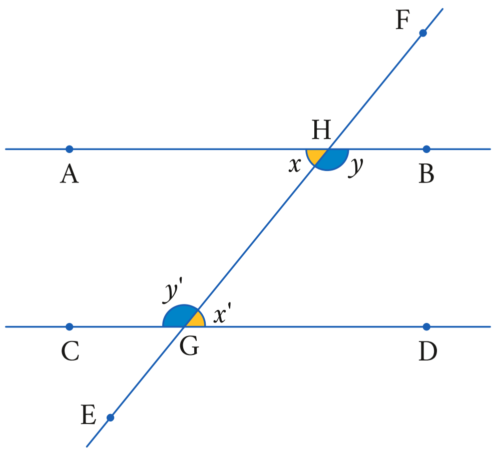


Figure 4. $\overleftrightarrow{EF}$ is a transversal.

A **transversal** is a line that intersects two other lines at two distinct points. In Figure 4, $\overleftrightarrow{EF}$ is a transversal. ∠x and ∠x′ are called **alternate intertor angles** of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. The word "alternate," here, means that the angles are on different sides of the transversal, one angle formed with $\overleftrightarrow{AB}$ and the other formed with $\overleftrightarrow{CD}$. The word "interior" means that they are between the two lines. Notice that they form the letter "Z"(Figure 5). y and y′ are also alternate intertor angles. They also form a "Z" though it is stretched out and backwards. Viewed from the side, the letter "Z" may also look like an "N."

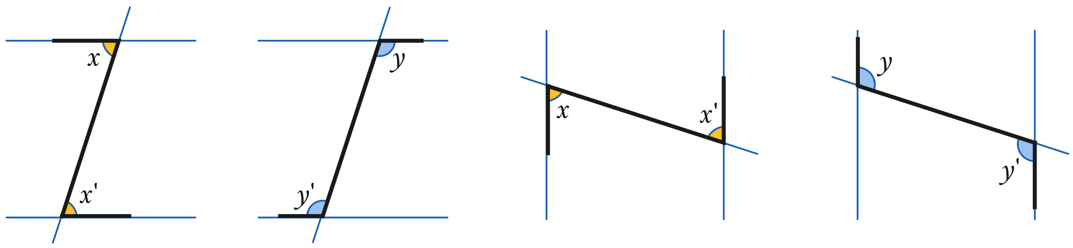


Figure 5. Alternate interior angles form the letter of "Z" or "N". The letters may be stretched out or backwards.

Alternate interior angles are important because of the following theorem:

THEOREM 1 (THE "Z" THEOREM)

If two lines are parallel then their alternate interior angles are equal. If the alternate interior angles of two lines are equal then the lines must be parallel.

In Figure 6, $\overleftrightarrow{AB}$ must be parallel to $\overleftrightarrow{CD}$ because the alternate interior angles are both 30°. Notice that the other pair of alternate interior angles, ∠y and ∠y′, are also equal. They are both 150°. In Figure 7, the lines are not parallel and none of the alternate interior angles are equal.

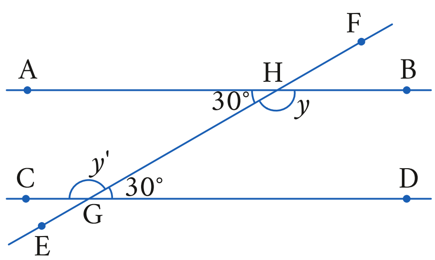


Figure 6. The lines are parallel and their alternate interior angles are equal.

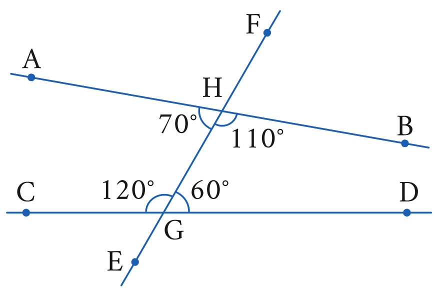
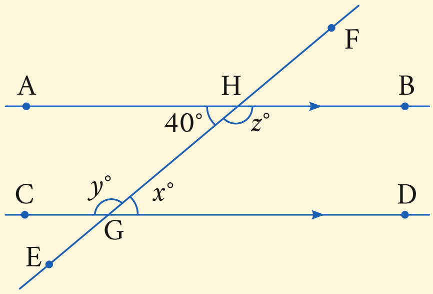


Figure 7. The lines are not parallel and their alternate interior angles are not equal.

The proof of **Theorem 1** is complicated and will be deferred to the appendix.

? EXAMPLE A

Find x, y and z:



SOLUTION

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ since the arrows indicate parallel lines. x = 40° because alternate interior angles of parallel lines are equal. y° = z°= 180° − 40° = 140°.

ANSWER : x = 40°, y = 140°, z = 140°.

Corresponding angles of two lines are two angles which are on the same side of the two lines and the same side of the transversal. In Figure 8, ∠w and ∠w′ are corresponding angles of lines AB and CD. They form the letter "F". ∠x and ∠x′, ∠y and ∠y′ and ∠z and ∠z′ are other pairs of corresponding angles of AB and CD. They all form the letter "F", though it might be a backwards or upside down "F" (Figure 9)

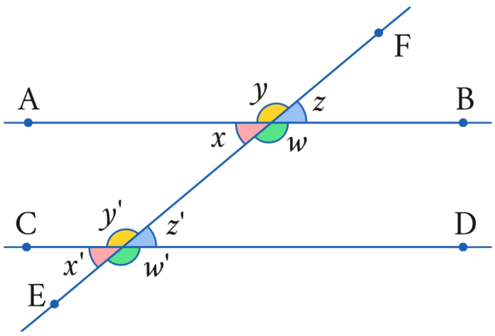


Figure 8. Four pairs of corresponding angles are illustrated.

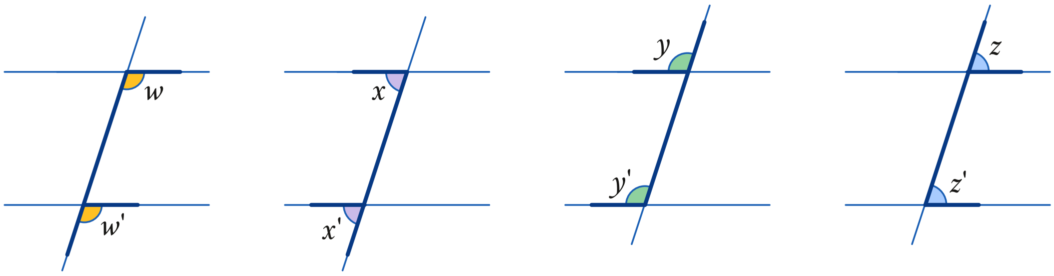


Figure 9. Corresponding angles form the letter "F", though it may be a backwards or upside down"F".

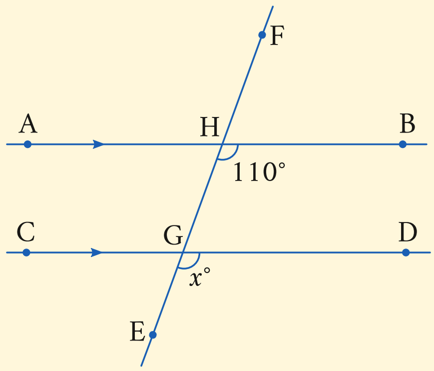
Corresponding angles are important because of the following theorem:

THEOREM 2 (THE "F" THEOREM)

If two lines are parallel then their corresponding angles are equal. If the corresponding angles of two lines are equal then the lines must be parallel.

EXAMPLE B

Find x:



SOLUTION

The arrow indicate $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Therefore $x = 110°$ because x° and 110° are the measures of corresponding angles of the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

ANSWER : x = 110.

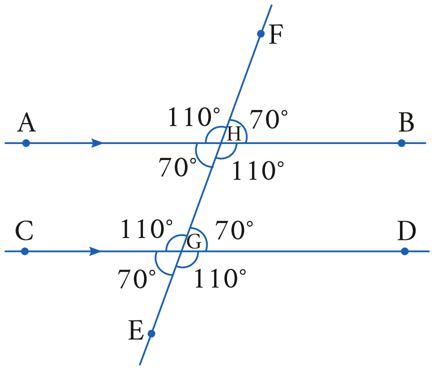


Figure 10. Each pair of corresponding angles is equal.

Notice that we can now find all the other angles in Example B. Each one is either supplementary to one of the $110°$ angles or forms equal vertical angles with one of them (Figure 10). Therefore all the corresponding angles are equal. Also each pair of alternate interior angles is equal. It is not hard to see that if just one pair of corresponding angles or one pair of alternate interior angles are equal then so are all other pairs of corresponding and alternate interior angles.

Proof of Theorem 2: The corresponding angles will be equal if the alternate interior angles are equal and vice versa. Therefore **Theorem 2** follows directly from **Therorem 1**.

In Figure 11, $\angle x$ and $\angle x'$ are called **interior angles on the same side of the transversal.*** $\angle y$ and $\angle y'$ are also interior angles on the same side of the transversal. Notice that each pair of angles forms the letter "C". Compare Figure 11 with Figure 10 and also with Example A. The following theorem is then apparent:

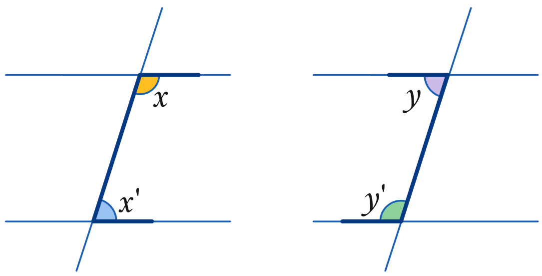


Figure 11. Interior angles on the same side of the transversal form the letter "C". It may also be a backwards "C".

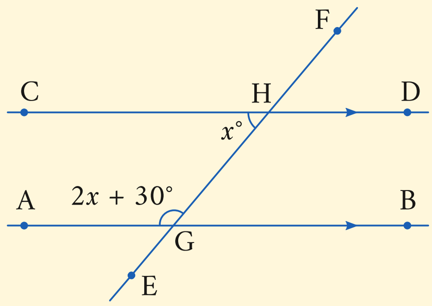
*In some textbooks, interior angles on the same side of the transversal are called **cointerior angles**.

THEOREM 3 (THE "C" THEOREM)

If two lines are parallel then the interior angles on the same side of the transversal are supplementary (they add up to 180°). If the interior angles of two lines on the same side of the transversal are supplementary then the lines must be parallel.

? EXAMPLE C

Find x and the marked angles:



SOLUTION

The lines are parallel so by **Theorem 3** the two labelled angles must be supplementary.

$$\begin{aligned} x + 2x + 30 &= 180 \\ 3x + 30 &= 180 \\ 3x &= 180 - 30 \\ 3x &= 150 \\ x &= 50 \end{aligned}$$

$\angle CHG = x = 50°$.
$\angle AGH = 2x + 30 = 2(50) + 30 = 100 + 30 = 130°$

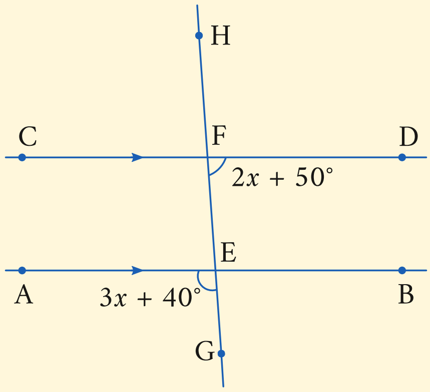
CHECK

$$\begin{array}{r|l} x + 2x + 30 & = 180° \\ 50 + 2(50) + 30 & \\ 50 + 130 & \\ 180 & \end{array}$$

ANSWER: x = 50, $\angle CHG = 50°$, $\angle AGH = 130°$

? EXAMPLE D

Find x and the marked angles:



SOLUTION

$\angle BEF = 3x + 40°$ because vertical angles are equal, $\angle BEF$ and $\angle DFE$ are interior angles on the same side of the transversal, and therefore are supplementary because the lines are parallel.

$$\begin{aligned} 3x + 40 + 2x + 50 &= 180 \\ 5x + 90 &= 180 \\ 5x &= 180 - 90 \\ 5x &= 90 \\ x &= 18 \end{aligned}$$

$\angle AEG = 3x + 40 = 3(18) + 40 = 54 + 40 = 94°.$
$\angle DFE = 2x + 50 = 2(18) + 50 = 36 + 50 = 86°.$

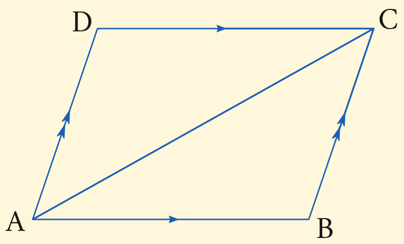
CHECK

$$\begin{array}{r|} 3x + 40 + 2x + 50 = 180 \\ 3(18) + 40 + 2(18) + 50 \\ 54 + 40 + 36 + 50 \\ 94 + 86 \\ 180 \end{array}$$

ANSWER : $x = 18°$, $\angle AEG = 94°$, $\angle DFE = 86°$.

? EXAMPLE E

List all pairs of alternate interior angles in the diagram. (The single arrow indicates $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and the double arrow indicates $\overleftrightarrow{AD}$ is parallel to $\overleftrightarrow{BC}$.)



SOLUTION

We see if a letter Z or N can be formed using the line segments in the diagram (Figure 12).

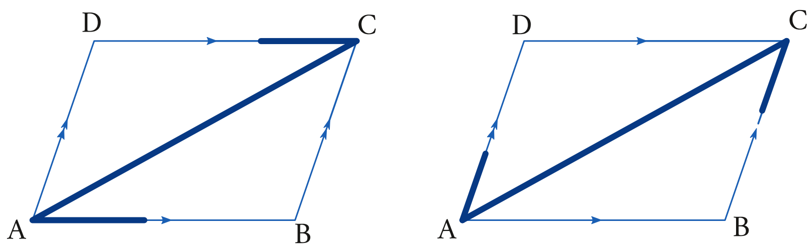
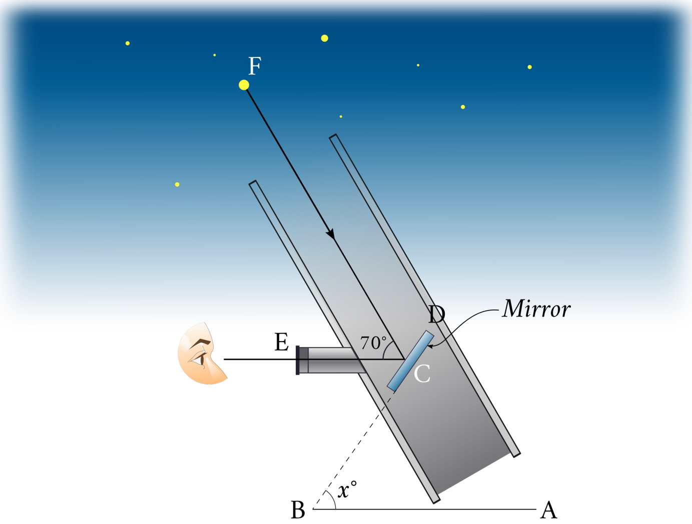


Figure 12. Forming the letter Z or N from the line segments in the diagram.

ANSWER : $\angle$DCA and $\angle$CAB are alternate interior angles of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. $\angle$DAC and $\angle$ACB are alternate interior angles of lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$.

? EXAMPLE F

A telescope is pointed at a star 70° above the horizon. What angle x° must the mirror BD make with the horizontal so that the star can be seen in the eyepiece E?
(See the next page for the figure.)



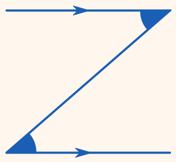
SOLUTION

$x° = \angle BCE$ because they are alternate interior angles of parallel lines AB and CE.
$\angle DCF = \angle BCE = x°$ because of the laws of physics.
Therefore

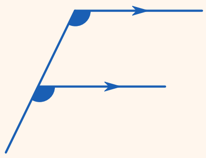
$$\begin{aligned} x + 70 + x &= 180 \\ 2x + 70 &= 180 \\ 2x &= 180 - 70 \\ 2x &= 110 \\ x &= 55 \end{aligned}$$

ANSWER : 55°

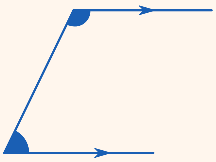
SUMMARY



Alternate interior angles of parallel lines are equal. They form letter "Z".



Corresponding angles of parallel lines are equal. They form letter "F".



Interior angles on the same side of the transversal of parallel lines are supplementary. They form the letter "C."

HISTORICAL NOTE

The parallel postulate given earlier in this section is the equivalent of the fifth postulate of Euclid's *Elements*. Euclid was correct in assuming it as a postulate rather than trying to prove it as a theorem. However this did not become clear to the mathematical world until the nineteenth century, 2200 years later. In the interim, scores of prominent mathematicians attempted unsuccessfully to give a satisfactory proof of the parallel postulate. They felt that it was not as self-evident as a postulate should be, and that it required some formal justification.

In 1826, N. I. Lobachevsky, a Russian mathematician, presented a system of geometry based on the assumption that through a given point more than one straight line can be drawn parallel to a given line (see Figure 13). In 1854, the German mathematician Georg Bernhard Riemann proposed a system of geometry in which there are no parallel lines at all. A geometry in which the parallel postulate has been replaced by some other postulate is called a **non-Euclidean** geometry. The existence of these geometries shows that the parallel postulate need not necessarily be true. Indeed Einstein used the geometry of Riemann as the basis for his theory of relativity.

Of course our original parallel postulate makes the most sense for ordinary applications, and we use it throughout this book. However, for applications where great distances are involved, such as in astronomy, it may well be that a non-Euclidean geometry gives a better approximation of physical reality.

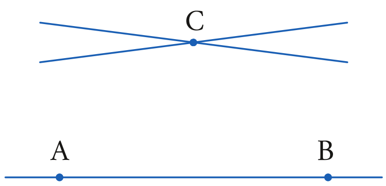


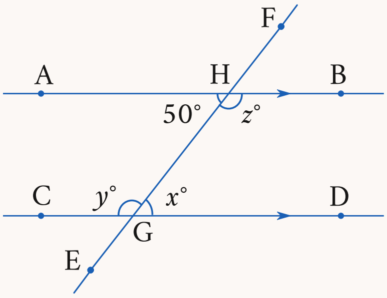
Figure 13. In the geometry of Lobachevsky, more than one line can be drawn through C parallel to AB.

PROBLEMS

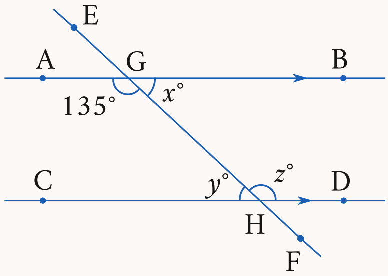
For each of the following, state the theorem(s) used in obtaining your answer (for example, "alternate interior angles of parallel lines are equal"). Lines marked with the same arrow are assumed to be parallel.

1-2. Find x, y, and z :

1.

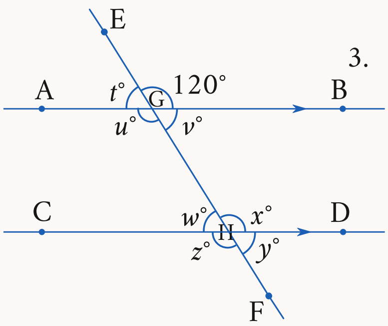


2.

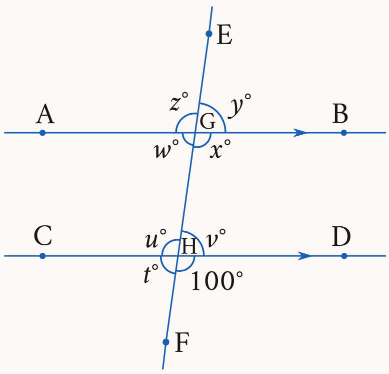


3-4. Find t, u, v, w, x, y and z:

4.

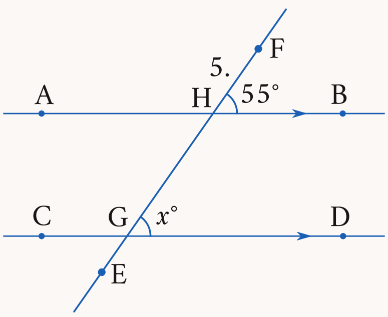


3.

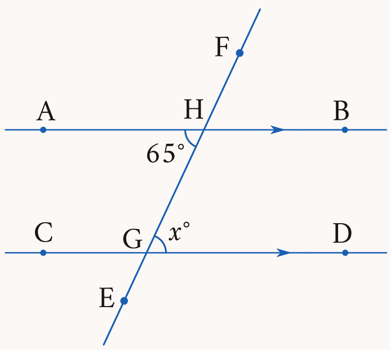


5-10. Find x:

6.



5.



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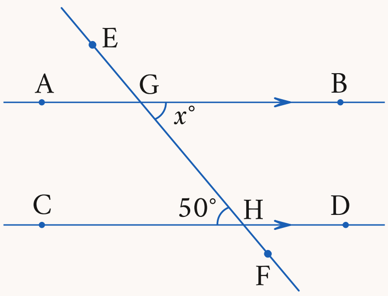

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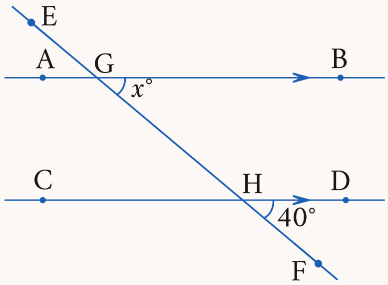

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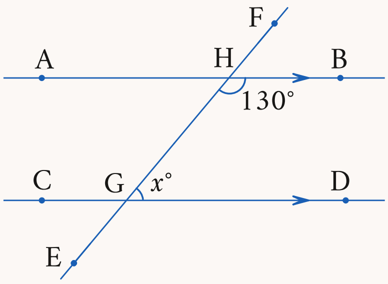

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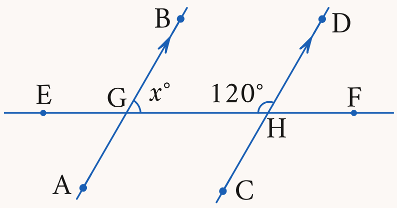
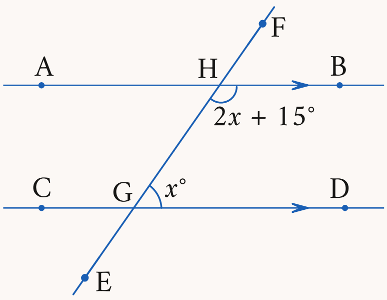
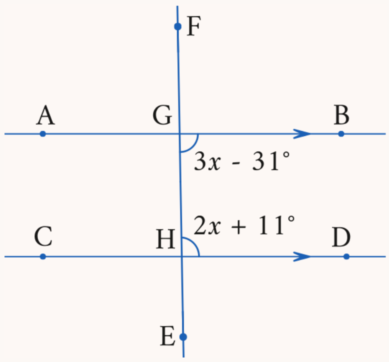


11 - 18. Find x and the marked angles:

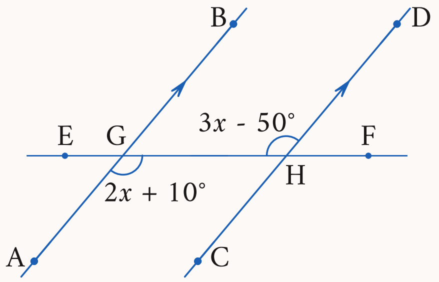
11.



12.



13.



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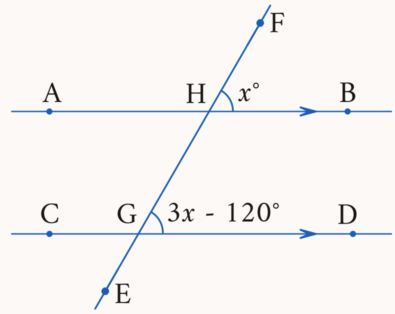

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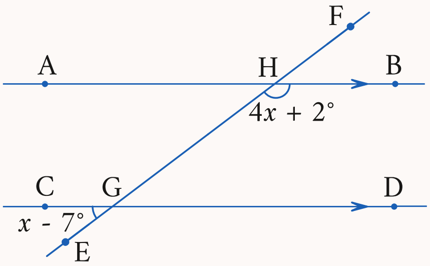

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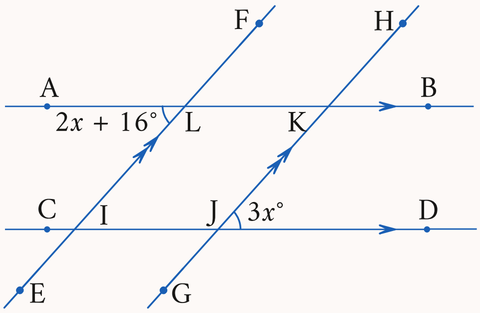

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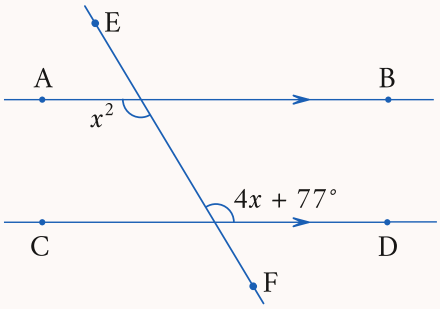

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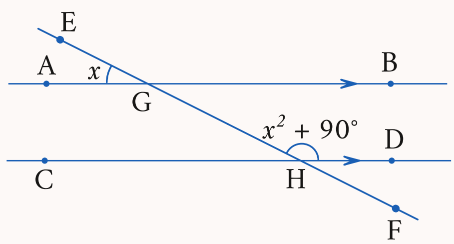
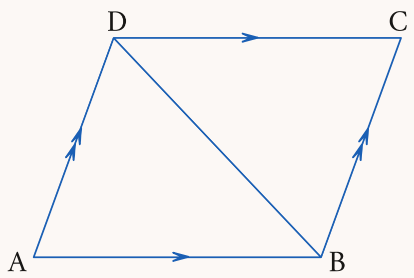


19 - 26. For each of the following, list all pairs of alternate interior angles and corresponding angles. If there are none, then list all pairs of interior angles on the same side of the transversal. Indicate the parallel lines which form each pair of angles.

19.



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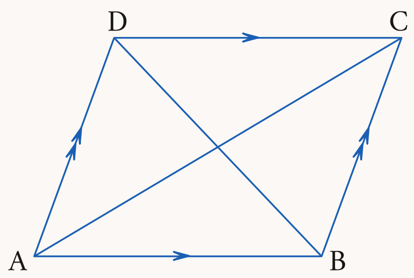

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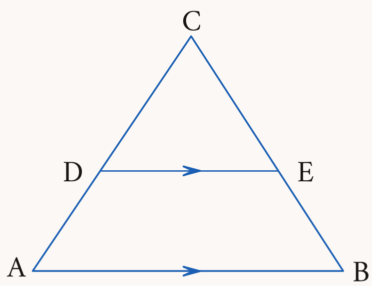

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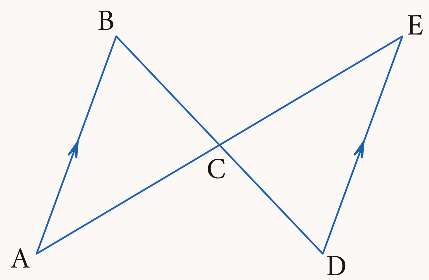

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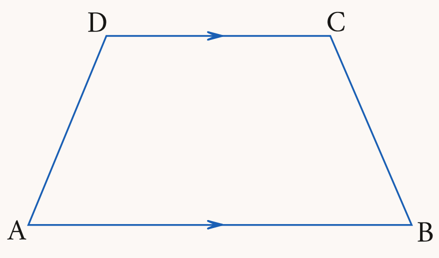

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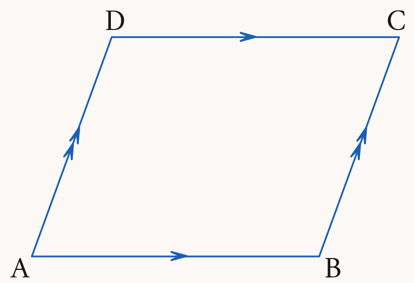

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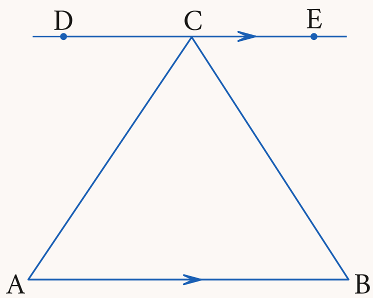

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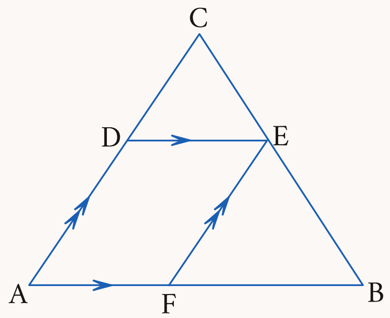
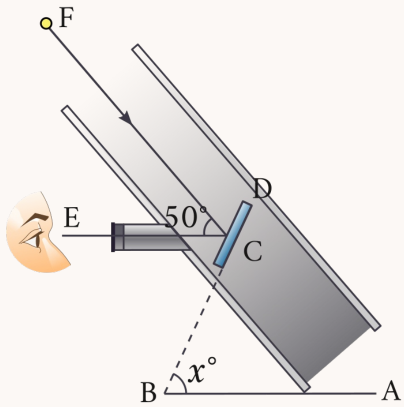


27. A telescope is pointed at a star 50° above the horizon. What angle x° must the mirror BD make with the horizontal so that the star can be seen in the eyepiece E?



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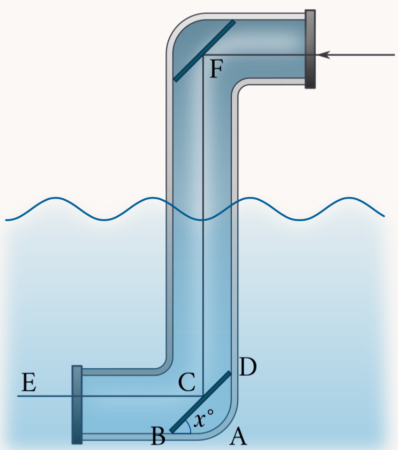


A periscope is used by sailors in a submarine to see objects on the surface of the water, If ∠ECF = 90°, what angle x° does the mirror BD make with the horizontal?

1.5 TRIANGLES

A **triangle** is formed when three straight line segments bound a portion of the plane, The line segments are called the **sides of the triangle**. A point where two sides meet is called a **vertex of the triangle**, and the angle formed is called an **angle of the triangle**. The symbol for triangle is $\triangle$.

The triangle in Figure 1 is denoted by $\triangle ABC$ (or $\triangle BCA$ or $\triangle CAB$, etc.). Its sides are AB, AC, and BC. Its vertices are A, B, and C. Its angles are $\angle A$, $\angle B$, and $\angle C$.

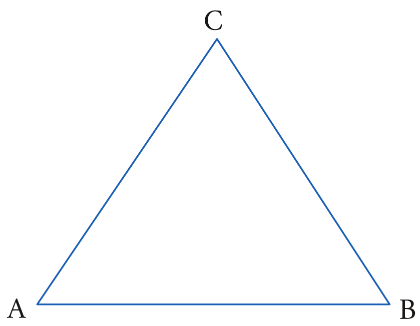


Figure 1. Triangle ABC

The triangle is the most important figure in plane geometry. This is because figures with more than 3 sides can always be divided into triangles (Figure 2). If we know the properties of a triangle, we can extend this knowledge to the study of other figures as well.

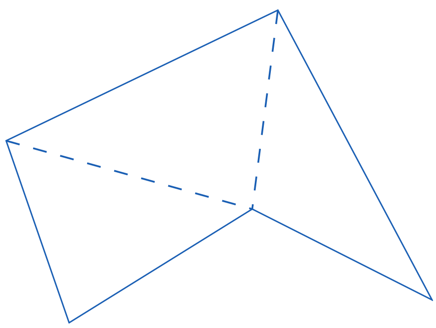


Figure 2. A closed figure formed by more than 3 straight lines can be divided into triangles.

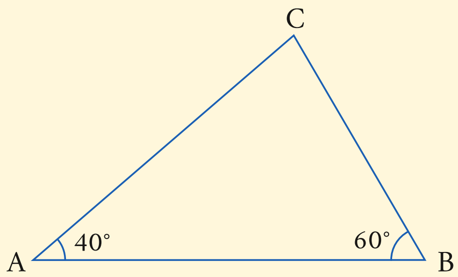
A fundamental property of triangles is the following:

THEOREM 1

The sum of the angles of a triangle is 180°. In $\triangle ABC$ of Figure 1;
$\angle \mathbf{A} + \angle \mathbf{B} + \angle \mathbf{C} = \mathbf{180°}$.

? EXAMPLE A

Find $\angle C$:



SOLUTION

$$\begin{aligned} \angle A + \angle B + \angle C &= 180^\circ \\ 40^\circ + 60^\circ + \angle C &= 180^\circ \\ 100^\circ + \angle C &= 180^\circ \\ \angle C &= 180^\circ - 100^\circ \\ \angle C &= 80^\circ \end{aligned}$$

ANSWER : $\angle C = 80^\circ$

Proof of Theorem 1 :

Through C draw DE parallel to AB (see Figure 3). Note that we are using the parallel postulate here. $\angle 1 = \angle A$ and $\angle 3 = \angle B$ because they are alternate interior angles of parallel lines.
$\angle A + \angle B + \angle C = \angle 1 + \angle 3 + \angle 2 = 180^\circ$.

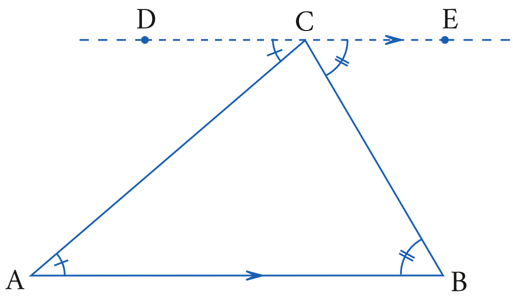
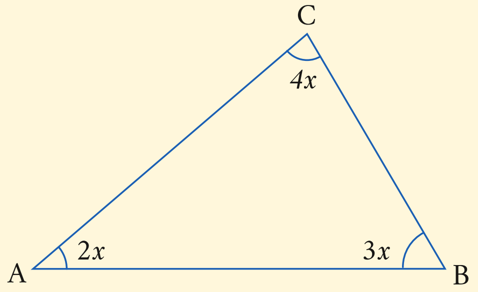


Figure 3. Through C draw DE parallel to AB.

We may verify **Theorem 1** by measuring the angles of a triangle with a protractor and taking the sum, However no measuring instrument is perfectly accurate, It is reasonable to expect an answer such as 179°, 182°, 180.5°, etc. The purpose of our mathematical proof is to assure us that the sum of the angles of every triangle must be exactly 180°.

? EXAMPLE B

Find x:



SOLUTION

$$\angle A + \angle B + \angle C = 180^\circ$$
$$2x + 3x + 4x = 180^\circ$$
$$9x = 180^\circ$$
$$x = 20^\circ$$

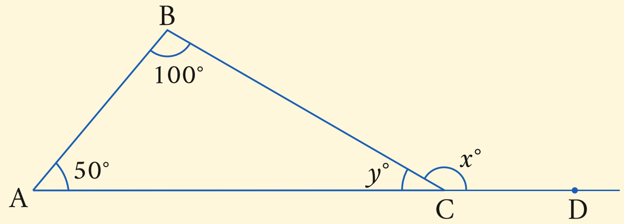
CHECK

$$\angle A + \angle B + \angle C = 180^\circ$$
$$2x + 3x + 4x$$
$$2(20^\circ) + 3(20^\circ) + 4(20^\circ)$$
$$40^\circ + 60^\circ + 80^\circ$$
$$180^\circ$$

ANSWER : 20°

? EXAMPLE C

Find y and x:



$$50 + 100 + y = 180$$
$$150 + y = 180$$
$$y = 180 - 150$$
$$y = 30$$

$$x = 180 - 30 = 150$$

ANSWER : y = 30, x = 150

In Figure 4, $\angle x$ is called an **exterior angle** of $\triangle ABC$. $\angle A$, $\angle B$ and $\angle y$ are called the **interior angles** of $\triangle ABC$. $\angle A$ and $\angle B$ are said to be the interior angles **remote** from the exterior angle $\angle x$. The results of Example C suggest the following theorem:

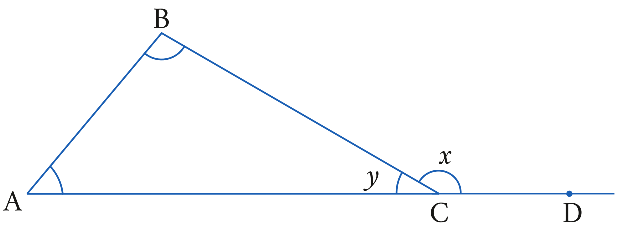


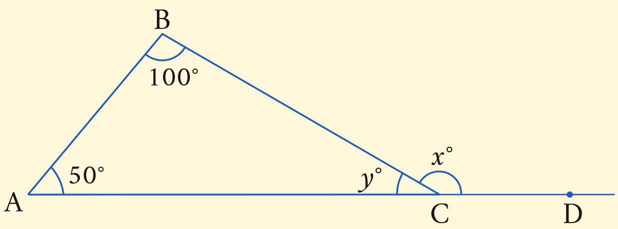
Figure 4. $\angle x$ is an exterior angle of $\triangle ABC$

THEOREM 2

An exterior angle is equal to the sum of the two remote interior angles.
In Figure 4, $\angle x = \angle A + \angle B$.

? EXAMPLE D

Find x:



SOLUTION

Using **Theorem 2**, $x° = 100° + 50° = 150°$.

ANSWER : $x = 150$

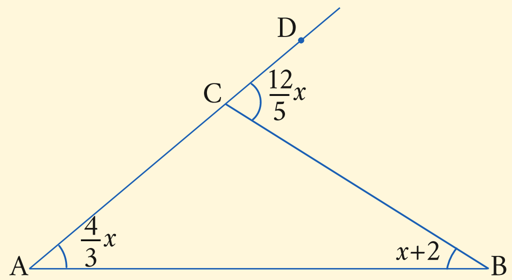
Proof of Theorem 2 :

We present this proof in double-column form, with statements in the left column and the reason for each statement in the right column. The last statement is the theorem we wish to prove.

	Statements	Reasons
1	$\angle A + \angle B + \angle y = 180°$	The sum of the angles of a triangle is 180°.
2	$\angle A + \angle B = 180° - \angle y$	Subtract $\angle y$ from both sides of the equation. (Statement 1)
3	$\angle x = 180° - \angle y$	$\angle x$ and $\angle y$ are supplementary.
4	$\angle x = \angle A + \angle B$	Both $\angle x$ (Statement 3) and $\angle A + \angle B$ (Statement 2) equal $180° - \angle y$

? EXAMPLE E

Find x:



SOLUTION

$\angle BCD$ is an exterior angle with remote interior angles $\angle A$ and $\angle B$. By **Theorem 2,**
$\angle BCD = \angle A + \angle B$

$$\frac{12}{5}x = \frac{4}{3}x + x + 2$$

The lowest common denominator (l. c. d.) is 15.

$$\overset{3}{(\cancel{15})}\frac{12}{\cancel{5}}x = \overset{5}{(\cancel{15})}\frac{4}{\cancel{3}}x + (15)x + (15)(2)$$

$$\begin{aligned} 36x &= 20x + 15x + 30 \\ 36x &= 35x + 30 \\ 36x - 35x &= 30 \\ x &= 30 \end{aligned}$$

ANSWER : $x = 30$

CHECK

$$\begin{array}{r|l} \angle BCD & = \angle A + \angle B \\ \frac{12}{5}x & \frac{4}{3}x + x + 2 \\ \frac{12}{5}(30) & \frac{4}{3}(30) + 30 + 2 \\ 72° & 40° + 32° \\ & 72° \end{array}$$

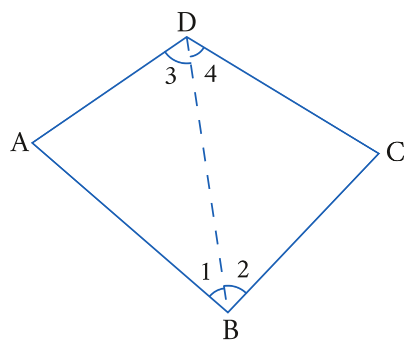
Our work on the sum of the angles of a triangle can easily be extended to other figures:

? EXAMPLE F

Find the sum of the angles of a quadrilateral (four-sided figure).

SOLUTION

Divide the quadrilateral into two triangles as illustrated:



$$
\begin{aligned}
\angle A + \angle B + \angle C + \angle D &= \angle A + \angle 1 + \angle 3 + \angle 2 + \angle 4 + \angle C \\
&= 180^\circ + 180^\circ \\
&= 360^\circ
\end{aligned}
$$

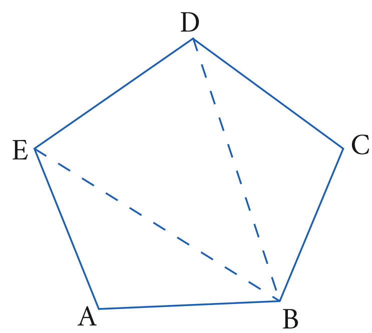
ANSWER : 360°

? EXAMPLE G

Find the sum of the angles of a pentagon (five-sided figure).

SOLUTION

Divide the pentagon into three triangles as illustrated. The sum is equal to the sum of the angles of the three triangles = $(3)(180^\circ) = 540^\circ$.



ANSWER : $x = 540^\circ$

There is one more simple principle which we will derive from **Theorem 1**. Consider the two triangles in Figure 5.

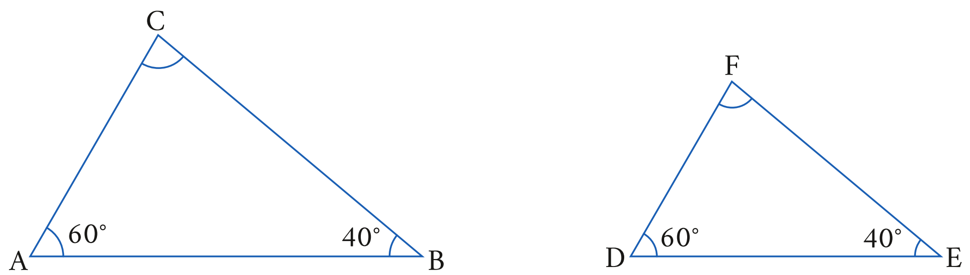


Figure 5. Each triangle has an angle of 60° and 40°.

We are given that $\angle A = \angle D = 60°$ and $\angle B = \angle E = 40°$. A short calculation shows that we must also have $\angle C = \angle F = 80°$. This suggests the following theorem:

THEOREM 3

If two angles of one triangle are equal respectively to two angles of another triangle, then their remaining angles are also equal.

In Figure 6, if $\angle A = \angle D$ and $\angle B = \angle E$ then $\angle C = \angle F$.

Proof of Theorem 3 :

$\angle C = 180° - (\angle A + \angle B) = 180° - (\angle D + \angle E) = \angle F$

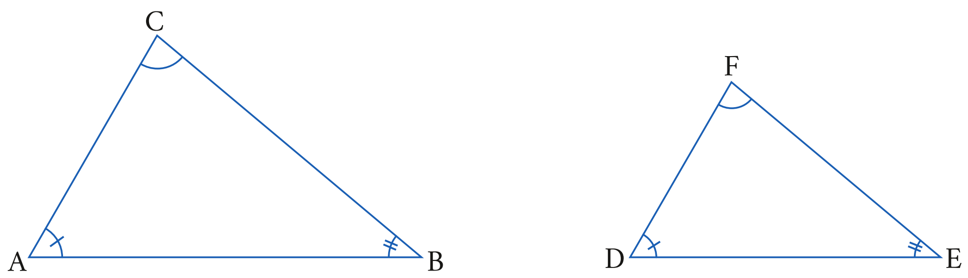


Figure 6. $\angle A = \angle D$ and $\angle B = \angle E$

HISTORICAL NOTE

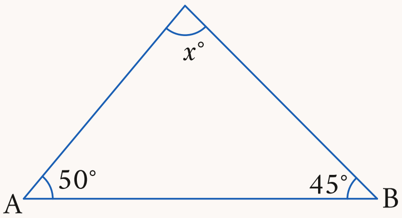
Our Theorem 1, which states that the sum of the angles of a triangle is $180°$, is one of the most important consequences of the parallel postulate, Therefore, one way of testing the truth of the parallel postulate (see the **Historical Note** in section 1.4) is to test the truth of Theorem 1. This was actually tried by the German mathematician, astronomer and physicist, Karl Friedrich Gauss (1777 - 1855). (This is the same Gauss whose name is used as a unit of measurement in the theory of magnetism). Gauss measured the sum of the angles of the triangle formed by three mountain peaks in Germany. He found the sum of the angles to be 14.85 seconds more than $180°$ (60 seconds 1 minute, 60 minutes = 1 degree). However this small excess could have been due to experimental error, so the sum might actually have been $180°$.

Aside from experimental error, there is another difficulty involved in verifying the angle sum theorem. According to the non-Euclidean geometry of Lobachevsky, the sum of the angles of a triangle is always **less** than $180°$. In the non-Euclidean geometry of Riemann, the sum of the angles is always **more** than $180°$. However in both cases the difference from $180°$ is insignificant unless the triangle is very large. Neither theory tells us exactly how large such a triangle should be. Even if we measured the angles of a very large triangle, like one formed by three stars, and found the sum to be indistinguishable from $180°$, we could only say that the angle sum theorem and parallel postulate are apparently true for these large distances. These distances still might be too small to enable us to determine which geometric system best describes the universe as a whole.

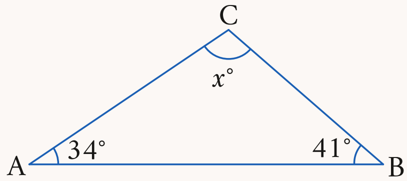
PROBLEMS

1 - 12. Find x and all the missing angles of each triangle:

1.



2.



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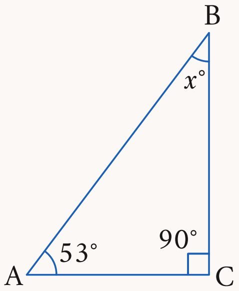

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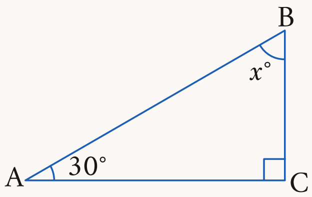

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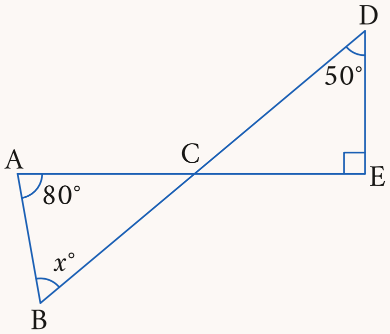

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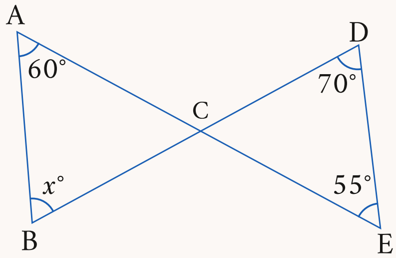

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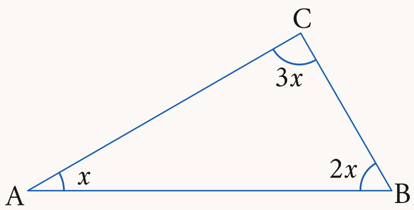

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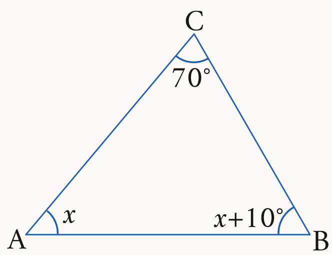

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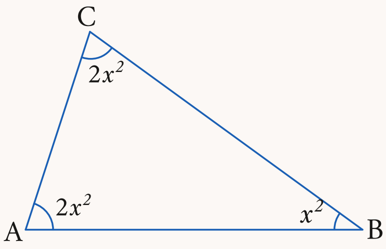

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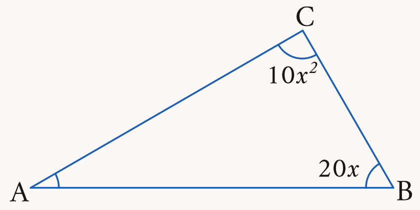

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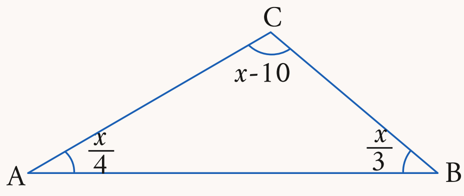

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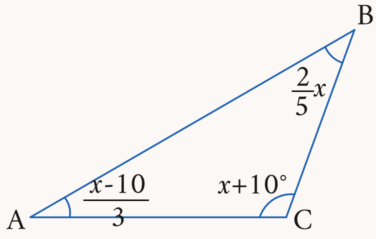
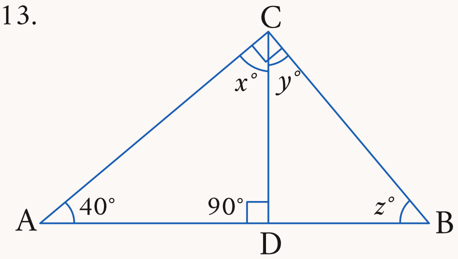


13 - 14. Find x, y and z:

13.



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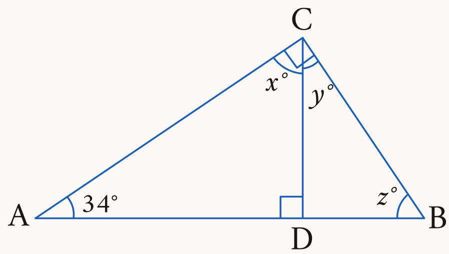
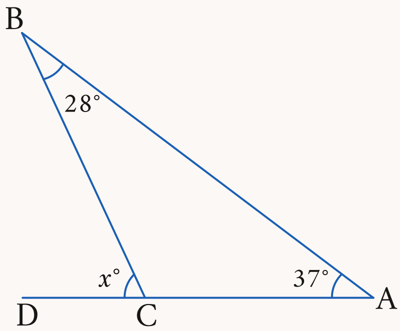


15 - 20. Find x:

15.



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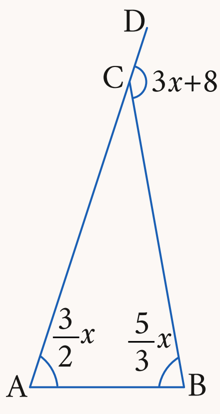

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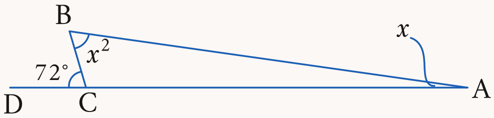

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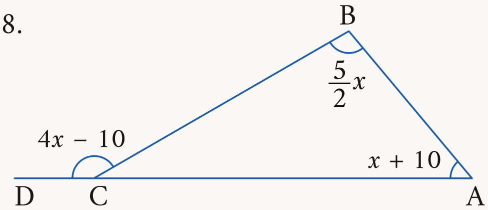

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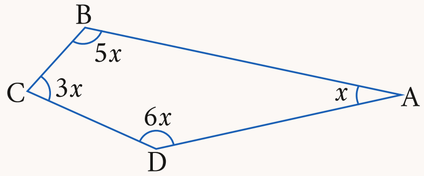

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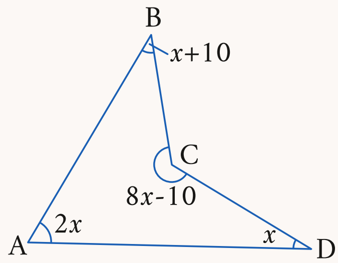
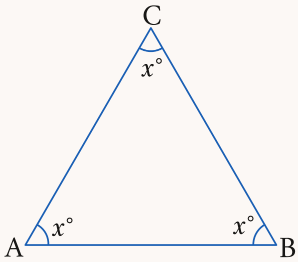


21. Find the sum of the angles of a hexagon (6-sided figure).
22. Find the sum of the angles of an octagon (8-sided figure).
23 - 26. Find x:

23.



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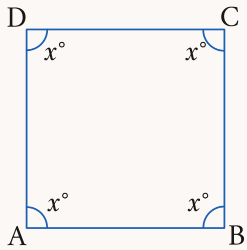
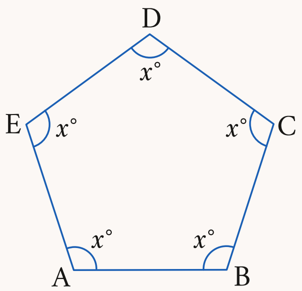


25.



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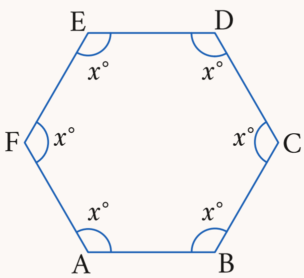


1.6 TRIANGLE CLASSIFICATIONS

Triangles may be classified according to the relative lengths of their sides:
An **equilateral** triangle has three equal sides.
An **isosceles** triangle has two equal sides.
A **scalene** triangle has no equal sides.

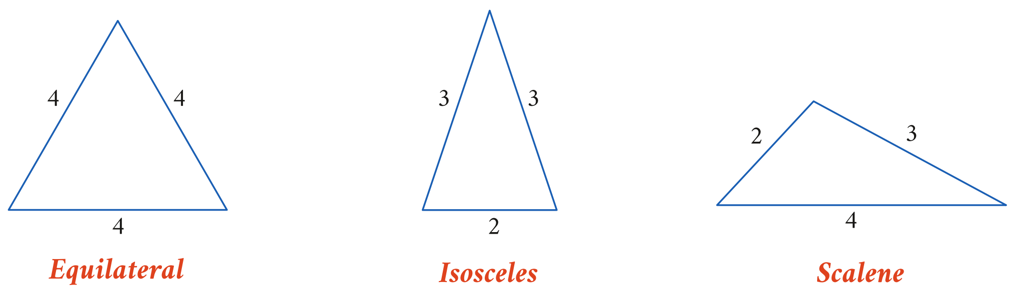


Figure 1. Triangles classified according to their sides.

Triangles may also be classified according to the measure of their angles:
An **acute** triangle is a triangle with three acute angles.
An **obtuse** triangle is a triangle with one obtuse angle.
An **equiangular** triangle is a triangle with three equal angles.
Each angle of an equiangular trialngle must be 60°. We will show in section 2.5 that equiangular triangles are the same as equilateral triangles.

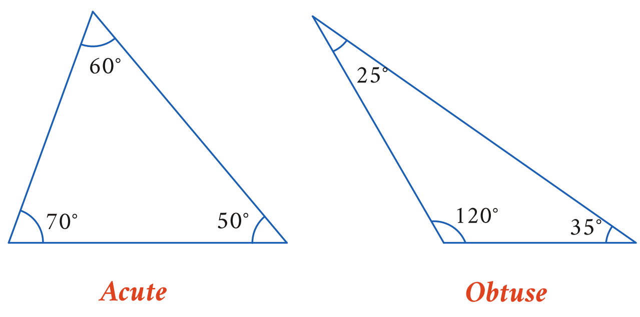


Figure 2. Triangles classified according to their angles.

A **right** triangle is a triangle with one right angle. The sides of the right angle are called the **legs** of the triangle and the remaining side is called the **hypotenuse**.

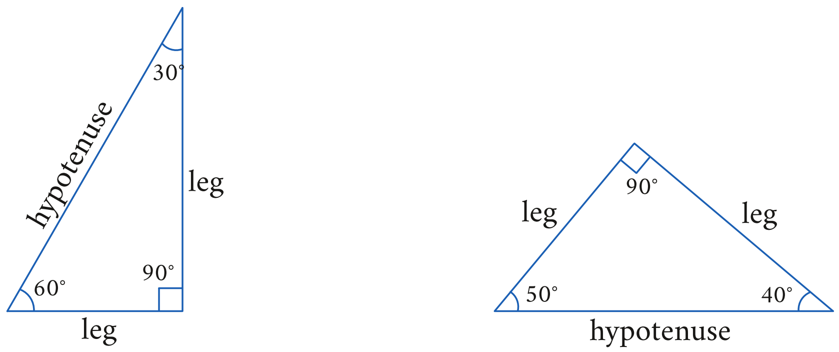
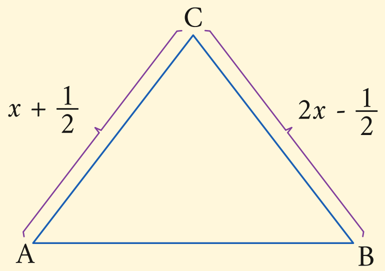


Figure 3. Right triangles.

? EXAMPLE A

Find x if △ABC is isosceles with AC = BC:



SOLUTION

$$
\begin{aligned}
x + \frac{1}{2} &= 2x - \frac{1}{2} \\
(2)\left(x + \frac{1}{2}\right) &= (2)\left(2x - \frac{1}{2}\right) \\
2x + 2\left(\frac{1}{2}\right) &= (2)(2x) - (2)\left(\frac{1}{2}\right) \\
2x + 1 &= 4x - 1 \\
1 + 1 &= 4x - 2x \\
2 &= 2x \\
1 &= x
\end{aligned}
$$

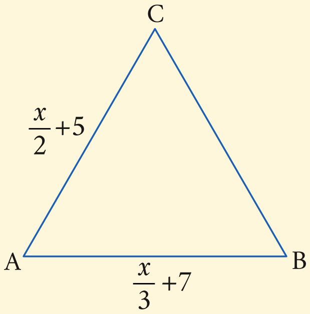
CHECK

$$
\begin{array}{r|l}
x + \frac{1}{2} \;=& 2x - \frac{1}{2} \\
1 + \frac{1}{2} & 2(1) - \frac{1}{2} \\
1\frac{1}{2} & 2 - \frac{1}{2} \\
 & 1\frac{1}{2}
\end{array}
$$

ANSWER : x = 1

? EXAMPLE B

$\triangle ABC$ is equilateral. Find x:



SOLUTION

$$\begin{aligned}
\frac{x}{2} + 5 &= \frac{x}{3} + 7 \\
(6)\left(\frac{x}{2} + 5\right) &= (6)\left(\frac{x}{3} + 7\right) \\
(6)\left(\frac{x}{2}\right) + (6)(5) &= (6)\left(\frac{x}{3}\right) + (6)(7) \\
3x + 30 &= 2x + 42 \\
3x - 2x &= 42 - 30 \\
x &= 12
\end{aligned}$$

CHECK

$$\frac{x}{2} + 5 = \frac{x}{3} + 7$$

$$\begin{array}{r|l}
\frac{12}{2} + 5 & \frac{12}{3} + 7 \\
6 + 5 & 4 + 7 \\
11 & 11
\end{array}$$

ANSWER : $x = 12$

An **altitude** of a triangle is a line segment from a vertex perpendicular to the opposite side. In Figure 4, CD and GH are altitudes. Note that altitude GH lies outside $\triangle EFG$ and side EF must be extended to meet it.

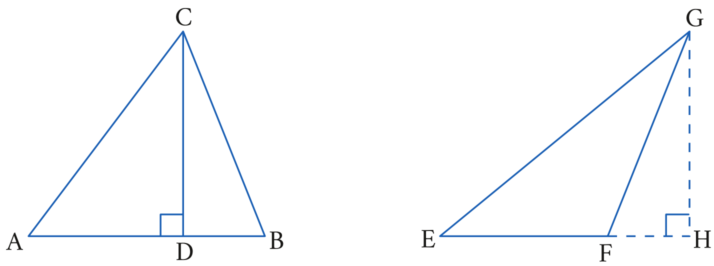


Figure 4. CD and GH are altitudes.

A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. In Figure 5, CD is a median.

An **angle bisector** is a ray which divides an angle into two eaual angles. In Figure 6, $\overrightarrow{CD}$ is an angle bisector.

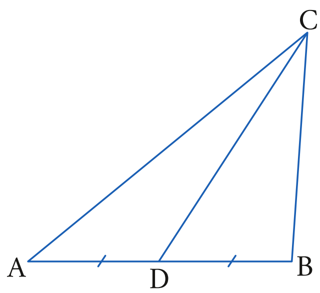


Figure 5. CD is a median.

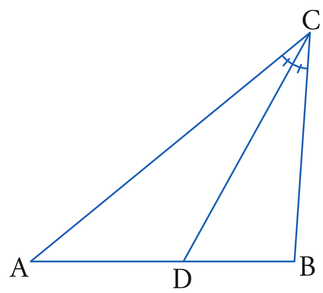
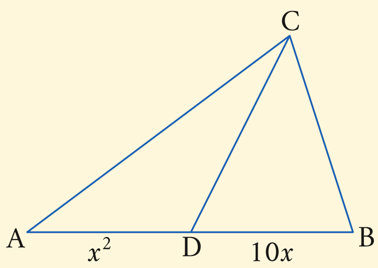


Figure 6. $\overrightarrow{CD}$ is an bisector of $\angle ACB$.

? EXAMPLE C

Find AB if CD is a median:



SOLUTION

$$AD = DB$$
$$x^2 = 10x$$
$$x^2 - 10x = 0$$
$$x(10 - x) = 0$$
$$x = 0 \text{ or } x - 10 = 0$$
$$x = 10$$

CHECK

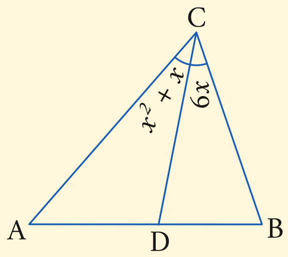
x = 0		**x = 10**	
AD	= DB	AD	= DB
x^2	$10x$	x^2	$10x$
0^2	$10(0)$	10^2	$10(10)$
0	0	100	100

We reject the answer $x = 0$ because the length of a line segment must be greater than 0. Therefore $AB = AD + DB = 100 + 100 = 200$.

ANSWER : $AB = 200$.

EXAMPLE D

Find $\angle ACB$ if $\overrightarrow{CD}$ is an angle bisector:



SOLUTION

$$
\begin{aligned}
\angle ACD &= \angle BCD \\
x^2 + x &= 6x \\
x^2 + x - 6x &= 0 \\
x^2 - 5x &= 0 \\
x(x-5) &= 0 \\
x = 0 \quad \text{or} \quad x - 5 &= 0 \\
x &= 5
\end{aligned}
$$

CHECK

x = 0

$\angle ACD = \angle BCD$

$x^2 + x$	$6x$
$0^2 + 0$	$6(0)$
0	0

x = 5

$\angle ACD = \angle BCD$

$x^2 + x$	$6x$
$5^2 + 5$	$6(5)$
30	30

We reject the answer $x = 0$ because the measures of $\angle ACD$ and $\angle BCD$ must be greater than 0.
Therefore $\angle ACB = \angle ACD + \angle BCD = 30° + 30° = 60°$.

ANSWER : $\angle ACB = 60°$.

The **perimeter** of a triangle is the sum of the lengths of the sides. The perimeter of $\triangle ABC$ in Figure 7 is $3 + 4 + 5 = 12$.

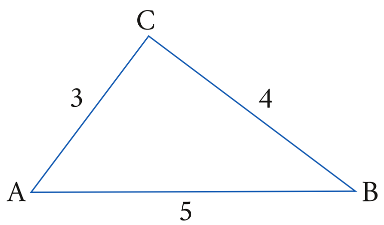


Figure 7. The perimeter of $\triangle ABC$ is 12.

THEOREM 1

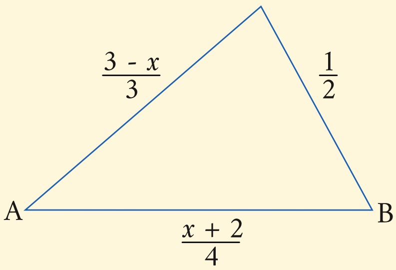
The sum of any two sides of a triangle is greater than the remaining side.

For example, in Figure 7, AC + BC= 3 + 4 > AB = 5.

Proof of Theorem 1 :
This follows from the postulate that the shortest distance between two points is along a straight line. For example, in Figure 7, the length AB(a straight line segment) must be less than the combined lengths of AC and CB (not on a straight line from A to B).

? EXAMPLE E

Find the perimeter of the triangle in terms of x. Then find the perimeter if x = 1:



SOLUTION

$$\textbf{\textit{Perimeter}} = \frac{3 - x}{3} + \frac{x + 2}{4} + \frac{1}{2} \quad (\text{l. c. d.} = 12)$$

$$= \frac{(4)(3 - x)}{(4)(3)} + \frac{(3)(x + 2)}{(3)(4)} + \frac{(6)1}{(6)2}$$

$$= \frac{12 - 4x}{12} + \frac{3x + 6}{12} + \frac{6}{12}$$

$$= \frac{12 - 4x + 3x + 6 + 6}{12}$$

$$= \frac{24 - x}{12}$$

If x = 1 $\quad \frac{24 - x}{12} = \frac{24 - 1}{12} = \frac{23}{12}$

ANSWER : $\frac{24 - x}{12}$, $\frac{23}{12}$

CHECK

$$\frac{3 - x}{3} + \frac{x + 2}{4} + \frac{1}{2} = \frac{24 - x}{12}$$

$$\frac{3 - 1}{3} + \frac{1 + 2}{4} + \frac{1}{2} \qquad \frac{24 - 1}{12}$$

$$\frac{2}{3} + \frac{3}{4} + \frac{1}{2} \qquad \frac{23}{12}$$

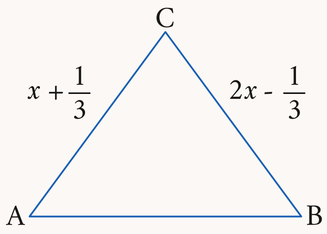
$$\frac{8}{12} + \frac{9}{12} + \frac{6}{12}$$

$$\frac{23}{12}$$

PROBLEMS

1 - 2. Find x if △ABC is isosceles with AC = BC :

1.



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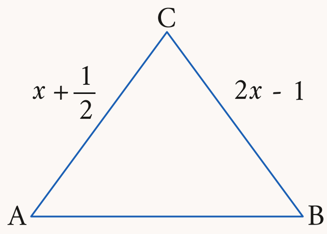
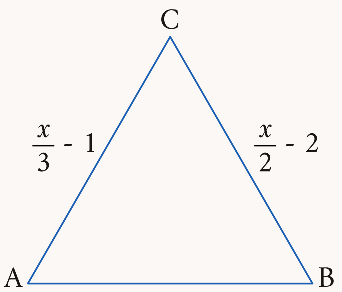


3 - 4. Find x if △ABC is equilateral :

3.



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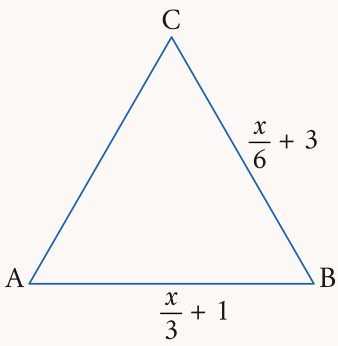
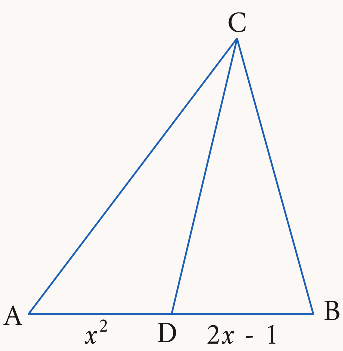


5 - 6. Find AB if CD is a median :

5.



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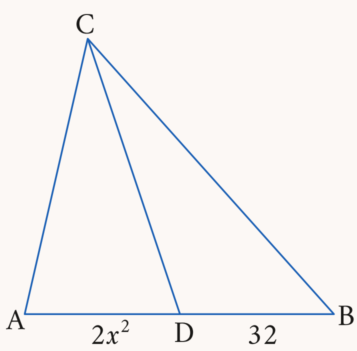
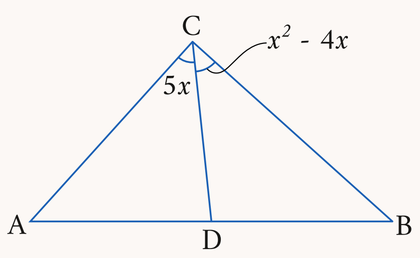


7 - 8. Find $\angle ACB$ if $\overrightarrow{CD}$ is an angle bisector:

7.



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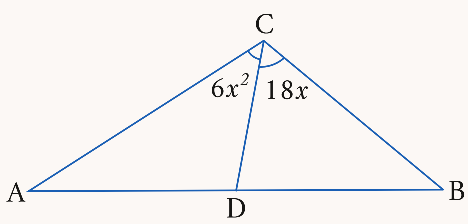
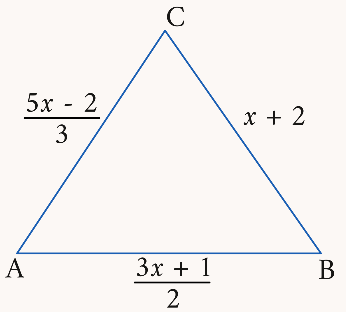


9 - 10. Find the perimeter of the triangle in terms of x.
Then find the perimeter if x=4 :

9.



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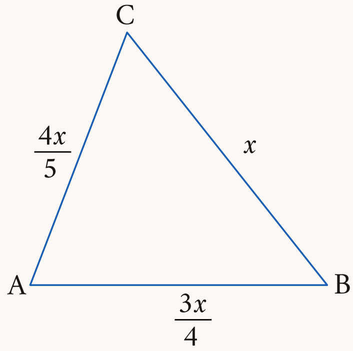
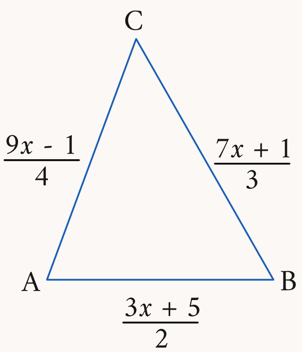
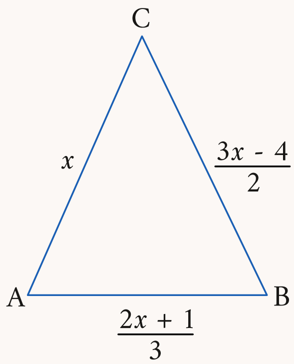


11. Find x if the perimeter of $\triangle ABC$ is 33.



12. Find x if the perimeter of $\triangle ABC$ is 11.



CHAPTER 2

CONGRUENT TRIANGLES

2.1 THE CONGRUENCE STATEMENT

Two triangles are said to be **congruent** if one can be placed over the other so that they coincide (fit together). This means that congruent triangles are exact copies of each other and when fitted together the sides and angles which coincide, called **corresponding sides** and **angles**, are equal.

In Figure 1, $\triangle ABC$ is congruent to $\triangle DEF$. The symbol for congruence is $\cong$ and we write $\triangle ABC \cong \triangle DEF$. $\angle A$ corresponds to $\angle D$, $\angle B$ corresponds to $\angle E$ and $\angle C$ corresponds to $\angle F$. Side AB corresponds to DE, BC corresponds to EF and AC corresponds to DF.

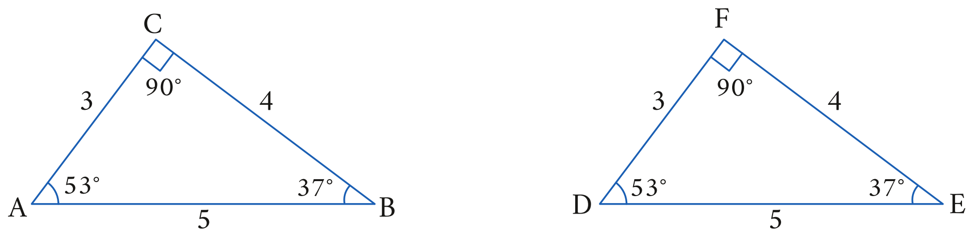


Figure 1. $\triangle ABC$ is congruent to $\triangle DEF$.

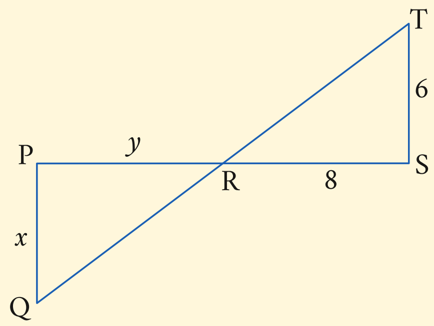
In this book the congruence statement $\triangle ABC \cong \triangle DEF$ will always be written so that corresponding vertices appear in the same order. For the triangles in Figure 1, we might also write $\triangle BAC \cong \triangle EDF$ or $\triangle ACB \cong \triangle DFE$ but **never** for example $\triangle ABC \cong \triangle EDF$ nor $\triangle ACB \cong \triangle DEF$.*

Therefore we can always tell which parts correspond just from the congruence statement. For example, given that $\triangle ABC \cong \triangle DEF$, side AB corresponds to side DE because each consists of the first two letters. AC corresponds to DF because each consists of the first and last letters. BC corresponds to EF because each consists of the last two letters.

* Be warned that not all textbooks follow this practice. Many authors will write the letters without regard to the order. If that is the case then we cannot tell which parts correspond from the congruence statement.

? EXAMPLE A

If $\triangle PQR \cong \triangle STR$
(1) list the corresponding angles and sides;
(2) find x and y.



SOLUTION

(1) $\underline{\triangle PQR}$ $\underline{\triangle STR}$

$\angle P = \angle S$ *first letter of each triangle in congruence statement*
$\angle Q = \angle T$ *second letter*
$\angle PRQ = \angle SRT$ *third letter. We don't write "$\angle R = \angle R$" since each $\angle R$ is different.*

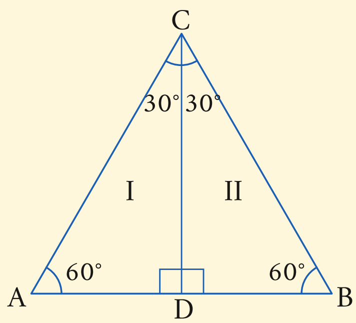
$PQ = ST$ *first two letters*
$PR = SR$ *first and last letters*
$QR = TR$ *last two letters*

(2) $x = PQ = ST = 6.$
$y = PR = SR = 8.$

ANSWER : $x = 6, y = 8$

? EXAMPLE B

Assuming $\triangle I \cong \triangle II$, write a congruence statement for $\triangle I$ and $\triangle II$:



SOLUTION

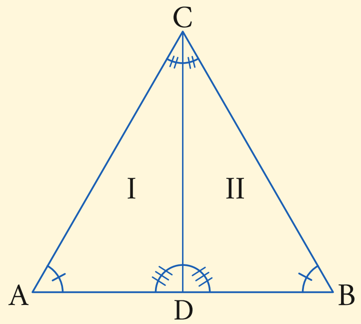
$\triangle I$		$\triangle II$	
$\angle A$	=	$\angle B$	*both* = 60°
$\angle ACD$	=	$\angle BCD$	*both* = 30°
$\angle ADC$	=	$\angle BDC$	*both* = 90°

Therefore $\triangle ACD \cong \triangle BCD$ (30°, 60°, 90°)

ANSWER: $\triangle ACD \cong \triangle BCD$

? EXAMPLE C

Assuming $\triangle I \cong \triangle II$, write a congruence statement for $\triangle I$ and $\triangle II$:



SOLUTION

The angles that are marked the same way are assumed to be equal.

$\triangle I$		$\triangle II$	
$\angle A$	=	$\angle B$	*both marked with one stroke*
$\angle ACD$	=	$\angle BCD$	*both marked with two strokes*
$\angle ADC$	=	$\angle BDC$	*both marked with three strokes*

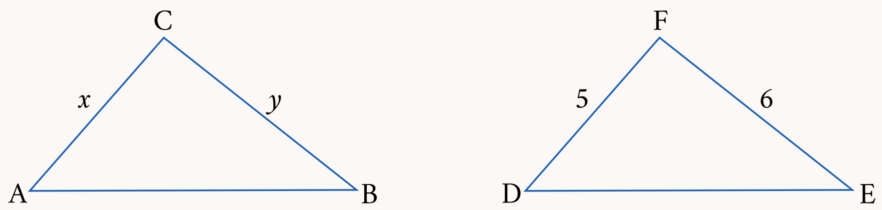
The relationships are the same as in *Example B*.

ANSWER: $\triangle ACD \cong \triangle BCD$

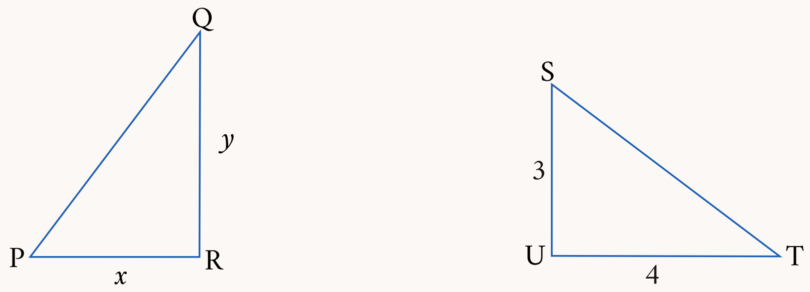
PROBLEMS

1 - 4. For each pair of congruent triangles.
a. list the corresponding sides and angles;
b. find x and y,

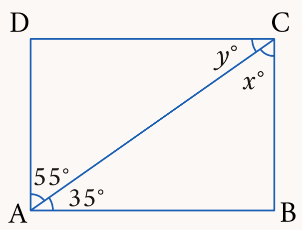
1. $\triangle ABC \cong \triangle DEF$:



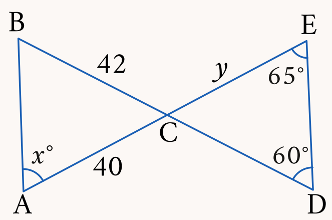
2. $\triangle PQR \cong \triangle STU$:



3. $\triangle ABC \cong \triangle CDA$:

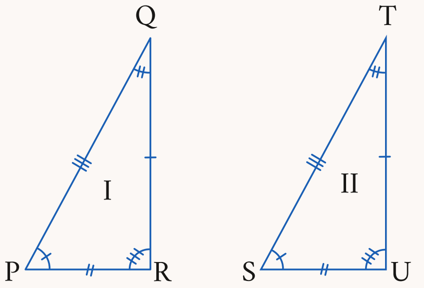


4. $\triangle ABC \cong \triangle EDC$:



5 - 10. Write a congruence statement for each of the following. Assume the triangles are congruent and that angles or sides marked in the same way are equal.

5.



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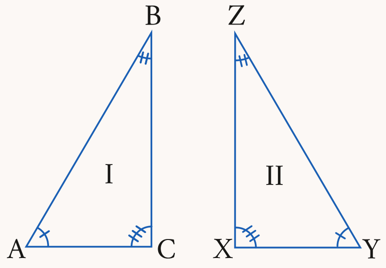

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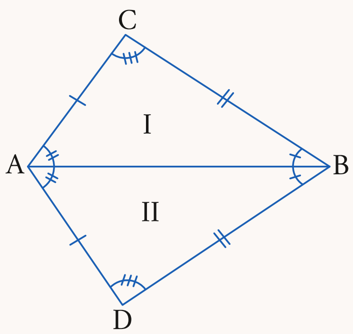

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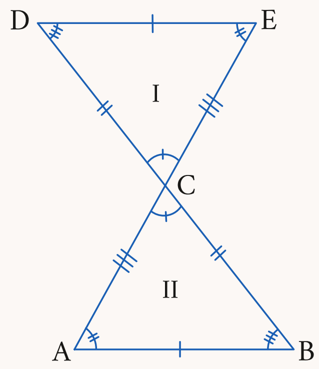

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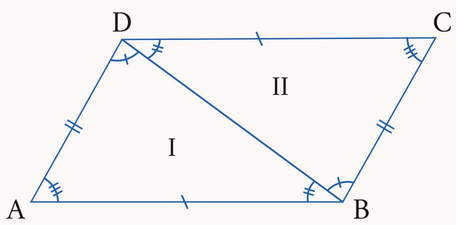

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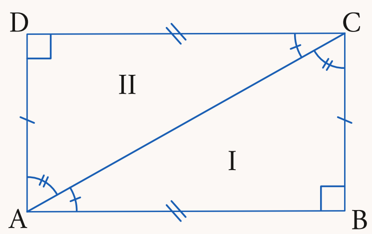


2.2 THE SAS THEOREM

We have said that two triangles are congruent if all their corresponding sides and angles are equal. However in some cases, it is possible to conclude that two triangles are congruent, with only partial information about their sides and angles.

Suppose we are told that △ABC has ∠A = 53°, AB = 5 inches, and AC = 3 inches. Let us attempt to sketch △ABC. We first draw an angle of 53° with a protractor and label it ∠A. Using a ruler, we find the point 5 inches from the vertex on one side of the angle and label it B. On the other side of the angle, we find the point 3 inches from the vertex and label it C. See Figure 1. There is now only one way for us to complete our sketch of △ABC, and that is to connect points B and C with a line segment. We could now measure BC, ∠B, and ∠C to find the remaining parts of the triangle.

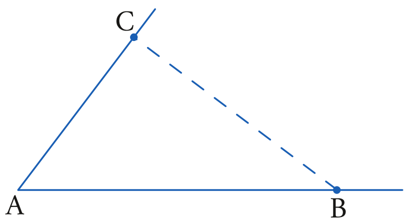


Figure 1. Sketching △ABC.

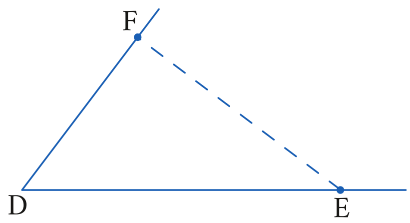


Figure 2. Sketching △DEF.

Suppose now ∠DEF were another triangle, with ∠D = 53°, DE= 5 inches, and DF = 3 inches. We could sketch △DEF just as we did △ABC, and then measure EF, ∠E, and ∠F (Figure 2). It is clear that we must have BC = EF, ∠B = ∠E, and ∠C = ∠F because both triangles were drawn in exactly the same way. Therefore △ABC ≅ △DEF.

In △ABC, we say that ∠A is the angle **included** between sides AB and AC.

In △DEF, we say that ∠D is the angle included between sides DE and DF.

Our discussion suggests the following theorem:

THEOREM 1 (SAS OR SIDE - ANGLE - SIDE THEOREM)

Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

In Figures 1 and 2, △ABC ≅ △DEF because AB, AC, and ∠A are equal respectively to DE, DF, and ∠D. We sometimes abbreviate **Theorem 1** by simply writing SAS = SAS.

? EXAMPLE A

In $\triangle PQR$ name the angle included between sides

(1) PQ and QR,

(2) PQ and PR,

and (3) PR and QR.

SOLUTION

Note that the included angle is named by the letter that is common to both sides.
For (1), the letter "Q" is common to PQ and QR and so $\angle Q$ is included between sides PQ and QR. Similarly for (2) and (3).

ANSWER : (1) $\angle Q$ (2) $\angle P$ (3) $\angle R$.

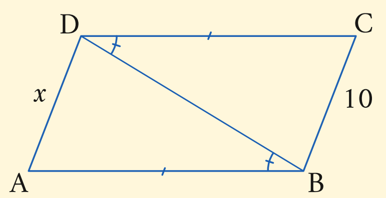
? EXAMPLE B

For the two triangles in the diagram

(1) list two sides and an included angle of each triangle that are respectively equal, using the information given in the diagram,

(2) write the congruence statement,

and (3) find x by identifying a pair of corresponding sides of the congruent triangles.



SOLUTION

(1) The angles and sides that are marked the same way in the diagram are assumed to be equal. So $\angle B$ in $\triangle ABD$ is equal to $\angle D$ in $\triangle CDB$.
Therefore "B" corresponds to "D." We also have AB = CD, Therefore "A" must correspond to "C". Thus, if the triangles are congruent, the correspondence must be

$$\triangle ABD \cong \triangle CDB$$

Finally, BD (the same as DB) is a side common to both triangles. Summaryzing,

	$\triangle ABD$		$\triangle CDB$	
Side	AB	=	CD	(marked = in diagram)
Included angle	$\angle B$	=	$\angle D$	(marked = in diagram)
Side	BD	=	DB	(common side)

(2) $\triangle ABD \cong \triangle CDB$ because of the **SAS Theorem** (SAS = SAS).

(3) x = AD = CB = 10 because AD and CB are corresponding sides (first and third letters in the congruence statement) and corresponding sides of congruent triangles are equal.

ANSWER :

(1) AB, $\angle B$, BD of $\triangle ABD$ = CD, $\angle D$, DB of $\triangle CDB$,

(2) $\triangle ABD \cong \triangle CDB$

(3) x = AD = CB = 10

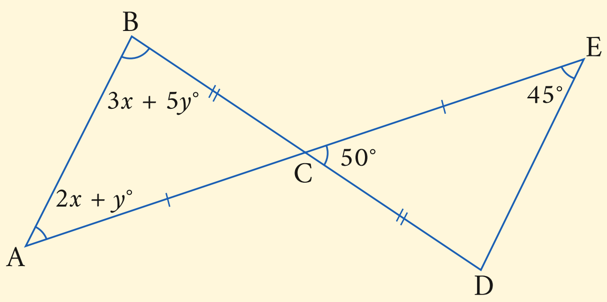
? EXAMPLE C

For the two triangles in the diagram

(1) list two sides and an included angle of each triangle that arerespectively equal, using the information given in the diagram,

(2) write the congruence statement,

and (3) find x and y by identifying a pair of corresponding sides of the congruent triangles.



SOLUTION

(1) AC = CE and BC = CD because they are marked the same way. We also know that $\angle ACB = \angle ECD = 50°$ because vertical angles are equal.

Therefore "C" in $\triangle ABC$ corresponds to "C" in $\triangle CDE$. Since AC = CE, we must have that "A" in $\triangle ABC$ corresponds to "E" in $\triangle CDE$. Thus, if the triangles are congruent, the correspondence must be

$$\triangle ABC \cong \triangle EDC$$

We summarize,

	$\triangle ABC$		$\triangle EDC$	
Side	AC	=	EC	(marked = in diagram)
Included angle	$\angle ACB$	=	$\angle ECD$	(vertical angles are =)
Side	BC	=	DC	(marked = in diagram)

(2) $\triangle ABC \cong \triangle EDC$ because of the **SAS Theorem** (SAS = SAS).

(3) $\angle A = \angle E$ and $\angle B = \angle D$ because they are corresponding angles of the congruent triangles. $\angle D = 85°$ because the sum of the angles of $\triangle EDC$ must be $180°$.
($\angle D = 180° - (50° + 45°) = 180° - 95° = 85°$). We obtain a system of two equations in the two unknowns x and y :

$$\begin{array}{lcrcrl}
\angle A = \angle E & \longrightarrow & 2x + y = 45 & \xrightarrow{-5} & -10x - 5y & = -225 \\
\angle B = \angle D & \longrightarrow & 3x + 5y = 85 & \longrightarrow & 3x + 5y & = \;\; 85 \\
\hline
 & & & & -7x & = -140 \\
 & & & & x & = 20
\end{array}$$

Substituting for x in the first original equation,

$$\begin{aligned}
2x + y &= 45 \\
2(20) + y &= 45 \\
40 + y &= 45 \\
y &= 45 - 40 \\
y &= 5
\end{aligned}$$

CHECK

$$\begin{array}{r|l}
\multicolumn{2}{c}{\angle A = \angle E} \\
2x + y & 45° \\
2(20) + 5 & \\
40 + 5 & \\
45° &
\end{array}
\qquad\qquad
\begin{array}{r|l}
\multicolumn{2}{c}{\angle B = \angle D} \\
3x + 5y & 85° \\
3(20) + 5(5) & \\
60 + 25 & \\
85° &
\end{array}$$

ANSWER :

(1) AC, $\angle ACB$, BC of $\triangle ABC$ = EC, $\angle ECD$, DC of $\triangle EDC$,

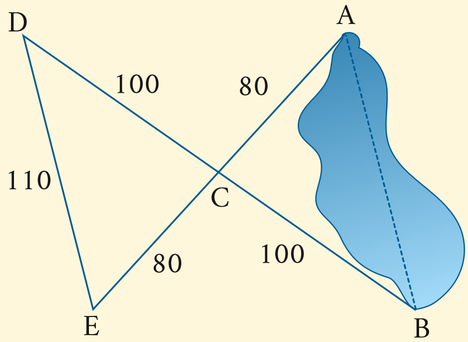
(2) $\triangle ACB \cong \triangle EDC$

(3) $x = 20$, $y = 5$.

? EXAMPLE D

The following procedure was used to measure the distance AB across a pond: From a point C, AC and BC were measured and found to be 80 and 100 feet respectively. Then AC was extended to E so that AC = CE and BC was extended to D so that BC = CD. Finally, DE was found to be 110 feet.

(1) Write the congruence statement.
(2) Give a reason for (1).
(3) Find AB.



SOLUTION

(1) ∠ACB = ∠ECD because vertical angles are equal. Therefore the "C's" correspond.
AC = EC so A must correspond to E. We have

$$\triangle ABC \cong \triangle EDC$$

(2) SAS = SAS. Sides AC, BC, and included angle C of △ABC are equal respectively to sides EC, DC, and included angle C of △EDC.
(3) AB = ED because they are corresponding sides of congruent triangles.
Since ED = 110, AB= 110.

ANSWER :

(1) △ABC ≅ △EDC .
(2) SAS = SAS: AC, ∠C, BC of △ABC = EC, ∠C, DC of △EDC.
(3) AB = 110 feet.

HISTORICAL NOTE

The SAS Theorem is Proposition 4 in Euclid's *Elements*. Both our discussion and Euclid's proof of the SAS Theorem implicitly use the following principle: If a geometric construction is repeated in a different location (or what amounts to the same thing is "moved" to a different location) then the size and shape of the figure remain the same. There is evidence that Euclid used this principle reluctantly, and many mathematicians have since questioned its use in formal proofs. They feel that it makes too strong an assumption about the nature of physical space and is an inferior form of geometric reasoning. Bertrand Russell (1872 - 1970), for example, has suggested that we would be better off assuming the SAS Theorem as a postulate. This is in fact done in a system of axioms for Euclidean geometry devised by David Hilbert (1862 - 1943), a system that has gained much favor with modern mathematicians. Hilbert was the leading exponent of the "formalist school," which sought to discover exactly what assumptions underlie each branch of mathematics and to remove all logical ambiguities. Hilbert's system, however, is too formal for an introductory course in geometry.

PROBLEMS

1 - 4. For each of the following (1) draw the triangle with the two sides and the included angle and (2) measure the remaining side and angles:

1. AB = 2 inches, AC = 1 inch, $\angle A = 60°$.
2. DE = 2 inches, DF = 1 inch, $\angle D = 60°$.
3. AB = 2 inches, AC = 3 inches, $\angle A = 40°$.
4. DE = 2 inches, DF = 3 inches, $\angle D = 40°$.

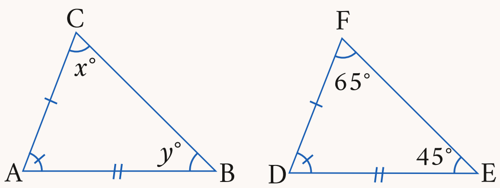
5 - 8. Name the angle included between sides

5. AB and BC in $\triangle ABC$.
6. XY and YZ in $\triangle XYZ$.
7. DE and DF in $\triangle DEF$.
8. RS and TS in $\triangle RST$.

9 - 22. For each of the following,

(1) list two sides and an included angle of each triangle that are respectively equal, using the information given in the diagram,
(2) write the congruence statement,
and (3) find x, or x and y,
Assume that angles or sides marked in the same way are equal.

9.



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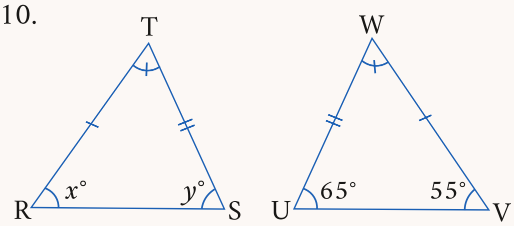

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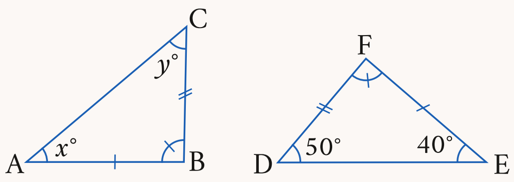

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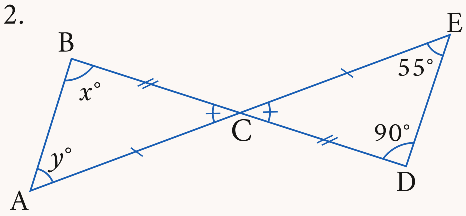

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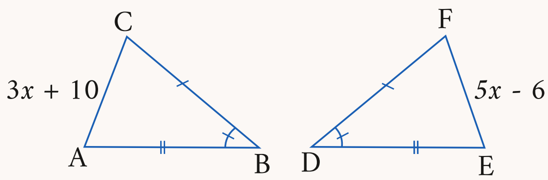

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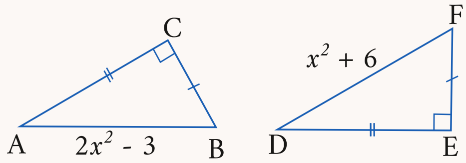

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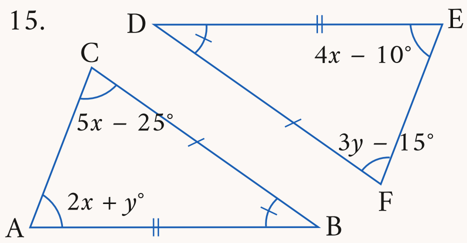

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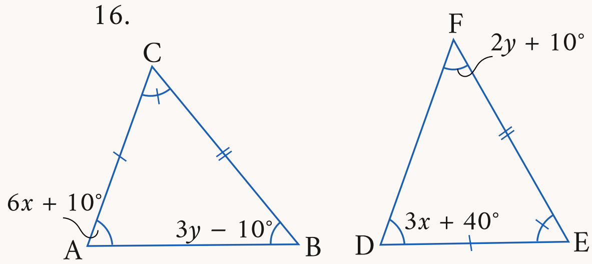

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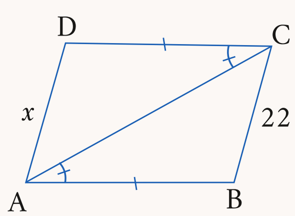

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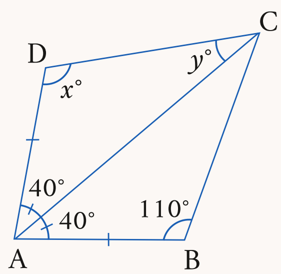

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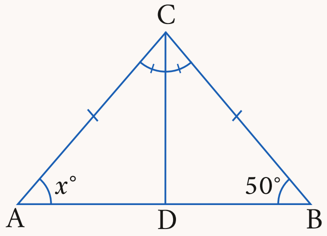

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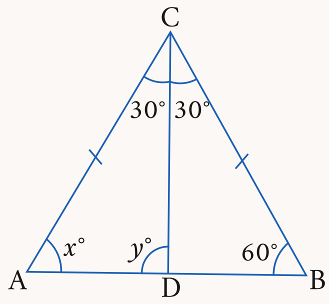

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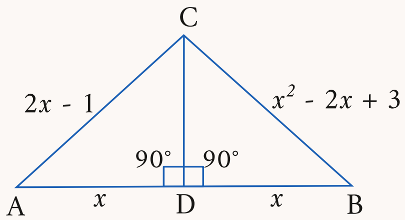

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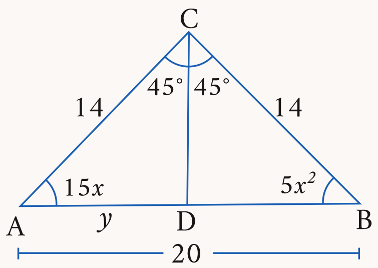

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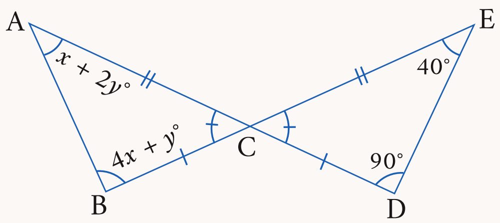

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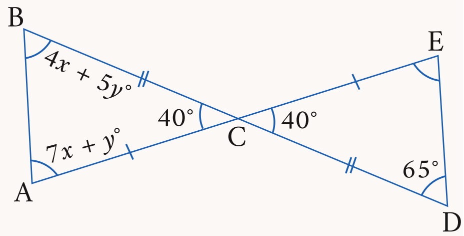

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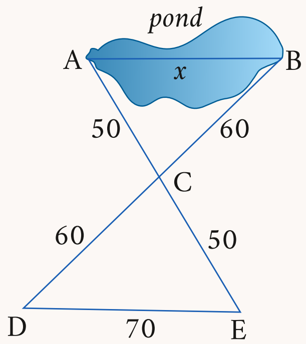

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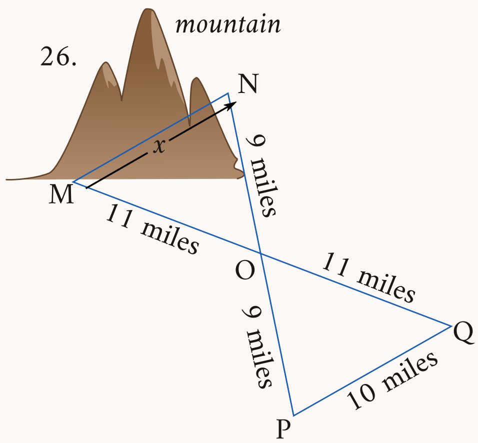


2.3 THE ASA and AAS THEOREMS

In this section we will consider two more cases where it is possible to conclude that triangles are congruent with only partial information about their sides and angles.
Suppose we are told that $\triangle ABC$ has $\angle A = 30°$, $\angle B = 40°$, and $AB = 2$ inches. Let us attempt to sketch $\triangle ABC$. We first draw a line segment of 2 inches and label it AB. With a protractor we draw an angle of $30°$ at A and an angle of $40°$ at B(Figure 1). We extend the lines forming $\angle A$ and $\angle B$ until they meet at C. We could now measure AC, BC, and $\angle C$ to find the remaining parts of the triangle.

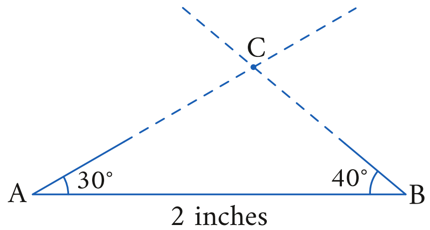


Figure 1. Sketching $\triangle ABC$.

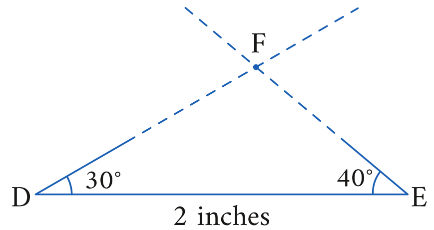


Figure 2. Sketching $\triangle DEF$.

Let $\triangle DEF$ be another triangle, with $\angle D = 30°$, $\angle E = 40°$, and $DE = 2$ inches. We could sketch $\triangle DEF$ just as we did $\triangle ABC$, and then measure DF, EF, and $\angle F$(Figure 2). It is clear that we must have $AC = DF$, $BC = EF$, and $\angle C = \angle F$, because both triangles were drawn in exactly the same way. Therefore $\triangle ABC \cong \triangle DEF$.
In $\triangle ABC$ we say that AB is the side **included** between $\angle A$ and $\angle B$. In $\triangle DEF$ we would say that DE is the side included between $\angle D$ and $\angle E$. Our discussion suggests the following theorem:

THEOREM 1 (ASA OR ANGLE - SIDE - ANGLE THEOREM)

Two triangles are congruent if two angles and an included side of one are equal respectively to two angles and an included side of the other.

In Figure 1 and 2, $\triangle ABC \cong \triangle DEF$ because $\angle A$, $\angle B$, and AB are equal respectively to $\angle D$, $\angle E$, and DE.
We sometimes abbreviate **Theorem 1** by simply writing ASA = ASA.

? EXAMPLE A

In $\triangle PQR$, name the side included between

(1) $\angle P$ and $\angle Q$,

(2) $\angle P$ and $\angle R$,

(3) $\angle Q$ and $\angle R$.

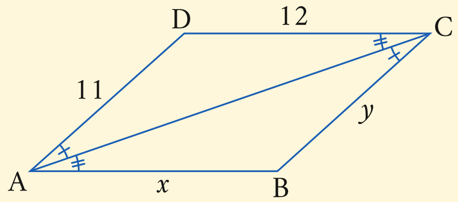
SOLUTION

Note that the included side is named by the two letters representing each of the angles. Therefore, for(1), the side included between $\angle P$ and $\angle Q$ is named by the letters P and Q -- that is, side PQ. Similarly for (2) and (3).

ANSWER : (1)PQ, (2)PR , (3)QR.

? EXAMPLE B

For the two triangles in the diagram
(1) write the congruence statement,
(2) give a reason for (1),
and (3) find x and y.



SOLUTION

(1) From the diagram $\angle A$ in $\triangle ABC$ is equal to $\angle C$ in $\triangle ADC$. Therefore, "A" corresponds to "C". Also $\angle C$ in $\triangle ABC$ is equal to $\angle A$ in $\triangle ADC$. So "C" corresponds to "A". We have

$$\triangle ABC \cong \triangle CDA$$

(2) $\angle A$, $\angle C$, and included side AC of $\triangle ABC$ are equal respectively to $\angle C$, $\angle A$, and included side CA of $\triangle CDA$. (AC = CA because they are just different names for the identical line segment. We sometimes say AC=CA because of **identity**.)
Therefore $\triangle ABC \cong \triangle CDA$ because of the **ASA Theorem** (ASA = ASA).

	$\triangle ABC$		$\triangle CDA$	
Angle	$\angle BAC$	=	$\angle DCA$	(marked = in diagram)
Included angle	AC	=	CA	(identity)
Side	$\angle BCA$	=	$\angle DAC$	(marked = in diagram)

(3) AB = CD and BC = DA because they are corresponding sides of the congruent triangles. Therefore x = AB = CD = 12 and y = BC = DA = 11.

ANSWER :

(1) $\triangle ABC \cong \triangle CDA$

(2) ASA = ASA: $\angle A$, AC, $\angle C$ of $\triangle ABC$ = $\angle C$, CA, $\angle A$ of $\triangle CDA$.

(3) $x = 12$, $y = 11$.

Let us now consider $\triangle ABC$ and $\triangle DEF$ in Figure 3.

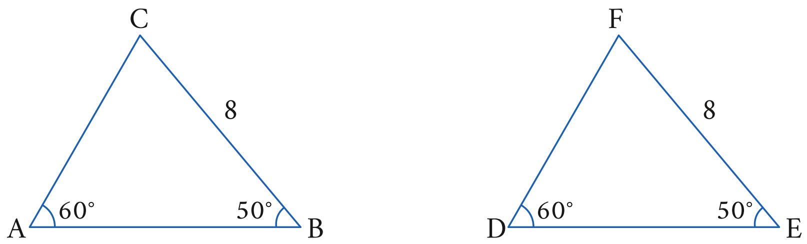


Figure 3. Two angles and an unincluded side of $\triangle ABC$ are equal respectively to two angles and an unincluded side of $\triangle DEF$.

$\angle A$ and $\angle B$ of $\triangle ABC$ are equal respectively to $\angle D$ and $\angle E$ of $\triangle DEF$, yet we have no information about the sides included between these angles, AB and DE. Instead we know that the unincluded side BC is equal to the corresponding unincluded side EF. Therefore, as things stand, we cannot use ASA = ASA to conclude that the triangles are congruent. However we may show $\angle C$ equals $\angle F$ as in **Theorem 3**, section 1.5 ($\angle C = 180° - (60° + 50°) = 180° - 110° = 70°$ and $\angle F = 180° - (60° + 50°) = 180° - 110° = 70°$).
Then we can apply the ASA Theorem to angles B and C and their included side BC and the corresponding angles E and F with included side EF. These remarks lead us to the following theorem:

THEOREM 2 (AAS OR ANGLE - ANGLE - SIDE THEOREM)

Two triangles are congruent if two angles and an unincluded side of one triangle are equal respectively to two angles and the corresponding unincluded side of the other triangle (AAS = AAS).

In Figure 4, if $\angle A = \angle D$, $\angle B = \angle E$ and $BC = EF$ then $\triangle ABC \cong \triangle DEF$.

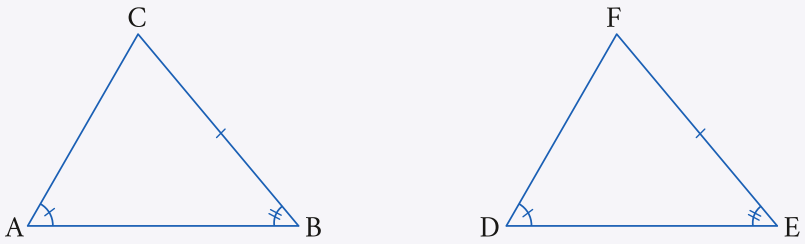


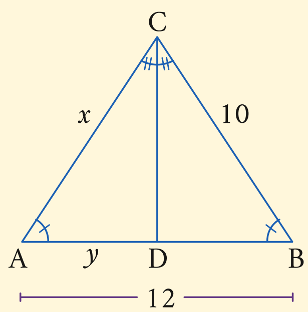
Figure 4. These two triangles are congruent by AAS = AAS.

Proof of Theorem 2 :

$\angle C = 180° - (\angle A + \angle B) = 180° - (\angle D + \angle E) = \angle F$. The triangles are then congruent by ASA = ASA applied to $\angle B$. $\angle C$ and BC of $\triangle ABC$ and $\angle E$, $\angle F$ and EF of $\triangle DEF$.

? EXAMPLE C

For the two triangles in the diagram
(1) write the congruence statement,
(2) give a reason for (1),
and (3) find x and y.



SOLUTION

(1) $\triangle ACD \cong \triangle BCD$

(2) AAS = AAS since $\angle A$, $\angle C$ and unincluded side CD of $\triangle ACD$ are equal respectively to $\angle B$, $\angle C$ and unincluded side CD of $\triangle BCD$.

	$\triangle ACD$		$\triangle BCD$	
Angle	$\angle A$	=	$\angle B$	(marked = in diagram)
Angle	$\angle ACD$	=	$\angle BCD$	(marked = in diagram)
Unincluded side	CD	=	CD	(identity)

(3) AC = BC and AD = BD since they are corresponding sides of the congruent triangles.
Therefore x = AC = BC = 10 and y = AD = BD.
Since AB = AD + BD = y + y = 2y =12, we must have y=6.

ANSWER :

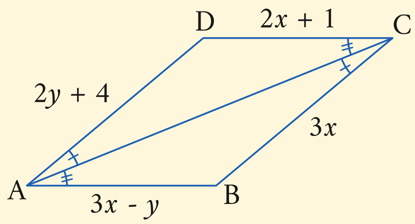
(1) $\triangle ACD \cong \triangle BCD$.

(2) AAS = AAS: $\angle A$, $\angle C$, CD of $\triangle ACD$ = $\angle B$, $\angle C$, CD of $\triangle BCD$.

(3) x = 10, y = 6.

? EXAMPLE D

For the two triangles in the diagram
(1) write the congruence statement,
(2) give a reason for (1),
and (3) find x and y.



SOLUTION

Part(1) and part(2) are identical to ***Example B***.

(3)

$$\begin{aligned} AB &= CD \\ 3x - y &= 2x + 1 \\ 3x - 2x - y &= 1 \\ x - y &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} BC &= DA \\ 3x &= 2y + 4 \\ 3x - 2y &= 4 \end{aligned}$$

We solve these equations simultaneously for x and y:.

$$\begin{aligned} x - y &= 1 \\ 3x - 2y &= 4 \end{aligned} \quad \begin{aligned} &\xrightarrow{-2} \\ &\longrightarrow \end{aligned} \quad \begin{aligned} -2x + 2y &= -2 \\ +\ \ 3x - 2y &= 4 \\ \hline x \quad &= 2 \end{aligned}$$

$$\begin{aligned} x - y &= 1 \\ 2 - y &= 1 \\ -y &= 1 - 2 \\ -y &= -1 \\ y &= 1 \end{aligned}$$

CHECK

AB = CD

3x - y	2x + 1
3(2) - 1	2(2) + 1
6 - 1	4 + 1
5	5

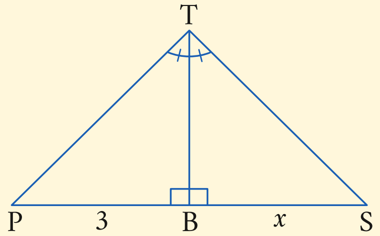
BC = DA

3x	2y + 4
3(2)	2(1) + 4
6	2 + 4
	6

ANSWER : (1) and (2) same as Example B. (3) x = 2, y = 1.

? EXAMPLE E

From the top of a tower T on the shore, a ship S is sighted at sea. A point P along the coast is also sighted from T so that ∠PTB = ∠STB. If the distance from P to the base of the tower B is 3 miles, how far is the ship from point B on the shore?.



SOLUTION

△PTB ≅ △STB by ASA = ASA. Therefore x = SB = FB = 3.

ANSWER : 3 miles.

HISTORICAL NOTE

The method of finding the distance of ships at sea described in Example E has been attributed to the Greek philosopher Thales (c. 600 B.C.). We know from various authors that the ASA Theorem has been used to measure distances since ancient times. There is a story that one of Napoleon's officers used the ASA Theorem to measure the width of a river his army had to cross, (see Problem 25 below.)

PROBLEMS

1-4. For each of the following (1) draw the triangle with the two angles and the included side and (2) measure the remaining sides and angle:

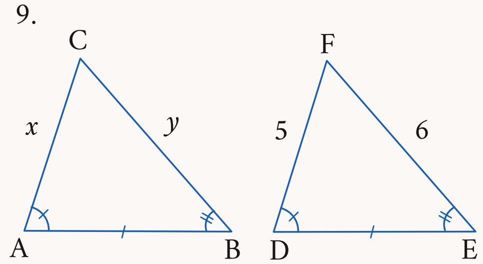
1. △ABC with ∠A = 40°, ∠B = 50° and AB = 3 inches,
2. △DEF with ∠D = 40°, ∠E = 50° and DE = 3 inches,
3. △ABC with ∠A = 50° ∠B = 40° and AB = 3 inches,
4. △DEF with ∠D = 50°, ∠E = 40° and DE = 3 inches.

5-8. Name the side included between the angles:

5. $\angle A$ and $\angle B$ in $\triangle ABC$.
6. $\angle X$ and $\angle Y$ in $\triangle XYZ$.
7. $\angle D$ and $\angle F$ in $\triangle DEF$.
8. $\angle S$ and $\angle T$ in $\triangle RST$.

9-22. For each of the following
(1) write the congruence statement for the two triangles,
(2) give a reason for (1) (SAS, ASA, or AAS Theorems),
and (3) find x, or x and y.

9.



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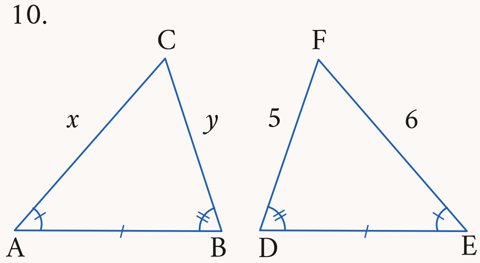

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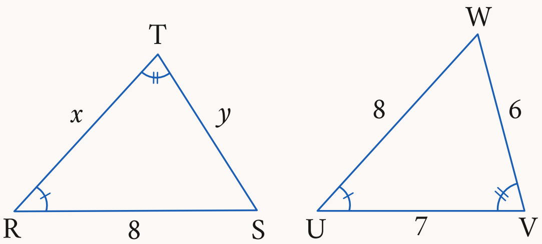

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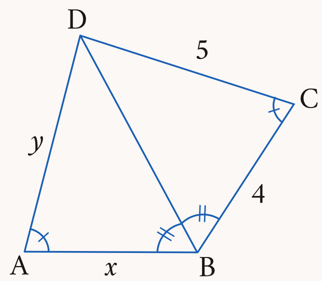

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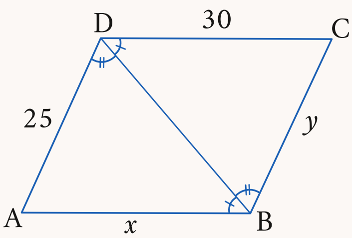

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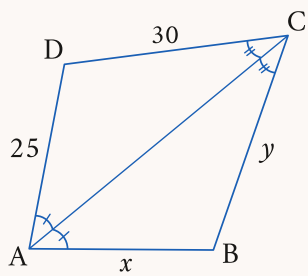

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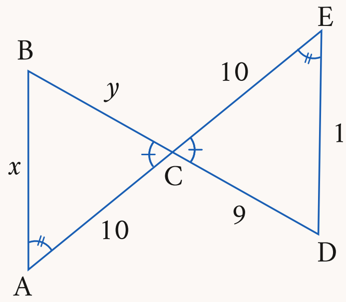

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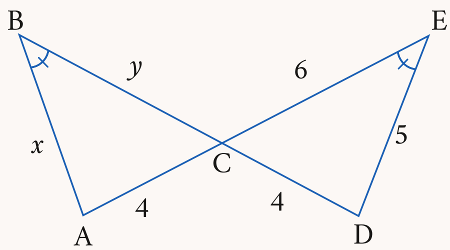

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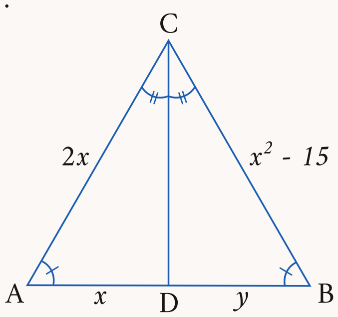

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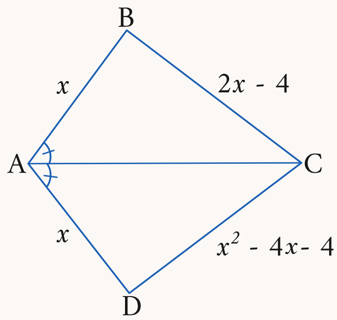

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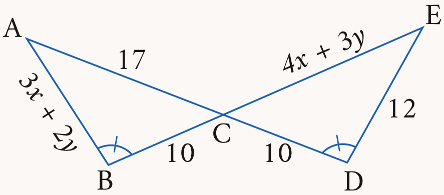

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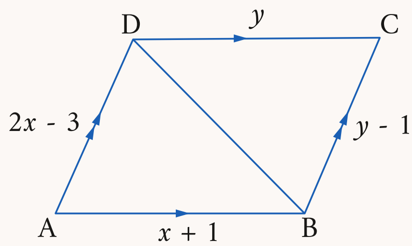

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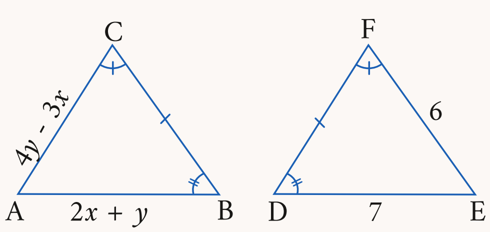

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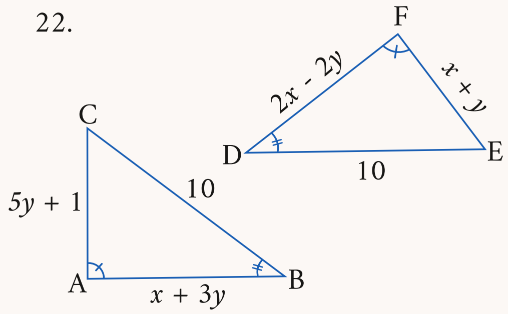
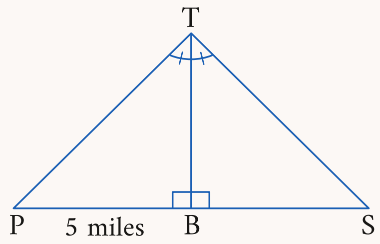
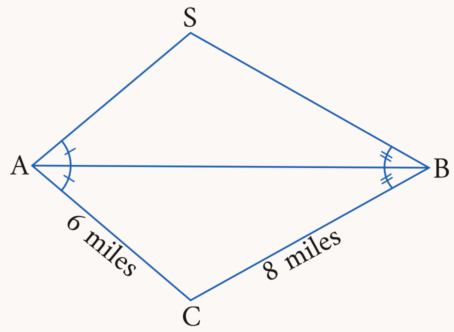


23-26. For each of the following, include the congruence statement and the reason as part of your answer:

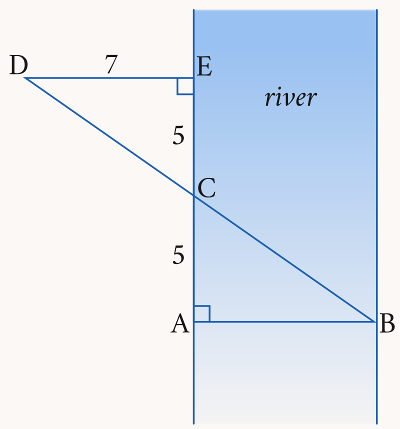
23. In the diagram how far is the ship S from the point P on the coast?



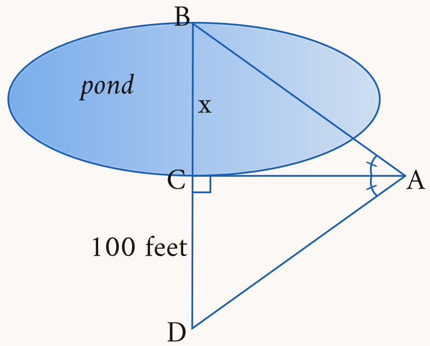
24. Ship S is observed from points A and B along the coast. Triangle ABC is then constructed and measured as in the diagram. How far is the ship from point A?



25. Find the distance AB across a river if AC = CD = 5 and DE = 7 as in the diagram?



26. What is the distance across the pond?

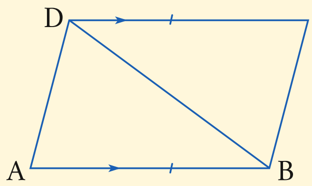


2.4 PROVING LINES and ANGLES EQUAL

We can prove lines and angles equal if we can show they are corresponding parts of congruent triangles. We find it convenient to present these proofs in **double-column form** with statements in the left column and the reason for each statement in the right.

? EXAMPLE A

Given AB || CD and AB = CD prove AD = BC.



SOLUTION

	Statements		Reasons
1	AB = CD	1	Given.
2	∠ADB = ∠CDB	2	Alternate interior angles of parallel lines (AB || CD) are equal.
3	BD = DB	3	Identity.
4	△ABD ≅ △CDB	4	SAS = SAS: AB, ∠B, BD of △ABD = CD, ∠D DB of △CDB.
5	AD = BC	5	Corresponding sides of congruent triangles are equal.

Explanation: Each of the first three statements says that a side or angle of △ABD is equal to the corresponding side or angle of △CDB. To arrive at these statements, we should first write the congruence statement using the methods of the previous sections. We then select three pairs of corresponding sides or angles which are equal because of one of the following reasons:

Reasons lines are equal

(1) *Given*. This means we are asked to assume the lines are equal at the beginning of the exercise. For example, the problem will state "given AB = CD " or AB and CD will be marked the same way in the diagram.

(2) *Identity*. This means the identical line segment appears in both triangles. For example, BD and DB represent the same line segment, Of course the length of a line segment is equal to itself.

Reasons angles are equal

(1) *Given.*
(2) *Identity.*
(3) *Alternate interior angles of parallel lines are equal. To apply this reason we must be given that the lines are parallel.*
(4) *Corresponding angles of parallel lines equal.*
(5) *Vertical angles are equal.*

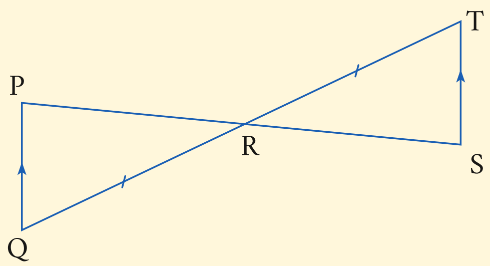
These are not the only possible reasons but they are all that we will use at first. We should also select the three pairs of equal sides or angles so that one of the reasons SAS = SAS, ASA = ASA, or AAS = AAS can be used to justify the congruence statement in statement 4. In sections 2.6 and 2.7, we will give some additional reasons for two triangles to be congruent.

Statement 5 is the one we wish to prove. The reason is that corresponding sides (or angles) of congruent triangles are equal. We can use this reason here because the triangles have already been proven congruent in **statement 4**.

One final comment. Notice how the solution of Example A conforms with our original definition of **proof**. Each new statement is shown to be true by using previous statements and reasons which have already been established. Let us give another example:

? EXAMPLE B

Given $QP \parallel ST$ and QR = TR prove PR = SR.

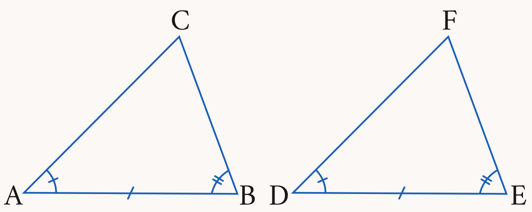


SOLUTION

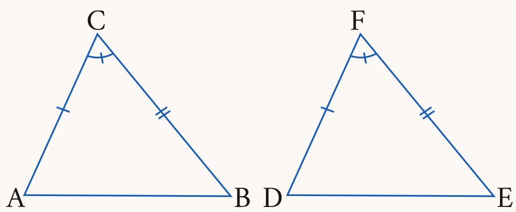
Statements	*Reasons*
1. QR = TR.	1. Given.
2. ∠Q = ∠T	2. Alternate interior angles of parallel lines (QP ‖ ST) are equal.
3. ∠PRQ = ∠SRT	3. Vertical angles are equal.
4. △PQR ≅ △STR	4. ASA = ASA: ∠Q, QR, ∠R of △PQR = ∠T, TR, ∠R of △STR.
5. PR = SR	5. Corresponding sides of congruent triangles are equal.

PROBLEMS

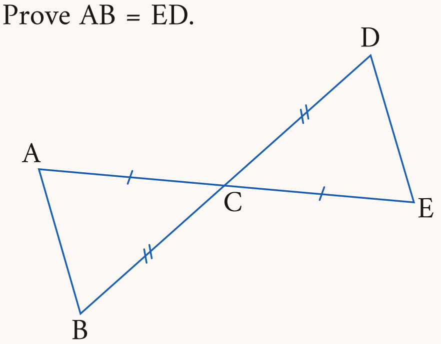
1. Given ∠A = ∠D, ∠B = ∠E, AB = DE.
   Prove AC = DF.



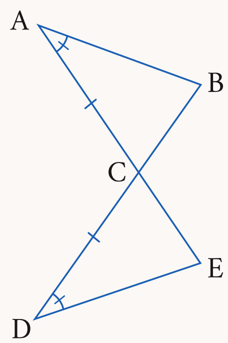
2. Given AC = DF, BC = EF, ∠C = ∠F.
   Prove AB = DE .



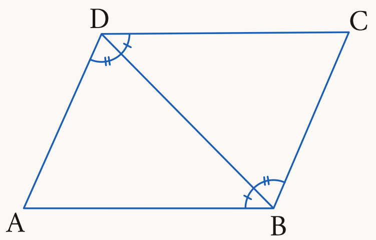
3. Given AC = EC and BC = DC.
   Prove AB = ED.



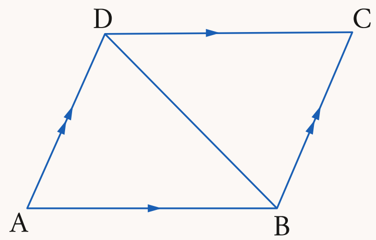
4. Given AC = DC, ∠A = ∠D.
   Prove BC = EC.



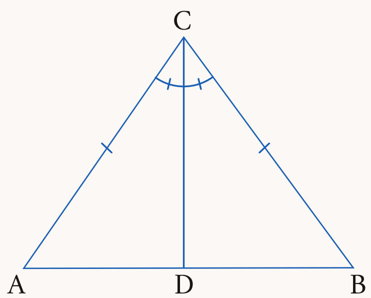
5. Given $\angle ABD = \angle CDB$ and $\angle ADB = \angle CBD$. Prove $AB = CD$.



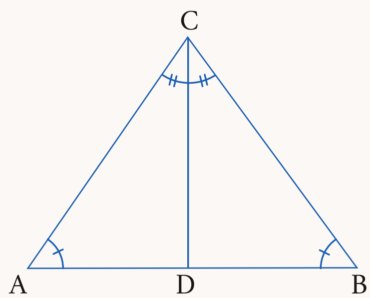
6. Given $AB \parallel CD$ and $AD \parallel CB$. Prove $AB = CD$.



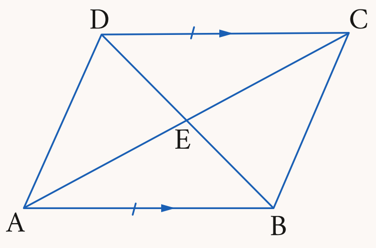
7. Given $AC = BC$ and $\angle ACD = \angle BCD$. Prove $\angle A = \angle B$.



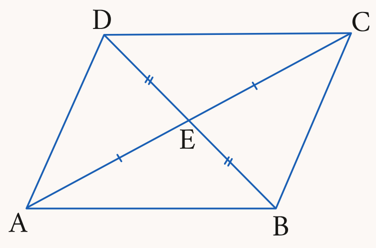
8. Given $\angle A = \angle B$, $\angle ACD = \angle BCD$. Prove $AC = BC$.



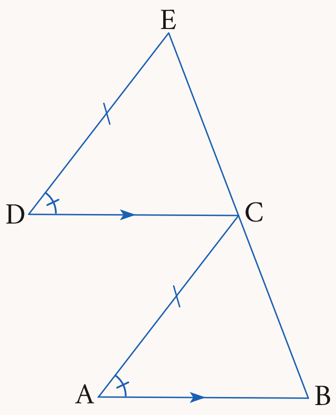
9. Given $AB \parallel CD$ and $AB = CD$. Prove $AE = CE$. (Hint: Show $\triangle ABE \cong \triangle CDE$)



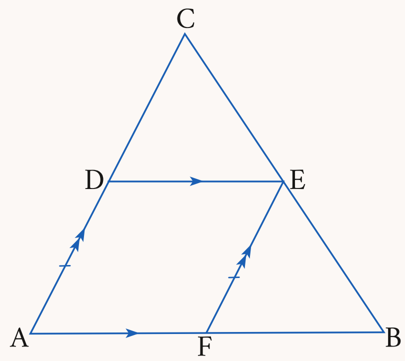
10. Given $AE = CE$ and $BE = DE$. Prove $\angle BAC = \angle CDB$



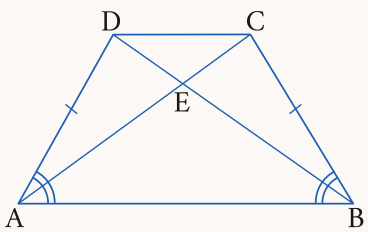
11. Given $\angle A = \angle D$, $AC = DE$, $AB \parallel DC$.
Prove $BC = CE$.



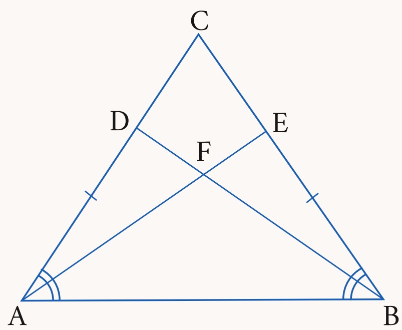
12. Given $AB \parallel DE$, $AC \parallel FE$ and $DC = FE$.
Prove $BE = EC$.



13. Given $AD = BC$ and $\angle BAD = \angle ABC$.
Prove $AC = BD$.
(Hint: Show $\triangle ABD \cong \triangle BAC$).



14. Given $AD = BE$, $\angle BAC = \angle ABC$.
Prove $AE = BD$.



2.5 ISOSCELES TRIANGLES

In section 1.6, we defined a triangle to be **isosceles** if two of its sides are equal. Figure 1 shows an isosceles triangle $\triangle ABC$ with $AC = BC$. In $\triangle ABC$ we say that $\angle A$ is **opposite** side BC and $\angle B$ is opposite side AC.

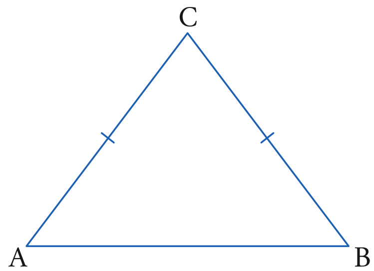


Figure 1. $\triangle ABC$ is isosceles with $AC = BC$.

The most important fact about isosceles triangles is the following:.

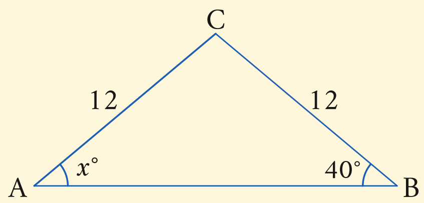
THEOREM 1

If two sides of a triangle are equal the angles opposite these sides are equal.

Theorem 1 means that if $AC = BC$ in $\triangle ABC$ then $\angle A = \angle B$.

EXAMPLE A

Find x:



SOLUTION

$AC = BC$ so $\angle A = \angle B$. Therefore, $x = 40$.

ANSWER : $x = 40$.

In $\triangle ABC$ if $AC = BC$ then side AB is called the **base** of the triangle and $\angle A$ and $\angle B$ are called the **base angles**. Therefore **Theorem 1** is sometimes stated in the following way: "The base angles of an isosceles triangle are equal."

***Proof of Theorem 1*:**

Draw CD, the angle bisector of $\angle ACB$(Figure 2). The rest of the proof will be presented in double-column form. We have given that $AC = BC$ and $\angle ACD = \angle BCD$.

We must prove $\angle A = \angle B$.

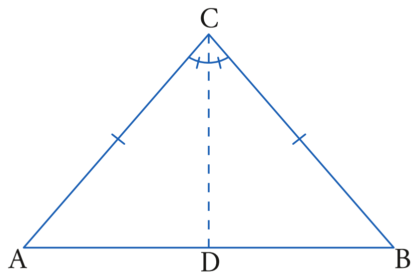
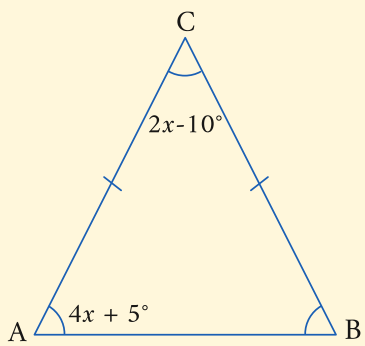


Figure 2. Draw CD, the angle bisector of $\angle ACB$.

	Statements		Reasons
1	$AC = BC$	1	Given, $\triangle ABC$ is isosceles.
2	$\angle ACD = \angle BCD$	2	Given, CD is the bisector of $\angle ACB$.
3	$CD = CD$	3	Identity.
4	$\triangle ACD \cong \triangle BCD$	4	SAS = SAS: AC, $\angle C$, CD of $\triangle ACD$ = BC, $\angle C$ CD of $\triangle BCD$.
5	$\angle A = \angle B$.	5	Corresponding angles of congruent triangles are equal.

? EXAMPLE B

Find x, $\angle A$, $\angle B$ and $\angle C$:



SOLUTION

$\angle B = \angle A = 4x + 5°$ by **Theorem 1**. We have

$$\begin{aligned} \angle A + \angle B + \angle C &= 180° \\ 4x + 5 + 4x + 5 + 2x - 10 &= 180 \\ 10x &= 180 \\ x &= 18 \end{aligned}$$

$\angle A = \angle B = 4x + 5° = 4(18) + 5° = 72° + 5° = 77°$
$\angle C = 2x - 10° = 2(18) - 10° = 36 - 10° = 26°$

CHECK

$$\begin{array}{r|l} \angle A + \angle B + \angle C = & 180° \\ 4x + 5 + 4x + 5 + 2x - 10 & \\ 4(18) + 5 + 4(18) + 5 + 2(18) - 10 & \\ 77° + 77° + 26° & \\ 180° & \end{array}$$

ANSWER : $x = 18$, $\angle A = 77°$, $\angle B = 77°$, $\angle C = 26°$

In Theorem 1 we assumed AC = BC and proved $\angle A = \angle B$. We will now assume $\angle A = \angle B$ and prove AC = BC. When the assumption and conclusion of a statement are interchanged the result is called the **converse** of the original statement.

THEOREM 2 (THE CONVERSE OF THEOREM 1)

If two angles of a triangle are equal the sides opposite these angles are equal.

In Figure 3, if $\angle A = \angle B$ then AC = BC.

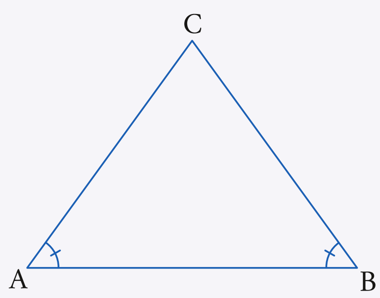
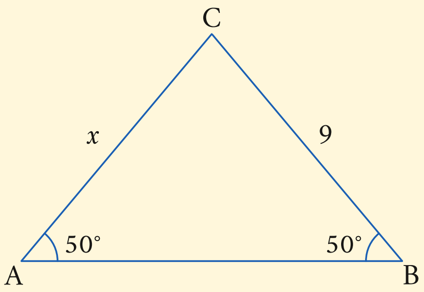


Figure 3. $\angle A = \angle B$

? EXAMPLE C

Find x:



SOLUTION

$\angle A = \angle B$ so $x = AC = BC = 9$ by **Theorem 2**.

ANSWER : $x = 9$

Proof of Theorem 2 :
Draw $\overrightarrow{CD}$ the angle bisector of $\angle ACB$ (Figure 4). We have $\angle ACD = \angle BCD$ and $\angle A = \angle B$. We must prove $AC = BC$.

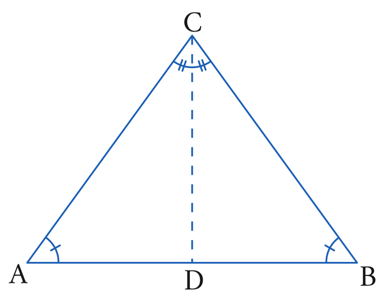


Figure 4. Draw $\overrightarrow{CD}$, the angle bisector of $\angle ACB$.

Statements	***Reasons***
1. $\angle A = \angle B$.	1. Given.
2. $\angle ACD = \angle BCD$	2. Given.
3. $CD = CD$.	3. Identity.
4. $\triangle ACD \cong \triangle BCD$	4. AAS = AAS: $\angle A$, $\angle C$, CD of $\triangle ACD = \angle B$, $\angle C$, CD of $\triangle BCD$.
5. $AC = BC$.	5. Corresponding sides of congruent triangles are equal.

The following two theorems are **corollaries** (immediate consequences) of the two preceding theorems:

THEOREM 3

An equilateral triangle is equiangular.
In Figure 5, if AB = AC = BC then $\angle A = \angle B = \angle C$.

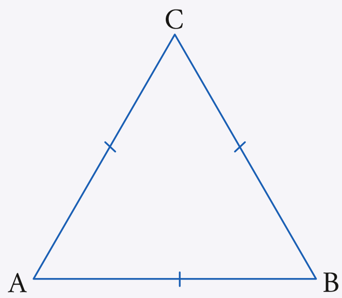


Figure 5. $\triangle ABC$ is eqilateral.

Proof :
AC = BC so by **Theorem 1** $\angle A = \angle B$. AB = AC so by **Theorem 1** $\angle B = \angle C$. Therefore $\angle A = \angle B = \angle C$.
Since the sum of the angle is 180° we must have in fact that $\angle A = \angle B = \angle C = 60°$.

THEOREM 4 (THE CONVERSE OF THEOREM 3)

An equiangular triangle is equilateral.
In Figure 6, if $\angle A = \angle B = \angle C$ then AB = AC = BC.

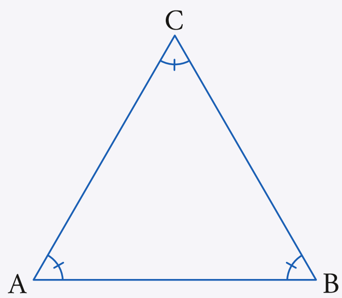
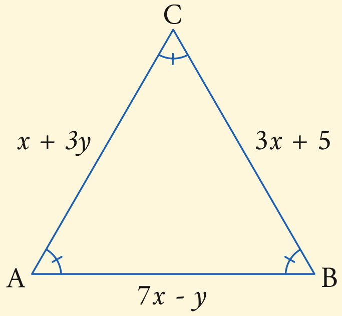


Figure 6. $\triangle ABC$ is eqiangular.

Proof :
$\angle A = \angle B$ so by **Theorem 2**, AC = BC. $\angle B = \angle C$ so by **Theorem 2**, AB = AC.
Therefore AB = AC = BC.

? EXAMPLE D

Find x, y and AC:



SOLUTION

△ABC is equiangular and so by **Theorem 4** is equilateral. Therefore:

$$\begin{aligned} AC &= AB \\ x + 3y &= 7x - y \\ x - 7x + 3y + y &= 0 \\ -6x + 4y &= 0 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} AB &= BC \\ 7x - y &= 3x + 5 \\ 7x - 3x - y &= 5 \\ 4x - y &= 5 \end{aligned}$$

We have a system of two equations in two unknowns to solve:

$$\begin{aligned} -6x + 4y &= 0 \\ 4x - y &= 5 \end{aligned} \quad \begin{aligned} &\longrightarrow \\ &\xrightarrow{4} \end{aligned} \quad \begin{aligned} -6x + 4y &= 0 \\ + \quad 16x - 4y &= 20 \\ \hline 10x &= 20 \\ x &= 2 \end{aligned} \qquad \begin{aligned} 4x - y &= 5 \\ 4(2) - y &= 5 \\ 8 - y &= 5 \\ -y &= 5 - 8 \\ -y &= -3 \\ y &= 3 \end{aligned}$$

CHECK

$$\begin{array}{r|l} AC & = \; AB \\ x + 3y & 7x - y \\ 2 + 3(3) & 7(2) - 3 \\ 2 + 9 & 14 - 3 \\ 11 & 11 \end{array} \qquad \begin{array}{r|l} AB & = \; BC \\ 7x - y & 3x + 5 \\ 7(2) - 3 & 3(2) + 5 \\ 14 - 3 & 6 + 5 \\ 11 & 11 \end{array}$$

ANSWER : x = 2, y = 3, AC = 11.

HISTORICAL NOTE

Theorem 1, the isosceles triangle theorem, is believed to have first been proven by Thales (c. 600 B,C,) - it is Proposition 5 in Euclid's Elements. Euclid's proof is more complicated than ours because he did not want to assume the existence of an angle bisector. Euclid's proof goes as follows:

Given $\triangle ABC$ with $AC = BC$ (*as in Figure 1 at the beginning of this section*) extend CA to D and CB to E so that $AD = BE$ (*Figure 7*). Then $\triangle DCB \cong \triangle ECA$ by SAS = SAS. The corresponding sides and angles of the congruent triangles are equal, so $DB = EA$, $\angle 3 = \angle 4$ and $\angle 1 + \angle 5 = \angle 2 + \angle 6$. Now $\triangle ADB \cong \triangle BEA$ by SAS = SAS. This gives $\angle 5 = \angle 6$ and finally $\angle 1 = \angle 2$.

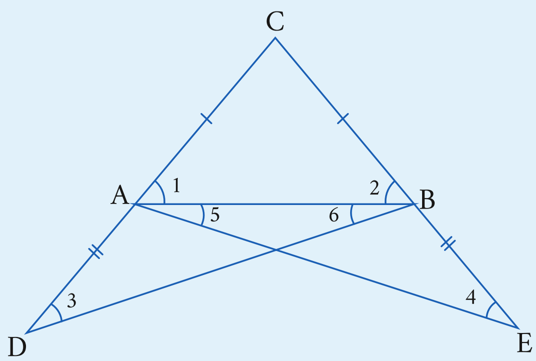


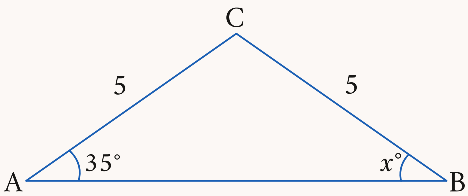
Figure 7. The "Bridge of Fools".

This complicated proof discouraged many students from further study in geometry during the long period when the Elements was the standard text. Figure 7 resembles a bridge which in the Middle Ages became known as the "bridge of fools". This was supposedly because a fool could not hope to cross this bridge and would abandon geometry at this point.

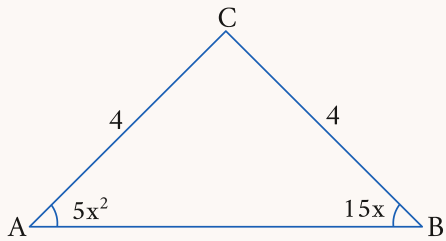
PROBLEMS

1. For each of the following state the theorem(s) used in obtaining your answer:

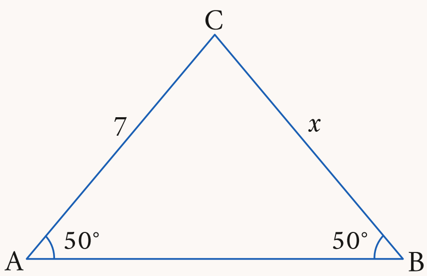
1. Find x:



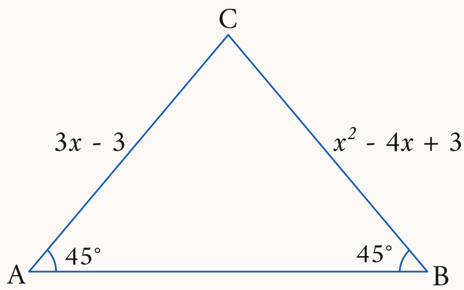
2. Find x, ∠A and ∠B:



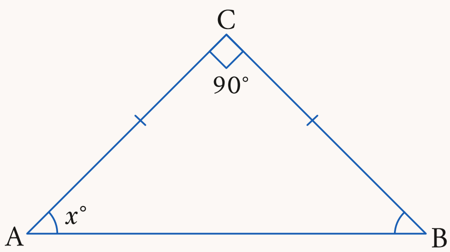
3. Find x:



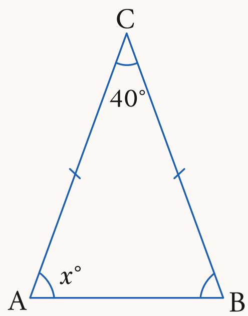
4. Find x, AC and BC:



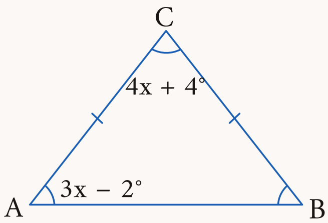
5. Find x:



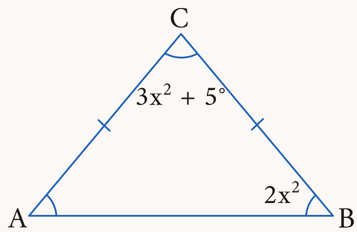
6. Find x:



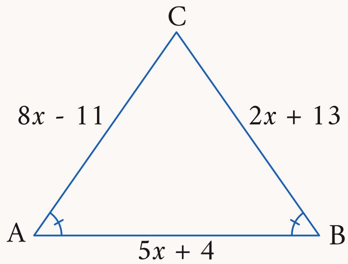
7. Find x, ∠A, ∠B and ∠C:



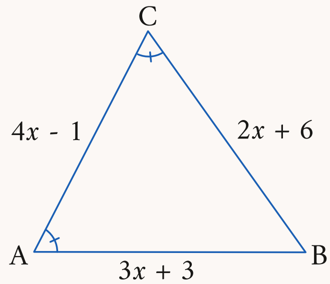
8. Find x, ∠A, ∠B and ∠C:



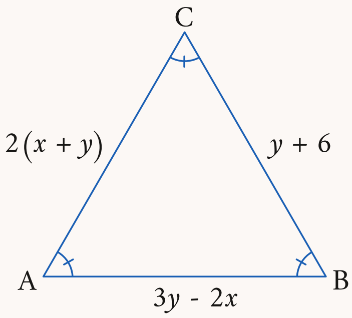
9. Find x, AB, AC and BC:



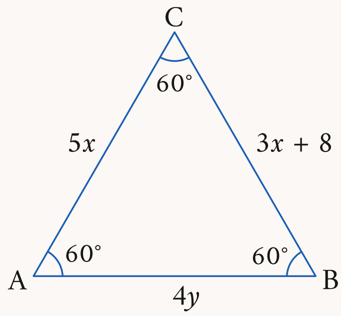
10. Find x, AB, AC and BC:



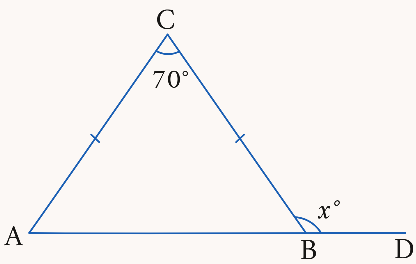
11. Find x, y and AC:



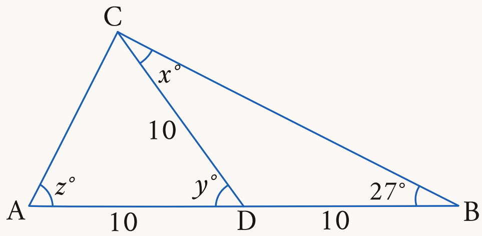
12. Find x, y and AC:



13. Find x:



14. Find x, y and z:



2.6 THE SSS THEOREM

We now consider the case where the side of two triangles are known to be of the same length.

THEOREM 1 (SSS OR SIDE - SIDE - SIDE THEOREM)

Two triangles are congruent if three sides of one are equal respectively to three sides of the other(SSS = SSS).

In Figure 1, if a = d, b = e and c = f then $\triangle ABC \cong \triangle DEF$.

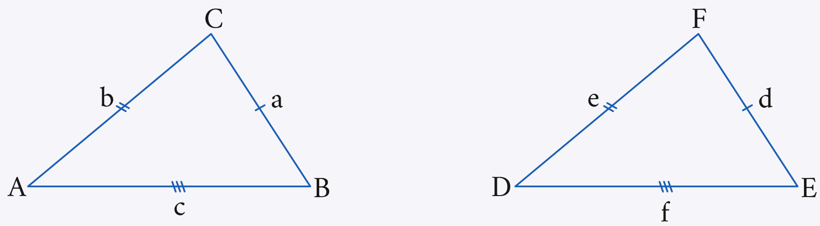
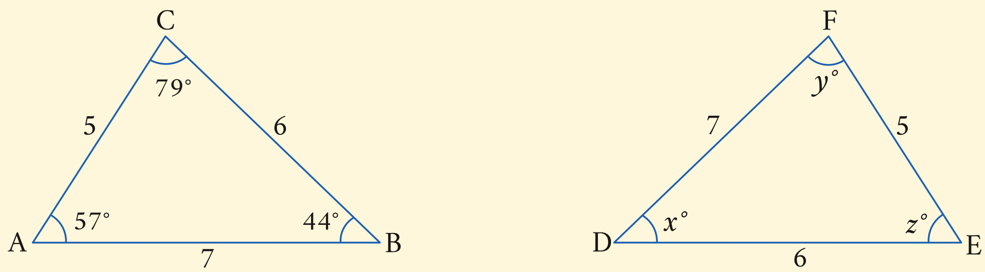


Figure 1. $\triangle ABC \cong \triangle DEF$ because SSS = SSS.

? EXAMPLE A

Find x, y and z:



SOLUTION

AB = 7 = DF. Therefore $\angle C$, the angle opposite AB must correspond to $\angle E$, the angle opposite DF. In the same way $\angle A$ corresponds to $\angle F$ and $\angle B$ corresponds to $\angle D$. We have $\triangle ABC \cong \triangle FDE$ by SSS=SSS. So

$x^\circ = \angle D = \angle B = 44^\circ$

$y^\circ = \angle F = \angle A = 57^\circ$

$z^\circ = \angle E = \angle C = 79^\circ$

ANSWER : x = 44, y = 57, z = 79.

Proof of Theorem 1:

In Figure 1, place △ABC and △DEF so that their longest sides coincide, in this case AB and DE. This can be done because AB = c = f = DE. Now draw CF, forming angles 1, 2, 3, and 4(Figure 2). The rest of the proof will be presented in double-column form:

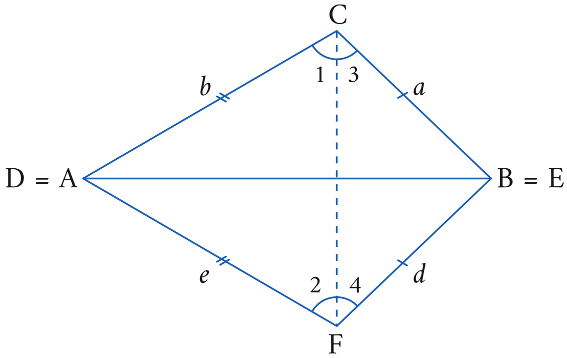
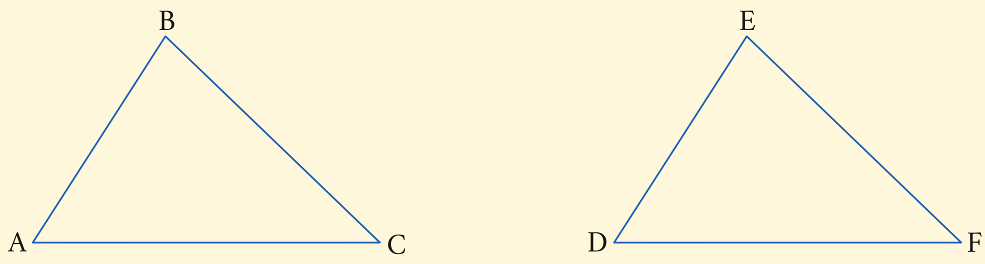


Figure 2. Place △ABC and △DEF so that AB and DE coincide and draw CF.

	Statements		Reasons
1	∠1 = ∠2	1	The base angles of isosceles triangle CAF are equal (Theorem 1, section 2.5).
2	∠3 = ∠4	2	The base angles of isosceles triangle CBF are equal.
3	∠C = ∠F	3	∠C = ∠1 + ∠3 = ∠2 + ∠4 = ∠F.
4	AC = DF	4	Given, AC = b = e = DF.
5	BC = EF	5	Given, BC = a = d = EF.
6	△ABC ≅ △DEF	6	SAS = SAS: AC, ∠C, BC of △ABC = DF, ∠F, EF of △DEF.

? EXAMPLE B

Given AB = DE, BC = EF and AC = DF. Prove ∠C = ∠F:



SOLUTION

Statements	***Reasons***
1. AB = DE	1. Given.
2. BC = EF	2. Given.
3. AC = DF.	3. Given.
4. △ABC ≅ △DEF	4. SSS = SSS: AB, BC, AC of △ABC = DE, EF, DF of △DEF.
5. ∠C = ∠F.	5. Corresponding angles of congruent triangles are equal.

Application:
The SSS Theorem is the basis of an important principle of construction and engineering called **triangular bracing**. Imagine the line segments in Figure 3 to be beans of wood or steel joined at the endpoints by nails or screws. If pressure is applied to one of the sides, ABCD will collapse and look like A′ B′ C′ D′.

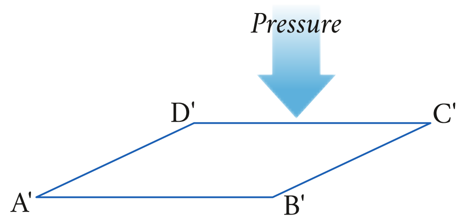


Figure 3. ABCD collapses into A′B′C′D′ when pressure is applied.

Now suppose points A and C are joined by a new beam, called a brace (Figure 4). The structure will not collapse as long as the beans remain unbroken and joined together. It is impossible to deform ABCD into any other shape A′B′C′D′ because if AB = A′B′, BC = B′C′ and AC = A′C′ then △ABC would be congruent to △A′B′C′ by SSS = SSS.

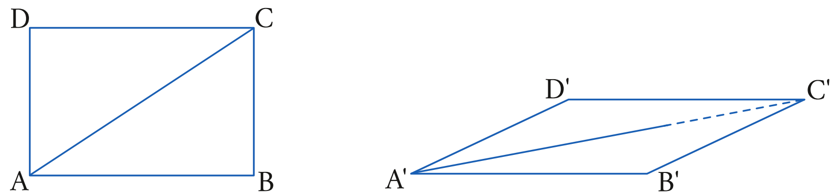


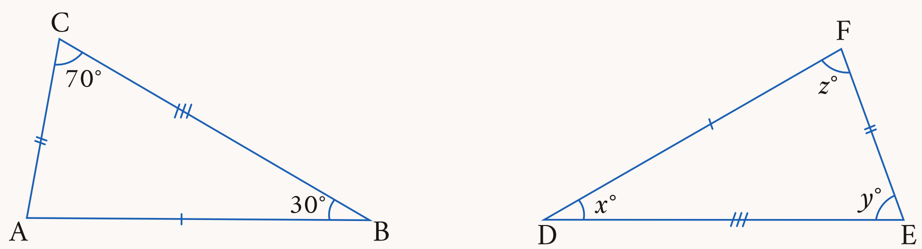
Figure 4. ABCD cannot collapse into A'B'C'D' as long as the beams remain unbroken and joined together.

We sometimes say that a triangle is a **rigid** figure. Once the sides of a triangle are fixed the angles cannot be changed. Thus in Figure 4, the shape of △ABC cannot be changed as long as the lengths of its sides remain the same.

PROBLEMS

1-8. For each of the following
(1) write the congruence statement,
(2) given the reason for (1) (SAS, ASA, AAS or SSS Theorems),
and (3) find x or x and y or x, y and z.

1.



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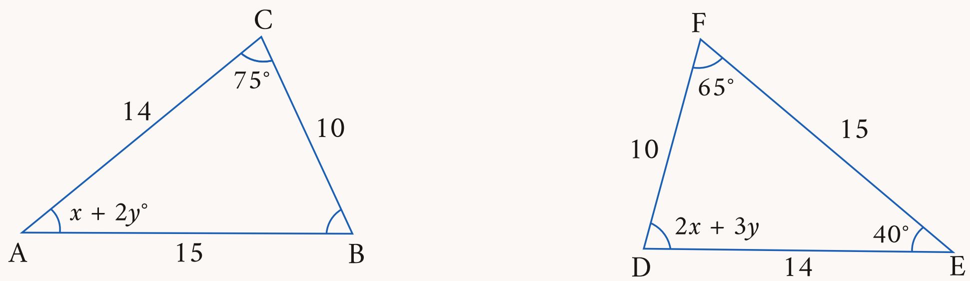

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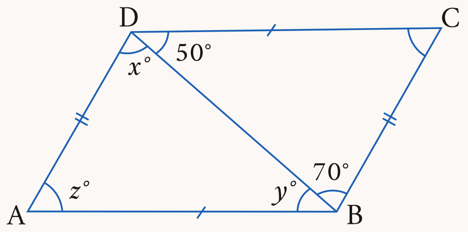

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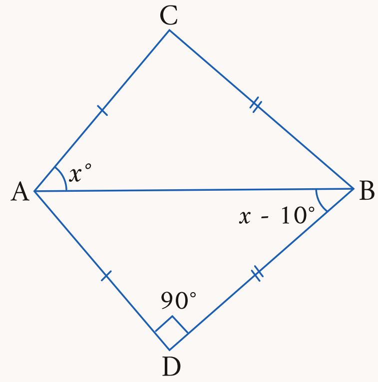

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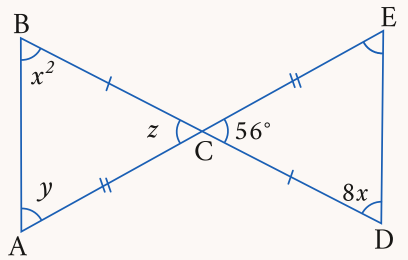

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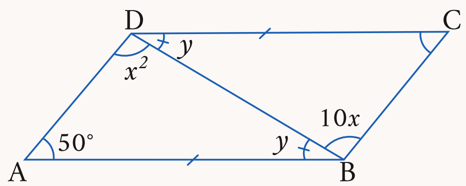

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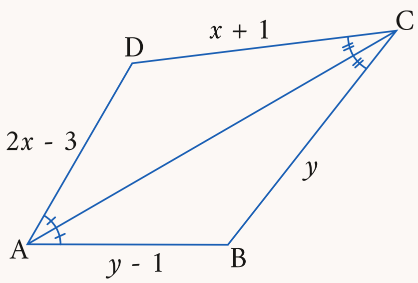

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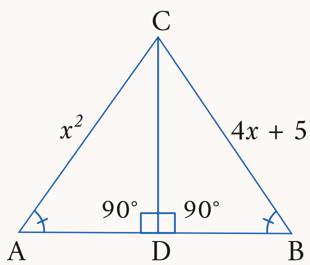
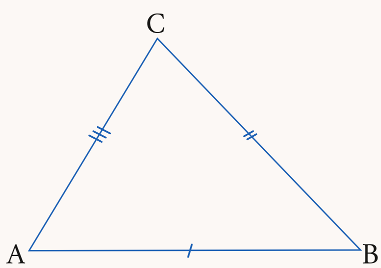
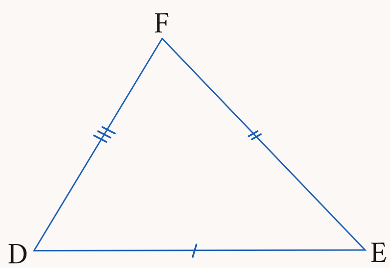
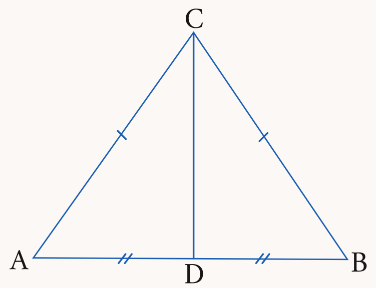


9. Given AB = DE, BC = EF and AC = DF. Prove $\angle A = \angle D$.

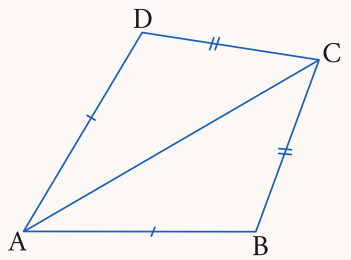




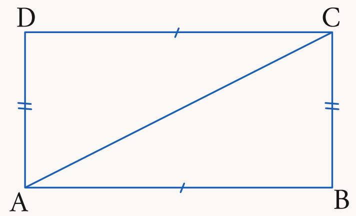
10. Given AC = BC, AD = BD. Prove ∠ADC = ∠BDC.



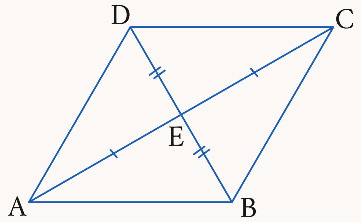
11. Given AB = AD, BC = DC.
Prove ∠BAC = ∠CAD.



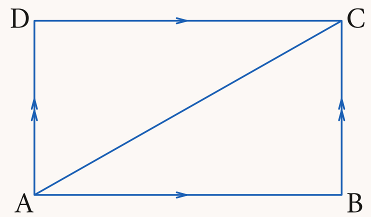
12. Given AB = CD, BC = DA.
Prove ∠BAC = ∠DCA.



13. Given AE = CE, BE = ED.
Prove AB = CD.



14. Given AB ∥ CD, AD ∥ BC.
Prove AB = CD.



2.7 THE HYP - LEG THEOREM and OTHER CASES

We give one more reason for two triangles to be congruent. Note that the following reason applies to **right triangles** only:

THEOREM 1 (HYPOTENUSE - LEG THEOREM)

Two right triangles are congruent if the hypotenuse and a leg of one triangle are respectively equal to the hypotenuse and a leg of the other triangle (Hyp - Leg = Hyp - Leg).

In Figure 1, if AB = DE and BC = BF then $\triangle ABC \cong \triangle DEF$.

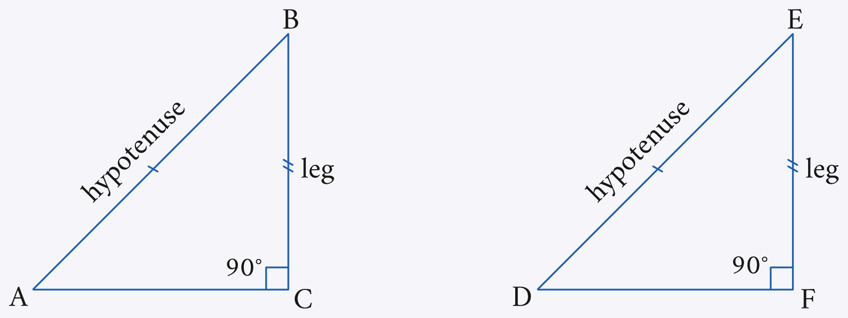
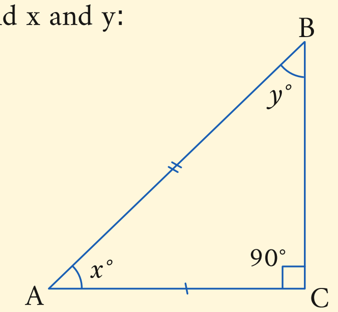
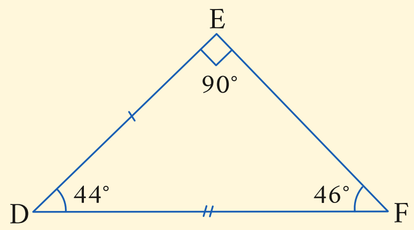


Figure 1. $\triangle ABC \cong \triangle DEF$ because Hyp - Leg = Hyp - Leg.

EXAMPLE A

Find x and y:





SOLUTION

The hypotenuse of $\triangle ABC$ = AB = hypotenuse of $\triangle DBF$ = DF.
A leg of $\triangle ABC$ = AC = A leg of $\triangle DEF$ = DE. Therefore
$\triangle ABC \cong \triangle DFE$ by Hyp - Leg = Hyp - Leg. So
$x^\circ = \angle A = \angle D = 44^\circ$, $y^\circ = \angle B = \angle F = 46^\circ$.

ANSWER : x = 44, y = 46

***Proof of Theorem 1*:**

In Figure 1, place △DEF so that BC and EF coincide (see Figure 2). Then ∠ACD = 180° so AD is a straight line segment. △ABD is isosceles with AB = DE. Therefore ∠A = ∠D because they are the base angles of isosceles triangle ABD (Theorem 1, section2.5).
Then ∠ABC ≅ ∠DEF by AAS = AAS.

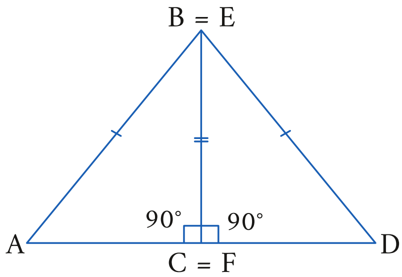
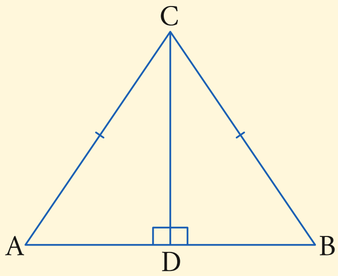


Figure 2. Place △DEF so that BC and EF coincide.

? EXAMPLE B

Given AC = BC, CD ⊥ AB. Prove AD = BD.



SOLUTION

Statements	***Reasons***
1. AC = BC	1. Given.
2. CD = CD	2. Identity
3. ∠ADC = ∠BDC = 90°	3. Given CD ⊥ AB.
4. △ACD ≅ △BCD	4. Hyp-Leg = Hyp-Leg: Hyp AC, Leg CD of △ACD = Hyp BD, Leg CD of △BCD.
5. AD = BD.	5. Corresponding sides of congruent triangles are equal.

At this point the student might be ready to conclude that two triangles are congruent whenever any three corresponding sides or angles are equal. However this is not true in the following two cases:

1. There may be two triangles that are not congruent but have two equal sides and an equal unincluded angle (SSA = SSA).
In Figure 3, AC = DF, BC = BF and ∠A = ∠D but none of the other angles or sides are equal.

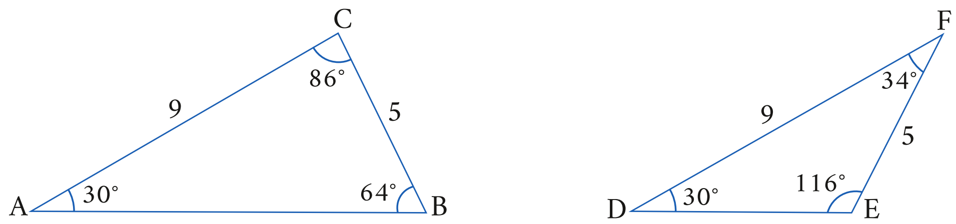


Figure 3. These two triangles satisfy SSA = SSA but are not congruent.

2. There are many triangles that are not congruent but have the same three angles. (AAA = AAA)
In Figure 4, the corresponding angles are equal but the corresponding sides are not.

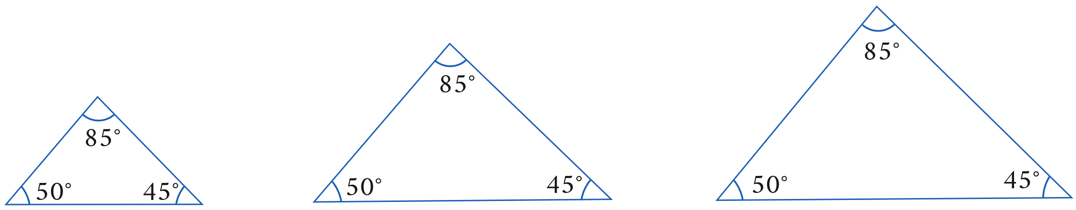
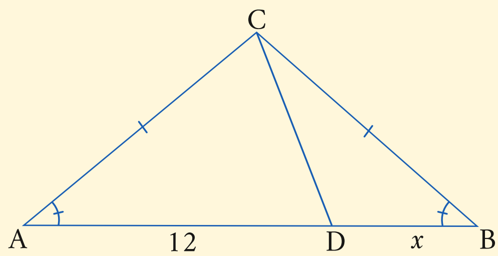


Figure 4. These triangles satisfy AAA = AAA but are not congruent.

If AAA = AAA the triangles are sald to be **similar**. Similar triangles are discussed in **Chapter 4.**

? EXAMPLE C

Determine if the triangles are congruent. If so write the congruence statement and find x.



SOLUTION

From the diagram AC = BC, CD = CD, and $\angle A = \angle B$. These are the only pairs of sides and angles which can be proven equal. $\angle A$ is not included between sides AC and CD and $\angle B$ is not included between sides BC and CD. Therefore we have only SSA = SSA. We cannot conclude the triangles are congruent and we cannot find x.

ANSWER : The triangles cannot be proven congruent.

SUMMARY

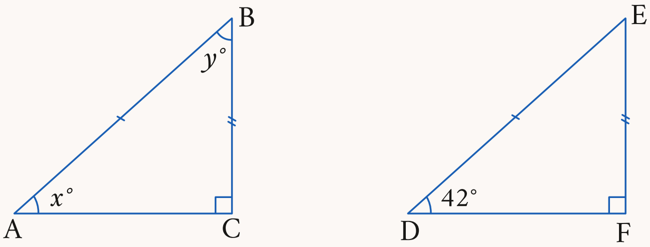
Valid reasons for congruence	*Invalid reasons for congruence*
SAS = SAS	SSA = SSA
ASA = ASA	AAA = AAA
AAS = AAS	
SSS = SSS	

Hyp - Leg = Hyp - Leg (right triangles only).

PROBLEMS

1 - 16. For each of the following, determine if the triangles are congruent. If so
(1) write the congruence statement,
(2) give the reason for (1),
and (3) find x or x and y or x, y and z.
Otherwise write "the triangles cannot be proven congruent".

1.



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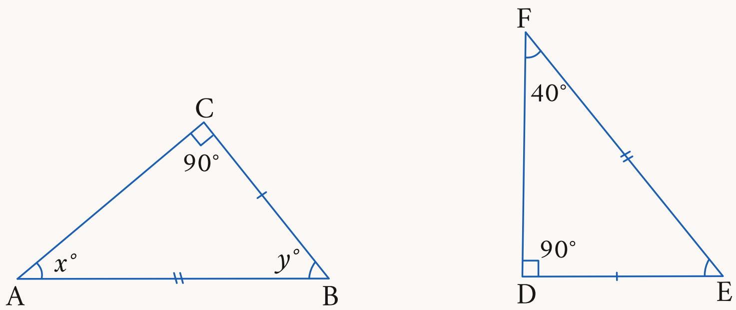

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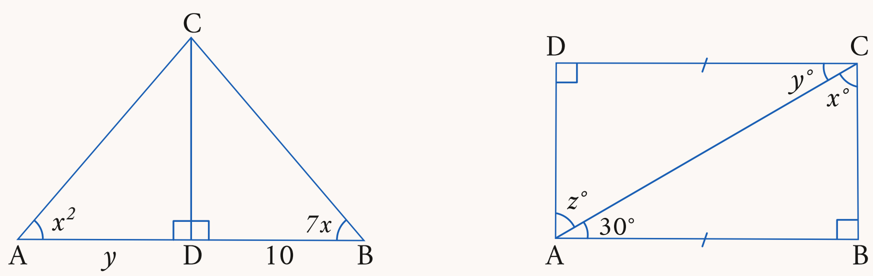

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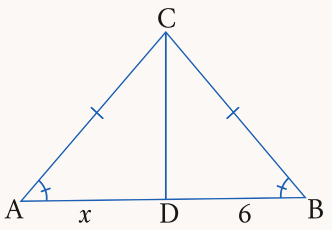

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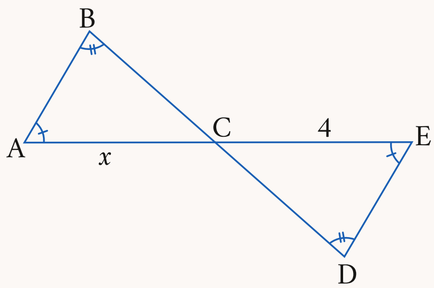

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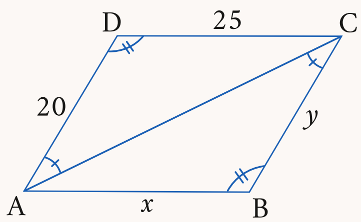

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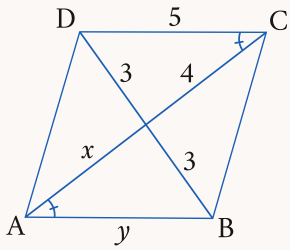

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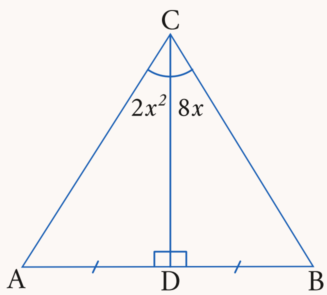

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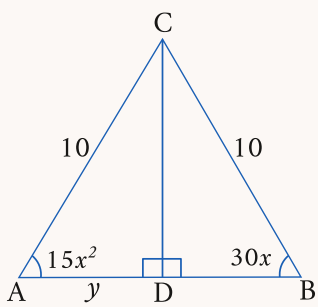

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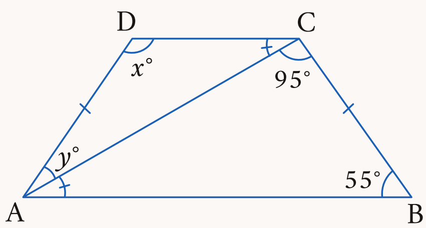

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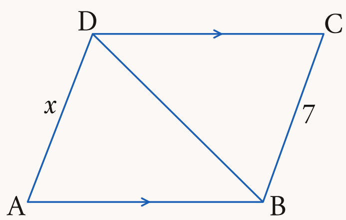

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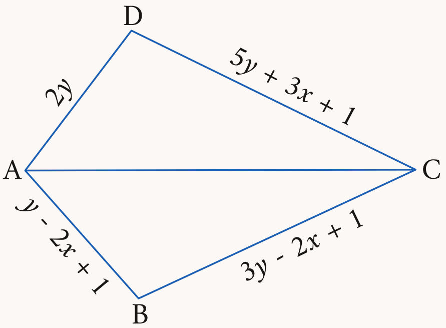

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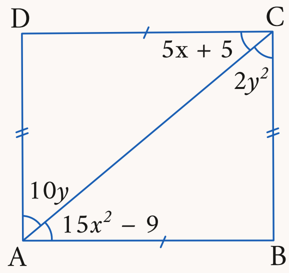

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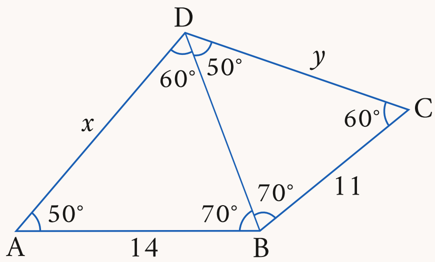

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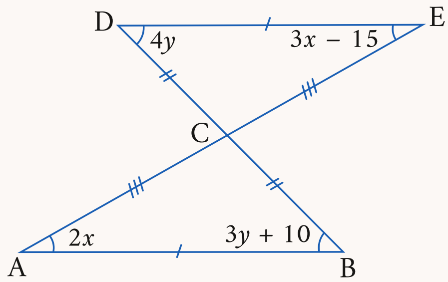
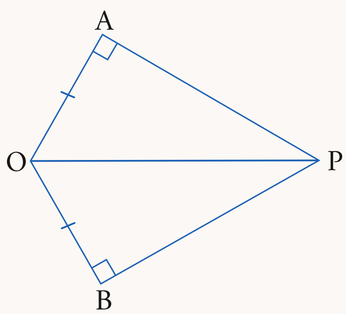
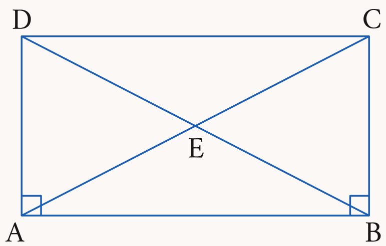


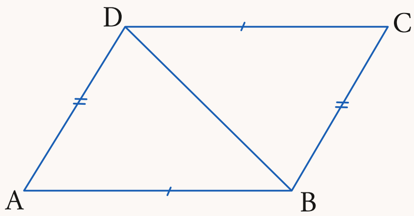
17. Given $OA = OB$ and $\angle A = \angle B = 90°$.
Prove $AP = BP$.



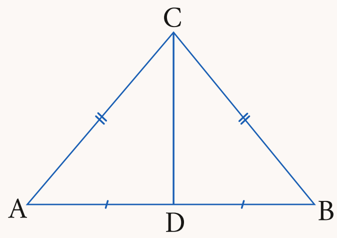
18. Given $AC = BD$ and $\angle A = \angle B = 90°$.
Prove $AD = BC$.
(Hint: Show $\triangle ABC \cong \triangle BAD$)



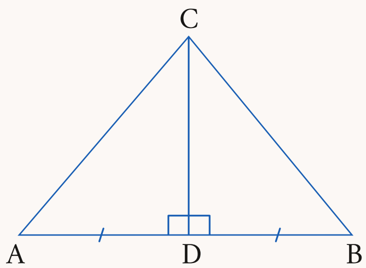
19. Given AB = CD and AD = CB.
    Prove $\angle A = \angle C$.



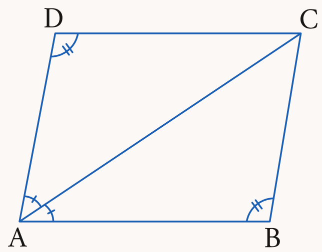
20. Given AC = BC and AD = BD.
    Prove $\angle A = \angle B$.



21. Given AD = BD and $AB \perp CD$.
    Prove $\angle A = \angle B$.



22. Given $\angle BAC = \angle DAC$ and $\angle B = \angle D$.
    Prove AB = AD.



CHAPTER 3

QUADRILATERALS

3.1 PARALLELOGRAMS

A **polygon** is a figure formed by line segments which bound a portion of the plane (Figure 1). The bounding line segments are called the **sides** of the polygon. The angles formed by the sides are the **angles** of the polygon and the vertices of these angles are the **vertices** of the polygon. The simplest polygon is the triangle, which has 3 sides. In this chapter we will study the **quadrilateral**, the polygon with 4 sides (Figure 2). Other polygons are the **pentagon** (5 sides), the **hexagon**(6 sides), the **octagon**(8 sides) and the **decagon** (10 sides).

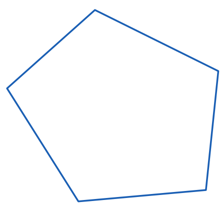


Figure 1
A polygon

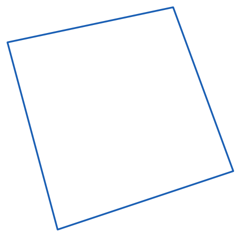


Figure 2
A quadrilateral

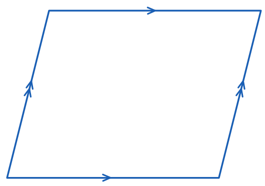


Figure 3
A parallelogram

A **parallelogram** is a quadrilateral in which the opposite sides are parallel (Figure 3). To discover its properties, we will draw a **diagonal**, a line connecting the opposite vertices of the parallelogram. In Figure 4, AC is a diagonal of parallelogram ABCD . We will now prove $\triangle ABC \cong \triangle CDA$:

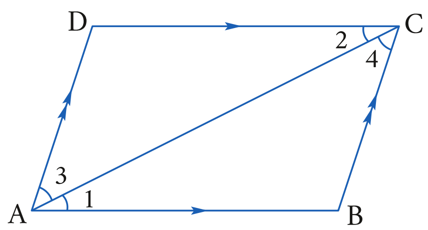


Figure 4. Diagonal AC divides parallelogram ABCD into two congruent triangles.

Statements		Reasons	
1	$\angle 1 = \angle 2$	1	The alternate interior angles of parallel lines AB and CD are equal.
2	$\angle 3 = \angle 4$	2	The alternate interior angles of parallel lines BC and AD are equal.
3	$AC = AC$	3	Identity.
4	$\triangle ABC \cong \triangle CDA$	4	ASA = ASA.
5	$AB = CD$, $BC = DA$	5	The corresponding sides of congruent triangles are equal.
6	$\angle B = \angle D$	6	The corresponding angles of congruent triangles are equal.
7	$\angle A = \angle C$	7	$\angle A = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle C$. (Add statements 1 and 2).

We have proved the following theorem:

THEOREM 1

The opposite sides and opposite angles of a parallelogram are equal.
In parallelogram ABCD of Figure 5, $AB = CD$, $AD = BC$, $\angle A = \angle C$, and $\angle B = \angle D$.

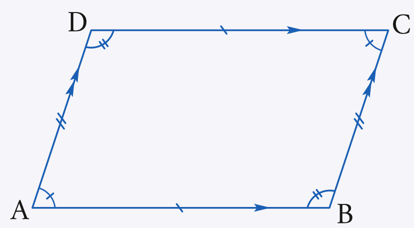
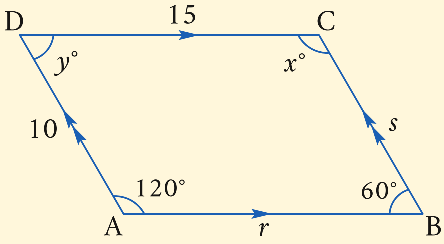


Figure 5. The opposite sides and opposite angles of a parallelogram are equal.

? EXAMPLE A

Find x, y, r and s:



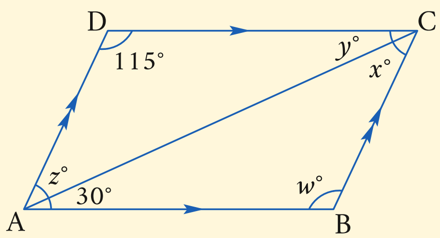
SOLUTION

By **Theorem 1**, the opposite sides and opposite angles are equal.
Hence $x° = 120°$, $y° = 60°$, $r = 15$ and $s = 10$.

ANSWER : $x = 120$, $y = 60$, $r = 15$, $s = 10$.

? EXAMPLE B

Find w, x, y and z:



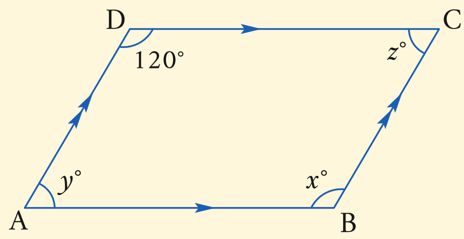
SOLUTION

$w° = 115°$ since the opposite angles of a parallelogram are equal.
$x° = 180° - (w° + 30°) = 180° - (115° + 30°) = 180° - 145° = 35°$, because the sum of the angles of $\triangle ABC$ is $180°$, $y° = 30°$ and $z° = x° = 35°$ because they are alternate interior angles of parallel lines.

ANSWER : $w = 115$, $x = z = 35$, $y = 30$.

? EXAMPLE C

Find x, y and z:



SOLUTION

x = 120 and y = z because the opposite angles are equal. ∠A and ∠D are supplementary because they are interior angles on the same side of the transversal of parallel lines. (They form the letter "C". **Theorem 3**, *Section 1.4*)

ANSWER : x = 120, y = z = 60.

In Example C, ∠A and ∠B, ∠B and ∠C, ∠C and ∠D and ∠D and ∠A are called the **successive** angles of parallelogram ABCD. Example C suggests the following theorem:

THEOREM 2

The successive angles of a parallelogram are supplementary.
In Figure 6, ∠A + ∠B = ∠B + ∠C = ∠C + ∠D = ∠D + ∠A = 180°.

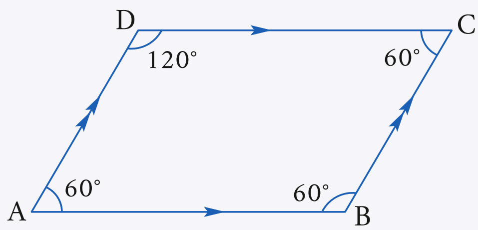
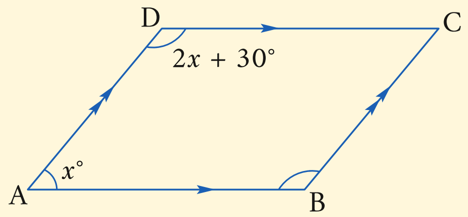


Figure 6. The successive angles of parallelogram ABCD are supplementary.

? EXAMPLE D

Find x, $\angle A$, $\angle B$, $\angle C$ and $\angle D$:



SOLUTION

$\angle A$ and $\angle D$ are supplementary by **Theorem 2**:

$$\begin{aligned} \angle A + \angle D &= 180^\circ \\ x + 2x + 30 &= 180 \\ 3x + 30 &= 180 \\ 3x &= 180 - 30 \\ 3x &= 150 \\ x &= 50 \end{aligned}$$

CHECK

$$\begin{aligned} \angle A + \angle D &= 180^\circ \\ x + 2x + 30 & \\ 50 + 2(50) + 30 & \\ 50 + 130 & \\ 180 & \end{aligned}$$

$\angle A = x^\circ = 50^\circ$, $\angle C = \angle A = 50^\circ$.
$\angle D = 2x + 30^\circ = 2(50) + 30^\circ = 100 + 30^\circ = 130^\circ$.
$\angle B = \angle D = 130^\circ$.

ANSWER : $x = 50$, $\angle A = 50^\circ$, $\angle B = 130^\circ$, $\angle C = 50^\circ$, $\angle D = 130^\circ$

Suppose now that both diagonals of parallelogram are drawn Figure 7:

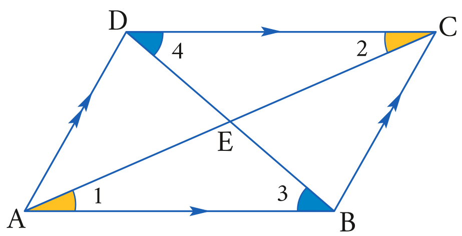


Figure 7. Parallelogram ABCD with diagonals AC and BD.

We have $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$(both pairs of angles are alternate interior angles of parallel lines AB and CD. Also AB = CD from **Theorem 1**. Therefore $\triangle ABE \cong \triangle CDE$ by ASA = ASA. Since corresponding sides of congruent triangles are equal, AE = CE and DE = BE. We have proven:

THEOREM 3

The diagonals of a parallelogram bisect each other (cut each other in half).
In parallelogram ABCD of Figure 8, AE = CE and BE = DE.

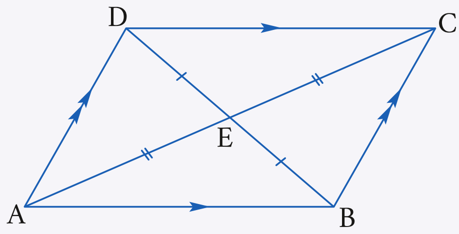
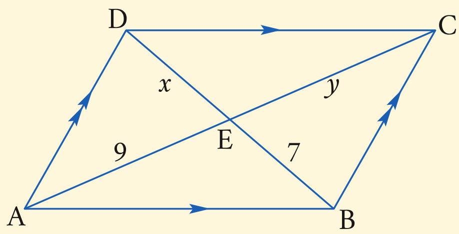


Figure 8. The diagonals of parallelogram ABCD bisect each other.

? EXAMPLE E

Find x, y, AC and BD:



SOLUTION

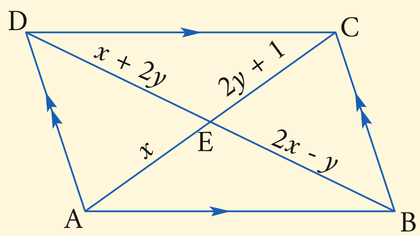
By **Theorem 3** the diagonals bisect each other,

$$
\begin{aligned}
x &= 7 \\
y &= 9 \\
AC &= 9 + 9 = 18 \\
BD &= 7 + 7 = 14
\end{aligned}
$$

ANSWER : x = 7, y = 9, AC = 18, BD = 14

? EXAMPLE F

Find x, y, AC and BD:



SOLUTION

By **Theorem 3** the diagonals bisect each other.

$$\begin{aligned} AE &= CE \\ x &= 2y + 1 \\ x - 2y &= 1 \end{aligned}$$

$$\begin{aligned} BE &= DE \\ 2x - y &= x + 2y \\ 2x - y - x - 2y &= 0 \\ x - 3y &= 0 \end{aligned}$$

$$\begin{array}{rcl} x - 2y = 1 & \longrightarrow & x - 2y = 1 \\ x - 3y = 0 & \xrightarrow{-1} & -x + 3y = 0 \\ & & y = 1 \end{array}$$

$x = 2y + 1 = 2(1) + 1 = 3$

$AC = AE + CE = 3 + 3 = 6$ $BD = BE + DE = 5 + 5 = 10$

CHECK

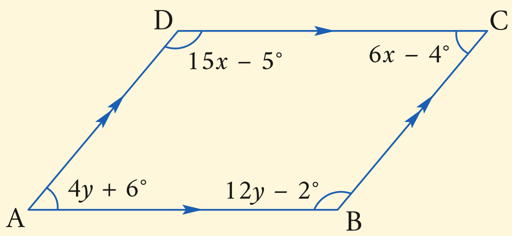
$$\begin{array}{r|l} AE & = CE \\ x & 2y + 1 \\ 3 & 2(1) + 1 \\ & 3 \end{array}$$

$$\begin{array}{r|l} BE & = DE \\ 2x - y & x + 2y \\ 2(3) - 1 & 3 + 2(1) \\ 5 & 5 \end{array}$$

ANSWER : $x = 3$, $y = 1$, $AC = 6$, $BD = 10$.

? EXAMPLE 6

Find x, y, $\angle A$, $\angle B$, $\angle C$ and $\angle D$:



SOLUTION

By **Theorem** 2,

$$\begin{aligned} \angle A + \angle B &= 180° \\ 4y + 6 + 12y - 2 &= 180 \\ 16y + 4 &= 180 \\ 16y &= 180 - 4 \\ 16y &= 176 \\ y &= 11 \end{aligned}$$

and

$$\begin{aligned} \angle C + \angle D &= 180° \\ 6x - 4 + 15x - 5 &= 180 \\ 21x - 9 &= 180 \\ 21x &= 180 + 9 \\ 21x &= 189 \\ x &= 9 \end{aligned}$$

CHECK

$$\angle A + \angle B = 180^\circ$$

$$\begin{array}{r|}
4y + 6 + 12y - 2 \\
4({\color{red}11}) + 6 + 12({\color{red}11}) - 2 \\
50^\circ + 130^\circ \\
180^\circ
\end{array}$$

$$\angle C + \angle D = 180^\circ$$

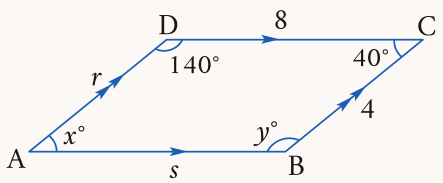
$$\begin{array}{r|}
6x - 4 + 15x - 5 \\
6({\color{red}9}) - 4 + 15({\color{red}9}) - 5 \\
50^\circ + 130^\circ \\
180^\circ
\end{array}$$

ANSWER : x = 9, y = 11, ∠A = ∠C = 50°, ∠B = ∠D = 130°

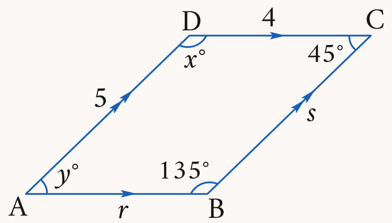
PROBLEMS

For each of the following state any theorem used in obtaining your answer(s):

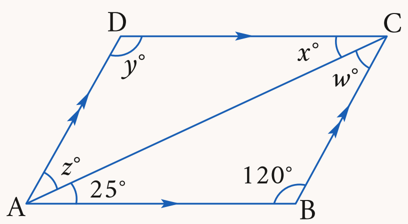
1. Find x, y, r and s:



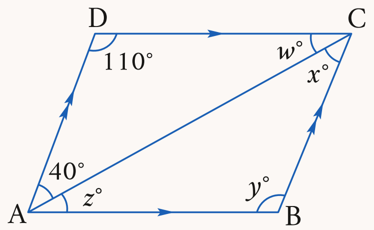
2. Find x, y, r and s:



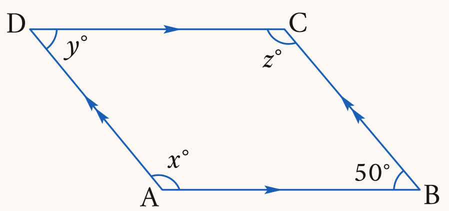
3. Find w, x, y and z:



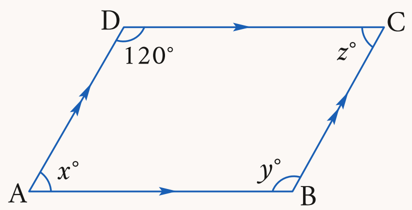
4. Find w, x, y and z:



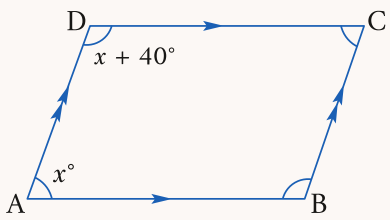
5. Find x, y and z:



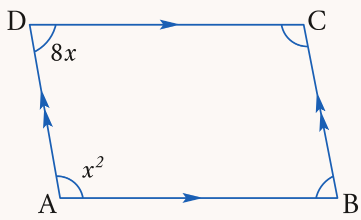
6. Find x, y and z:



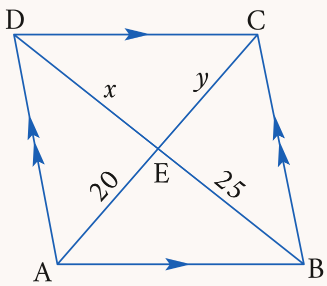
7. Find x, ∠A, ∠B, ∠C and ∠D:



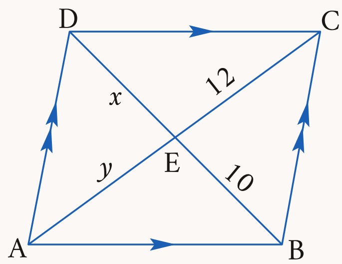
8. Find x, ∠A, ∠B, ∠C and ∠D:



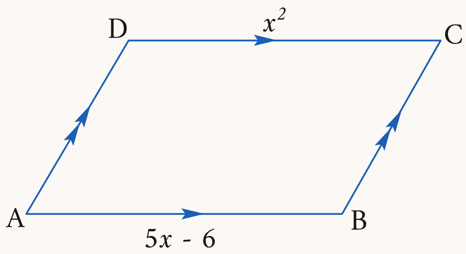
9. Find x, ∠A, ∠B, ∠C and ∠D:



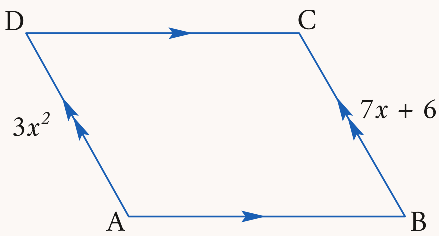
10. Find x, ∠A, ∠B, ∠C and ∠D:



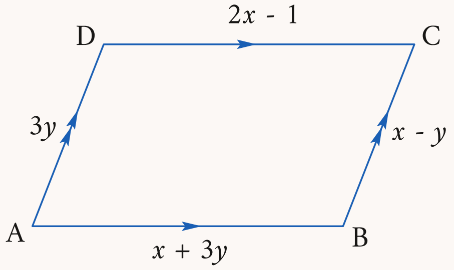
11. Find x, AB and CD:



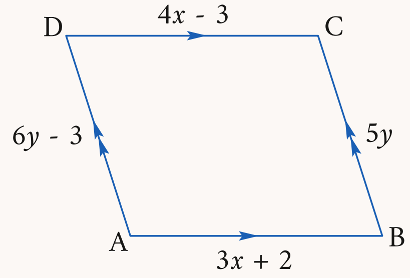
12. Find x, AD and BC:



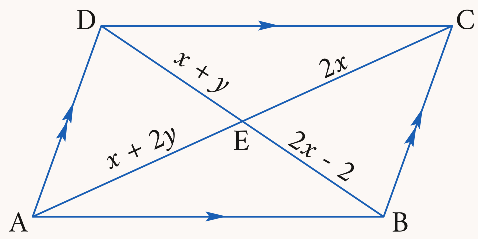
13. Find x, y, AB, BC, CD and AD:



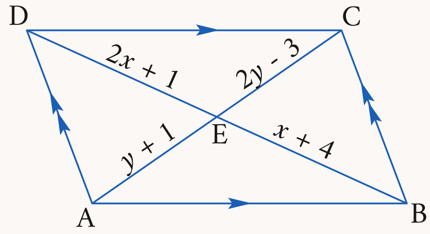
14. Find x, y, AB, BC, CD and AD:



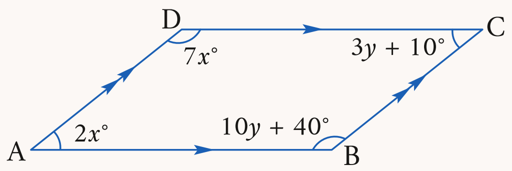
15. Find x, y, AC and BD:



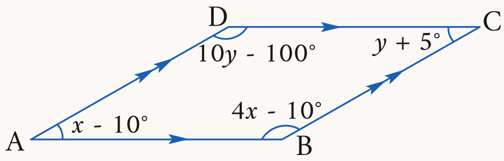
16. Find x, y, AC and BD:



17. Find x, y, ∠A, ∠B, ∠C and ∠D:



18. Find x, y, ∠A, ∠B, ∠C and ∠D:



3.2 OTHER QUADRILATERALS

In this section we will consider other quadrilaterals with special properties: the rhombus, the rectangle, the square and the trapezoid.

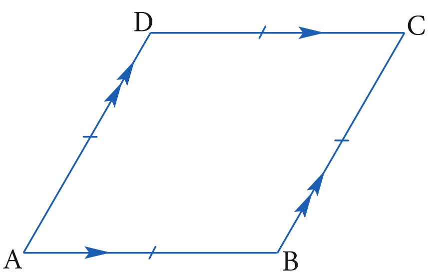


Figure 1. A rhombus.

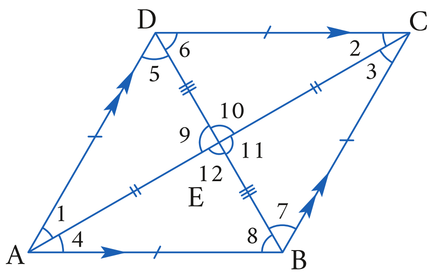


Figure 2. A rhombus with diagonals.

A **rhombus** is a parallelogram in which all sides are equal (Figure 1). It has all the properties of a parallelogram plus some additional ones as well. Let us draw the diagonals AC and BD (Figure 2). By Theorem 3 of section 3.1 the diagonals bisect each other. Hence $\triangle ADE \cong \triangle CDE \cong \triangle CBE \cong \triangle ABE$ by SSS = SSS. The corresponding angles of the congruent triangles are equal:
$\angle 1 = \angle 2 = \angle 3 = \angle 4$, $\angle 5 = \angle 6 = \angle 7 = \angle 8$ and $\angle 9 = \angle 10 = \angle 11 = \angle 12$. $\angle 9$ and $\angle 10$ are supplementary in addition to being equal, hence $\angle 9 = \angle 10 = \angle 11 = \angle 12 = 90°$. We have proven the following theorem:

THEOREM 1

The diagonals of a rhombus are perpendicular and bisect the angles. See Figure 3.

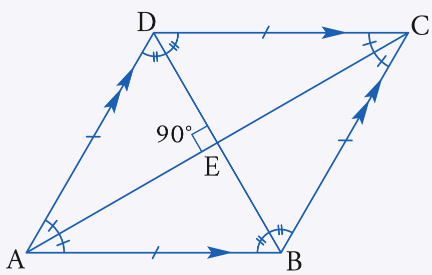
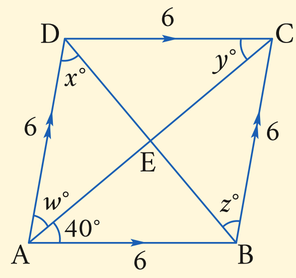


Figure 3. The diagonals of a rhombus are perpendicular and bisect the angles.

? EXAMPLE A

Find w, x, y and z:



SOLUTION

ABCD is a rhombus since it is a parallelogram all of whose sides equal 6. According to **Theorem 1**, the diagonals are perpendicular and bisect the angles. Therefore w° = 40° since AC bisects ∠BAD. ∠AED = 90° so x° = 180° − (90° + 40°) = 180° − 130° = 50° (the sum of the angles of △AED is 180°).
Finally y° = w° = 40° (compare with Figure 3) and z° = x° = 50°.

ANSWER : w = 40, x = 50, y = 40, z = 50

Figure 4 shows rhombus ABCD of Example A with all its angles identified.

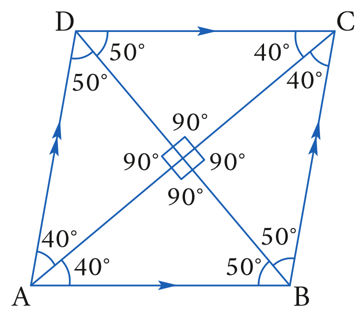


Figure 4. The rhombus of Example A with all angles identified.

A **rectangle** is a parallelogram in which all the angles are right angles (Figure 5). It has all the properties of a parallelogram plus some additional ones as well. It is not actually necessary to be told that all the angles are right angles:

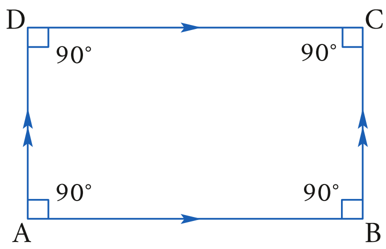


Figure 5. A rectangle.

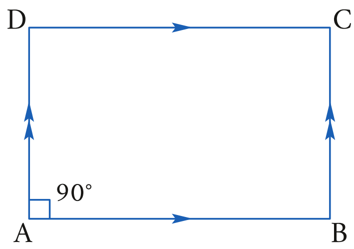


Figure 6. A parallelogram with just one right angle must also be a rectangle.

THEOREM 2

A parallelogram with just one right angle must be a rectangle.

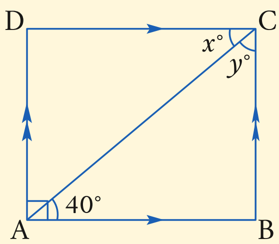
In Figure 6, if $\angle A$ is a right angle then all the other angles must be right angles too.

Proof :

In Figure 6, $\angle C = \angle A = 90^\circ$ because the opposite angles of a parallelogram are equal (**Theorem 1**, *Section 3.1*). $\angle B = 90^\circ$ and $\angle D = 90^\circ$ because the successive angles of a parallelogram are supplementary (**Theorem 2**, *Section 3.1*).

EXAMPLE B

Find x and y:



SOLUTION

By **Theorem 2**, ABCD is a rectangle. $x^\circ = 40^\circ$ because alternate interior angles of parallel lines AB and CD must be equal.

Since the figure is a rectangle, $\angle BCD = 90^\circ$ and $y^\circ = 90^\circ - x^\circ = 90^\circ - 40^\circ = 50^\circ$.

ANSWER : $x = 40$, $y = 50$

Let us draw the diagonals of rectangle ABCD (Figure 7).

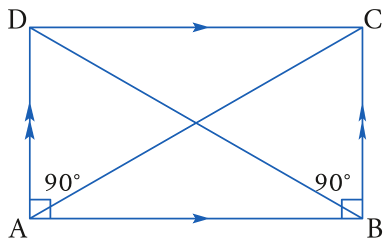


Figure 7. Rectangle wIth diagonals drawn.

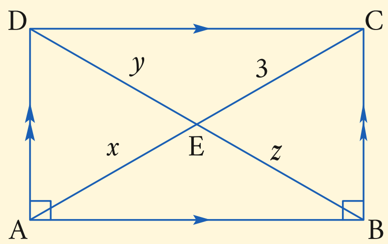
We will show $\triangle ABC \cong \triangle BAD$. AB = BA because of identity. $\angle A = \angle B = 90°$. BC = AD because the opposite sides of a parallelogram are equal. Then $\triangle ABC \cong \triangle BAD$ by SAS = SAS. Therefore diagonal AC = diagonal BD because they are corresponding sides of congruent triangles. We have proven:

THEOREM 3

The diagonals of a rectangle are equal. In Figure 7, AC = BD.

? EXAMPLE C

Find w, x, y, z, AC and BD:



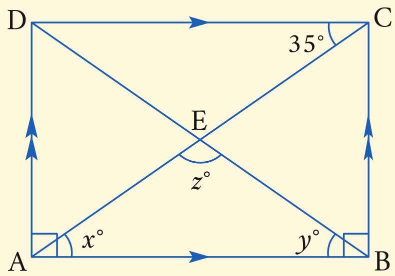
SOLUTION

x = 3 because the diagonals of a parallelogram bisect each other.
So AC = 3 + 3 = 6. BD = AC = 6 since the diagonals of a rectangle are equal (Theorem 3).
Therefore y = z = 3 since diagonal BD is bisected by diagonal AC.

ANSWER : x = y = z = 3 and AC = BD = 6.

? EXAMPLE D

Find x, y and z:



SOLUTION

x° = 35°, because alternate interior angles of parallel lines are equal. y° = x° = 35° because they are base angles of isosceles triangle ABE (AE = BE because the diagonals of a rectangle are equal and bisect each other).
z° = 180° − (x° + y°) = 180° − (35° + 35°) = 180° − 70° = 110°.
Figure 8 shows rectangle ABCD with all the angles identified.

ANSWER: x = y = 35, z = 110.

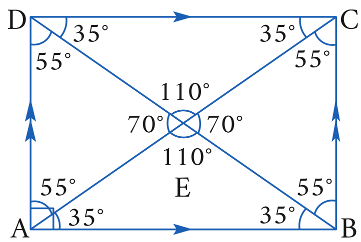


Figure 8. The rectangle of Example D with all the angles identified.

A **square** is a rectangle with all its sides equal. It is therefore also a rhombus. So it has all the properties of the rectangle and all the properties of the rhombus.

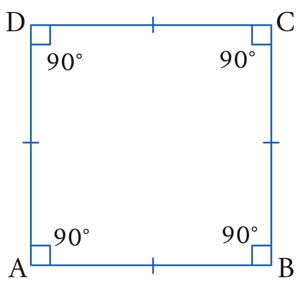


Figure 9. A square.

A **trapezoid** is a quadrilateral with two and only two sides parallel. The parallel sides are called **bases** and the other two sides are called **legs**. In Figure 10, AB and CD are the bases and AD and BC are the legs. ∠A and ∠B are a pair of **base angles**. ∠C and ∠D are another pair of base angles.

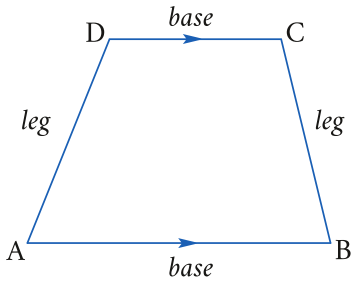


Figure 10. A trapezoid.

An **isosceles trapezoid** is a trapezoid in which the legs are equal. In Figure 11, ABCD is an isosceles trapezoid with AD = BC. An isosceles trapezoid has the following property:

THEOREM 4

The base angles of an isosceles trapezoid are equal.
In Figure 11, ∠A = ∠B and ∠C = ∠D.

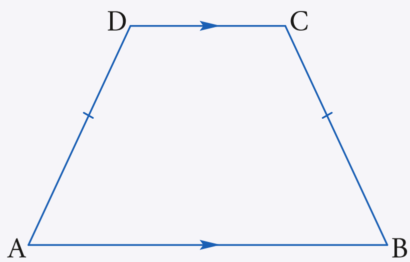
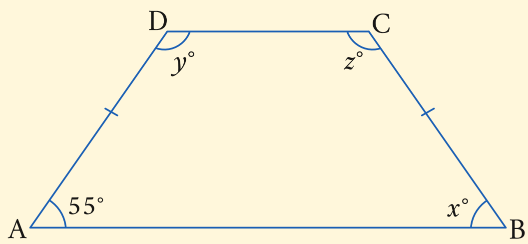


Figure 11. An isosceles trapezoid.

? EXAMPLE E

Find x, y and z:



SOLUTION

$x° = 55°$ because $\angle A$ and $\angle B$, the base angles of isosceles trapezoid ABCD, are equal. Now the interior angles of parallel lines on the same side of the transversal are supplementary (**Theorem 3**, *section 1.4*).
Therefore $y° = 180° - x° = 180° - 55° = 125°$ and $z° = 180° - 55° = 125°$.

ANSWER : $x = 55$, $y = z = 125$

***Proof of Theorem 4*:**
Draw DE parallel to CB as in Figure 12. $\angle 1 = \angle B$ because corresponding angles of parallel lines are equal. DE = BC because they are the opposite sides of parallelogram BCDE. Therefore AD = DE. So $\triangle ADE$ is isosceles and its base angles, $\angle A$ and $\angle 1$ are equal.

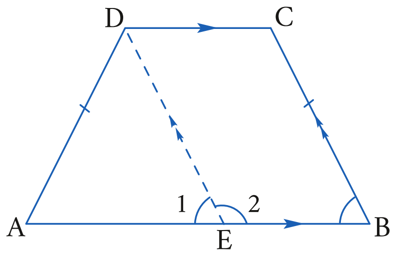


Figure 12. Draw DE parallel to CB.

We have proven $A = \angle 1 = \angle B$. To prove $\angle C = \angle D$, observe that they are both supplements of $\angle A = \angle B$ (**Theorem 3**, *Section 1.4*).

The isosceles trapezoid has one additional property:

THEOREM 5

The diagonals of an isosceles trapezoid are equal.
In Figure 13, AC = BD.

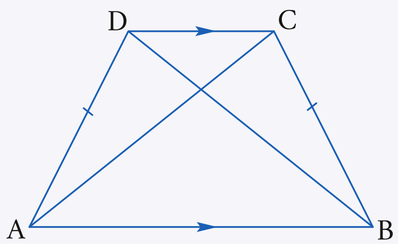
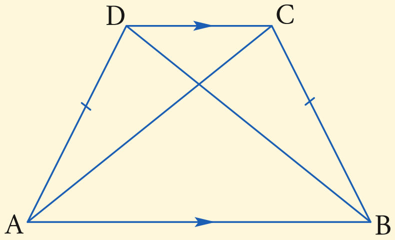


Figure 13. The diagonals AC and BD are equal.

***Proof*:** BC = AD, given. $\angle ABC = \angle BAD$ because they are the base angles of isosceles trapezoid ABCD (Theorem 4). AB = BA, identity. Therefore $\triangle ABC \cong \triangle BAD$ by SAS = SAS. So AC = BD because they are corresponding sides of the congruent triangles.

EXAMPLE F

Find x if $AC = \frac{2}{x}$ and $BD = 3 - x$:



SOLUTION

By **Theorem 5**,

$$\begin{aligned} AC &= BD \\ \frac{2}{x} &= 3 - x \\ (x)\frac{2}{x} &= (3 - x)(x) \\ 2 &= 3x - x^2 \\ x^2 - 3x + 2 &= 0 \\ (x - 1)(x - 2) &= 0 \end{aligned}$$

$$x - 1 = 0 \qquad x - 2 = 0$$

$$x = 1 \qquad x = 2$$

CHECK

$x = 1$:

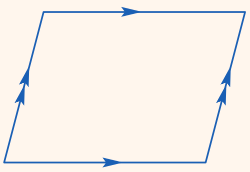
$$\begin{array}{c|c} \multicolumn{2}{c}{AC = BD} \\ \frac{2}{x} & 3 - x \\ \frac{2}{1} & 3 - 1 \\ 2 & 2 \end{array}$$

$x = 2$:

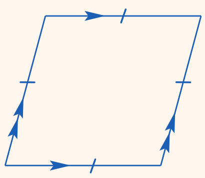
$$\begin{array}{c|c} \multicolumn{2}{c}{AC = BD} \\ \frac{2}{x} & 3 - x \\ \frac{2}{2} & 3 - 2 \\ 1 & 1 \end{array}$$

ANSWER : $x = 1$ or $x = 2$.

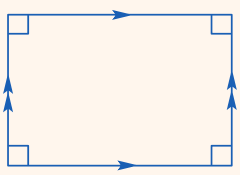
SUMMARY



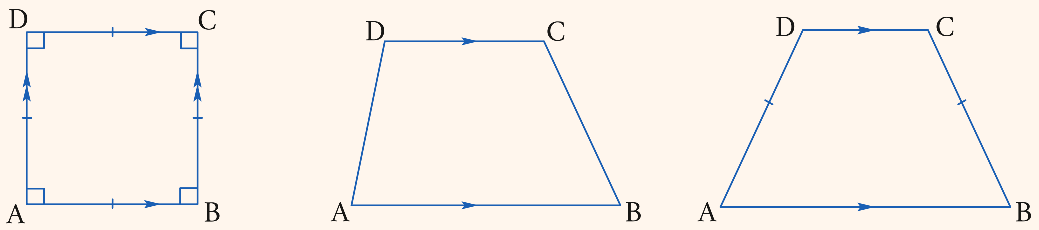
The Parallelogram
A quadrilateral in which the opposite sides are parallel.



The Rhombus
A parallelogram in which all of the sides are equal.



The Rectangle
A parallelogram in which all of the angles are equal to 90°.



The Square
A parallelogram which is both a rhombus and a rectangle.

The Trapezoid
A quadrilateral with just one pair of parallel sides.

The Isosceles Trapezoid
A trapezoid in which the non-parallel sides are equal.

PROPERTIES OF QUADRILATERALS

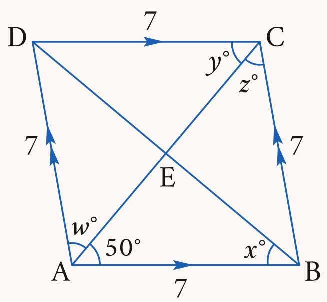
	Opposite sides are parallel	Opposite sides are equal	Opposite angles are equal	Diagonals bisect each other	Diagonals are equal	Diagonals are perpendicular	Diagonals bisect the angles	All sides are equal	All angles are equal
Parallelogram	Yes	Yes	Yes	Yes	-	-	-	-	-
Rhombus	Yes	Yes	Yes	Yes	-	Yes	Yes	Yes	-
Rectangle	Yes	Yes	Yes	Yes	Yes	-	-	-	Yes
Square	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Trapezoid	*	-	-	-	-	-	-	-	-
Isosceles Trapezoid	*	*	-	-	Yes	-	-	-	-

*One pair only.

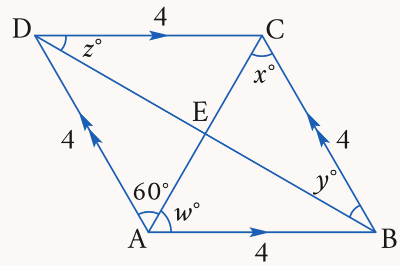
PROBLEMS

For each of the following state any theorems used in obtaining your answer.

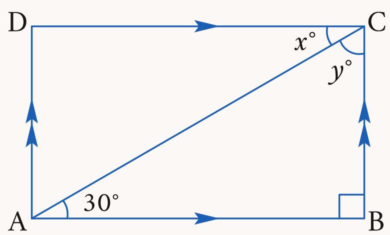
1. Find w, x, y and z:



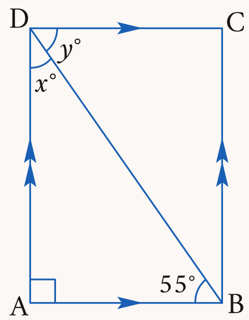
2. Find w, x, y and z:



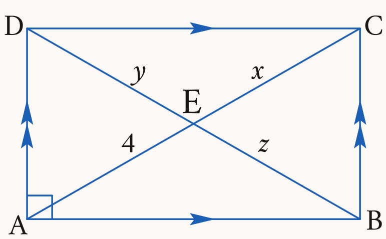
3. Find x and y:



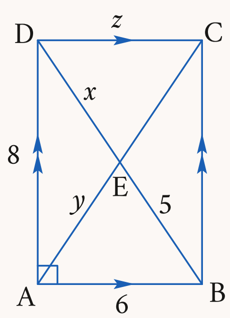
4. Find x and y:



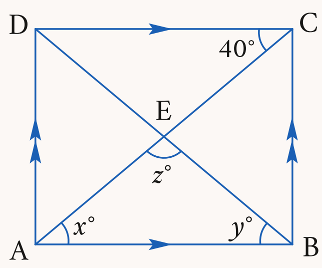
5. Find x, y, z, AC and BD:



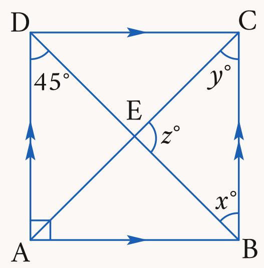
6. Find x, y and z:



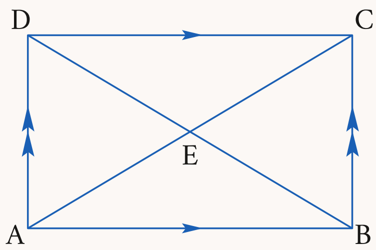
7. Find x, y and z:



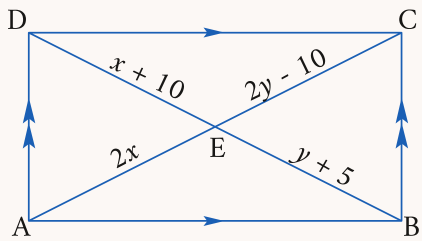
8. Find x, y and z:



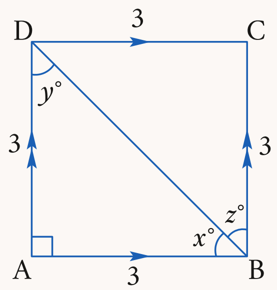
9. Find x if AC = $\frac{3}{x}$ and BD = $4x-1$:



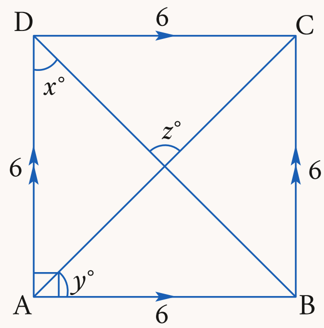
10. Find x and y :



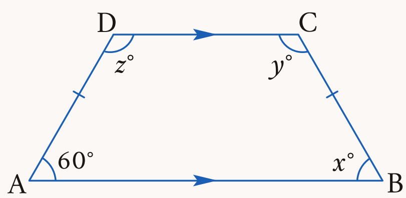
11. Find x, y and z:



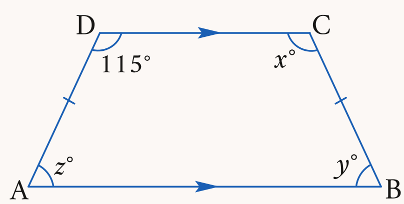
12. Find x, y and z:



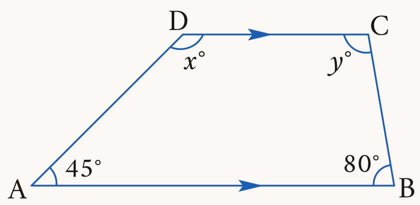
13. Find x, y and z :



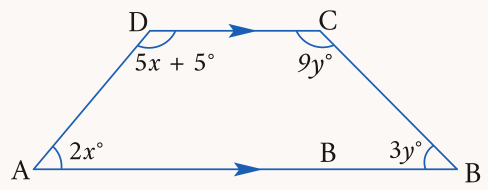
14. Find x, y and z:



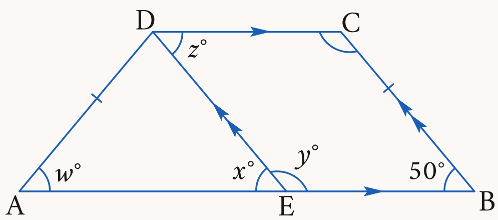
15. Find x and y:



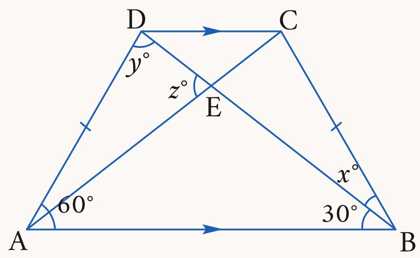
16. Find x, y, ∠A, ∠B, ∠C and ∠D:



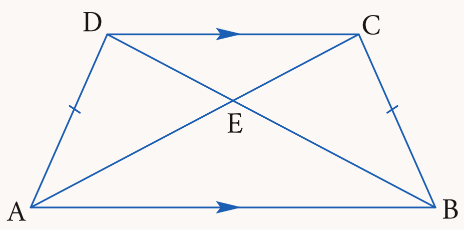
17. Find w, x, y and z:



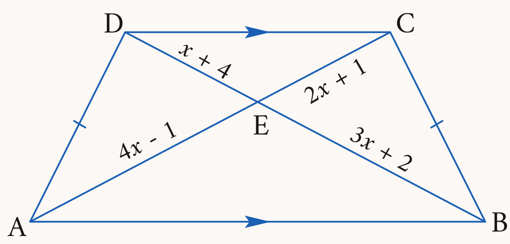
18. Find x, y and z:



19. Find x if AC = $x^2 - 13$ and BD = $2x + 2$.



20. Find x, AC and BD.



CHAPTER 4

SIMILAR TRIANGLES

4.1 PROPORTIONS

In our discussion of similar triangles the idea of a proportion will play an important role. In this section we will review the important properties of proportions.

A **proportion** is an equation which states that two fractions are equal. For example $\frac{2}{6} = \frac{4}{12}$ is a proportion. We sometimes say "2 is to 6 as 4 is to 12 ."

This is also written $2 : 6 = 4 : 12$. The **extremes** of this proportion are the numbers 2 and 12 and the **means** are the numbers 6 and 4. Notice that the product of the means $6 \times 4 = 24$ is the same as the product of the extremes $2 \times 12 = 24$.

THEOREM 1

If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$. Conversely, if $ad = bc$ then $\frac{a}{b} = \frac{c}{d}$.

(The product of the means is equal to the product of the extremes).

Examples

$\frac{2}{6} = \frac{4}{12}$ and $2 \times 12 = 6 \times 4$ are both true.

$\frac{2}{3} = \frac{6}{9}$ and $2 \times 9 = 3 \times 6$ are both true.

$\frac{1}{4} = \frac{4}{12}$ and $1 \times 12 = 4 \times 4$ are both false.

Proof of Theorem 1 :

If $\frac{a}{b} = \frac{c}{d}$, multiply both sides of the equation by bd:

$$\frac{a}{\cancel{b}}(\cancel{b}d) = \frac{c}{\cancel{d}}(b\cancel{d})$$

We obtain ad = bc.

Conversely, if ad = bc, divide both sides of the equation by bd:

$$\frac{a\cancel{d}}{(b\cancel{d})} = \frac{\cancel{b}c}{(\cancel{b}d)}$$

The result is $\frac{a}{b} = \frac{c}{d}$.

The following theorem shows that we can interchange the means or the extremes or both of them simultaneously and still have a valid proportion:

THEOREM 2

If one of the following is true then they are all true:

(1) $\frac{a}{b} = \frac{c}{d}$

(2) $\frac{a}{c} = \frac{b}{d}$

(3) $\frac{d}{b} = \frac{c}{a}$

(4) $\frac{d}{c} = \frac{b}{a}$

Proof :
If any of these proportions is true then ad = bc by **Theorem 1**. The remaining proportions can be obtained from ad = bc by dvision, as in **Theorem 1**.

Example :

$\frac{2}{6} = \frac{4}{12}$, $\frac{2}{4} = \frac{6}{12}$, $\frac{12}{6} = \frac{4}{2}$, $\frac{12}{4} = \frac{6}{2}$ are all true because $2 \times 12 = 6 \times 4$.

The process of converting a proportion $\frac{2}{6} = \frac{4}{12}$ to the equivalent equation $2 \times 12 = 6 \times 4$ is sometimes called **cross multiplication.**

The idea is conveyed by the following notation:

$$\frac{2}{6} \quad \frac{4}{12}$$

? EXAMPLE A

Find x:

$$\frac{3}{x} = \frac{4}{20}$$

SOLUTION

By "cross multiplication"

$$\begin{aligned} 3(20) &= 4x \\ 60 &= 4x \\ 15 &= x \end{aligned}$$

CHECK

$$\frac{3}{x} = \frac{3}{15} = \frac{1}{5}$$

$$\frac{4}{20} = \frac{1}{5}$$

ANSWER : $x = 15.$

? EXAMPLE B

Find x:

$$\frac{x - 1}{x - 3} = \frac{2x + 2}{x + 1}$$

SOLUTION

$$\begin{aligned} (x - 1)(x + 1) &= (x - 3)(2x + 2) \\ x^2 - 1 &= 2x^2 - 4x - 6 \\ 0 &= x^2 - 4x - 5 \\ 0 &= (x - 5)(x + 1) \end{aligned}$$

$$\begin{aligned} 0 &= x - 5 \qquad & 0 &= x + 1 \\ 5 &= x & -1 &= x \end{aligned}$$

CHECK

$x = 5$:

$$\frac{x - 1}{x - 3} = \frac{5 - 1}{5 - 3} = \frac{4}{2} = 2 \qquad \frac{2x + 2}{x + 1} = \frac{2(5) + 2}{5 + 1} = \frac{12}{6} = 2$$

$x = -1$:

$$\frac{x - 1}{x - 3} = \frac{-1 - 1}{-1 - 3} = \frac{-2}{-4} = \frac{1}{2} \qquad \frac{2x + 2}{x + 1} = \frac{2(-1) + 2}{-1 + 1} = \frac{0}{0}$$

Since $\frac{0}{0}$ is undefined, we reject the answer $x = -1$.

ANSWER : $x = 5.$

PROBLEMS

1 - 12. Find x:

1. $\frac{6}{x} = \frac{18}{3}$

2. $\frac{4}{x} = \frac{2}{6}$

3. $\frac{x}{4} = \frac{9}{3}$

4. $\frac{x}{8} = \frac{9}{6}$

5. $\frac{7}{1} = \frac{x}{3}$

6. $\frac{10}{2} = \frac{25}{x}$

7. $\frac{x + 5}{x} = \frac{5}{4}$

8. $\frac{x - 6}{4} = \frac{5}{10}$

9. $\frac{3 + x}{x} = \frac{3}{2}$

10. $\frac{x}{x + 3} = \frac{4}{x}$

11. $\frac{3x - 3}{2x + 6} = \frac{x - 1}{x}$

12. $\frac{3x - 6}{x - 2} = \frac{2x + 2}{x - 1}$

4.2 SIMILAR TRIANGLES

Two triangles are said to be **similar** if they have equal sets of angles. In Figure 1, △ABC is similar to △DEF. The angles which are equal are called **corresponding angles**. In Figure 1, ∠A corresponds to ∠D, ∠B corresponds to ∠E and ∠C corresponds to ∠F. The sides joining corresponding vertices are called **corresponding sides**. In Figure 1, AB corresponds to DE, BC corresponds to EF, and AC corresponds to DF. The symbol for similar is ∼. The **similarity statement** △ABC ∼ △DEF will always be written so that corresponding vertices appear in the same order.

For the triangles in Figure 1, we could also write △BAC ∼ △EDF or △ACB ∼ △DFE but **never** △ABC ∼ △EDF nor △ACB ∼ △DEF.

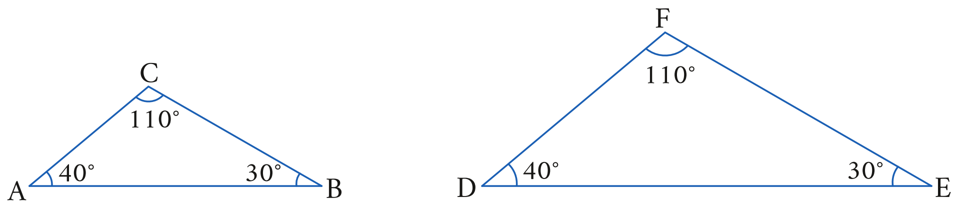
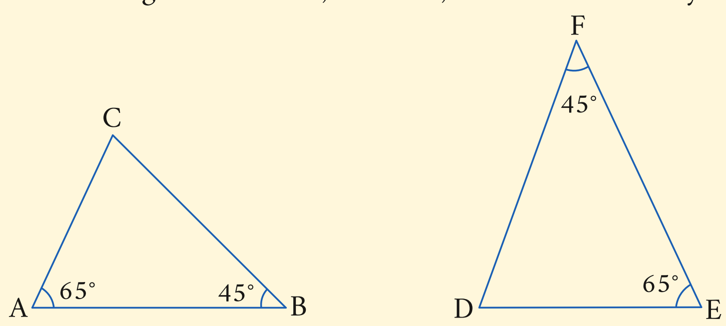


Figure 1. △ABC is similar to △DEF.

We can tell which sides correspond from the similarity statement. For example, if △ABC∼△DEF, then side AB corresponds to side DE because both are the first two letters. BC corresponds to EF because both are the last two letters, AC corresponds to DF because both consist of the first and last letters.

? EXAMPLE A

Determine if the triangles are similar, and if so, write the similarity statement:



SOLUTION

∠C = 180° − (65° + 45°) = 180° − 110° = 70°.

∠D = 180° − (65° + 45°) = 180° − 110° = 70°.

Therefore both triangles have the same angles and △ABC ∼ △EFD.

ANSWER : △ABC ∼ △EFD

Example A suggests that to prove similarity it is only necessary to know that **two** of the corresponding angles are equal:

THEOREM 1

Two triangles are similar if two angles of one equal two angles of the other (AA = AA). In Figure 2, $\triangle ABC \sim \triangle DEF$ because $\angle A = \angle D$ and $\angle B = \angle E$.

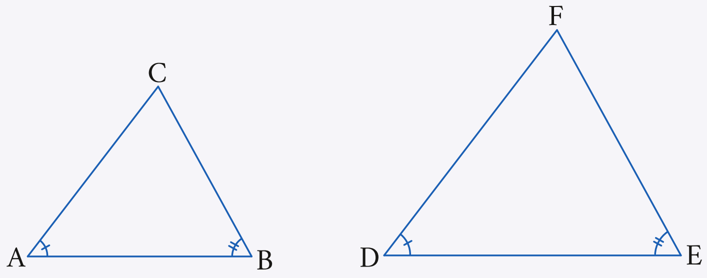
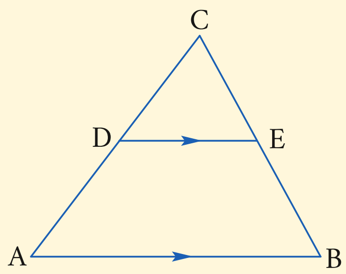


Figure 2. $\triangle ABC \sim \triangle DEF$ *because* AA = AA.

Proof:
$\angle C = 180° - (\angle A + \angle B) = 180° - (\angle D + \angle E) = \angle F$.

EXAMPLE B

Determine which triangles are similar and write a similarity statement:



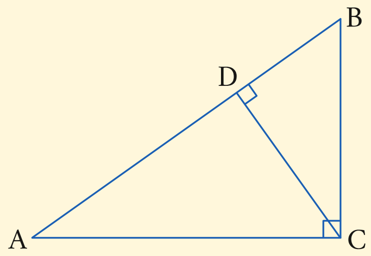
SOLUTION

$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by AA = AA.

ANSWER : $\triangle ABC \sim \triangle DEC$.

? EXAMPLE C

Determine which triangles are similar and write a similarity statement:



SOLUTION

$\angle A = \angle A$ identity. $\angle ACB = \angle ADC = 90°$. Therefore

$$\triangle ABC \sim \triangle ACD$$

Also $\angle B = \angle B$ identity. $\angle BDC = \angle BCA = 90°$. Therefore

$$\triangle ABC \sim \triangle CBD$$

ANSWER : $\triangle ABC \sim \triangle ACD \sim \triangle CBD$

Similar triangles are important because of the following theorem:

THEOREM 2

The corresponding sides of similar triangles are proportional.
This means that if $\triangle ABC \sim \triangle DEF$ then

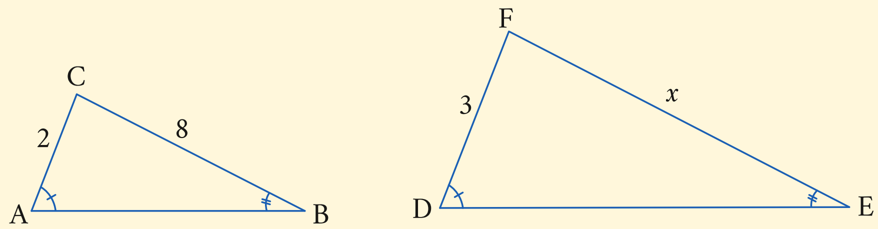
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

That is, the first two letters of $\triangle ABC$ are to the first two letters of $\triangle DEF$ as the last two letters of $\triangle ABC$ are to the last two letters of $\triangle DEF$ as the first and last letters of $\triangle ABC$ are to the first and last letters of $\triangle DEF$.

Before attempting to prove **Theorem 2**, we will give several examples of how it is used:

? EXAMPLE D

Find x:



SOLUTION

$\angle A = \angle D$ and $\angle B = \angle E$ so $\triangle ABC \sim \triangle DEF$. By **Theorem 2**

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

We will ignore $\frac{AB}{DE}$ here since we do not know and do not have to find either AB or DE.

$$\frac{BC}{EF} = \frac{AC}{DF}$$

$$\frac{8}{x} = \frac{2}{3}$$

$$24 = 2x$$

$$12 = x$$

ANSWER : $x = 12$

CHECK

$$\frac{BC}{EF} = \frac{AC}{DF}$$

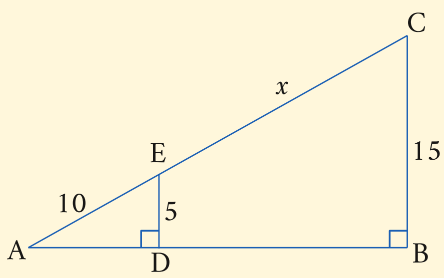
$$\frac{8}{x} \quad \Bigg| \quad \frac{2}{3}$$

$$\frac{8}{12}$$

$$\frac{2}{3}$$

? EXAMPLE E

Find x:



SOLUTION

$\angle A = \angle A$ and $\angle ADE = \angle ABC$, so $\triangle ADE \sim \triangle ABC$ by AA = AA.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}$$

We ignore $\frac{AD}{AB}$.

$$\frac{DE}{BC} = \frac{AE}{AC}$$

$$\frac{5}{15} = \frac{10}{10 + x}$$

$$5(10 + x) = 15(10)$$
$$50 + 5x = 150$$
$$5x = 150 - 50$$
$$5x = 100$$
$$x = 20$$

ANSWER : $x = 20$

CHECK

$$\frac{DE}{BC} = \frac{AE}{AC}$$

$$\frac{5}{15} \quad \Big| \quad \frac{10}{10 + x}$$

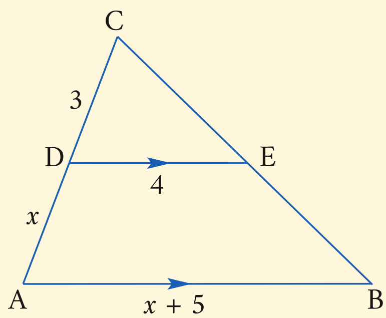
$$\frac{1}{3} \quad \Big| \quad \frac{10}{10 + 20}$$

$$\frac{10}{30}$$

$$\frac{1}{3}$$

? EXAMPLE F

Find x:



SOLUTION

$\angle A = \angle CDE$ because they are corresponding angles of parallel lines. $\angle C = \angle C$ because of identity. Therefore $\triangle ABC \sim \triangle DEC$ by AA = AA.

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC}$$

We ignore $\frac{BC}{EC}$:

$$\frac{AB}{DE} = \frac{AC}{DC}$$

$$\frac{x+5}{4} = \frac{x+3}{3}$$

$$(x+5)(3) = 4(x+3)$$
$$3x + 15 = 4x + 12$$
$$15 - 12 = 4x - 3x$$
$$3 = x$$

ANSWER : $x = 3$

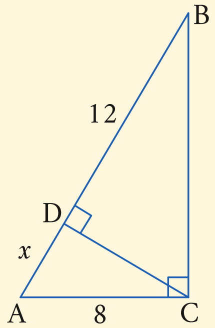
CHECK

$$\frac{AB}{DE} = \frac{AC}{DC}$$

$\frac{x+5}{4}$	$\frac{x+3}{3}$
$\frac{3+5}{4}$	$\frac{3+3}{3}$
$\frac{8}{4}$	$\frac{6}{3}$
2	2

? EXAMPLE G

Find x :



SOLUTION

$\angle A = \angle A$, $\angle ACB = \angle ADC = 90°$, $\triangle ABC \sim \triangle ACD$.

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{x+12}{8} = \frac{8}{x}$$

$$(x+12)(x) = (8)(8)$$
$$x^2 + 12x = 64$$
$$x^2 + 12x - 64 = 0$$
$$(x-4)(x+16) = 0$$

$x = 4$, $x = -16$ We reject the answer $x = -16$ because $AD = x$ cannot be negative.

ANSWER : $x = 4$

CHECK

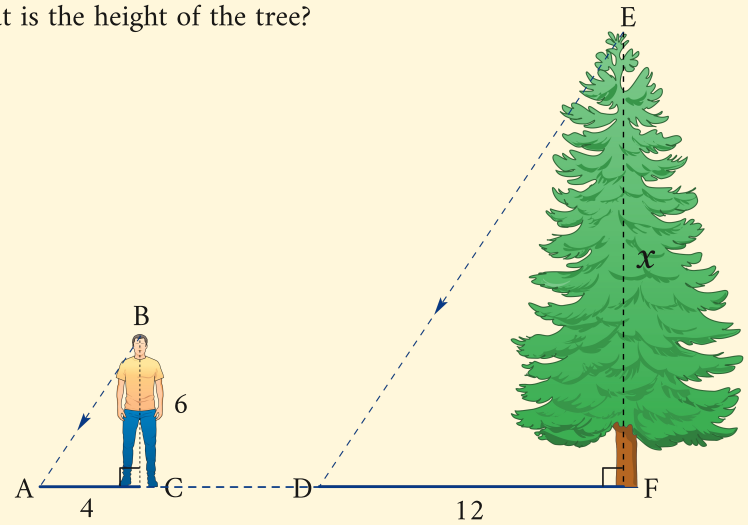
$x = 4$

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$\frac{x+12}{8}$	$\frac{8}{x}$
$\frac{4+12}{8}$	$\frac{8}{4}$
$\frac{16}{8}$	2
2	

? EXAMPLE H

A tree casts a shadow 12 feet long at the same time a 6 foot man casts a shadow 4 feet long. What is the height of the tree?



SOLUTION

In the diagram AB and DE are parallel rays of the sun. Therefore $\angle A = \angle D$ because they are corresponding angles of parallel lines with respect to the transversal AF.
Since also $\angle C = \angle F = 90°$, we have $\triangle ABC \sim \triangle DEF$ by AA = AA.

$$\frac{AC}{DF} = \frac{BC}{EF}$$

$$\frac{4}{12} = \frac{6}{x}$$

$$4x = 72$$

$$x = 18$$

ANSWER: x = 18 feet

Proof of Theorem 2 (The corresponding sides of similar triangles are proportional):
We illustrate the proof using the triangles of Example D (Figure 3). The proof for other similar triangles follows the same pattern.
Here we will prove that $x = 12$ so that $\frac{2}{3} = \frac{8}{x}$.

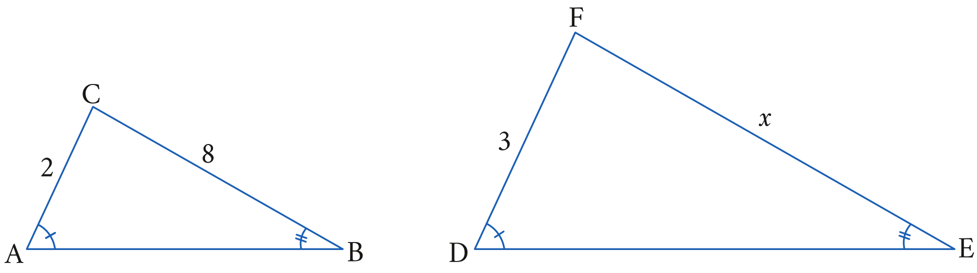


Figure 3. The triangles of Example D.

First draw lines parallel to the sides of △ABC and △DEF as shown in Figure 4. The corresponding angles of these parallel lines are equal and each of the parallelograms with a side equal to 1 has its opposite side equal to 1 as well. Therefore all of the small triangles with a side equal to 1 are congruent by AAS = AAS.

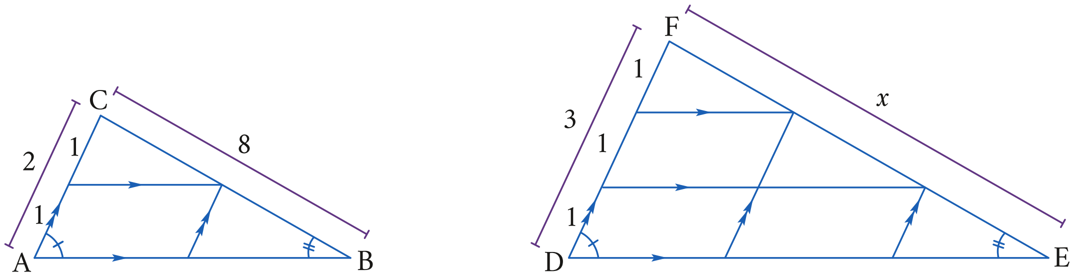


Figure 4. Draw lines parallel to the sides of △ABC and △DEF.

The corresponding sides of these triangles form side BC = 8 of △ABC (see Figure 5). Therefore each of these sides must equal 4 and x = EF = 4 + 4 + 4 = 12 (Figure 6).

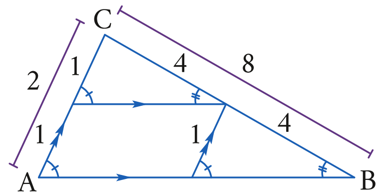


Figure 5. The small triangles are congruent hence the corresponding sides lying on BC must each be equal to 4.

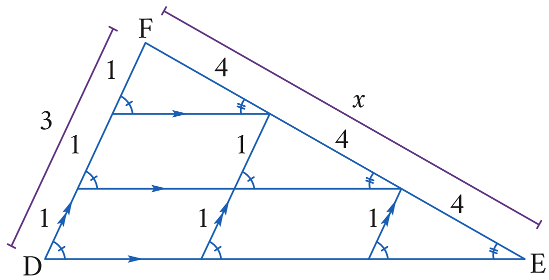


Figure 6. The small triangles of △DEF are congruent to the small triangles of △ABC hence x = EF = 4 + 4 + 4 = 12.

Note to instructor:
This proof can be carried out whenever the lengths of the sides of the triangles are rational numbers. However, since irrational numbers can be approximated as closely as necessary by rationals, the proof extends to that case as well.

HISTORICAL NOTE

Thales (c. 600 B.C.) used the proportionality of sides of similar triangles to measure the heights of the pyramids in Egypt. His method was much like the one we used in Example H to measure the height of trees.

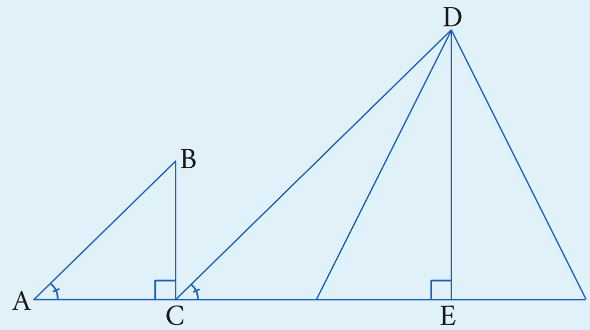


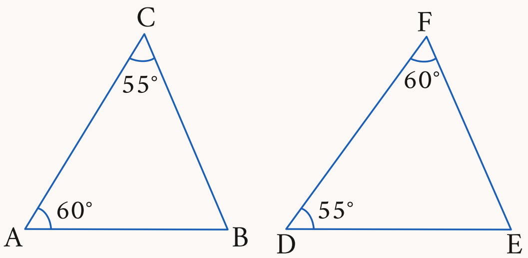
Figure 7. Using similar triangles to measure the height of a pyramid.

In Figure 7, DE represent the height of the pyramid and CE is the length of its shadow. BC represents a vertical stick and AC is the length of its shadow. We have $\triangle ABC \sim \triangle CDE$. Thales was able to measure directly the lengths AC, BC and CE. Substituting these values in the proportion $\frac{BC}{DE} = \frac{AC}{CE}$, he was able to find the height DE.

PROBLEMS

1 - 6. Determine which triangles are similar and write the similarity statement:

1.



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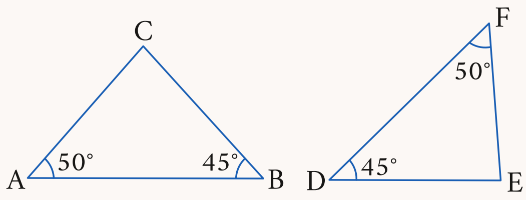
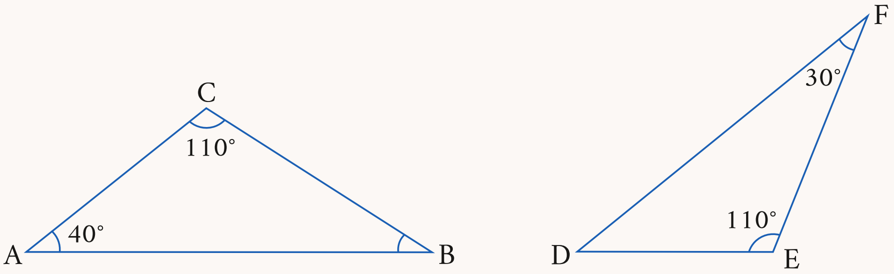


3.



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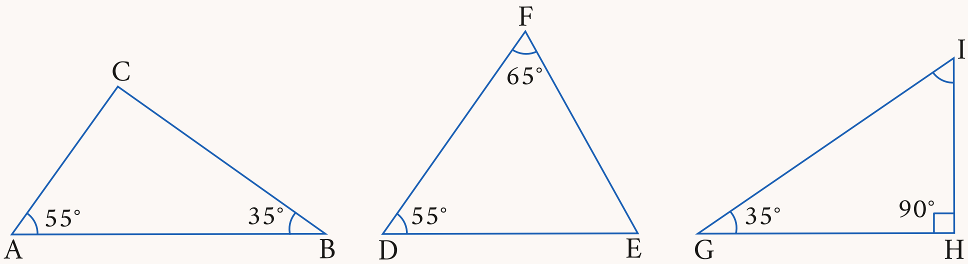

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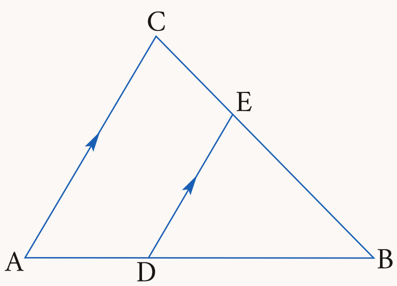

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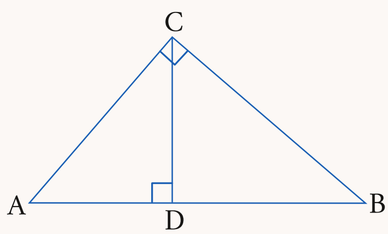
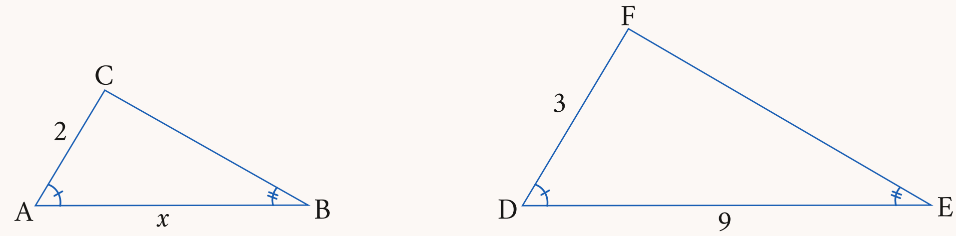


7 - 22. For each of the following

1. write the similarity statement
2. write the proportion between the corresponding sides,
3. solve for x or x and y.

7.



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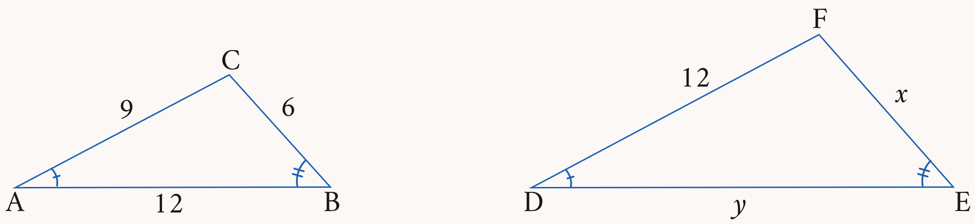

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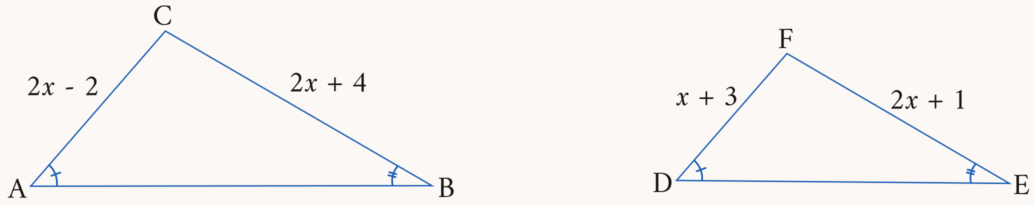

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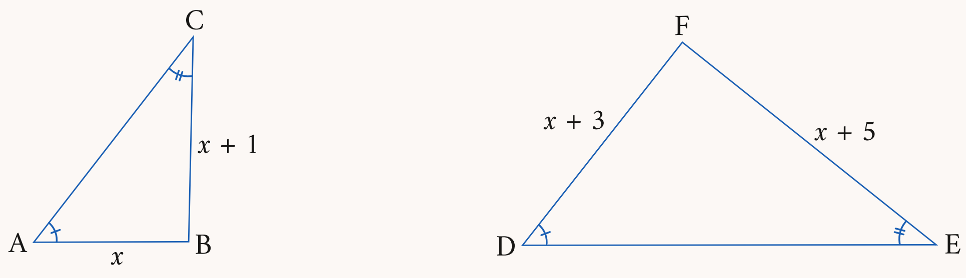

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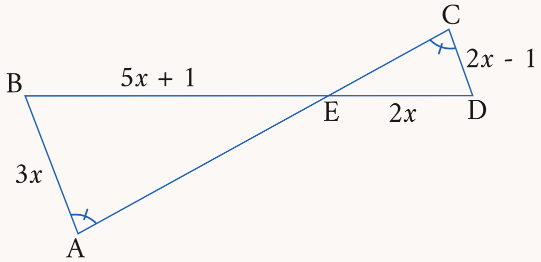

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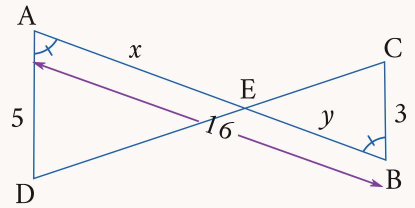

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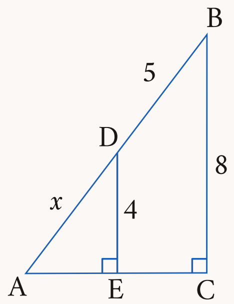

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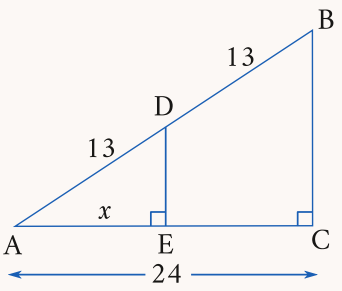

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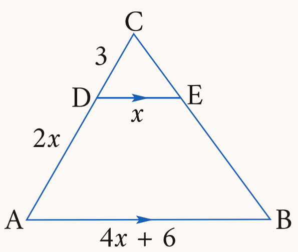

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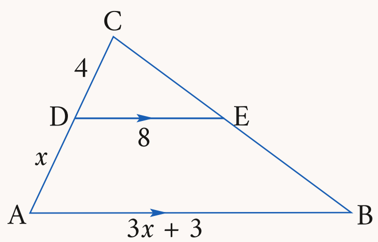

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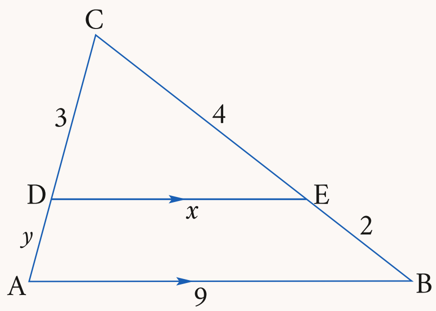

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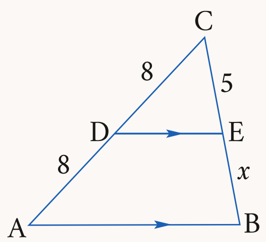

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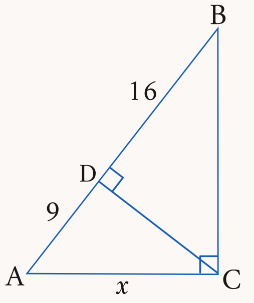

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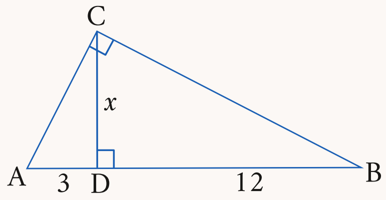

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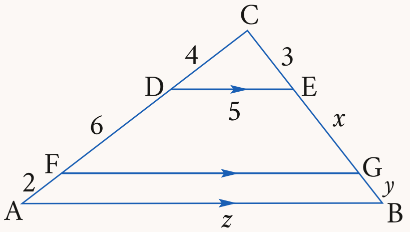

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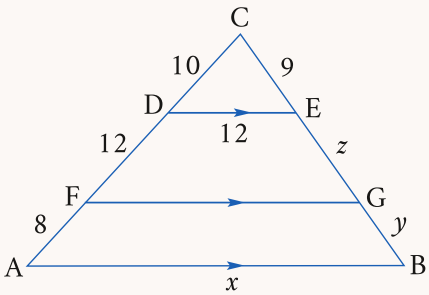
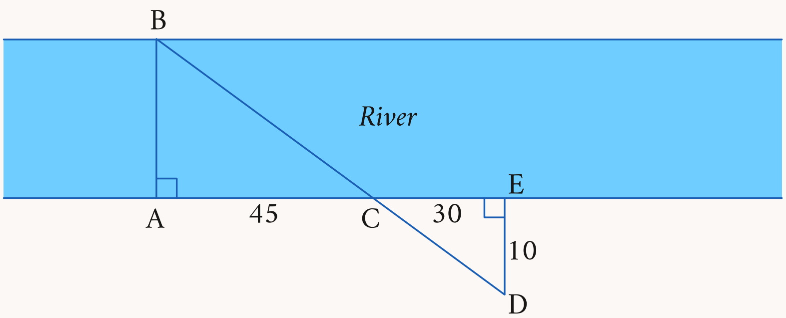


23. A flagpole casts a shadow 80 feet long at the same time a 5 foot boy casts a shadow 4 feet long. How tall is the flagpole?

24. Find the width AB of the river:



4.3 TRANSVERSALS TO THREE PARALLEL LINES

In Chapter 1 we defined a transversal to be a line which intersects two other lines. We will now extend the definition to a line which intersects **three** other lines. In Figure 1, AB is a transversal to three lines.

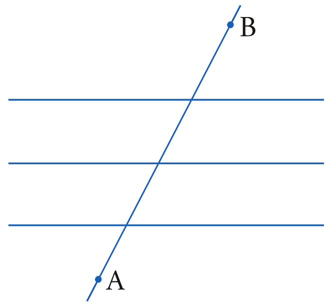


Figure 1. AB is a transversal to three lines.

If the three lines are parallel and we have two such transversals we may state the following theorem:

THEOREM 1

The line segments formed by two transversals crossing three parallel lines are proportional. In Figure 2,

$$\frac{a}{b} = \frac{c}{d}$$

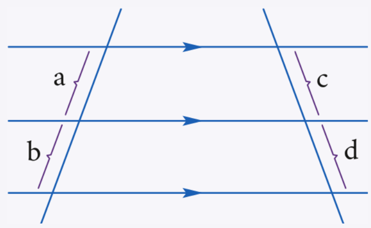
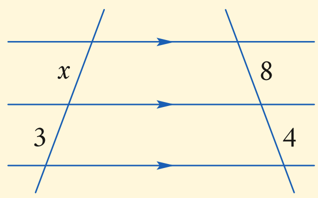


Figure 2. The line segments formed by the transversals are proportional.

EXAMPLE A

Find x :



SOLUTION

$$\frac{x}{3} = \frac{8}{4}$$
$$4x = 24$$
$$x = 6$$

ANSWER : $x = 6$

CHECK

$$\frac{x}{3} = \frac{8}{4}$$

$$\frac{6}{3} \;\Big|\; 2$$
$$2$$

***Proof of Theorem 1*:**
Draw GB and HC parallel to DF (Figure 3). The corresponding angles of the parallel lines are equal and so △BCH ~ △ABG.
Therefore:

$$\frac{BC}{AB} = \frac{CH}{BG}$$

Now CH = FE = c and BG = ED = d because they are the opposite sides of a parallelogram. Substituting, we obtain

$$\frac{a}{b} = \frac{c}{d}$$

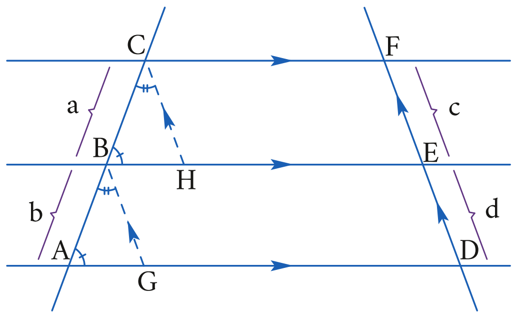
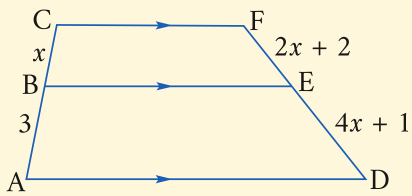


Figure 3. *Draw* GB *and* HC *parallel to* DF.

? EXAMPLE B

Find x :



SOLUTION

$$\frac{x}{3} = \frac{2x+2}{4x+1}$$

$$x(4x+1) = (3)(2x+2)$$

$$4x^2 + x = 6x + 6$$

$$4x^2 - 5x - 6 = 0$$

$$(x-2)(4x+3) = 0$$

$x - 2 = 0$ *or* $4x + 3 = 0$

$x = 2$ $\qquad$ $4x = -3$

$$x = -\frac{3}{4}$$

CHECK

$$\frac{x}{3} = \frac{2x+2}{4x+1}$$

$$\frac{2}{3} \;\Big|\; \frac{2(2)+2}{4(2)+1}$$

$$\frac{4+2}{8+1}$$

$$\frac{6}{9}$$

$$\frac{2}{3}$$

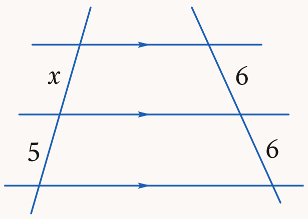
We reject $x = -\frac{3}{4}$ because BC = x cannot be negative.

ANSWER : $x = 2$

PROBLEMS

1 - 6. Find x:

1.



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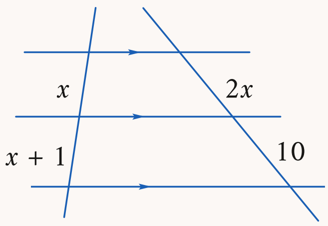

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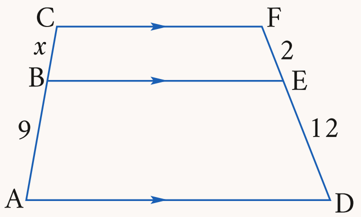

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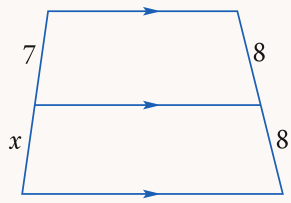

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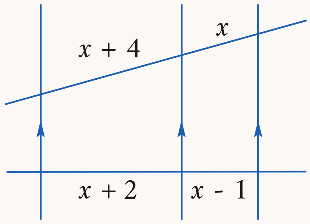

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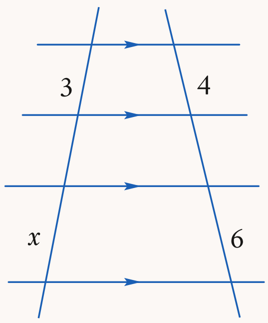


4.4 PYTHAGOREAN THEOREM

In a right triangle, the sides of the right angle are called the **legs** of the triangle and the remaining side is called the **hypotenuse**.
In Figure 1, side AC and BC are the legs and side AB is the hypotenuse.

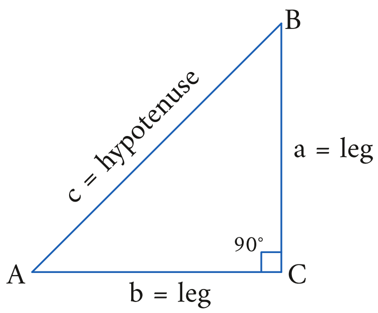


Figure 1. A right triangle.

The following is one of the most famous theorems in mathematics:

THEOREM 1 (PYTHAGOREAN THEOREM)

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

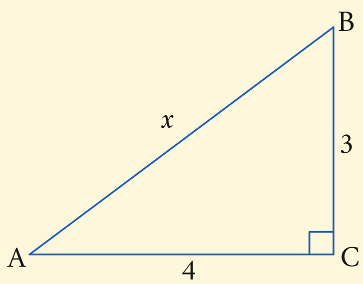
$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$

In Figure 1, $a^2 + b^2 = c^2$.

Before we prove **Theorem 1**, we will give several examples.

? EXAMPLE A

Find x :



SOLUTION

$$\begin{aligned} leg^2 + leg^2 &= hyp^2 \\ 3^2 + 4^2 &= x^2 \\ 9 + 16 &= x^2 \\ 25 &= x^2 \\ 5 &= x \end{aligned}$$

CHECK

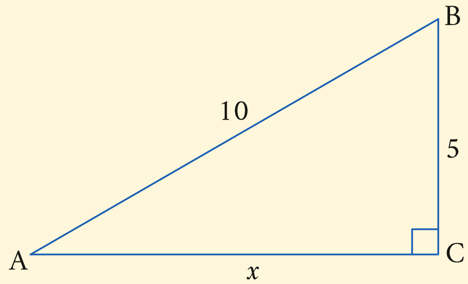
$$leg^2 + leg^2 = hyp^2$$

$3^2 + 4^2$	x^2
$9 + 16$	5^2
25	25

ANSWER : $x = 5$

? EXAMPLE B

Find x :



SOLUTION

$$\begin{aligned} leg^2 + leg^2 &= hyp^2 \\ 5^2 + x^2 &= 10^2 \\ 25 + x^2 &= 100 \\ x^2 &= 75 \\ x &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3} \end{aligned}$$

ANSWER : $x = 5\sqrt{3}$.

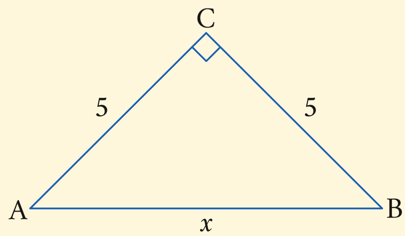
CHECK

$$leg^2 + leg^2 = hyp^2$$

$5^2 + x^2$	10^2
$25 + (5\sqrt{3})^2$	100
$25 + 25\sqrt{9}$	
$25 + 25(3)$	
$25 + 75$	
100	

? EXAMPLE C

Find x :



SOLUTION

$$\begin{aligned} leg^2 + leg^2 &= hyp^2 \\ 5^2 + 5^2 &= x^2 \\ 25 + 25 &= x^2 \\ 50 &= x^2 \\ x &= \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2} \end{aligned}$$

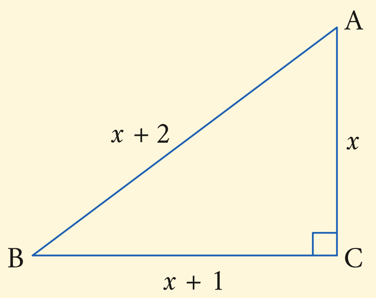
CHECK

$$leg^2 + leg^2 = hyp^2$$

$$\begin{array}{r|l} 5^2 + 5^2 & x^2 \\ 25 + 25 & (5\sqrt{2})^2 \\ 50 & 25\sqrt{4} \\ & 25(2) \\ & 50 \end{array}$$

? EXAMPLE D

Find x :



SOLUTION

$$\begin{aligned} leg^2 + leg^2 &= hyp^2 \\ x^2 + (x+1)^2 &= (x+2)^2 \\ x^2 + x^2 + 2x + 1 &= x^2 + 4x + 4 \\ x^2 + x^2 + 2x + 1 - x^2 - 4x - 4 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

$$\begin{array}{ll} x - 3 = 0 & \quad x + 1 = 0 \\ x = 3 & \quad x = -1 \end{array}$$

We reject $x = -1$ because $AC = x$ cannot be negative.

ANSWER : $x = 3$

CHECK

$$leg^2 + leg^2 = hyp^2$$

$$\begin{array}{r|l} x^2 + (x+1)^2 & (x+2)^2 \\ 3^2 + (3+1)^2 & (3+2)^2 \\ 9 + (4)^2 & 5^2 \\ 9 + 16 & 25 \\ 25 & \end{array}$$

We will now restate and prove **Theorem 1**:

THEOREM 1 (PYTHAGOREAN THEOREM)

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. That is,

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2$$

In Figure 1, $a^2 + b^2 = c^2$.

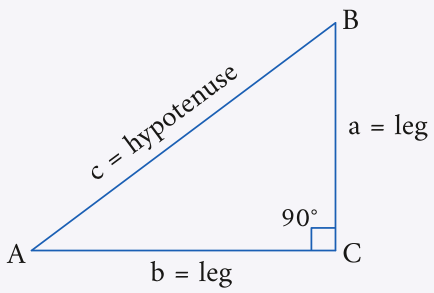


Figure 1. A right triangle

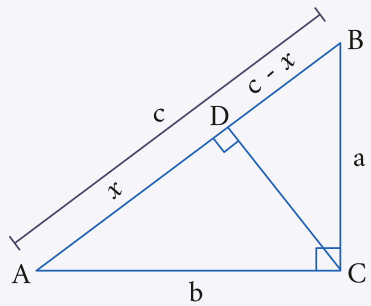


Figure 2. Draw CD perpendicular to AB

Proof:

In Figure 1, draw CD perpendicular to AB. Let x = AD. Then BD = c − x (Figure 2). As in Example C, section 4.2, △ABC ~ △ACD and △ABC ~ △CBD. From these two similarities we obtain two proportions:

△ABC ~ △ACD

$$\frac{AB}{AC} = \frac{AC}{AD}$$

$$\frac{c}{b} = \frac{b}{x}$$

$$cx = b^2$$

△ABC ~ △ACD

$$\frac{AB}{CB} = \frac{BC}{BD}$$

$$\frac{c}{a} = \frac{a}{c - x}$$

$$c(c - x) = a^2$$

$$c^2 - cx = a^2$$

$$c^2 - b^2 = a^2$$

$$c^2 = a^2 + b^2$$

The converse of the Pythagorean Theorem also holds:

THEOREM 2 (CONVERSE OF THE PYTHAGOREAN THEOREM)

In a triangle, if the square of one side is equal to the sun of the squares of the other two sides then the triangle is a right triangle.

In Figure 3, if $c^2 = a^2 + b^2$ then $\triangle ABC$ is a right triangle with $\angle C = 90°$.

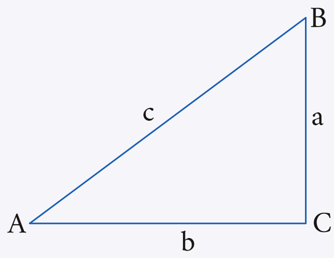


Figure 3. If $c^2 = a^2 + b^2$ then $\angle C = 90°$.

Proof:
Draw a new triangle, $\triangle DEF$ so that $\angle F = 90°$, $d = a$ and $e = b$ (Figure 4). $\triangle DEF$ is a right triangle, so by **Theorem 1**, $f^2 = d^2 + e^2$. We have $f^2 = d^2 + e^2 = a^2 + b^2 = c^2$ and therefore $f = c$. Therefore $\triangle ABC \cong \triangle DEF$ because SSS = SSS. Therefore, $\angle C = \angle F = 90°$:

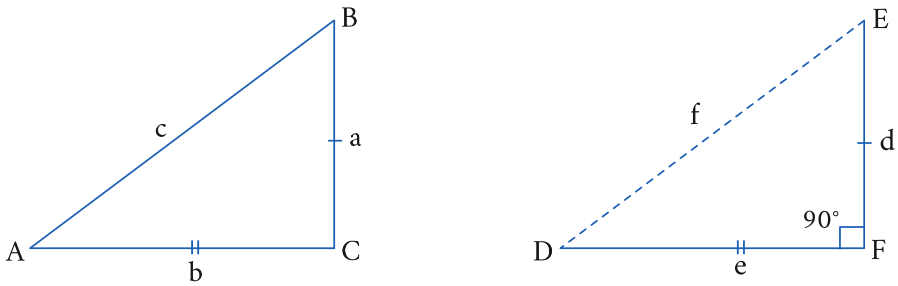
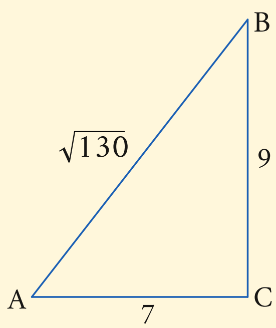


Figure 4. Given $\triangle ABC$, draw $\triangle DEF$ so that $\angle F = 90°$, $d = a$ and $e = b$.

? EXAMPLE E

Is $\triangle ABC$ a right triangle?



SOLUTION

$AC^2 = 7^2 = 49$

$BC^2 = 9^2 = 81$

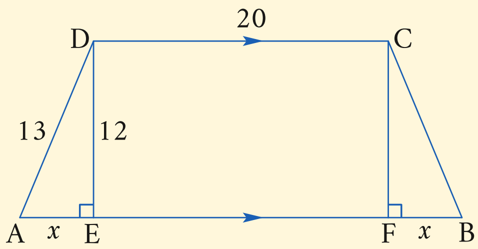
$AB^2 = (\sqrt{130})^2 = 130$

$49 + 81 = 130$, so by **Theorem 2**, $\triangle ABC$ is a right triangle.

ANSWER : Yes.

? EXAMPLE F

Find x and AB:



SOLUTION

$x^2 + 12^2 = 13^2$

$x^2 + 144 = 169$

$x^2 = 169 - 144$

$x^2 = 25$

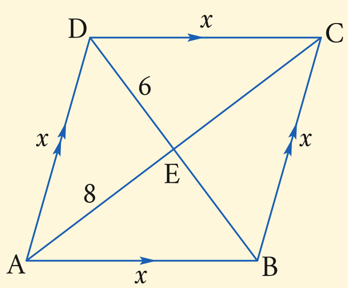
$x = 5$

CDEF is a rectangle so EF = CD = 20 and CF = DE = 12.
Therefore FB = 5 and AB = AE + EF + FB = 5 + 20 + 5 = 30.

ANSWER : x=5, AB = 30.

? EXAMPLE G

Find x, AC and BD:



SOLUTION

ABCD is a rhombus. The diagonals of a rhombus are perpendicular and bisect each other.

$6^2 + 8^2 = x^2$

$36 + 64 = x^2$

$100 = x^2$

$10 = x$

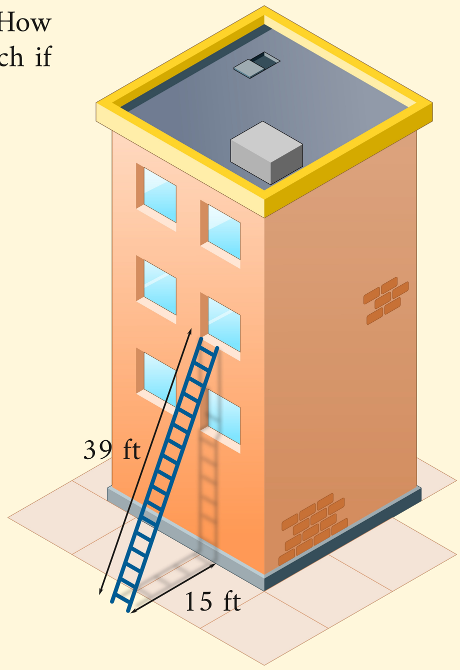
$AC = 8 + 8 = 16$

$BD = 6 + 6 = 12$

ANSWER : $x = 10$, $AC = 16$, $BD = 12$

? EXAMPLE H

A ladder 39 feet long leans against a building. How far up the side of the building does the ladder reach if the foot of the ladder is 15 feet from the building?



SOLUTION

$\text{leg}^2 + \text{leg}^2 = \text{hyp}^2$

$x^2 + 15^2 = 39^2$

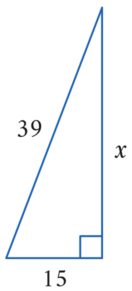
$x^2 + 225 = 1521$

$x^2 = 1521 - 225$

$x^2 = 1296$

$x = \sqrt{1296}$

$x = 36$



ANSWER : $x = 36$ feet

HISTORICAL NOTE

Pythagoras (c. 582 - 507 B.C.) was not the first to discover the theorem which bears his name. It was known long before his time by the Chinese, the Babylonians and perhaps also the Egyptians and the Hindus. According to tradition, Pythagoras was the first to give a proof of the theorem. His proof probably made use of areas, like the one suggested in Figure 5 below. (Each square contains four congruent right triangles with

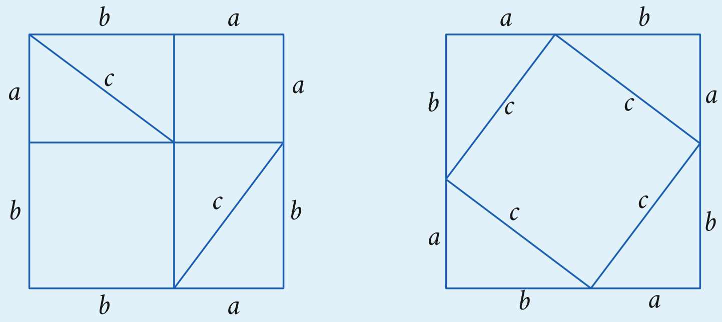


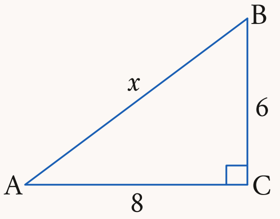
Figure 5. Pythagoras may have proved $a^2 + b^2 = c^2$ in this way.

sides of lengths a, b and c. In addition the square on the left contains a square with side a and a square with side b while the one on the right contains a square with side c.) Since the time of Pythagoras, at least several hundred different proofs of the Pythagorean Theorem have been proposed. Pythagoras was the founder of the Pythagorean school, a secret religious society devoted to the study of philosophy, mathematics and science. Its membership was a select group, which tended to keep the discoveries and practices of the society secret from outsiders. The Pythagoreans believed that numbers were the ultimate components of the universe and that all physical relationships could be expressed with whole numbers. This belief was prompted in part by their discovery that the notes of the musical scale were related by numerical ratios. The Pythagoreans made important contributions to medicine, physics, and astronomy. In geometry, they are credited with the angle sum theorem for triangles, the properties of parallel lines and the theory of similar triangles and proportions.

PROBLEMS

1 - 10. Find x. Leave answers in simplest radical form.

1.



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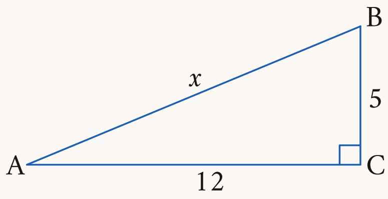

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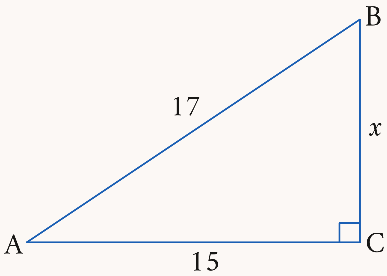

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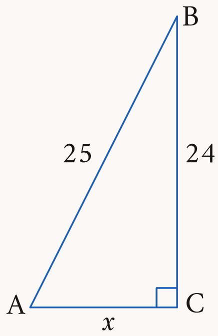

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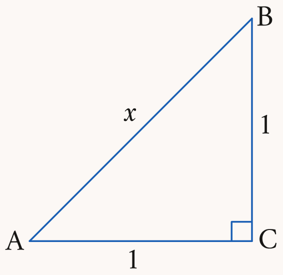

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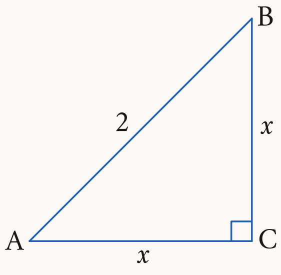

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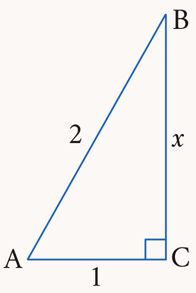

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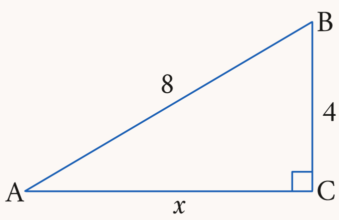

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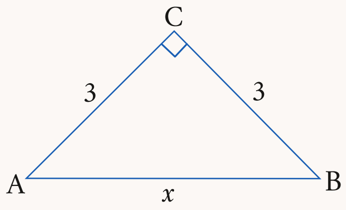

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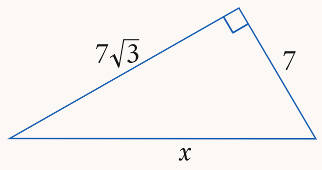
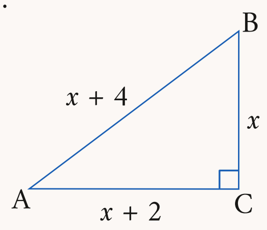


11 - 14. Find x and all sides of the triangle.

11.



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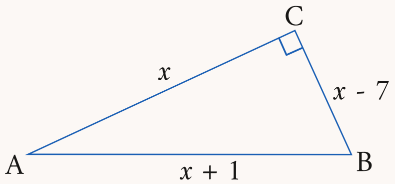

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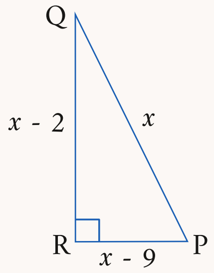

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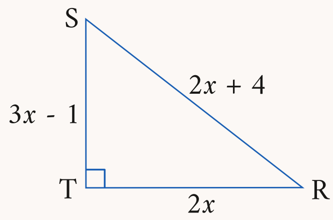
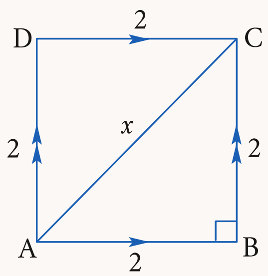


15 - 16. Find x:

15.



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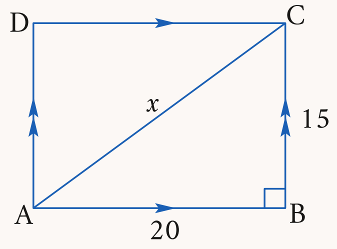
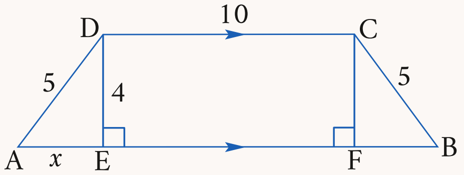
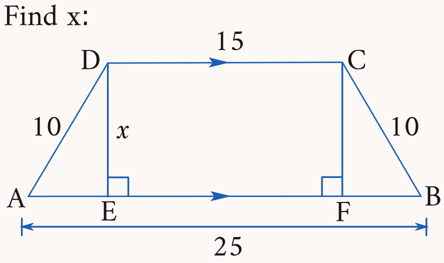


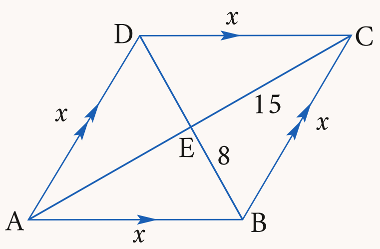
17. Find x and AB:



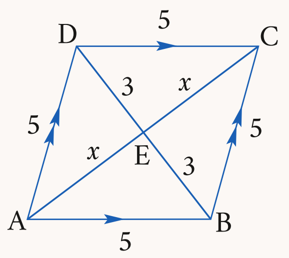
18. Find x:



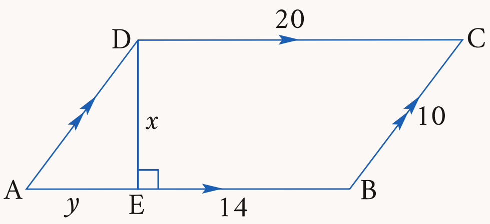
19. Find x, AC and BD:



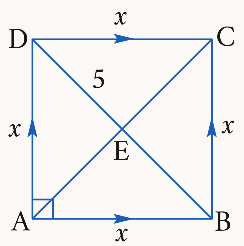
20. Find x, AC and BD:



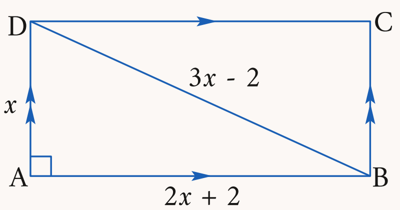
21. Find x and y:



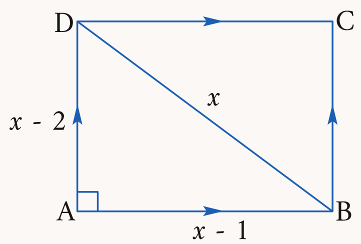
22. Find x, AC and BD:



23. Find x, AB and BD:

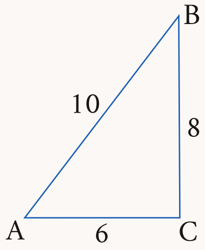


24. Find x, AB and AD:



25 - 30. Is $\triangle ABC$ a right triangle?

25.



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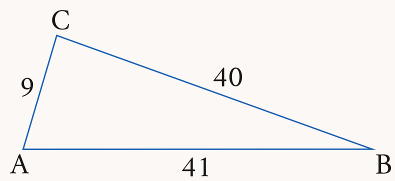

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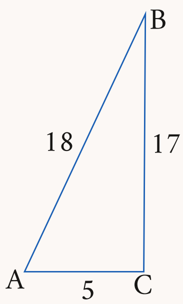

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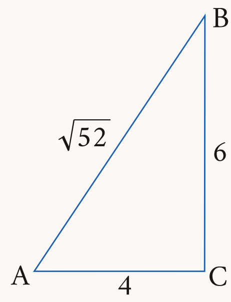

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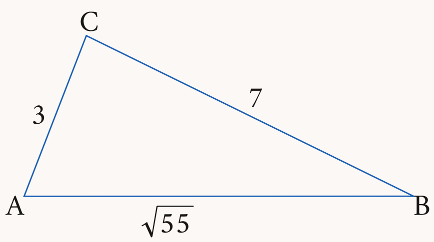
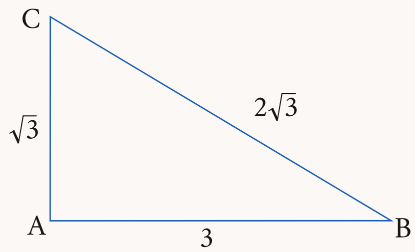


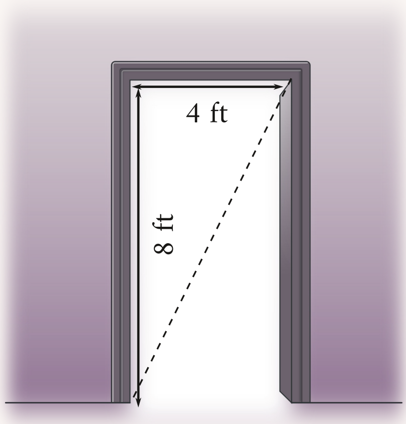
30.



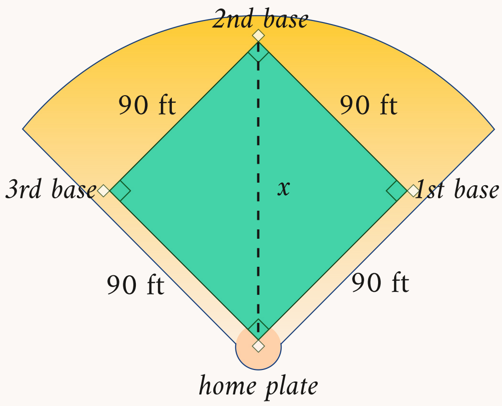
31. A ladder 25 feet long leans against a building. How far up the side of the building does the ladder reach if the foot of the ladder is 7 feet from the building?

32. A man travels 24 miles east and then 10 miles north. At the end of his journey how far is he from his starting point?

33. Can a table 9 feet wide (with its legs folded) fit through a rectangular doorway 4 feet by 8 feet?



34. A baseball diamond is a square 90 feet on each side. Find the distance from home plate to second base (leave answer in simplest radical form)?



4.5 SPECIAL RIGHT TRIANGLES

There are two kinds of right triangle which deserve special attention: the 30°- 60°- 90° right triangle and the 45° - 45°- 90° right triangle.
A triangle whose angles are 30°, 60° and 90° is called a **30° - 60° - 90° triangle**. △ABC in Figure 1 is a 30° - 60° - 90° triangle with side AC = 1.

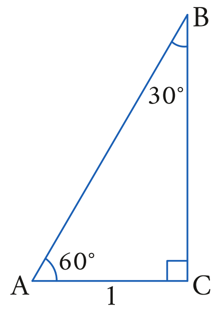


Figure 1. A 30° - 60° - 90° triangle.

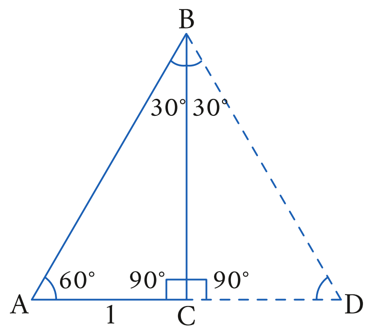


Figure 2. Draw BD and CD.

To learn more about this triangle let us draw lines BD and CD as in Figure 2.
△ABC ≅ △DBC by ASA = ASA so AC = DC = 1. △ABD is an equiangular triangle so all the sides must be equal to 2. Therefore AB = 2 (Figure 3).

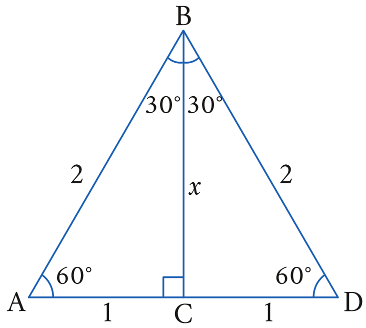
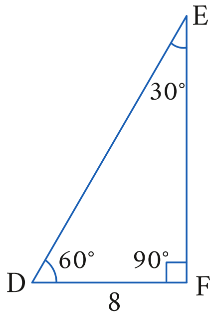


Figure 3. △ABD is equiangular with all sides equal to 2.

Let x = BC. Let us find x. Applying the **Pythagorean Theorem** to △ABC,

$$\text{leg}^2 + \text{leg}^2 = \text{hyp}^2$$
$$1^2 + x^2 = 2^2$$
$$1 + x^2 = 4$$
$$x^2 = 3$$
$$x = \sqrt{3}$$

Now suppose we are given another 30° - 60° - 90° triangle △DEF, with side DF = 8 (Figure 4). △DEF is similar to △ABC of Figure 3. Therefore



$$\frac{DF}{AC} = \frac{DE}{AB} \qquad \textit{and} \qquad \frac{DF}{AC} = \frac{EF}{BC}$$

$$\frac{8}{1} = \frac{DE}{2} \qquad\qquad \frac{8}{1} = \frac{EF}{\sqrt{3}}$$

$$16 = DE \qquad\qquad 8\sqrt{3} = EF$$

Figure 4. *△DEF is similar to △ABC of Figure 3.*

Our conclusions about triangles ABC and DEF suggest the following theorem:

THEOREM 1

In the 30° - 60° - 90° triangle the hypotenuse is always twice as large as the leg opposite the 30° angle (the **shorter leg**). The leg opposite the 60° angle (the **longer leg**) is always equal to the shorter leg times $\sqrt{3}$.

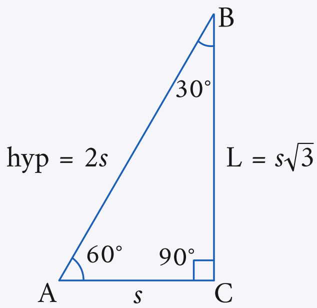


Figure 5. The hypotenuse is twice the shorter leg and the longer leg is equal to the shorter leg times the $\sqrt{3}$.

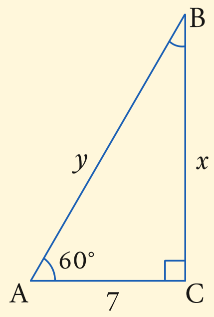
In Figure 5, s = shorter leg, L = longer leg and hyp = hypotenuse. **Theorem 1** says that

$$\boxed{\begin{array}{c} hyp = 2s \\ L = s\sqrt{3} \end{array}}$$

Note that the longer leg is always the leg opposite (furthest away from) the 60° angle and the shorter leg is always the leg opposite (furthest away from) the 30° angle.

? EXAMPLE A

Find x and y:



SOLUTION

$\angle B = 180^\circ - (60^\circ + 90^\circ) = 180^\circ - 150^\circ = 30^\circ$, so $\triangle ABC$ is a 30° - 60° - 90° triangle.
By **Theorem 1:**

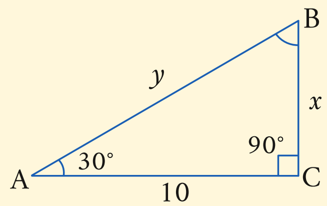
$\text{hyp} = 2s \qquad L = s\sqrt{3}$

$y = 2(7) = 14 \qquad x = 7\sqrt{3}$

ANSWER: $x = 7\sqrt{3}$, $y = 14$

? EXAMPLE B

Find x and y:



SOLUTION

$\angle B = 60^\circ$, so $\triangle ABC$ is a 30° - 60° - 90° triangle. By **Theorem 1**:

$$L = s\sqrt{3}$$

$$10 = x\sqrt{3}$$

$$\frac{10}{\sqrt{3}} = \frac{x\sqrt{3}}{\sqrt{3}}$$

$$x = \frac{10}{\sqrt{3}} = \frac{10}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

$$\text{hyp} = 2s$$

$$y = 2x = 2\left(\frac{10\sqrt{3}}{3}\right) = \frac{20\sqrt{3}}{3}$$

ANSWER: $x = \dfrac{10\sqrt{3}}{3}$, $y = \dfrac{20\sqrt{3}}{3}$

The second special triangle we will consider is the **45° - 45° - 90° triangle**. A triangle whose angles are 45°, 45° and 90° is called a 45° - 45° - 90° triangle or an **isosceles right triangle.** $\triangle ABC$ in Figure 6 is a 45° - 45° - 90° triangle with side $AC = 1$.

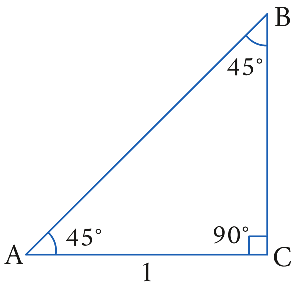


Figure 6. A 45° - 45° - 90° triangle.

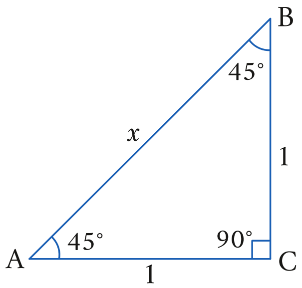


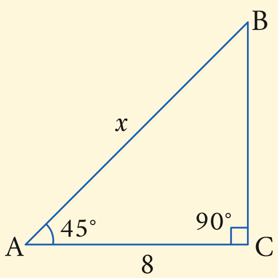
Figure 7. The legs of 45° - 45° - 90° triangle are equal.

Since $\angle A = \angle B = 45°$, the sides opposite these angles must be equal (**Theorem 2**, *Section 2.5*). Therefore $AC = BC = 1$. Let $x = AB$ (Figure 7). By the **Pythagorean Theorem**,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 1^2 + 1^2 &= x^2 \\ 1 + 1 &= x^2 \\ 2 &= x^2 \\ \sqrt{2} &= x \end{aligned}$$

? EXAMPLE C

Find x:



SOLUTION

$\angle B = 180° - (45° + 90°) = 180° - 135° = 45°$. So $\triangle ABC$ is a 45° - 45° - 90° triangle.
$AC = BC = 8$ because these sides are opposite equal angles. By the **Pythagorean Theorem**,

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 8^2 + 8^2 &= x^2 \\ 64 + 64 &= x^2 \\ 128 &= x^2 \\ \sqrt{128} &= x \end{aligned} \qquad x = \sqrt{128} = \sqrt{64}\sqrt{2} = 8\sqrt{2}$$

ANSWER : $8\sqrt{2}$

The triangles of Figure 6 and Example C suggest the following theorem:

THEOREM 2

In the 45° - 45° - 90° triangle the legs are equal and the hypotenuse is equal to either leg times $\sqrt{2}$.

In Figure 8, hyp is the hypotenuse and L is the length of each leg. Theorem 2 says that.

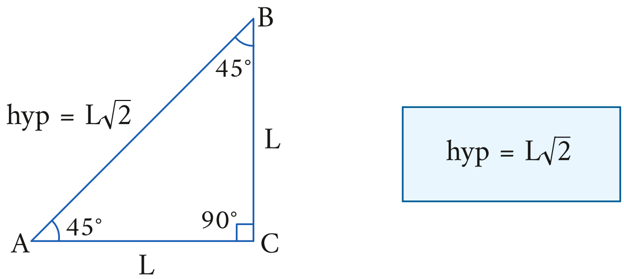
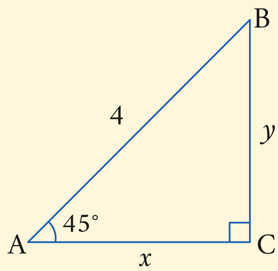


Figure 8. The legs are equal and the hypotenuse is equal to either leg times $\sqrt{2}$

EXAMPLE D

Find x and y:



SOLUTION

$\angle B = 45°$. So $\triangle ABC$ is an isosceles right triangle and $x = y$.

$$x^2 + y^2 = 4^2$$
$$x^2 + x^2 = 16$$
$$2x^2 = 16$$
$$x^2 = 8$$
$$x = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$$

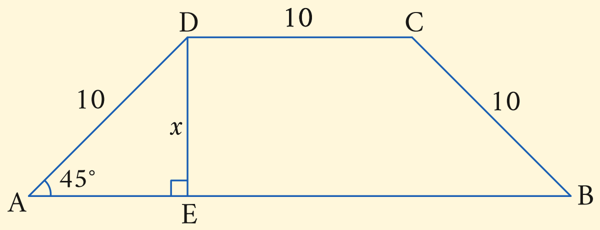
ANSWER: $x = y = 2\sqrt{2}$

Another method:

$$\text{hyp} = L\sqrt{2}$$
$$4 = x\sqrt{2}$$
$$\frac{4}{\sqrt{2}} = \frac{x\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$$
$$x = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

? EXAMPLE E

Find AB:



SOLUTION

$\triangle ADE$ is a 45° - 45° - 90° triangle. Hence

$$hyp = L\sqrt{2}$$

$$10 = x\sqrt{2}$$

$$\frac{10}{\sqrt{2}} = \frac{x\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \qquad x = \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \qquad AE = x = 5\sqrt{2}$$

Now draw CF perpendicular to AB (Figure 9). $\angle B = 45°$ since ABCD is an isosceles trapezoid (**Theorem 4**, *Section 3.2*).

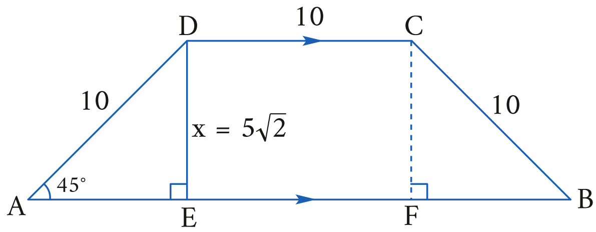


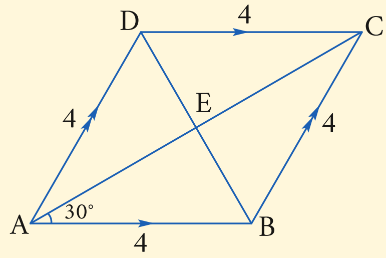
Figure 9. Draw CF perpendicular to AB.

So $\triangle BCF$ is a 45° - 45° - 90° triangle congruent to $\triangle ADE$ and therefore $BF = 5\sqrt{2}$. CDEF is a rectangle and therefore $EF = 10$.
We have $AB = AE + EF + FB = 5\sqrt{2} + 10 + 5\sqrt{2} = 10\sqrt{2} + 10$

ANSWER: $AB = 10\sqrt{2} + 10$

? EXAMPLE F

Find AC and BD:



SOLUTION

ABCD is a rhombus. The diagonals AC and BD are perpendicular and bisect each other. $\angle AEB = 90°$ and $\angle ABE = 180° - (90° + 30°) = 60°$. So $\triangle AEB$ is a $30°$ - $60°$ - $90°$ triangle.

$\text{hyp} = 2s$	$L = s\sqrt{3}$
$4 = 2(BE)$	$AE = 2\sqrt{3}$
$2 = BE$	$AC = 2\sqrt{3} + 2\sqrt{3}$
$BD = 2 + 2 = 4$	$AC = 4\sqrt{3}$

ANSWER : $AC = 4\sqrt{3}$, $BD = 4$.

HISTORICAL NOTE

The Pythagoreans believed that all physical relation ships could be expressed with whole numbers. However the sides of the special triangles described in this section are related by **irrational** numbers, $\sqrt{2}$ and $\sqrt{3}$. An irrational number is a number which can be approximated, but not expressed exactly, by a ratio of whole numbers. For example $\sqrt{2}$ can be approximated with increasing accuracy by such ratios as

$$1.4 = \frac{14}{10}, \quad 1.41 = \frac{141}{100}, \quad 1.414 = \frac{1414}{1000},$$

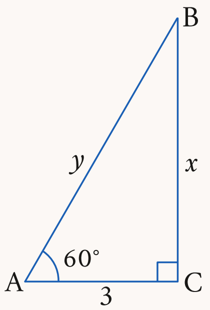
etc., but there is no fraction of whole numbers which is **exactly** equal to $\sqrt{2}$. (For more details and a proof, see the book by Richardson listed in the References). The Pythagoreans discovered that $\sqrt{2}$ was irrational in about the 5th century B.C. It was a tremendous shock to them that not all triangles could be measured "exactly." They may have even tried to keep this discovery a secret for fear of the damage it would do to their philosophical credibility.

The inability of the Pythagoreans to accept irrational numbers had unfortunate consequences for the development of mathematics. Later Greek mathematicians avoided giving numerical values to lengths of line segments. Problems whose algebraic solutions might be irrational numbers, such as those involving quadratic equations, were instead stated and solved geometrically. The result was that geometry flourished at the expense of algebra. It was left for the Hindus and the Arabs to resurrect the study of algebra in the Middle Ages. And it was not until the 19th century that irrational numbers were placed in the kind of logical framework that the Greeks had given to geometry 2000 years before.

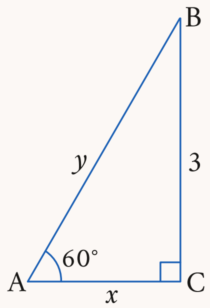
PROBLEMS

1 - 10. Find x and y:

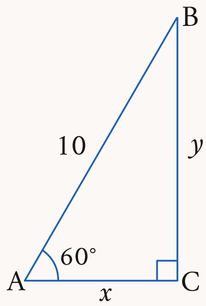
1.



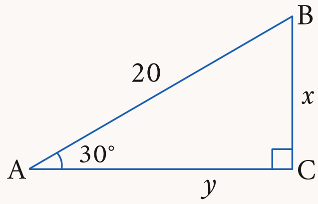
2.



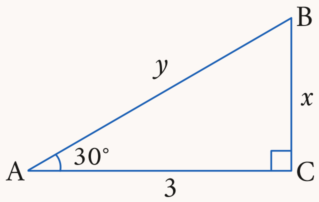
3.


B
10
y
60°
A
x
C

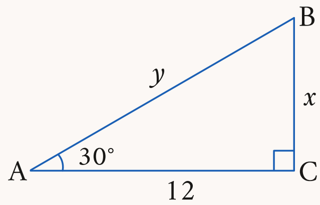
4.


B
20
x
30°
A
y
C

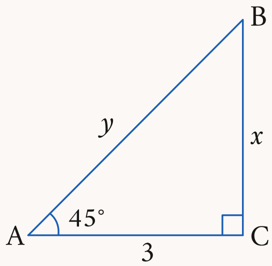
5.


B
y
x
30°
A
3
C

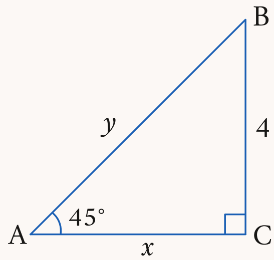
6.


B
y
x
30°
A
12
C

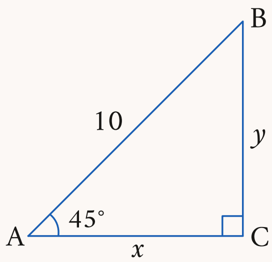
7.


B
y
x
45°
A
3
C

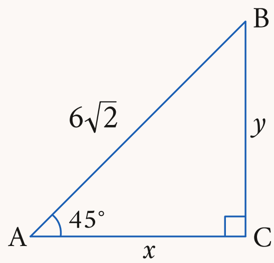
8.


B
y
4
45°
A
x
C

9.

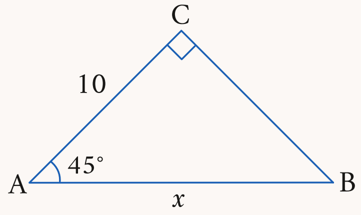

B
10
y
45°
A
x
C

10.

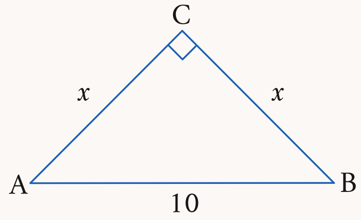

B
$6\sqrt{2}$
y
45°
A
x
C

11 - 14. Find x:

11.



12.



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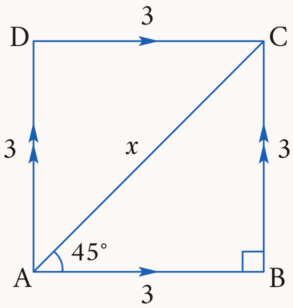
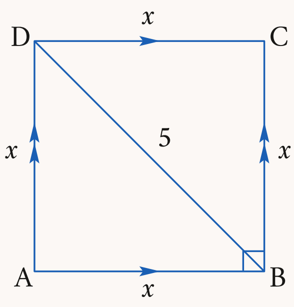
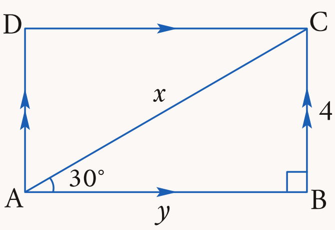


14.



15 - 20. Find x and y:

15.



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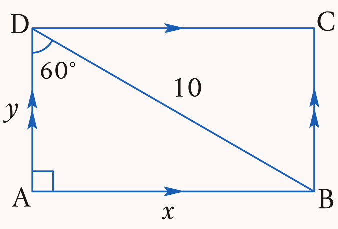
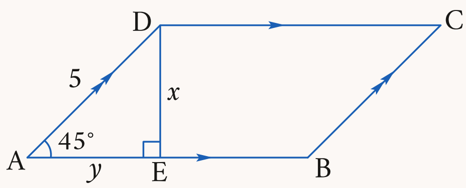


17.



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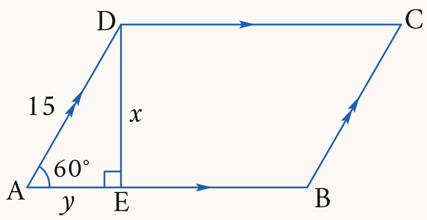

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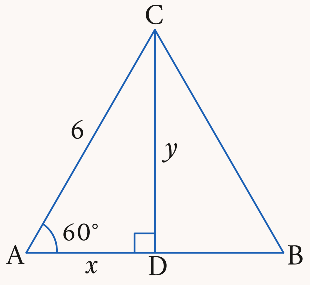
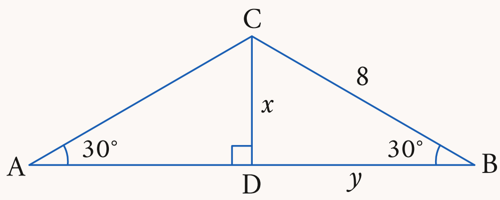
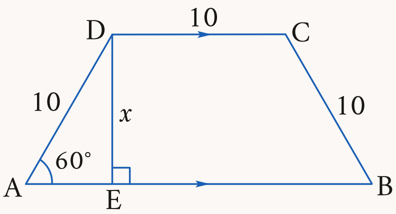


20.

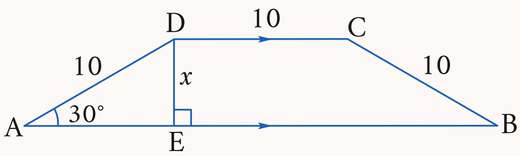


21 - 22. Find x and AB:

21.

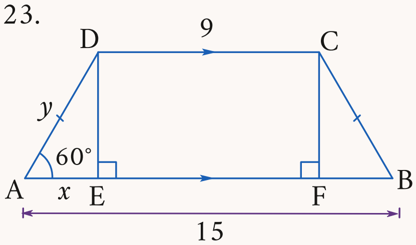


22.

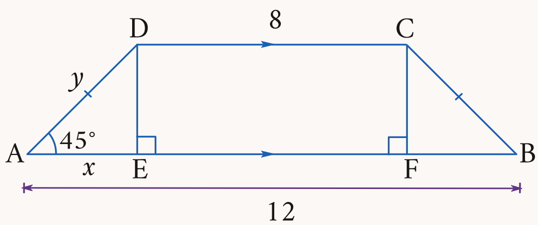


23 - 24. Find x and y:

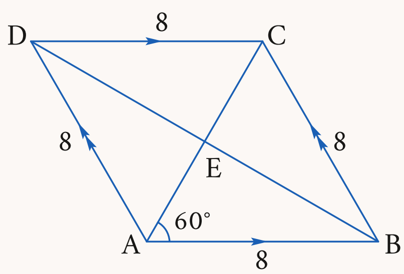
23.



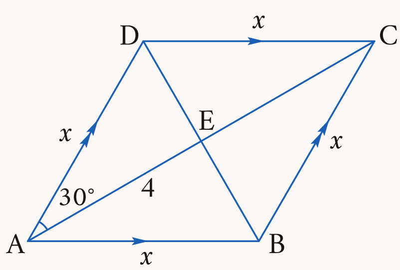
24.



25. Find AC and BD:



26. Find x, AC and BD:



4.6 DISTANCE FROM A POINT TO A LINE

Suppose we are given a point P and a line $\overleftrightarrow{AB}$ as in Figure 1. We would like to find the shortest line segment that can be drawn from P to $\overleftrightarrow{AB}$.

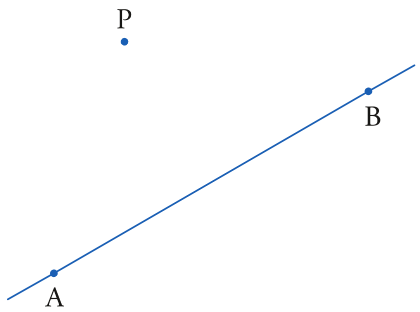


Figure 1. Point P and line $\overleftrightarrow{AB}$.

First we will prove a theorem:

THEOREM 1

In a right triangle the hypotenuse is larger than either leg. In Figure 2, $c > a$ and $c > b$. (The symbol ">" means "is greater than.")

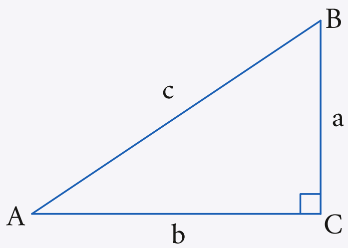


Figure 2. c is larger than either a or b.

Proof :

By the Pythagorean Theorem,

$$c^2 = \sqrt{a^2 + b^2} > \sqrt{a^2} = a$$

$$c^2 = \sqrt{a^2 + b^2} > \sqrt{b^2} = b$$

Now we can give the answer to our question:

THEOREM 2

The perpendicular is the shortest line segment that can be drawn from a point to a straight line.
In Figure 3 the shortest line segment from P to $\overleftrightarrow{AB}$ is PD. Any other line segment, such as PC, must be longer.

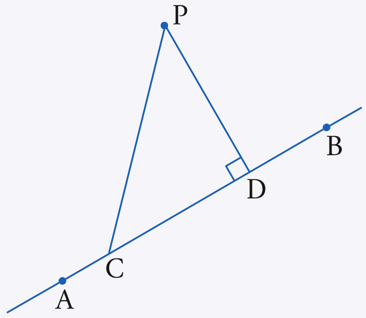


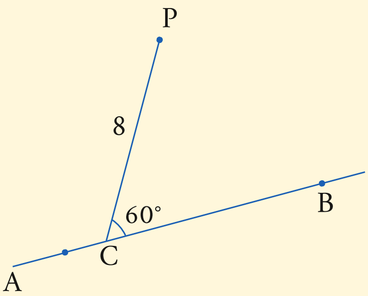
Figure 3. PD is the shortest line segment from P to $\overleftrightarrow{AB}$.

Proof :

PC is the hypotenuse of right triangle PCD. Therefore by **Theorem 1**, PC > PD.
We define the **distance from a point to a line** to be the length of the perpendicular.

EXAMPLE A

Find the distance from P to $\overleftrightarrow{AB}$:



SOLUTION

Draw PD perpendicular to $\overleftrightarrow{AB}$ (Figure 4). $\triangle$PCD is a 30° - 60° - 90° triangle.

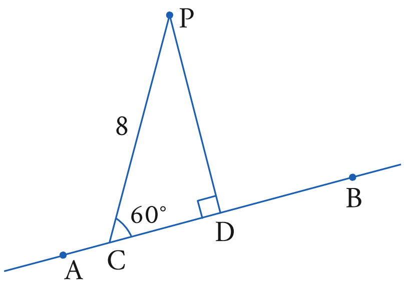


Figure 4. Draw PD perpendicular to $\overleftrightarrow{AB}$.

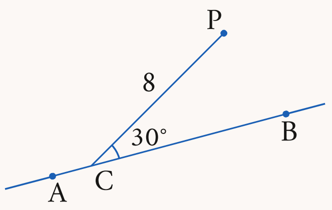
$$\text{hyp} = 2s$$
$$8 = 2(CD)$$
$$4 = CD$$
$$L = s\sqrt{3}$$
$$PD = 4\sqrt{3}$$

ANSWER : PD $= 4\sqrt{3}$

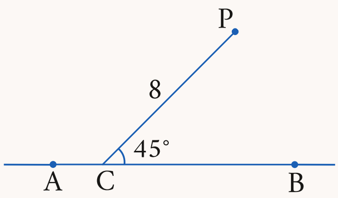
PROBLEMS

1 - 6. Find the distance from P to $\overleftrightarrow{AB}$:

1.



2.



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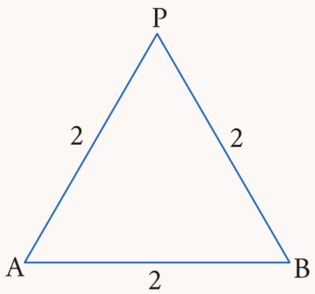
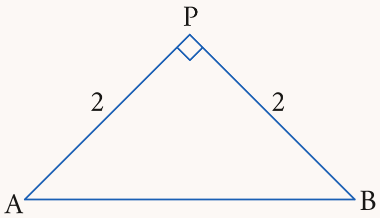
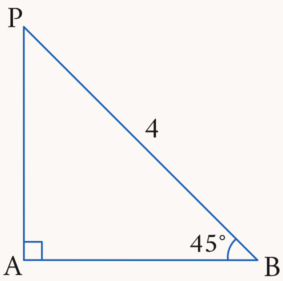


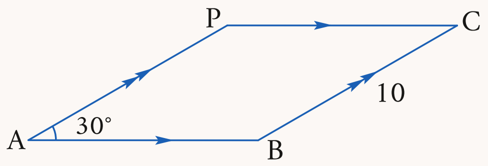
4.



5.



6.



CHAPTER 5

TRIGONOMETRY OF THE RIGHT TRIANGLE

5.1 THE TRIGONOMETRIC FUNCTIONS

Trigonometry (from Greek words meaning triangle-measure) is the branch of mathematics concerned with computing unknown sides and angles of triangles. For example, in Figure 1, we might want to measure the height of the tree without actually having to climb the tree. The methods of trigonometry will enable us to do this.

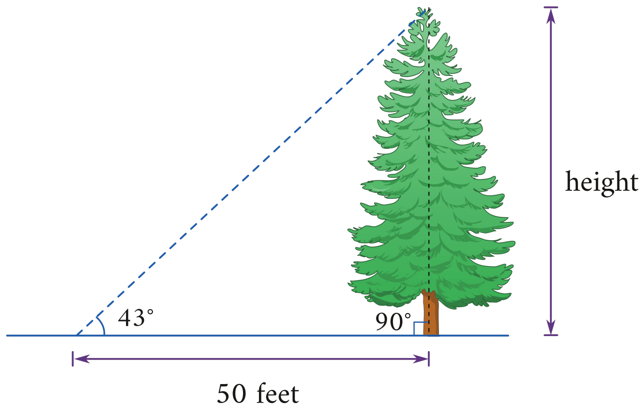


Figure 1. Trigonometry will enable us to measure the height of the tree without actually climbing the tree.

In this book we will consider just the trigonometry of the right triangle. In more advanced courses, trigonometry deals with other kinds of triangles as well. Here, however, the following definitions apply only to **right triangles**.

In right triangle ABC of Figure 2, AC is called the leg **adjacent** to ∠A. "Adjacent" means "next to." BC is called the leg **opposite** ∠A. "Opposite" here means "furthest away from."

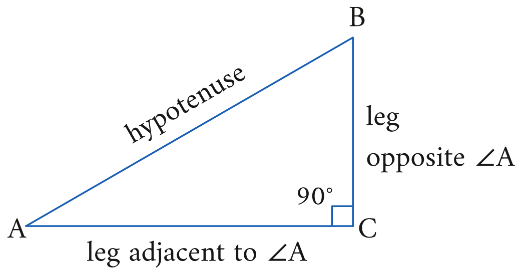


Figure 2. Right triangle ABC.

We define the **sine**, **cosine**, and **tangent** of an acute angle A in a right triangle as follows:

$$\text{sine A} = \frac{\text{leg opposite } \angle A}{\text{hypotenuse}} \qquad \left(\sin A = \frac{\text{opp}}{\text{hyp}}\right)$$

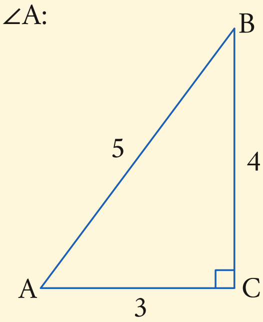
$$\text{cosine A} = \frac{\text{leg adjacent to } \angle A}{\text{hypotenuse}} \qquad \left(\cos A = \frac{\text{adj}}{\text{hyp}}\right)$$

$$\text{tangent A} = \frac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A} \qquad \left(\tan A = \frac{\text{opp}}{\text{adj}}\right)$$

The sine, cosine and tangent are called **trigonometric functions.**

? EXAMPLE A

Find the sine, cosine and tangent of ∠A:



SOLUTION

leg adjacent to ∠A = 3.

leg opposite ∠A = 4.

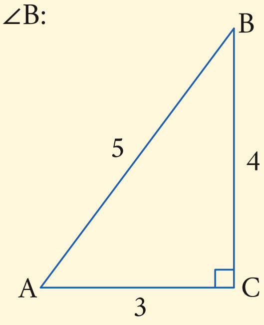
hypotenuse = 5.

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}. \quad \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}. \quad \tan A = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}.$$

ANSWER : $\sin A = \dfrac{4}{5}$, $\cos A = \dfrac{3}{5}$, $\tan A = \dfrac{4}{3}$.

? EXAMPLE B

Find the sine, cosine and tangent of $\angle B$:



SOLUTION

leg adjacent to $\angle B = 4$.

leg opposite $\angle B = 3$.

hypotenuse $= 5$.

$$\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}. \quad \cos B = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}. \quad \tan B = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}.$$

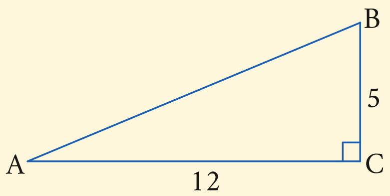
ANSWER : $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$, $\tan B = \frac{3}{4}$.

The definitions of sine, cosine and tangent should be memorized. It may be helpful to remember the mnemonic "**SOHCAHTOA:**"

Sin = **O**pp / **H**yp   **C**os = **A**dj / **H**yp   **T**an = **O**pp / **A**dj

? EXAMPLE C

Find the sin A, cos A and tan A:



SOLUTION

To find the hypotenuse, we use the **Pythagorean Theorem**:

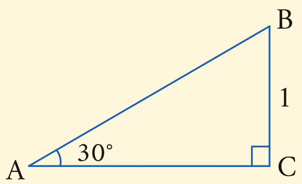
$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ 5^2 + 12^2 &= \text{hyp}^2 \\ 25 + 144 &= \text{hyp}^2 \\ 169 &= \text{hyp}^2 \\ 13 &= \text{hyp} \end{aligned}$$

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13} \qquad \tan A = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}$$

ANSWER : $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$

? EXAMPLE D

Find the sin A, cos A and tan A:

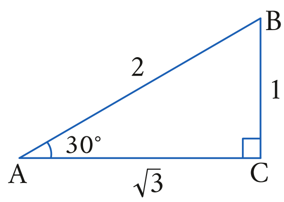


SOLUTION

$\triangle ABC$ is a 30° - 60° - 90° triangle so by **Theorem 1**, Section 4.5,
$AB = \text{hyp} = 2s = 2(1) = 2$ and $AC = L = s\sqrt{3} = (1)\sqrt{3} = \sqrt{3}$.

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2},$$

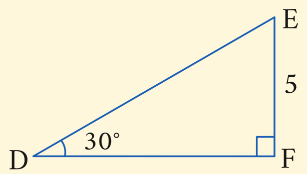
$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$



ANSWER : $\sin A = \frac{1}{2}$, $\cos A = \frac{\sqrt{3}}{2}$, $\tan A = \frac{\sqrt{3}}{3}$.

? EXAMPLE E

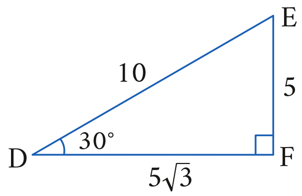
Find the sin D, cos D and tan D:



SOLUTION

Again using **Theorem 1**, Section 4.5,
DE = hyp = 2s = 2(5) = 10 and DF = L = $s\sqrt{3} = 5\sqrt{3}$.

$$\sin D = \frac{\text{opp}}{\text{hyp}} = \frac{5}{10} = \frac{1}{2} \qquad \cos D = \frac{\text{adj}}{\text{hyp}} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$$



$$\tan D = \frac{\text{opp}}{\text{adj}} = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

ANSWER : $\sin D = \frac{1}{2}$, $\cos D = \frac{\sqrt{3}}{2}$, $\tan D = \frac{\sqrt{3}}{3}$.

Notice that the answers to Example D and Example E were the same. This is because $\angle A = \angle D = 30°$. The values of the trigonometric functions for all 30° angles will be the same. The reason is that all right triangles with a 30° angle are similar. Therefore their sides are proportional and the trigonometric ratios are equal. What holds for 30° angles holds for other acute angles as well. We state this in the following theorem:

THEOREM 1

The values of the trigonometric functions for equal angles are the same.
In Figure 3, if $\angle A = \angle D = x°$ then sin A =sin D, cos A = cos D and tan A = tan D.

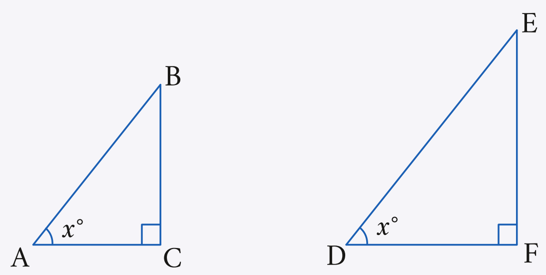


Figure 3. $\angle A = \angle D = x°$ so sin A = sin D, cos A = cos D and tan A = tan D.

Proof :
$\angle A = \angle D = x°$ and $\angle C = \angle F = 90°$ so $\triangle ABC \sim \triangle DEF$ by AA = AA. Therefore

$$\frac{BC}{EF} = \frac{AB}{DE} \quad \textit{and} \quad \frac{AC}{DF} = \frac{AB}{DE} \quad \textit{and} \quad \frac{BC}{EF} = \frac{AC}{DF}.$$

By **Theorem 2**, *Section 4.1*, we may interchange the means of each proportion:

$$\frac{BC}{AB} = \frac{EF}{DE} \quad \textit{and} \quad \frac{AC}{AB} = \frac{DF}{DE} \quad \textit{and} \quad \frac{BC}{AC} = \frac{EF}{DF}.$$

These proportions just state that
sin A = sin D and cos A = cos D and tan A = tan D.

Theorem 1 tells us that the trigonometric functions do not depend on the particular triangle chosen, only on the number of degrees in the angle. If we want to find the trigonometric values of an angle, we may chose any right triangle containing the angle which is convenient to use.

? EXAMPLE F

If $\sin A = \frac{12}{13}$ find cos A and tan A.

SOLUTION

If $\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{12}{13}$ then there is a right triangle ABC containing $\angle A$ with leg opposite $\angle A = 12$ and hypotenuse = 13 (see Figure 4).

Let b = leg adjacent to $\angle A$.

$$\begin{aligned} \text{leg}^2 + \text{leg}^2 &= \text{hyp}^2 \\ b^2 + 12^2 &= 13^2 \\ b^2 + 144 &= 169 \\ -144 \quad &\;\; -144 \\ b^2 &= 25 \\ b &= 5 \end{aligned}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{5}{13}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{12}{5}$$

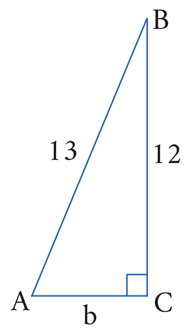


Figure 4.
$\triangle ABC$ with leg opposite $\angle A = 12$ and hypotenuse = 13.

ANSWER : $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$.

? EXAMPLE G

If $\tan A = 2$ find $\sin A$ and $\cos A$.

SOLUTION

$$\tan A = \frac{\text{opp}}{\text{adj}} = 2 = \frac{2}{1}.$$

Let $\triangle ABC$ be such that a = leg opposite $\angle A = 2$ and b = leg adjacent to $\angle A = 1$. See Figure 5.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 2^2 + 1^2 &= c^2 \\ 4 + 1 &= c^2 \\ 5 &= c^2 \\ \sqrt{5} &= c \end{aligned}$$

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

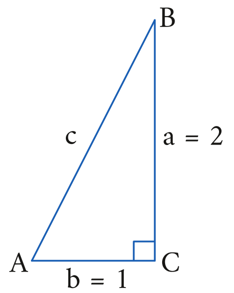


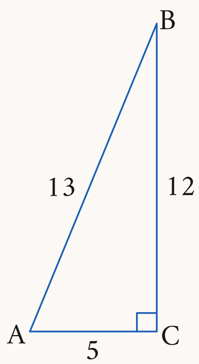
Figure 5.
$\triangle ABC$ with $a = 2$ and $b = 1$.

ANSWER: $\sin A = \frac{2\sqrt{5}}{5}$, $\cos A = \frac{\sqrt{5}}{5}$.

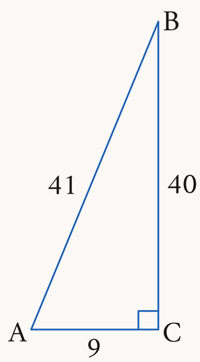
PROBLEMS

1 - 14. Find sin A, cos A, tan A, sin B, cos B and tan B:

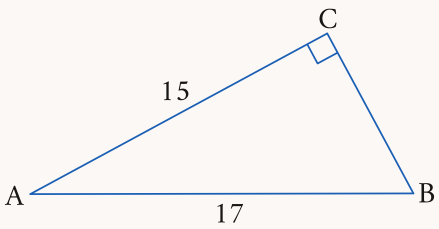
1.



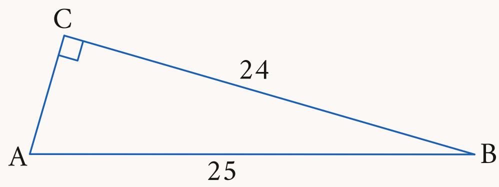
2.



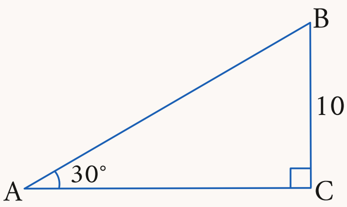
3.


C
15
A
17
B

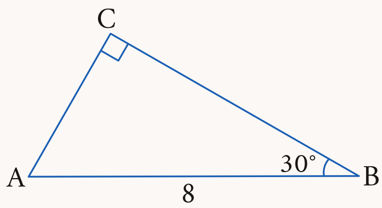
4.


C
24
A
25
B

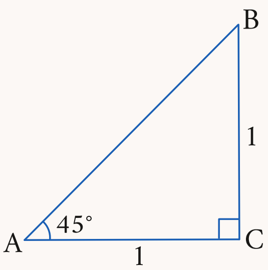
5.


B
10
A
30°
C

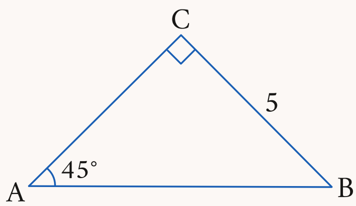
6.


C
A
30°
B
8

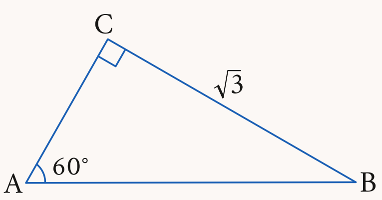
7.


B
1
A
45°
C
1

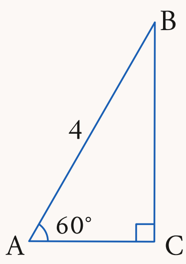
8.


C
5
A
45°
B

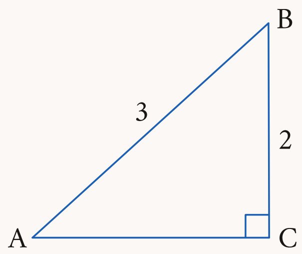
9.


C
$\sqrt{3}$
A
60°
B

10.


B
4
A
60°
C

11.



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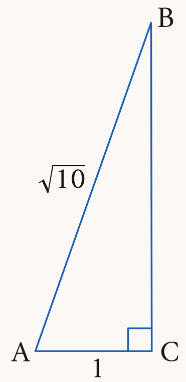
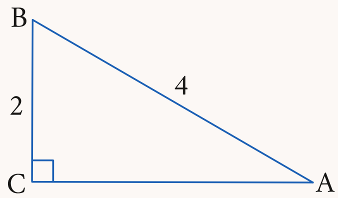
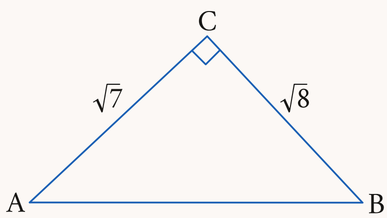


13.



14.



15. If $\sin A = \dfrac{4}{5}$ find $\cos A$ and $\tan A$.

16. If $\sin A = \dfrac{\sqrt{2}}{2}$ find $\cos A$ and $\tan A$.

17. If $\cos A = \dfrac{\sqrt{3}}{2}$ find $\sin A$ and $\tan A$.

18. If $\cos A = \dfrac{1}{3}$ find $\sin A$ and $\tan A$.

19. If $\tan A = 3$ find $\sin A$ and $\cos A$.

20. If $\tan A = 1$ find $\sin A$ and $\cos A$.

5.2 SOLUTION OF RIGHT TRIANGLES

In the preceding section we showed that all 30° angles have the same trigonometric values. If we compute each of these values to four decimal places, we obtain $\sin 30^\circ = \frac{1}{2} = 0.5000$, $\cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{1.73205}{2} = 0.8660$ and $\tan 30^\circ = \frac{\sqrt{3}}{3} = \frac{1.73205}{3} = 0.5774$. These numbers appear in the table of trigonometric values on page 303 in the row corresponding to 30°. As you can see, this table contains the trigonometric values of angles from 1° to 90°. It is impractical to compute most of these values directly, so we will use this table when we need them. A calculator with trigonometric functions may also be used.

? EXAMPLE A

Find $\sin 20^\circ$, $\cos 20^\circ$ and $\tan 20^\circ$.

SOLUTION

Look for 20° in the angle column of the table on page 303:

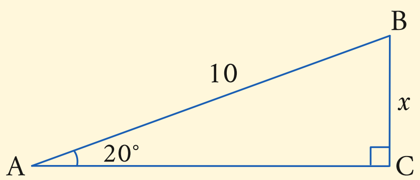
Angle	Sine	Cosine	Tangent
.	.	.	.
.	.	.	.
20°	0.3420	0.9397	0.3640
.	.	.	.

If you are using a calculator, first make sure that it is in degree mode. Then type the sin, cos or tan key followed by 20. Some calculators might require you to enter the 20 first followed by the sin, cos or tan key.

ANSWER : $\sin 20^\circ = 0.3420$, $\cos 20^\circ = 0.9397$, $\tan 20^\circ = 0.3640$

? EXAMPLE B

Find x to the nearest tenth:



SOLUTION

We wish to find the leg opposite 20° and we know the hypotenuse. We use the sine because it is the only one of the three trigonometric functions which involves both the opposite leg and the hypotenuse.

$$\sin 20^\circ = \frac{\text{opp}}{\text{hyp}}$$

$$0.3420 = \frac{x}{10}$$

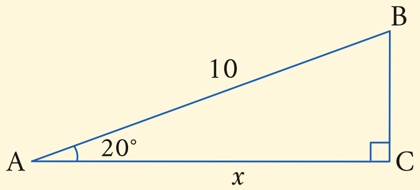
$$(10)(0.3420) = \frac{x}{\cancel{10}}(\cancel{10})$$

$$3.420 = x$$

ANSWER : $x = 3.4$.

? EXAMPLE C

Find x to the nearest tenth:



SOLUTION

We know the hypotenuse and we wish to find the leg adjacent to ∠A. We therefore use the cosine.

$$\cos 20^\circ = \frac{\text{adj}}{\text{hyp}}$$

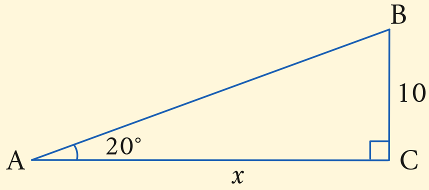
$$0.9397 = \frac{x}{10}$$

$$9.3970 = x$$

ANSWER : $x = 9.4$.

? EXAMPLE D

Find x to the nearest tenth:



SOLUTION

We know the leg opposite $\angle A$ and we wish to find the leg adjacent to $\angle A$. We therefore use the tangent.

$$\tan 20^\circ = \frac{\text{opp}}{\text{adj}}$$

$$0.3640 = \frac{10}{x}$$

$$(x)(0.3640) = \frac{10}{\cancel{x}}\cancel{(x)}$$

$$0.3640x = 10$$

$$\frac{0.3640x}{0.3640} = \frac{10}{0.3640}$$

$$x = \frac{10}{0.3640} = 27.47$$

ANSWER: $x = 27.5$.

There is an easier method to solve Example D. $\angle B = 90^\circ - 20^\circ = 70^\circ$ The leg opposite $\angle B$ is x and the leg adjacent to $\angle B$ is 10.

$$\tan 70^\circ = \frac{\text{opp}}{\text{adj}}$$

$$2.7475 = \frac{x}{10}$$

$$(2.7475)(10) = x$$

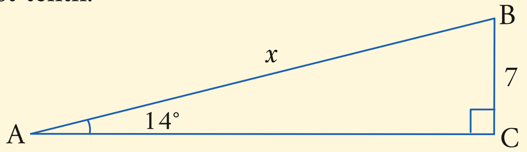
$$27.475 = x$$

$$27.5 = x$$

This method is easier because it involves multiplication rather than long division.

? EXAMPLE E

Find x to the nearest tenth:



SOLUTION

$$\sin 14^\circ = \frac{\text{opp}}{\text{hyp}}$$

$$0.2419 = \frac{7}{x}$$

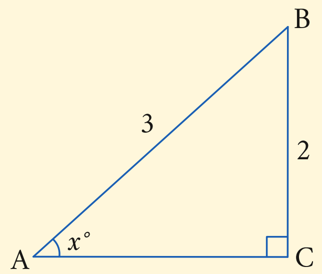
$$0.2419x = 7$$

$$x = \frac{7}{0.2419} = 28.9$$

ANSWER: $x = 28.9$

? EXAMPLE F

Find x to the nearest degree:



SOLUTION

$$\sin x^\circ = \frac{\text{opp}}{\text{hyp}}$$

$$\sin x^\circ = \frac{2}{3} = 0.6667$$

In the table we look in the sine column for the value closest to 0.6667:

Angle	Sine
.	.
.	.
41°	0.6561
42°	0.6691
.	.

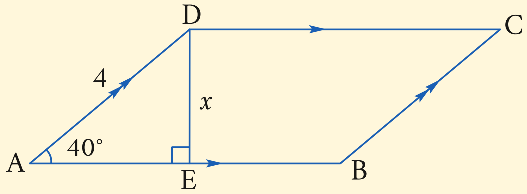
0.6667 is closest to 0.6691 because 0.6691 − 0.6667 = 0.0024 whereas 0.6667 − 0.6561 = 0.0106. Therefore $x^\circ = 42^\circ$, to the nearest degree.

If you are using a calculator, you will need to use the INV, SHIFT, or 2nd key followed by the sin key, depending on the calculator, followed by 2/3. Some calculators might have a built in sine inverse key. Some calculators might require you to enter the 2/3 first.

ANSWER : x = 42

? EXAMPLE G

Find x to the nearest tenth:



SOLUTION

$$\sin 40^\circ = \frac{\text{opp}}{\text{hyp}}$$

$$0.6428 = \frac{x}{4}$$

$$(4)(0.6428) = \frac{x}{\cancel{4}}(\cancel{4})$$

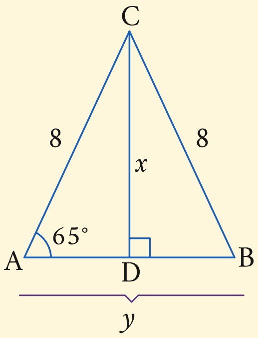
$$2.5712 = x$$

$$2.6 = x$$

ANSWER: $x = 2.6$.

EXAMPLE H

Find x and y to the nearest tenth:



SOLUTION

$$\sin 65^\circ = \frac{\text{opp}}{\text{hyp}}$$

$$0.9063 = \frac{x}{8}$$

$$(8)(0.9063) = \frac{x}{\cancel{8}}(\cancel{8})$$

$$7.2504 = x$$

$$7.3 = x$$

To find y we first find AD:

$$\cos 65^\circ = \frac{\text{adj}}{\text{hyp}}$$

$$0.4226 = \frac{AD}{8}$$

$$(8)(0.4226) = \frac{AD}{\cancel{8}}(\cancel{8})$$

$$3.3808 = AD$$

Since AC = BC = 8 we have $\angle A = \angle B = 65^\circ$. Therefore BD = AD = 3.3808.
y = AD + BD = 3.3808 + 3.3808 = 6.7616 = 6.8.

ANSWER: $x = 7.3$, $y = 6.8$.

HISTORICAL NOTE

The first table of trigonometric values was constructed by the Greek astronomer Hipparchus (c.180 - 125 B.C.). Hipparchus assumed the vertex of each angle to be the center of a circle, as ∠AOB is shown to be in the circle of Figure 1. Depending on the number of degrees in ∠AOB, his table would give the length of the chord AB relative to the radius of the circle. Today we would measure ∠AOC instead of ∠AOB and use the half chord AC instead of AB. The ratio AC/AO is then just the sine of ∠AOC.

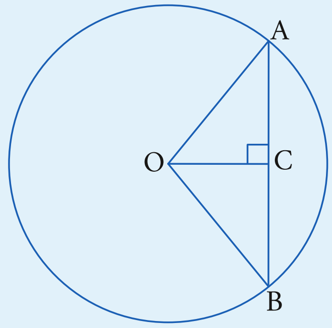


Figure 1. The table of Hipparchus gave the length of the chord AB relative to the radius AO for each angle AOB.

Hipparchus obtained some of the values for his table from the properties of special geometric figures, such as the $30^\circ-60^\circ-90^\circ$ triangle and the $45^\circ-45^\circ-90^\circ$ triangle. The rest of the values were obtained from those already known by using trigonometric identities and approximation. The identities he used were essentially the half-angle and sum and difference formulas which students encounter in modern trigonometry courses.

The trigonometry of the Greeks and later of the Hindus and the Arabs, was based primarily on the sine function. The Hindus replaced the table of chords of Hipparchus with a table of half chords. The term **sine** is derived from a Hindu word meaning "half-chord."

Gradually the right triangle replaced the chords of circles as the basis of trigonometric definitions. The **cosine** is just the sine of the complement of the angle in a right triangle. For example the complement of 60° is 30° and $\cos 60^\circ = \sin 30^\circ = 0.5$.

A **tangent** is a line which touches a circle at only one point (see Chapter 7). In trigonometry it refers to just that part of the tangent line intercepted by the angle, relative to the radius of the circle. In Figure 2 the tangent of ∠DOE is the segment DE divided by the radius OD.

The ancient Greeks were probably aware of the tangent function but the first known table of values was constructed by the Arabs in 10th century. The term "tangent" was adopted in the 16th century.

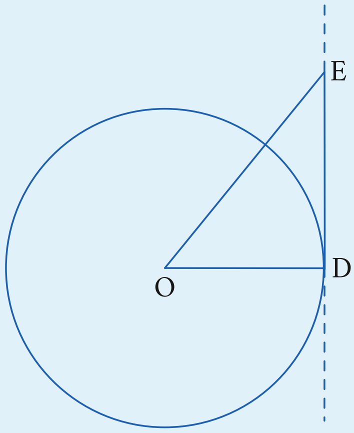


Figure 2. The tangent of ∠DOE is DE/OD.

Modern trigonometric tables are constructed from infinite series. These were first discovered in the 17th century by Newton, Leibniz and others. For example the infinite series for the sine function is

$$\text{sine } x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \ldots$$

where x is in radians, 1 radian = 57.296 degrees. A good approximation of the sine of an angle can be obtained from the infinite series by summing just the first few terms. This is also the method computers and calculators use to find trigonometric values. The derivation of these formulas is found in calculus textbooks.

PROBLEMS

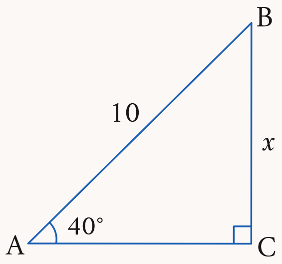
1 - 10. Find each of the following using the table:

1. $\sin 10^\circ$ 2. $\sin 30^\circ$ 3. $\cos 80^\circ$ 4. $\cos 60^\circ$ 5. $\tan 45^\circ$

6. $\tan 60^\circ$ 7. $\sin 18^\circ$ 8. $\cos 72^\circ$ 9. $\tan 50^\circ$ 10. $\tan 80^\circ$

11 - 30. Find x to the nearest tenth:

11.



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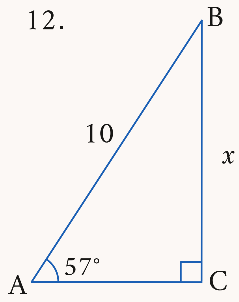

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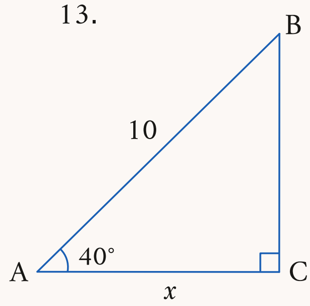

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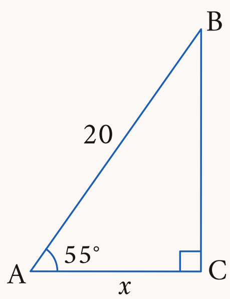

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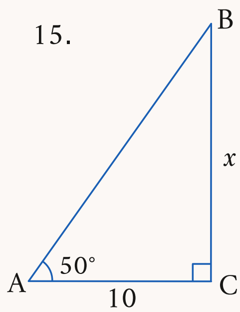

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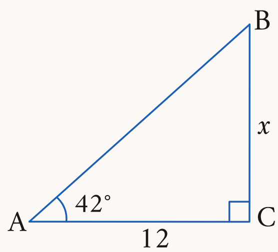
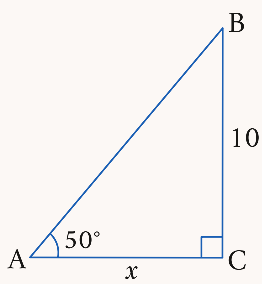


17.



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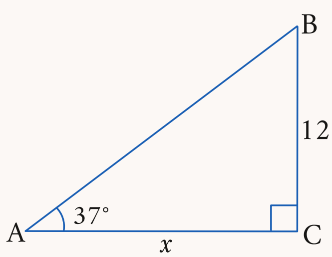

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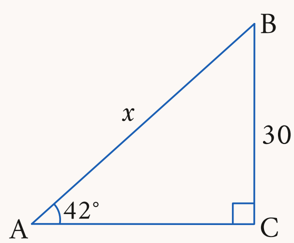

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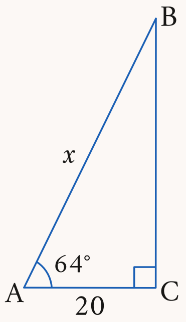

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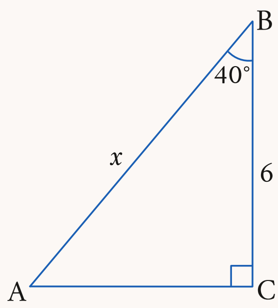

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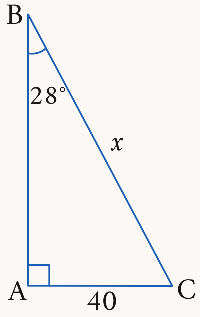

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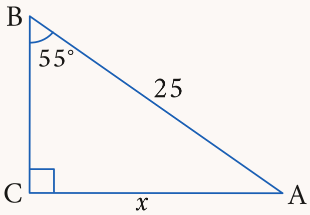

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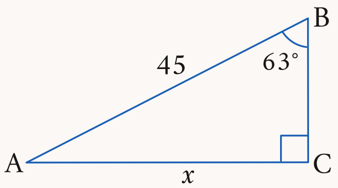

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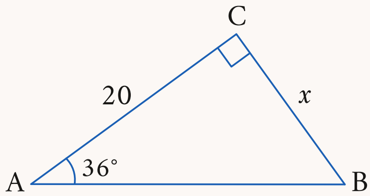

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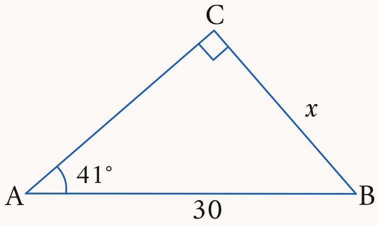

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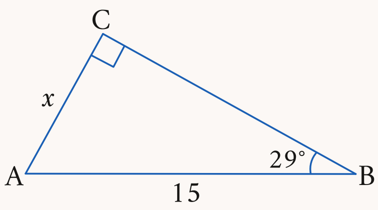

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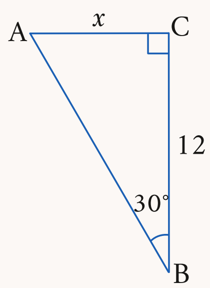

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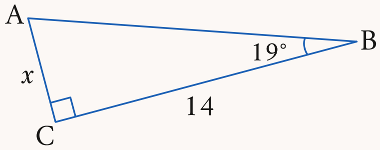

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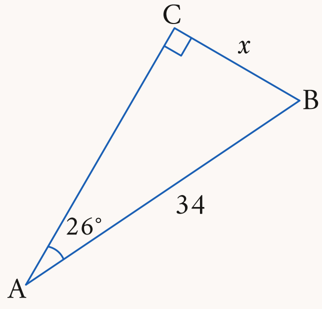
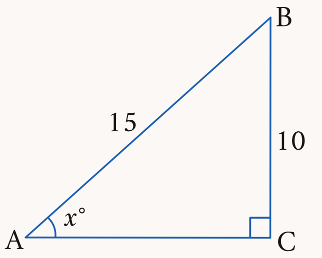


31 - 38. Find x to the nearest degree:

31.



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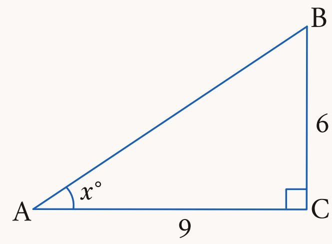

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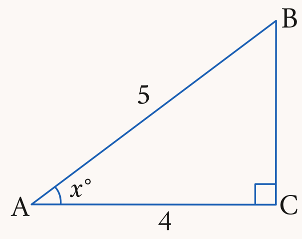

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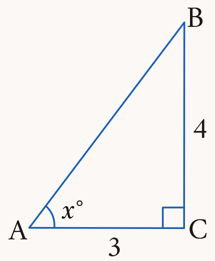

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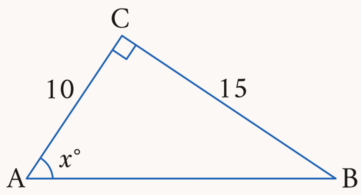

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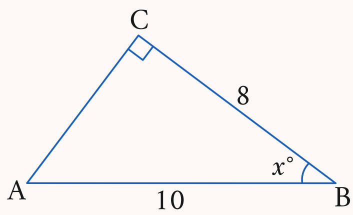

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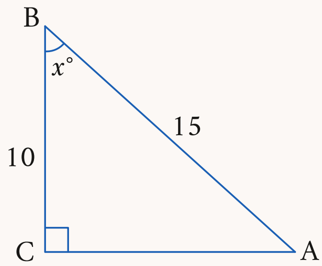

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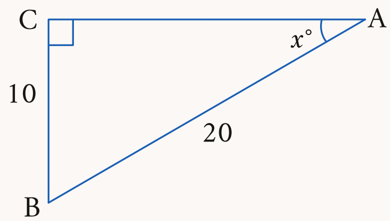
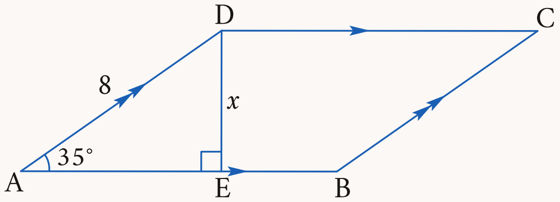


39 - 46. Find x or x and y to the nearest tenth:

39.



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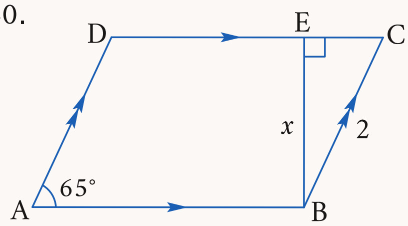

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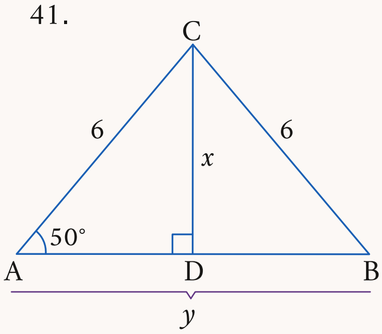

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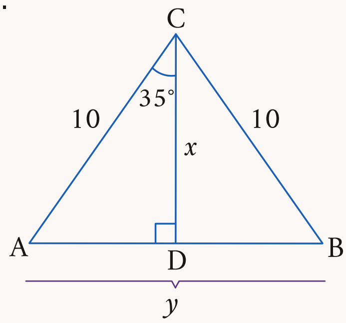

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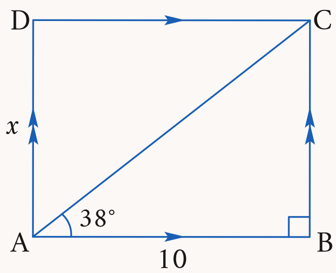

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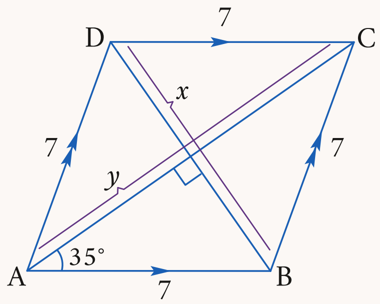

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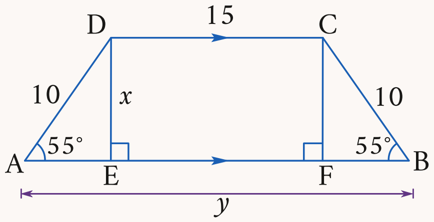

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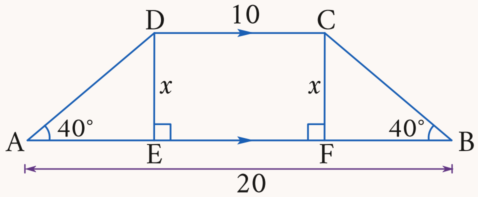


5.3 APPLICATIONS OF TRIGONOMETRY

Trigonometry has many applications in science and engineering. In this section we will present just a few examples from surveying and navigation.

The angle made by the line of sight of an observer on the ground to a point above the horizontal is called the **angle of elevation**. In Figure 1, ∠BAC is the angle of elevation.

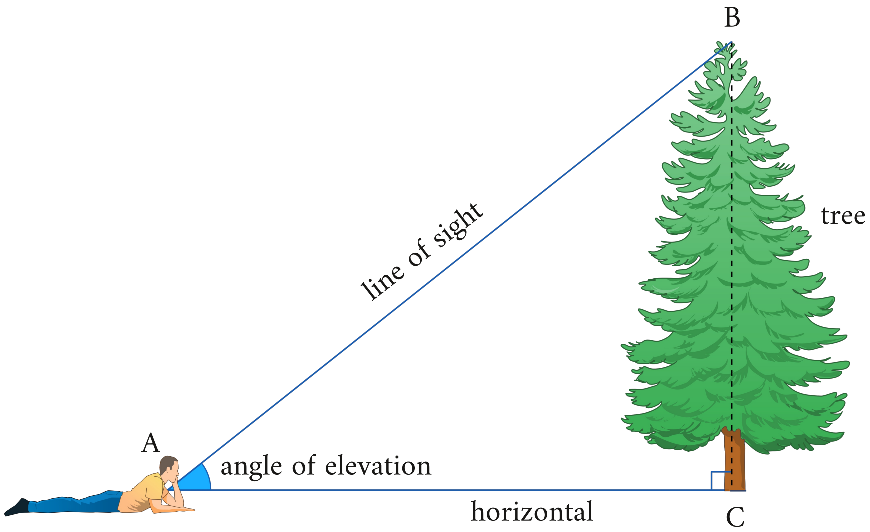
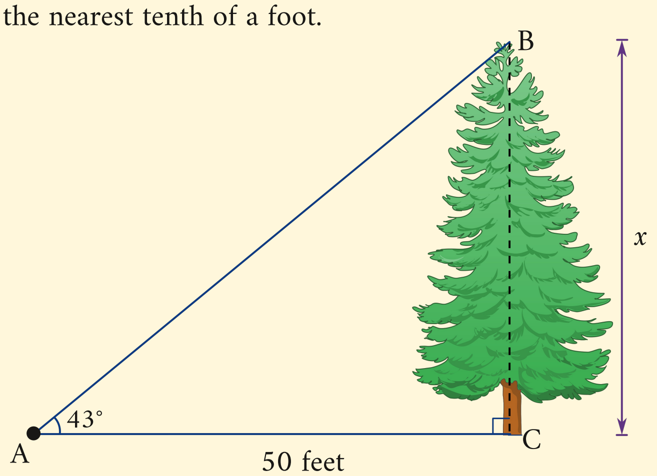


Figure 1. The angle of elevation.

? EXAMPLE A

At a point 50 feet from a tree the angle of elevation of the top of the tree is 43°. Find the height of the tree to the nearest tenth of a foot.



SOLUTION

Let x = height of tree.

$$\tan 43^\circ = \frac{x}{50}$$

$$0.9325 = \frac{x}{50}$$

$$(50)\,0.9325 = \frac{x}{\cancel{50}}\cancel{(50)}$$

$$46.6250 = x$$

$$46.6 = x$$

ANSWER : x = 46.6 feet.

The angle made by the line of sight of an observer above to a point on the ground is called the **angle of depression**. In Figure 2 ∠ABD is the angle of depression.

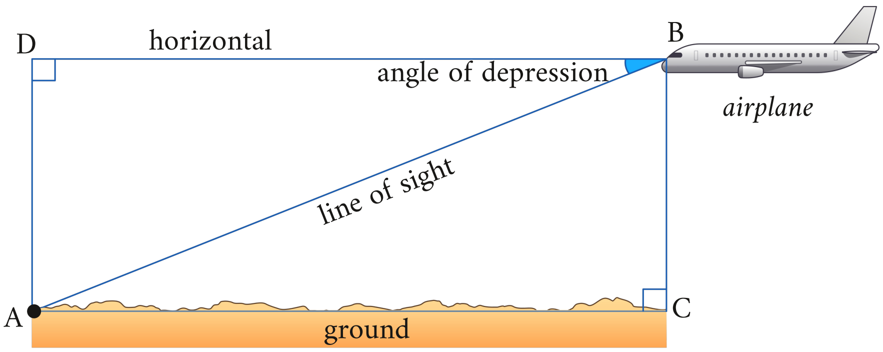
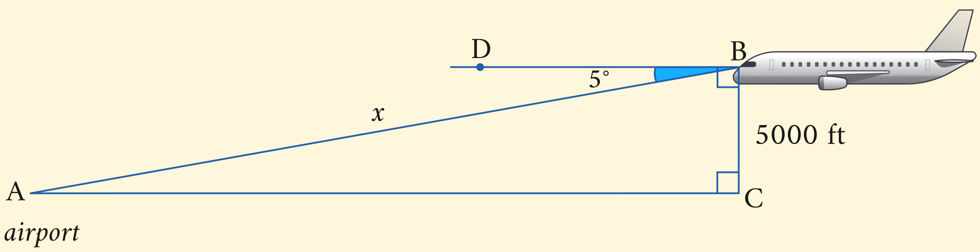


Figure 2. The angle of depression.

? EXAMPLE B

From an airplane 5000 feet above the ground the angle of depression of an airport is 5°. How far away is the airport to the nearest hundred feet?



SOLUTION

Let x = distance to airport. $\angle ABC = 85^\circ$.

$$\cos 85^\circ = \frac{5000}{x}$$

$$0.0872 = \frac{5000}{x}$$

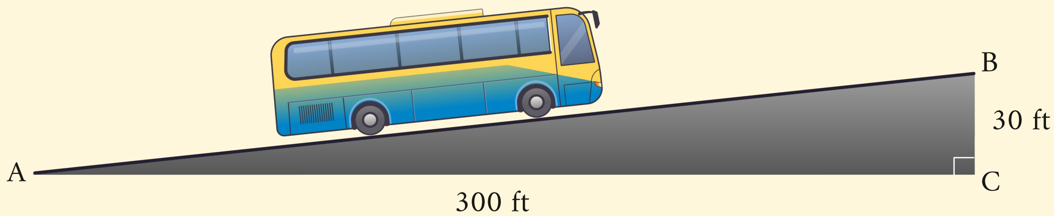
$$0.0872x = 5000$$

$$x = \frac{5000}{0.0872} = 57{,}300$$

ANSWER : x = 57,300 feet.

? EXAMPLE C

A road rises 30 feet in a horizontal distance of 300 feet. Find to the nearest degree the angle the road makes with the horizontal.



SOLUTION

$$\tan A = \frac{30}{300}$$

$$\tan A = 0.1000$$

$$\angle A = 6^\circ$$

ANSWER : 6°.

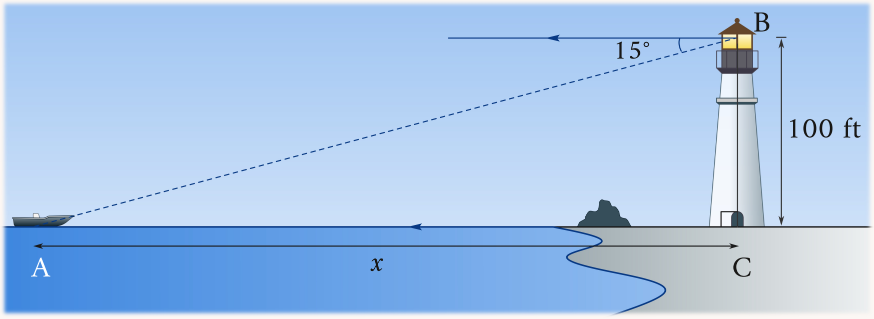
PROBLEMS

1. At a point 60 feet from a tree the angle of elevation of the top of the tree is 40°. Find the height of the tree to the nearest tenth of a foot.

2. At a point 100 feet from a tall building the angle of elevation of the top of the building is 65°. Find the height of the building to the nearest foot.

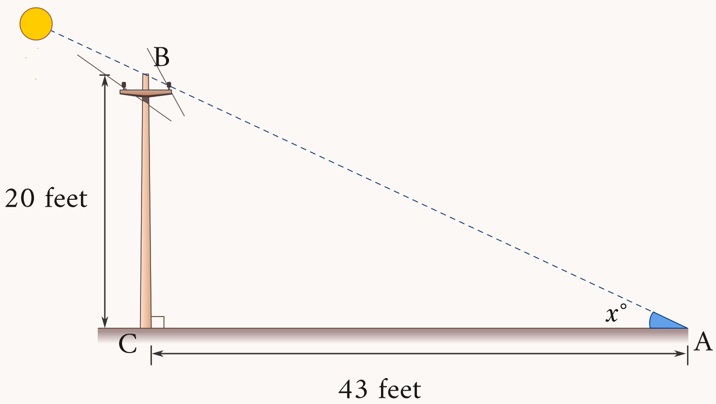
3. From a helicopter 1000 feet above the ground the angle of depression of a heliport is $10°$. How far away is the heliport to the nearest foot?

4. From the top of a 100 foot lighthouse the angle of depression of a boat is $15°$. How far is the boat from the bottom of the lighthouse (nearest foot)?



5. A road rises 10 feet in a horizontal distance of 400 feet. Find to the nearest degree the angle the road makes with the horizontal.

6. If a 20 foot telephone pole casts a shadow of 43 feet, what is the angle of elevation of the sun?



7. A 20 foot ladder is leaning against a wall. It makes an angle of $70°$ with the ground. How high is the top of the ladder from the ground (nearest tenth of a foot)?

8. The angle of elevation of the top of a mountain from a point 20 miles away is $6°$. How high is the mountain (nearest tenth of a mile)?

CHAPTER 6

AREA AND PERIMETER

6.1 THE AREA OF A RECTANGLE AND SQUARE

The measurement of the area of geometric figures is one of the most familiar ways mathematics is used in our daily lives. The floor space of a building, the size of a picture, the amount of paper in a roll of paper towels are all examples of items often measured in terms of area. In this chapter we will derive formulas for the areas of the geometric objects which we have studied.

Area is measured in square inches, square feet, square centimeters, etc. The basic unit of measurement is the **unit square**, the square whose sides are of length 1 (Figure 1). Its area is 1 square inch, 1 square foot, 1 square centimeter, etc., depending on which measurement of length is chosen. The **area** of any closed figure is defined to be the number of unit squares it contains.

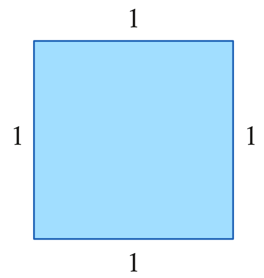


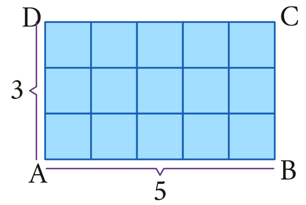
Figure 1. The unit square.

? EXAMPLE A

Find the area of a rectangle with length 5 and width 3.

SOLUTION

We see from the diagram that the area is $(5)(3) = 15$.



ANSWER: 15.

This suggests the following theorem:

THEOREM 1

The area of a rectangle is the length times its width.

$$\boxed{A = lw}$$

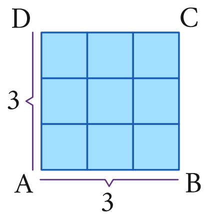
EXAMPLE B

Find the area of a square with side 3.

SOLUTION

Area $= (3)(3) = 3^2 = 9$.

ANSWER : 9.



The formula for a square is now self-evident:

THEOREM 2

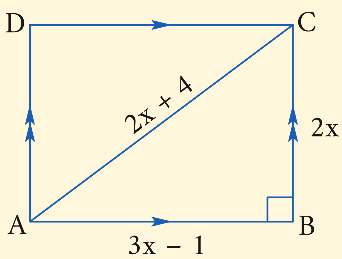
The area of a square is the square of one of its sides.

$$\boxed{A = s^2}$$

The **perimeter** of a polygon is the sum of the lengths of its sides. For instance the perimeter of the rectangle of Example A would be $5 + 5 + 3 + 3 = 16$.

EXAMPLE C

Find the area and perimeter of rectangle ABCD:



SOLUTION

We first use the **Pythagorean Theorem** to find x:

$$\begin{aligned} AB^2 + BC^2 &= AC^2 \\ (3x-1)^2 + (2x)^2 &= (2x+4)^2 \\ 9x^2 - 6x + 1 + 4x^2 &= 4x^2 + 16x + 16 \\ 9x^2 - 22x - 15 &= 0 \\ (9x+5)(x-3) &= 0 \end{aligned}$$

$$9x + 5 = 0 \qquad x - 3 = 0$$

$$x = -\frac{5}{9} \qquad x = 3$$

We reject the answer $x = -\frac{5}{9}$ because $BC = 2x = 2(-\frac{5}{9}) = -\frac{10}{9}$ would have negative length. Therefore x = 3.

$AB = 3x - 1 = 3(3) - 1 = 9 - 1 = 8.$ $\qquad BC = 2x = 2(3) = 6$

$AC = 2x + 4 = 2(3) + 4 = 6 + 4 = 10.$

CHECK :

$$AB^2 + BC^2 = AC^2$$

$8^2 + 6^2$	10^2
$64 + 36$	100
100	

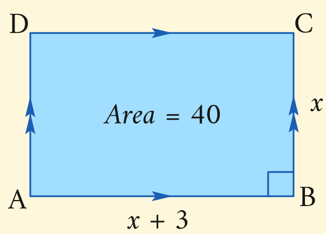
Area = $l \cdot w = (8)(6) = 48.$

Perimeter = $8 + 8 + 6 + 6 = 28.$

ANSWER : Area = 48, Perimeter = 28.

? EXAMPLE D

Find x:



SOLUTION

$$\text{Area} = l \cdot w$$
$$40 = (x + 3)(x)$$
$$40 = (x^2 + 3x)$$
$$0 = x^2 + 3x - 40$$
$$0 = (x - 5)(x + 8)$$
$$0 = x - 5 \qquad x = -8$$

We reject x = -8 because side BC = x of the rectangle would be negative.

CHECK :

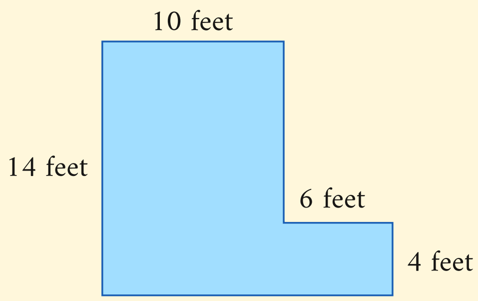
$x = 5$

$$\text{Area} = l \cdot w$$
$$\begin{array}{r|l} 40 & (x + 3)(x) \\ 40 & (5 + 3)(5) \\ 40 & (8)(5) \\ & 40 \end{array}$$

ANSWER : $x = 5$.

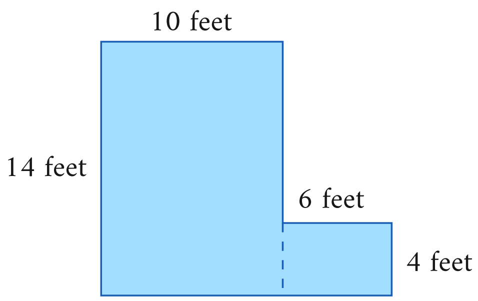
? EXAMPLE E

An L-shaped room has the dimensions indicated in the diagram. How many one by one foot tiles are needed to tile the floor?



SOLUTION

Divide the room into two rectangles as shown.



$$\begin{aligned}\text{Area of room} &= \text{Area of large rectangle} + \text{Area of small rectangle} \\ &= (14)(10) + (6)(4) \\ &= 140 + 24 \\ &= 164 \text{ square feet}\end{aligned}$$

ANSWER : 164.

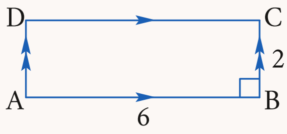
HISTORICAL NOTE

The need to measure land areas was one of the ancient problems which led to the development of geometry. Both the early Egyptians and Babylonians had formulas for the areas of rectangles, triangles and trapezoids, but some of their formulas were not entirely accurate. The formulas in this chapter were known to the Greeks and are found in Euclid's *Elements*.

PROBLEMS

1 - 14. Find the area and perimeter of ABCD:

1.



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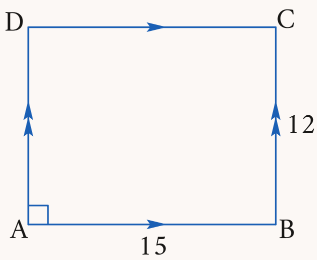

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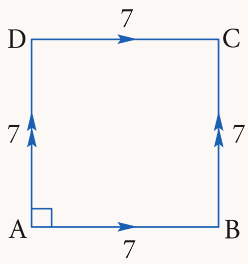

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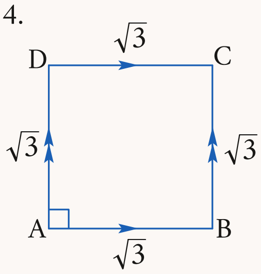

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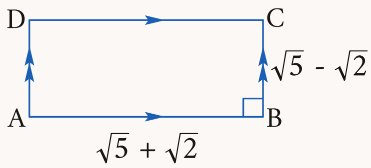

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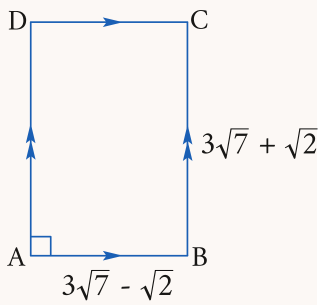

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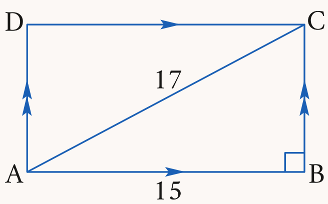

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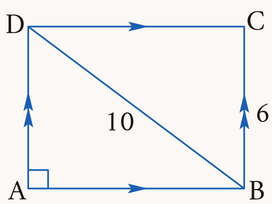

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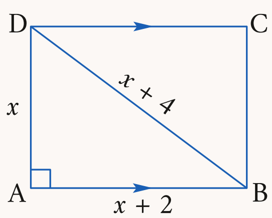

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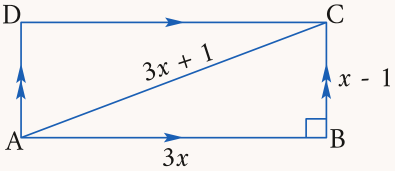

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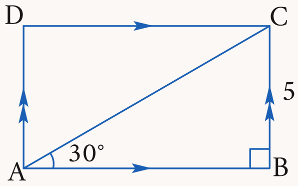

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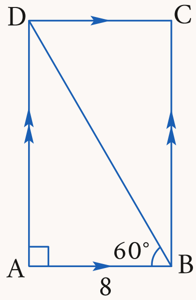
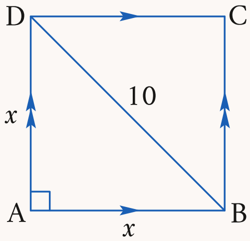


13.



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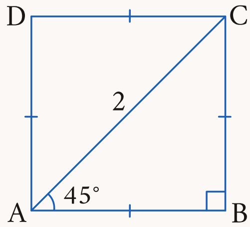
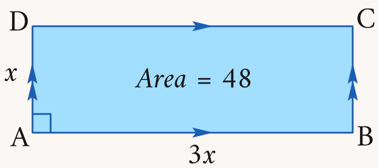


15 - 18. Find x:

15.



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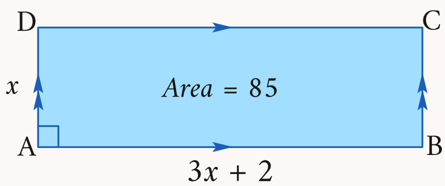

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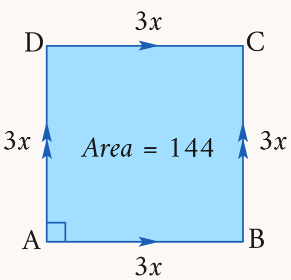

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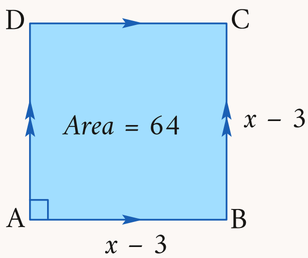
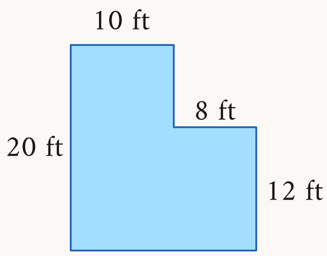


19. A football field has length 300 feet and width 160 feet. What is the area?

20. A tennis court is 78 feet long and 36 feet wide. What is the area?

21 - 24. How many one by one foot tiles are needed to tile each of the following rooms?

21.



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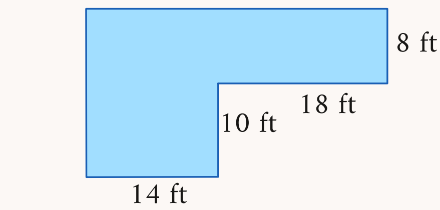

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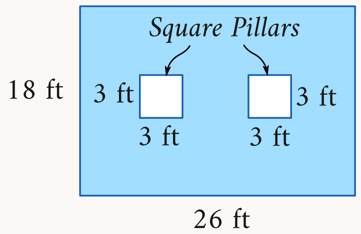

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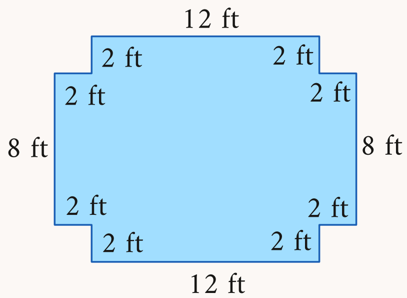


25. A concrete slab weighs 60 pounds per square foot. What is the total weight of a rectangular slab 10 feet long and 3 feet wide?

26. A rectangular piece of plywood is 8 by 10 feet. If the plywood weighs 3 pounds per square foot, what is the weight of the whole piece?

6.2 THE AREA OF A PARALELLOGRAM

In parallelogram ABCD of Figure 1, side AB is called the **base** and the line segment DE is called the **height** or **altitude**. The base may be any side of the parallelogram, though it is usually chosen to be the side on which the parallelogram appears to be resting. The height is a line drawn perpendicular to the base from the opposite side.

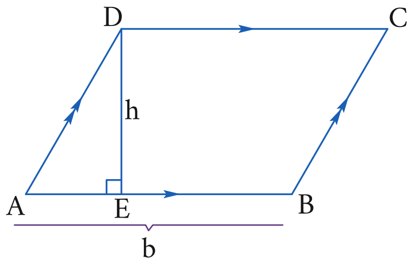


Figure 1. Parallelogram ABCD with base b and height h.

THEOREM 1

The area of a parallelogram is equal to its base times its height.

$$\boxed{A = bh}$$

Proof :
Draw BF and CF as shown in Figure 2. $\angle A = \angle CBF$, $\angle AED = \angle F = 90^\circ$ and AD = BC. Therefore $\triangle ADE \cong \triangle BCF$ and the area of $\triangle ADE$ equals the area of $\triangle BCF$. We have:

$$\begin{aligned} \text{Area of parallelogram ABCD} &= \text{Area of } \triangle ADE + \text{Area of trapezoid BCDE} \\ &= \text{Area of } \triangle BCF + \text{Area of trapezoid BCDE} \\ &= \text{Area of rectangle CDEF} \\ &= bh \end{aligned}$$

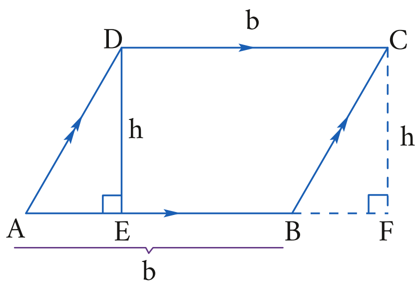
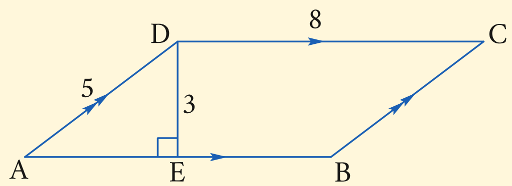


Figure 2. Draw BF and CF.

? EXAMPLE A

Find the area and perimeter of ABCD:



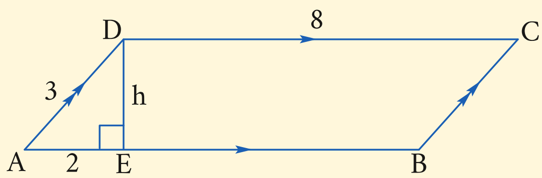
SOLUTION

b = AB = CD = 8, h=3.
Area = bh = (8)(3) = 24.
AB = CD = 8. BC = AD = 5.
Perimeter = 8 + 8 + 5 + 5 = 26.

ANSWER : Area = 24, Perimeter = 26.

? EXAMPLE B

Find the area and perimeter of ABCD:



SOLUTION

Apply the **Pythagorean theorem** to right triangle ADE:

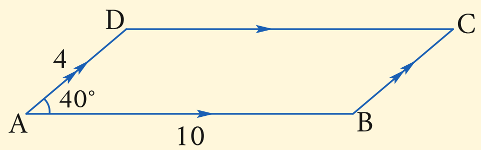
$$AE^2 + DE^2 = AD^2$$
$$2^2 + h^2 = 3^2$$
$$4 + h^2 = 9$$
$$h^2 = 5$$
$$h = \sqrt{5}$$

Area = bh = $(8)(\sqrt{5}) = 8\sqrt{5}$.
Perimeter = 8 + 8 + 3 + 3 = 22.

ANSWER : Area = $8\sqrt{5}$, Perimeter = 22.

? EXAMPLE C

Find the area and perimeter to the nearest tenth:



SOLUTION

To find the area we must first find the height h (Figure 3). Using trigonometry,

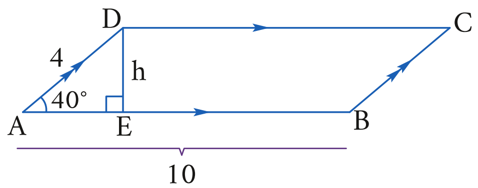


Figure 3. Draw in height h.

$$\sin 40° = \frac{h}{4}$$

$$(4)\ 0.6428 = \frac{h}{\cancel{4}}\ \cancel{(4)}$$

$$2.5712 = h$$

$$\text{Area} = bh = (10)(2.5712)$$

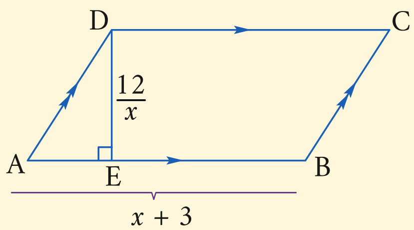
$$= 25.712 = 25.7$$

$$\text{Perimeter} = 10 + 10 + 4 + 4 = 28.$$

ANSWER: $A = 25.7$, $P = 28$.

? EXAMPLE D

Find x if the area is 21:



SOLUTION

$$A = bh$$

$$21 = (x + 3)\left(\frac{12}{x}\right)$$

$$(x)21 = (x + 3)\frac{12}{\cancel{x}}\ \cancel{(x)}$$

$$21x = 12x + 36$$

$$9x = 36$$

$$x = 4$$

CHECK

$$A = bh$$

$$21 \;\Big|\; (x + 3)\frac{12}{x}$$

$$(4 + 3)\frac{12}{4}$$

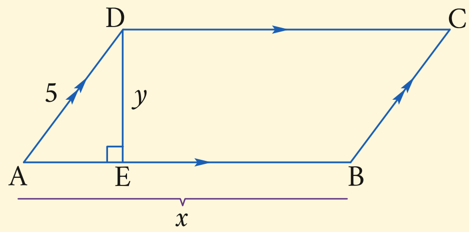
$$(7)(3)$$

$$21$$

ANSWER: $x = 4$.

? EXAMPLE E

The area of parallelogram ABCD is 48 and the perimeter is 34. Find x and y:



SOLUTION

Perimeter = AB + BC + CD + DA

$34 = x + 5 + x + 5$

$34 = 2x + 10$

$24 = 2x$

$12 = x$

Area = xy

$48 = 12y$

$4 = y$

CHECK

Perimeter = x + 5 + x + 5

$34 \mid 12 + 5 + 12 + 5$

$ \mid 34$

Area = xy

$48 \mid (12)(4)$

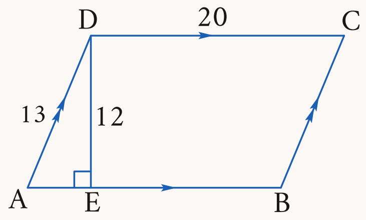
$ \mid 48$

ANSWER: $x = 12$, $y = 4$.

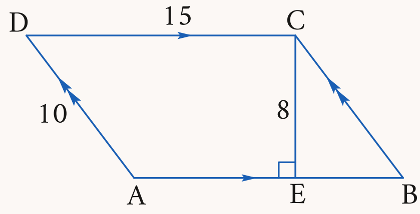
PROBLEMS

1 - 4. Find the area and perimeter of ABCD:

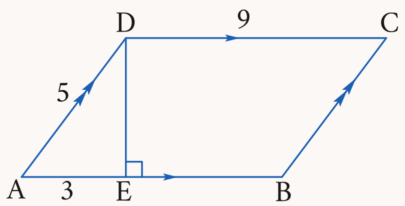
1.



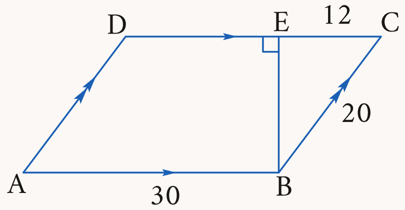
2.



3.

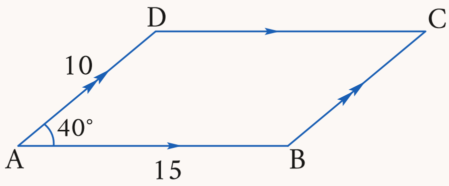


4.

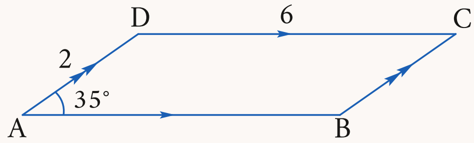


5 - 6. Find the area and perimeter to the nearest tenth:

5.

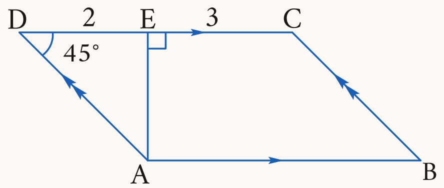


6.

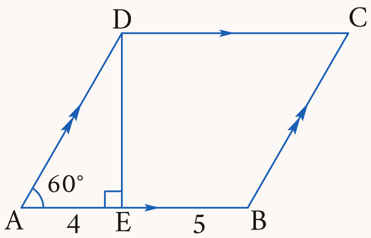


7 - 8. Find the area and perimeter. Leave answers in simplest radical form:

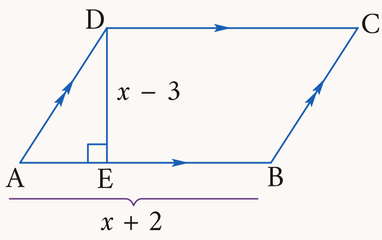
7.



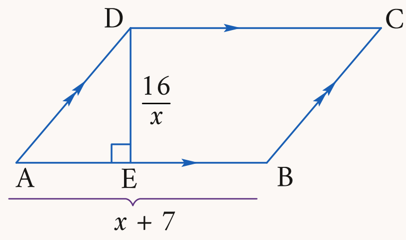
8.



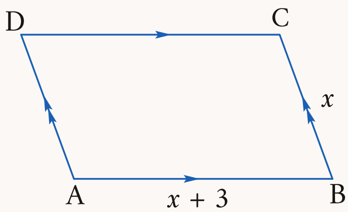
9. Find x if the area of ABCD is 36:



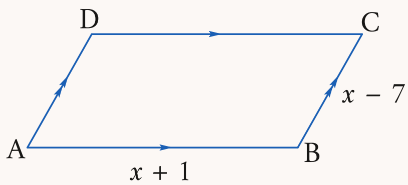
10. Find x if the area of ABCD is 72:



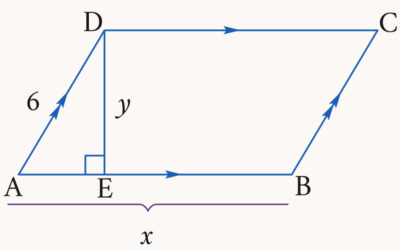
11. Find x if the perimeter is 22:



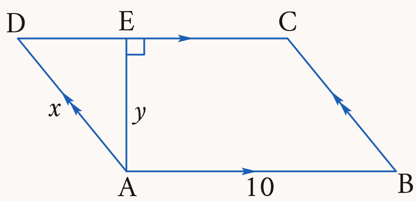
12. Find x if the perimeter is 40:



13. The area of ABCD is 40 and the perimeter is 28. Find x and y:



14. The area of ABCD is 40 and the perimeter is 30. Find x and y:



6.3 THE AREA OF A TRIANGLE

For each of the triangles in Figure 1, side AB is called the **base** and CD is called the **height** or **altitude** drawn to this base. The base can be any state of the triangle though it is usually chosen to be the side on which the triangle appears to be resting. The height is the line drawn perpendicular to the base from the opposite vertex. Note that the height may fall outside the triangle, if the triangle is obtuse, and that the height may be one of the legs, if the triangle is a right triangle.

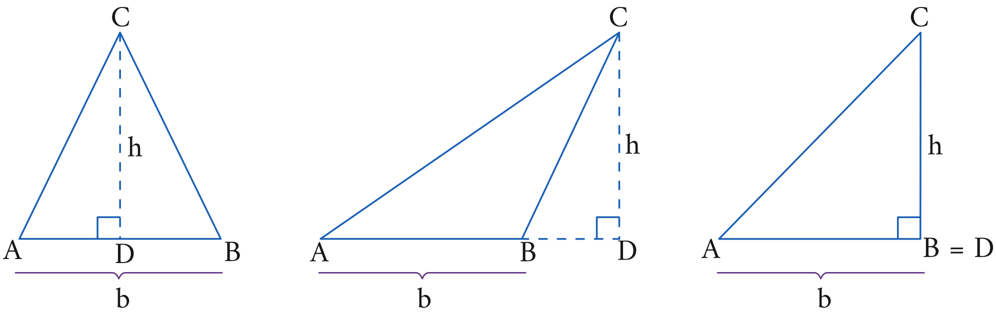


Figure 1. Triangles with base *b* and height *h*.

THEOREM 1

The area of a triangle is equal to one-half of its base times its height.

$$\boxed{A = \frac{1}{2}bh}$$

Proof :

For each of the triangles illustrated in Figure 1, draw AE and CE so that ABCE is a parallelogram (Figure 2). $\triangle ABC \cong \triangle CEA$ so area of $\triangle ABC$ = area of $\triangle CEA$. Therefore area of $\triangle ABC = \frac{1}{2}$ area of parallelogram ABCE $= \frac{1}{2}$ bh.

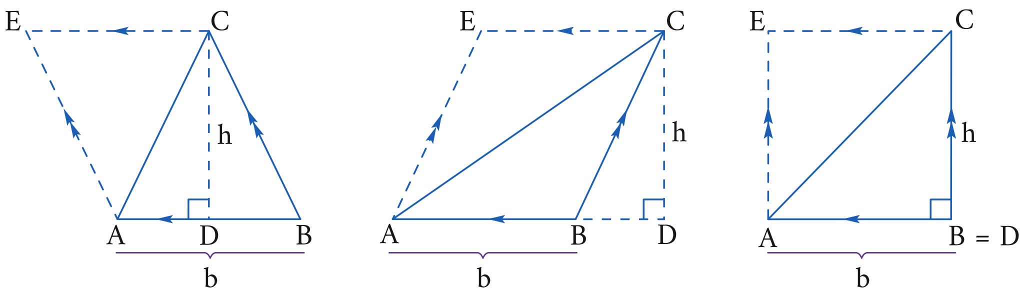
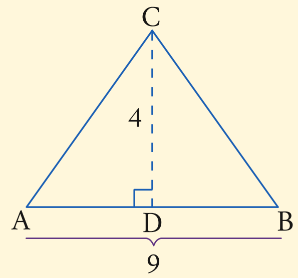


Figure 2. Draw AE and CE so that ABCE is a parallelogram.

? EXAMPLE A

Find the area:



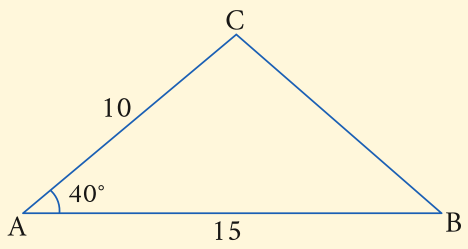
SOLUTION

$$A = \frac{1}{2}bh = \frac{1}{2}(9)(4) = \frac{1}{2}(36) = 18.$$

ANSWER : 18.

? EXAMPLE B

Find the area to the nearest tenth:



SOLUTION

Draw the height h as shown in Figure 3.

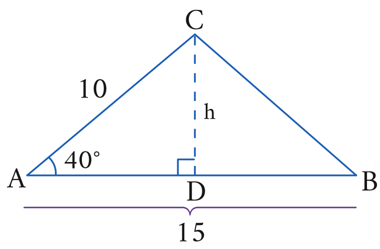


Figure 3. Draw height h.

$$\sin 40^\circ = \frac{h}{10}$$

$$0.6428 = \frac{h}{10}$$

$$(10)(0.6428) = \frac{h}{10}(10)$$

$$6.428 = h$$

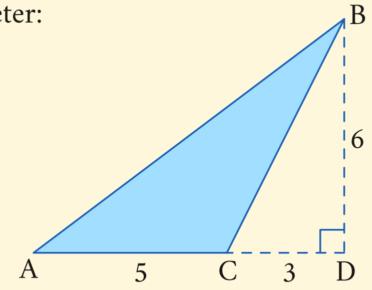
$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}(15)(6.428) = \frac{1}{2}(96.420) = 48.21 = 48.2$$

ANSWER : 48.2 .

? EXAMPLE C

Find the area and perimeter:



SOLUTION

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(5)(6) = \frac{1}{2}(30) = 15.$$

To find AB and BC we use the **Pythagorean theorem**:

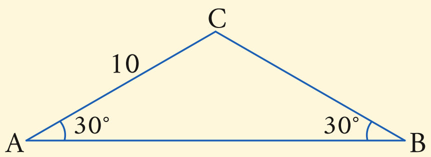
$$AD^2 + BD^2 = AB^2 \qquad CD^2 + BD^2 = BC^2$$
$$8^2 + 6^2 = AB^2 \qquad 3^2 + 6^2 = BC^2$$
$$64 + 36 = AB^2 \qquad 9 + 36 = BC^2$$
$$100 = AB^2 \qquad 45 = BC^2$$
$$10 = AB \qquad BC = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$$

$$\text{Perimeter} = AB + AC + BC = 10 + 5 + 3\sqrt{5} = 15 + 3\sqrt{5}$$

ANSWER : $A = 15,\ P = 15 + 3\sqrt{5}$

? EXAMPLE D

Find the area and perimeter:



SOLUTION

$\angle A = \angle B = 30°$ so $\triangle ABC$ is isosceles with $BC = AC = 10$. Draw height h as in Figure 4.

$\triangle ACD$ is a $30° - 60° - 90°$ triangle hence

hypotenuse = 2 (short leg)

$$10 = 2h$$

$$5 = h$$

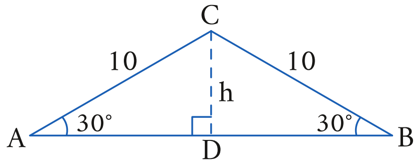


Figure 4. Draw height h.

long leg = (short leg)$\sqrt{3}$

AD = $h\sqrt{3} = 5\sqrt{3}$.

Similarly BD = $5\sqrt{3}$.

Area = $\frac{1}{2}bh = \frac{1}{2}(5\sqrt{3} + 5\sqrt{3})(5) = \frac{1}{2}(10\sqrt{3})(5) = \frac{1}{2}(50\sqrt{3}) = 25\sqrt{3}$.

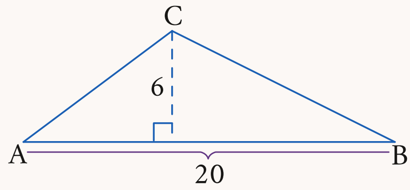
Perimeter = $10 + 10 + 5\sqrt{3} + 5\sqrt{3} = 20 + 10\sqrt{3}$.

ANSWER : $A = 25\sqrt{3}$, $P = 20 + 10\sqrt{3}$.

PROBLEMS

1 - 4. Find the area of $\triangle ABC$:

1.



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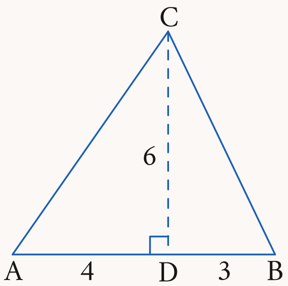

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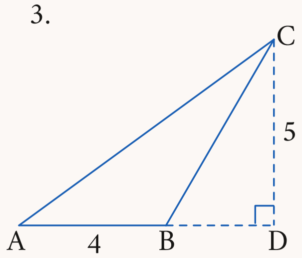

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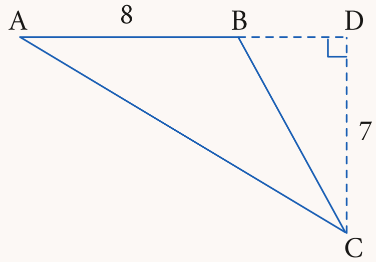
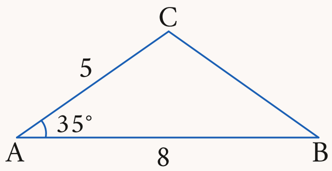
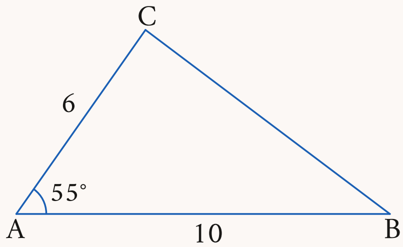


5 - 6. Find the area to the nearest tenth:

5.

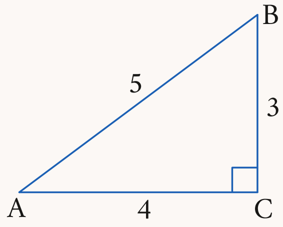


6.

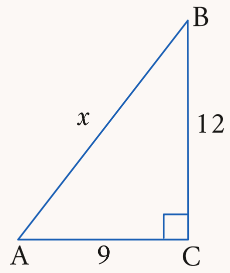


7 - 20. Find the area and perimeter of △ABC :

7.



8.



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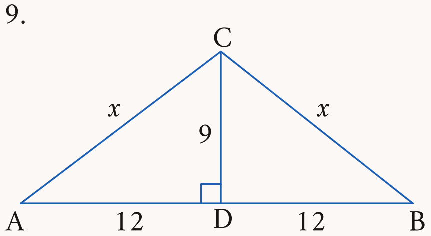

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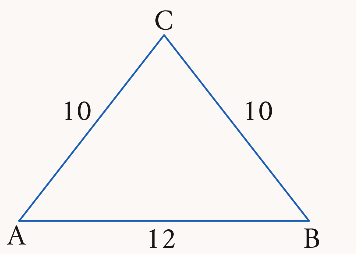

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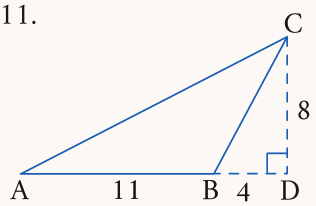

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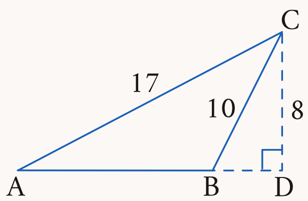
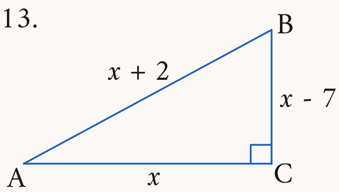


13.



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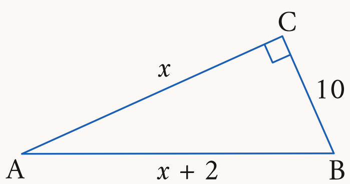

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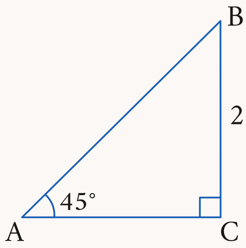

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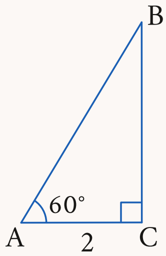

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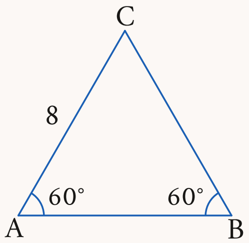

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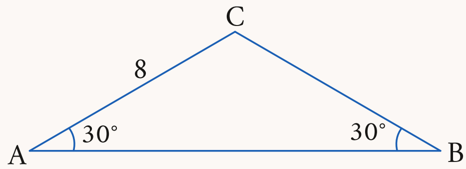
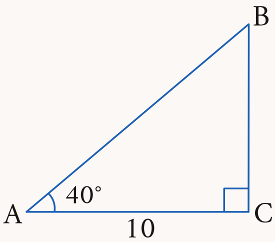
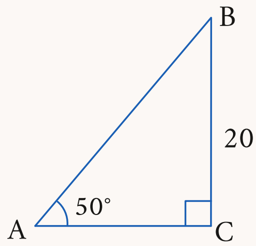


19 - 20. Find the area and perimeter to the nearest tenth:

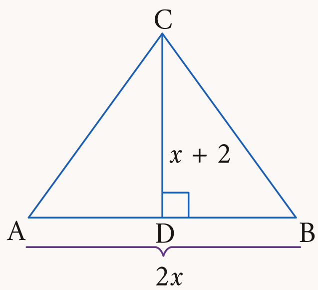
19.



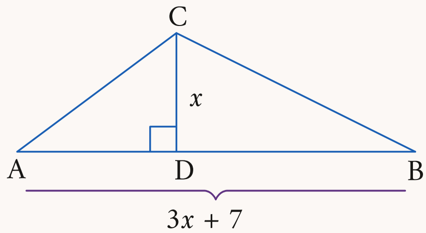
20.



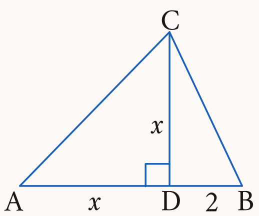
21. Find x if the area of △ABC is 35:



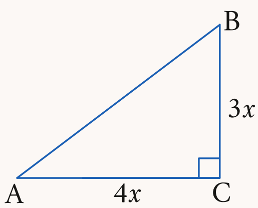
22. Find x if the area of △ABC is 24:



23. Find x if the area of △ABC is 12:



24. Find x if the area of △ABC is 108:



6.4 The Area of a Rhombus

The area of a rhombus can be found by using the formula for the area of a parallelogram, $A = bh$, since a rhombus is a special kind of parallelogram (Figure 1). However if the diagonals are known the following formula can be used instead (see Figure 2):

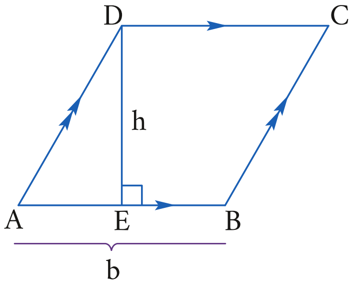


Figure 1.
The area of Rhombus ABCD is bh.

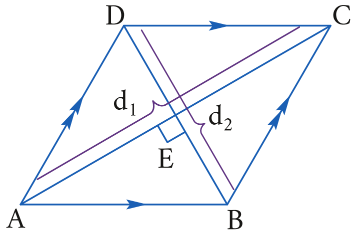


Figure 2.
The area of rhombus ABCD is $\frac{1}{2} d_1 d_2$.

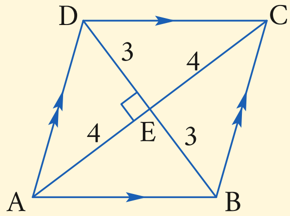
THEOREM 1

The area of a rhombus is one-half the product of the diagonals.

$$\boxed{A = \frac{1}{2} d_1 d_2}$$

? EXAMPLE A

Find the area of the rhombus:



SOLUTION

$$\text{Area} = \frac{1}{2} d_1 d_2 = \frac{1}{2}(8)(6) = \frac{1}{2}(48) = 24.$$

ANSWER : 24.

Proof of Theorem 1:

Referring to Figure 2,

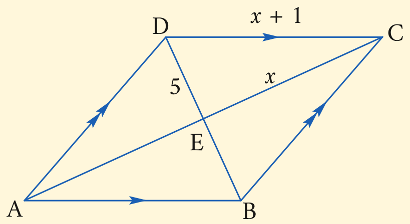
$$\text{Area of } \triangle ABC = \frac{1}{2}bh = \frac{1}{2}(AC)(BE) = \frac{1}{2}d_1\left(\frac{1}{2}d_2\right) = \frac{1}{4}d_1d_2 .$$

$$\text{Area of } \triangle ADC = \frac{1}{2}bh = \frac{1}{2}(AC)(DE) = \frac{1}{2}d_1\left(\frac{1}{2}d_2\right) = \frac{1}{4}d_1d_2 .$$

$$\text{Area of rhombus } ABDC = \text{Area of } \triangle ABC + \text{Area of } \triangle ADC = \frac{1}{4}d_1d_2 + \frac{1}{4}d_1d_2 = \frac{1}{2}d_1d_2.$$

? EXAMPLE B

Find the area and perimeter of the rhombus:



SOLUTION

The diagonals of a rhombus are perpendicular so $\triangle CDE$ is a right triangle. Therefore we can apply the Pythagorean theorem.

$$\begin{aligned} 5^2 + x^2 &= (x+1)^2 \\ 25 + x^2 &= x^2 + 2x + 1 \\ 24 &= 2x \\ 12 &= x \end{aligned}$$

$$\begin{aligned} d_1 &= 12 + 12 = 24. \\ d_2 &= 5 + 5 = 10. \\ CD &= x + 1 = 12 + 1 = 13. \end{aligned}$$

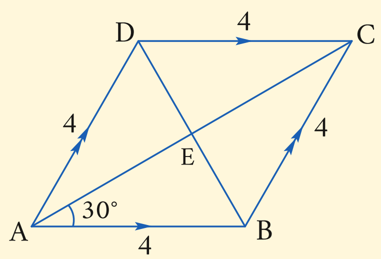
$$A = \frac{1}{2}d_1 \cdot d_2 = \frac{1}{2}(24)(10) = 120.$$

$$\text{Perimeter} = 13 + 13 + 13 + 13 = 52.$$

ANSWER : A= 120, P= 52.

? EXAMPLE C

Find the area of the rhombus:



SOLUTION

As in Example F of section 4.5, we obtain AC = $4\sqrt{3}$ and BD = 4.

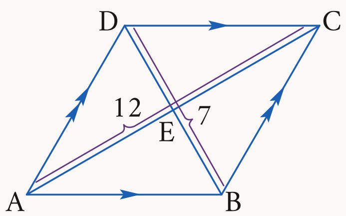
Area = $\frac{1}{2}\,d_1 d_2 = \frac{1}{2}\,(AC)(BD) = \frac{1}{2}\,(4\sqrt{3})(4) = 8\sqrt{3}$.

ANSWER : A = $8\sqrt{3}$.

PROBLEMS

1 - 2. Find the area of the rhombus:

1.



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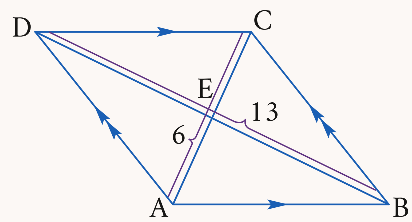
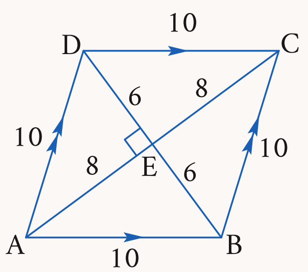


3 - 8. Find the area and perimeter of the rhombus:

3.



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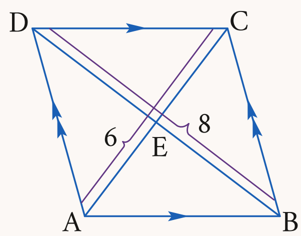

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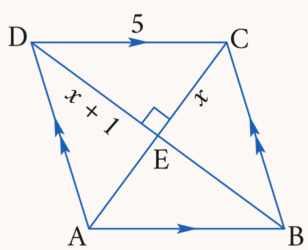

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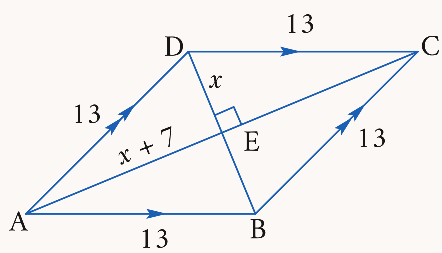

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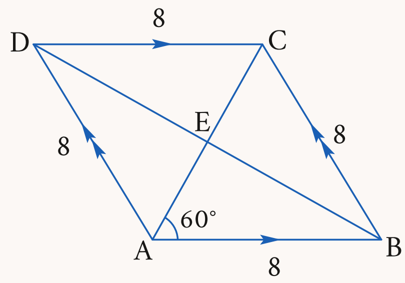

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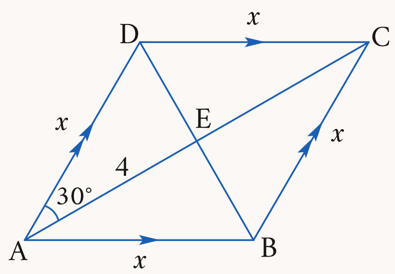
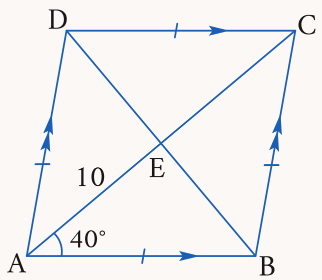


9 - 10. Find the area to the nearest tenth:

9.



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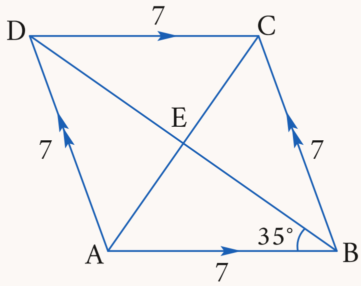


6.5 THE AREA OF A TRAPEZOID

In Figure 1, b_1 and b_2 are the **bases** of trapezoid ABCD and h is the **height** or **altitude**. The formula for the area is given in the following theorem:

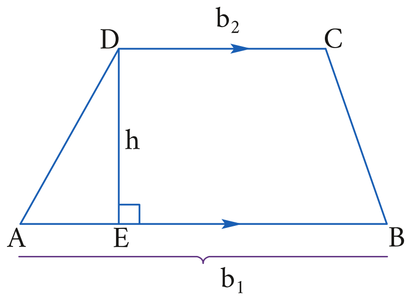


Figure 1. Trapezoid ABCD with bases b_1 and b_2 and height h.

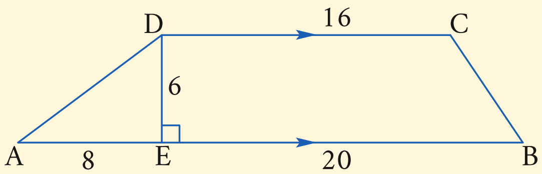
THEOREM 1

The area of a trapezoid is equal to one-half the product of its height and the sum of its bases.

$$\boxed{A = \frac{1}{2} h\,(b_1 + b_2)}$$

EXAMPLE A

Find the area:



SOLUTION

$$\text{Area} = \frac{1}{2} h\,(b_1 + b_2) = \frac{1}{2}(6)(28 + 16) = \frac{1}{2}(6)(44) = 132.$$

ANSWER : A = 132.

***Proof of Theorem 1*:**

In Figure 1 draw BD (see Figure 2). Note that CD = b_2 is the base and BF = h is the height of $\triangle$BCD.

Area of trapezoid ABCD = Area of $\triangle$ABD + Area of $\triangle$BCD.

Area of trapezoid ABCD = $\frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}h(b_1 + b_2)$.

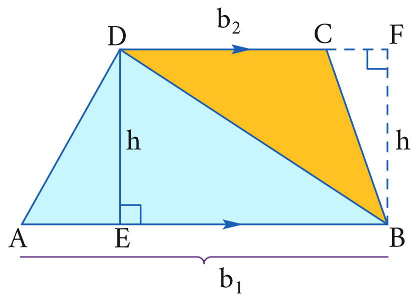
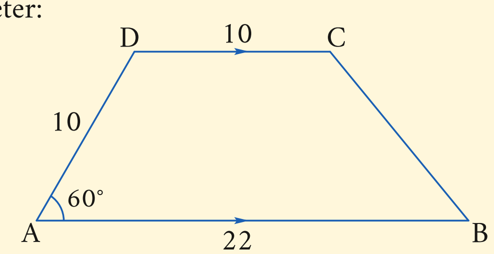


Figure 2. Draw BD. CD is the base and BF is the height of $\triangle$BCD.

? EXAMPLE B

Find the area and perimeter:



SOLUTION

Draw heights DE and CF (Figure 3). $\triangle$ADE is 30° - 60° - 90° triangle.

So AE = short leg = $\frac{1}{2}$hypotenuse = $\frac{1}{2}(10) = 5$ and

DE = long leg = (short leg)($\sqrt{3}$) = $5\sqrt{3}$.

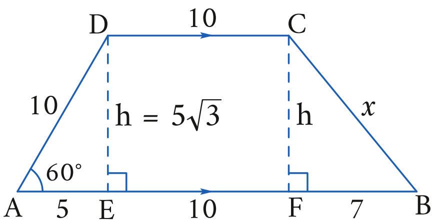


Figure 3. Draw heights DE and CF.

CDEF is a rectangle so EF = CD = 10.
Therefore BF = AB − AF = 22 − (10 + 5) = 7. Let x = BC.

$$CF^2 + BF^2 = BC^2$$
$$5\sqrt{3}^2 + 7^2 = x^2$$
$$75 + 49 = x^2$$
$$124 = x^2$$
$$x = \sqrt{124} = \sqrt{4}\sqrt{31} = 2\sqrt{31}.$$

$$\text{Area} = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(5\sqrt{3})(22 + 10) = \frac{1}{2}(5\sqrt{3})(32) = 80\sqrt{3}.$$

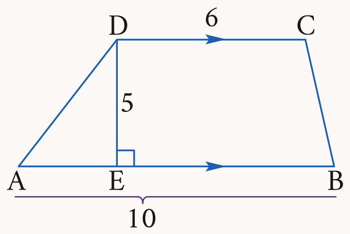
Perimeter = $22 + 10 + 10 + 2\sqrt{31} = 42 + 2\sqrt{31}$.

ANSWER : $A = 80\sqrt{3}$, $P = 42 + 2\sqrt{31}$.

PROBLEMS

1 - 2. Find the area of ABCD:

1.



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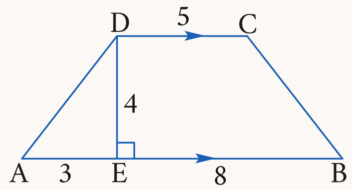
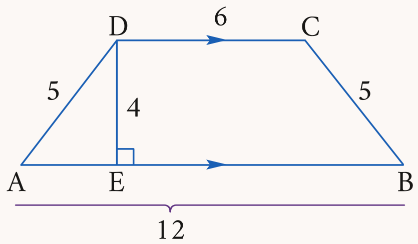


3 - 12. Find the area and perimeter of ABCD:

3.



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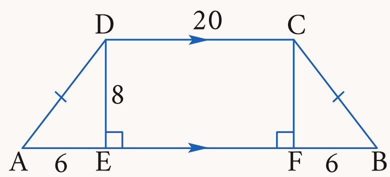

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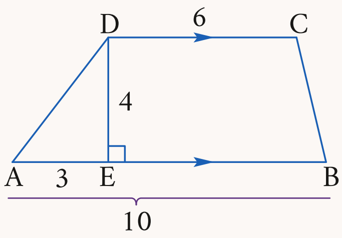

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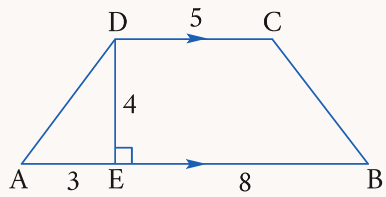

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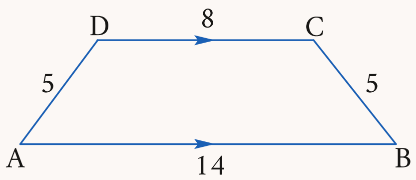

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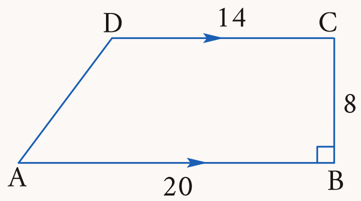

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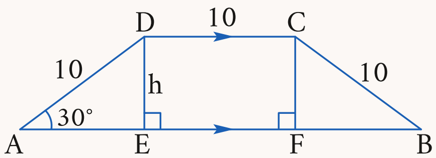

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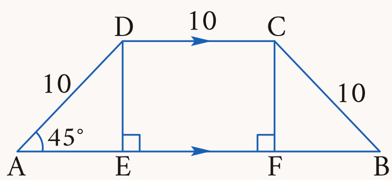

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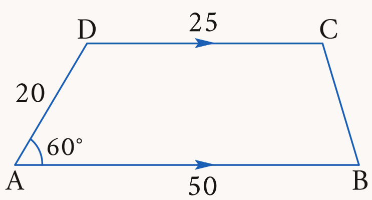

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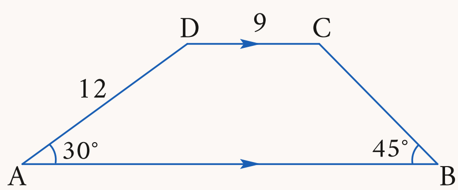
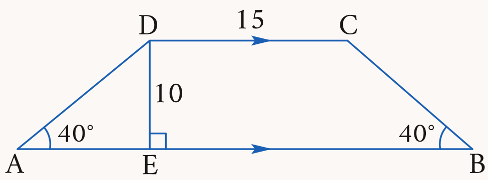


13 - 14. Find the area and perimeter to the nearest tenth of ABCD:

13.



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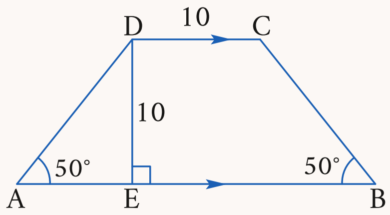
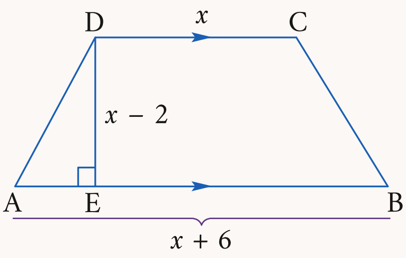
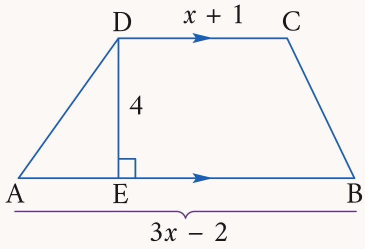


15. Find x if the area of ABCD is 50:



16. Find x if the area of ABCD is 30:

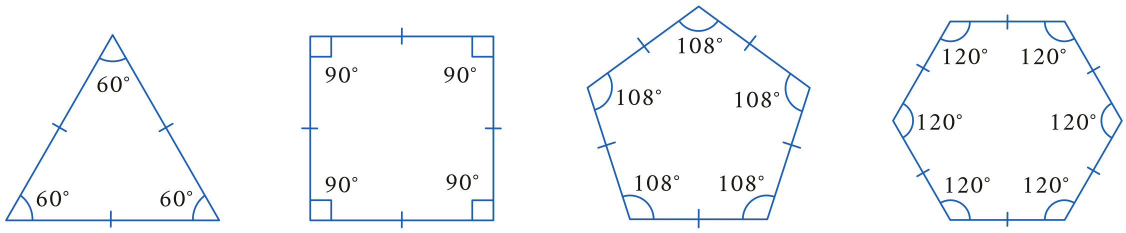


CHAPTER 7

REGULAR POLYGONS AND CIRCLES

7.1 REGULAR POLYGONS

A **regular polygon** is a polygon in which all sides are equal and all angles are equal. Examples of a regular polygon are the equilateral triangle (3 sides), the square (4 sides), the regular pentagon (5 sides) and the regular hexagon (6 sides). The angles of a regular polygon can easily be found using the methods of section 1.5.



Equilateral Triangle *Square* *Regular Pentagon* *Regular Hexagon*

Figure 1. Examples of regular polygons.

Suppose we draw the angle bisector of each angle of a regular polygon. We will find these angle bisectors all meet at the same point (Figure 2).

THEOREM 1

The angle bisectors of each angle of a regular polygon meet at the same point. This point is called the **center** of the regular polygon.

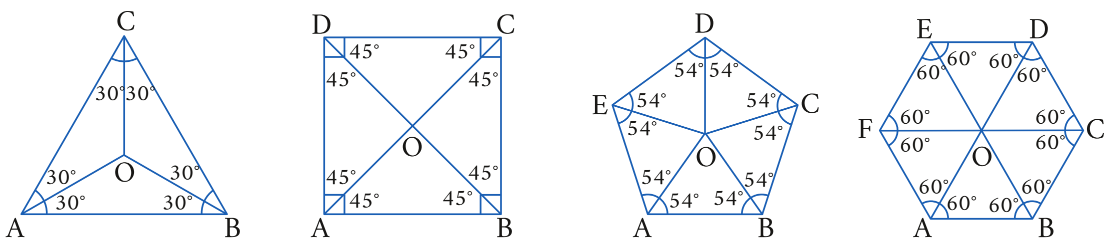


Figure 2. The angle bisectors of a regular polygon meet at the same point, O. O is called the center of the regular polygon.

In Figure 2, O is the center of each regular polygon. The segment of each angle bisector from the center to the vertex is called a **radius**. For example, OA, OB, OC, OD and OE are the five radii of regular pentagon ABCDE.

THEOREM 2

The radii of a regular polygon divide the polygon into congruent isosceles triangles. All the radii are equal.

In Figure 3, radii OA, OB, OC, OD and OE divide the regular pentagon into five isosceles triangles with OA = OB = OC = OD = OE.

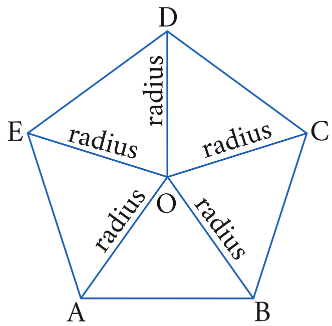
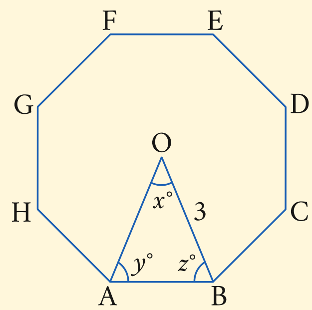


Figure 3. The five radii of a regular pentagon.

EXAMPLE A

Find the radius CA and the angles x°, y° and z° in the regular octagon (eight-sided figure):



SOLUTION

The radii divide the octagon into 8 congruent isosceles triangles. Therefore OA = OB = 3.

$$x^\circ = \frac{1}{8}(360^\circ) = 45^\circ$$

$$y^\circ = z^\circ = \frac{1}{2}(180^\circ - 45^\circ) = \frac{1}{2}(135^\circ) = 67.5^\circ$$

ANSWER: OA = 3, x° = 45°, y° = z° = 67.5°.

Theorem 1 and Theorem 2 appear to be true intuitively, but we verify them with a formal proof:

Proof of Theorem 1 and Theorem 2:

We will prove these theorems for the regular pentagon. The proof for other regular polygons is similar.

Draw the angle bisectors of $\angle A$ and $\angle B$ as in Figure 4 and call their point of intersection O. We will show OC, OD and OE are the angle bisectors of $\angle C$, $\angle D$ and $\angle E$ respectively.

$\angle EAB = \angle ABC$ since the angles of a regular pentagon are equal.

$\angle 1 = \angle 2 = \frac{1}{2}$ of $\angle EAB = \frac{1}{2}$ of $\angle ABC = \angle 3 = \angle 4$ since OA and OB are angle bisectors.

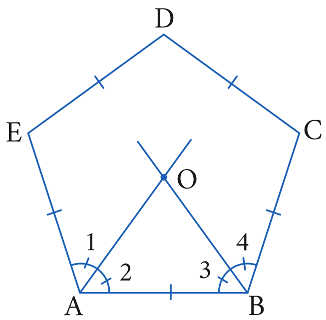


Figure 4. Draw the angle bisectors of $\angle A$ and $\angle B$ and call their point of intersection O.

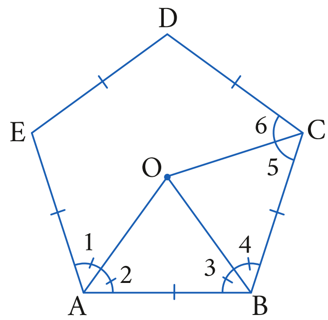


Figure 5. Draw OC.

Draw OC (Figure 5). AB = BC since the sides of a regular pentagon are equal. Therefore $\triangle AOB \cong \triangle COB$ by SAS = SAS. Therefore $\angle 5 = \angle 2 = \frac{1}{2}$ of $\angle EAB = \frac{1}{2}$ of $\angle BCD$. So OC is the angle bisector of $\angle BCD$.

Similarly we can show $\triangle BOC \cong \triangle DOC$, $\triangle COD \cong \triangle EOD$, $\triangle DOE \cong \triangle AOE$ and that OD and OE are angle bisectors. The triangles are all isosceles because their base angles are equal. This completes the proof.

A line segment drawn from the center perpendicular to the sides of a regular polygon is called an **apothem**. (see Figure 6).

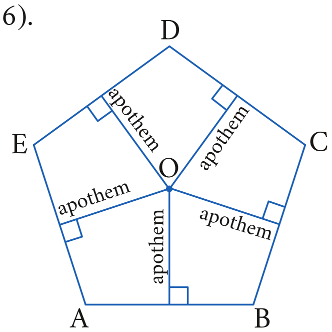


Figure 6. The apothems of a regular pentagon.

THEOREM 3

The apothems of a regular polygon are all equal. They bisect the sides of the regular polygon.

***Proof of Theorem 3*:**
The apothems are all equal because they are the heights of the congruent isosceles triangles formed by the radii (see Theorem 2). Each apothem divides the isosceles triangle into two congruent right triangles. Therefore each apothem bisects a side of the polygon, which is what we wanted to prove.

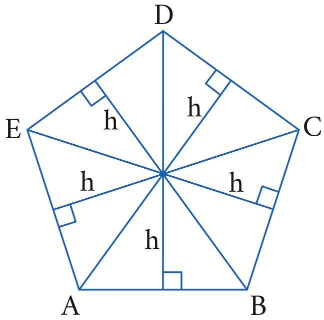


Figure 7. The apothems are the heights of the congruent isosceles triangles formed by the radii.

? EXAMPLE B

Find the apothem of a regular pentagon with side 20, to the nearest tenth.

SOLUTION In Figure 8,

$\angle AOB = \frac{1}{5}(360^\circ) = (72^\circ)$,

$\angle AOF = \frac{1}{2}\angle AOB = \frac{1}{2}(72^\circ) = 36^\circ$,

and $\angle OAF = 90^\circ - 36^\circ = 54^\circ$.

$$\tan 54^\circ = \frac{a}{10}$$

$$(10)1.3764 = \frac{a}{10}(10)$$

$$13.764 = a$$

$$13.8 = a$$

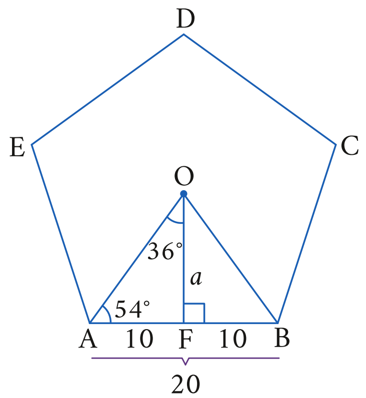


Figure 8. A regular pentagon with side 20.

ANSWER: 13.8

The apothem of a regular polygon is important because it is used to find the area:

THEOREM 4

The area of a regular polygon is one-half the product of the apothem and the perimeter.

$$\boxed{A = \frac{1}{2} aP}$$

EXAMPLE C

Find the area of a regular pentagon with side 20, to the nearest tenth.

SOLUTION

From Example B we know $a = 13.764$. The perimeter $P = (5)(20) = 100$.

Therefore $A = \frac{1}{2} aP = \frac{1}{2}(13.764)(100) = \frac{1}{2}(1376.4) = 688.2$.

ANSWER : 688.2.

Proof of Theorem 4:
We prove the theorem for the regular pentagon. The proof for other regular polygons is similar.
The radii of a regular pentagon divide the regular pentagon into five congruent triangles. The area of each triangle is $\frac{1}{2}as$, where s is the side of the pentagon (Figure 9). Therefore, area of the pentagon $= 5(\frac{1}{2}as) = \frac{1}{2}a(5s) = \frac{1}{2}aP$, which is the formula we wanted to prove.

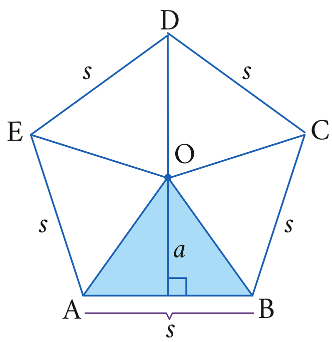


Figure 9. The area of $\triangle AOB$ is $\frac{1}{2}as$, where s is the side of the pentagon.

To find the perimeter of a regular polygon, all we have to do is to multiply the length of a side by the number of sides. For example, the pentagon of Figure 8 has perimeter $P = 5(20) = 100$. However it is also useful to have a formula for the perimeter when only the **radius** is known:

THEOREM 5

The perimeter of a regular polygon of n sides with radius r is given by the formula

$$\boxed{P = 2rn \sin \frac{180^\circ}{n}}.$$

EXAMPLE D

Find the perimeter of a regular pentagon with radius 10, to the nearest tenth.

SOLUTION

A pentagon has n = 5 sides. Using the formula of **Theorem 5**,

$$P = 2\text{rn} \sin \frac{180^\circ}{\text{n}} = 2(10)(5)\sin\frac{180^\circ}{5} = 100\sin 36^\circ = 100(0.5878) = 58.78 = 58.8.$$

ANSWER : 58.8.

Proof of Theorem 5:

Let us label the regular polygon as in Figure 10. Since the radii of the regular polygon divide the polygon into n congruent triangles (Theorem 2), we have

$$\angle AOB = \frac{1}{n}(360^\circ) = \frac{360^\circ}{n}$$

By **Theorem 3** apothem OC divides △AOB into two congruent right triangles, so

$$\angle AOC = \frac{1}{2}\angle AOB = \frac{1}{2}\left(\frac{360^\circ}{n}\right) = \frac{180^\circ}{n}.$$

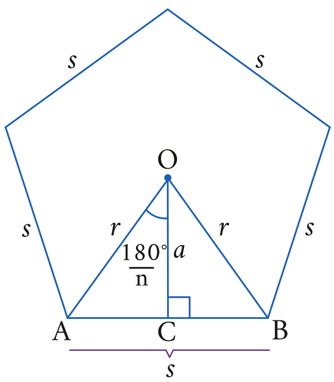


Figure 10. A regular polygon with radius r and side s.

Applying trigonometry to right triangle AOC we have

$$\sin\frac{180^\circ}{n} = \frac{AC}{r}$$

$$(r)\sin\frac{180^\circ}{n} = \frac{AC}{r}(r)$$

$$r\sin\frac{180^\circ}{n} = AC\ .$$

Since OC bisects AB,

$$s = 2(AC) = 2r\sin\frac{180^\circ}{n}$$

and therefore

$$P = ns = n(2r\sin\frac{180^\circ}{n}) = 2rn\sin\frac{180^\circ}{n}$$

which is the formula that we wish to prove.

We can also give explicit formulas for the various regular polygons, as in the following table:

Regular Figure	n	$n\sin\frac{180^\circ}{n}$	$P = 2rn\sin\frac{180^\circ}{n}$
Triangle	3	$3\sin 60^\circ = 2.5980$	5.1960 r
Square	4	$4\sin 45^\circ = 2.8284$	5.6568 r
Pentagon	5	$5\sin 36^\circ = 2.9390$	5.8780 r
Hexagon	6	$6\sin 30^\circ = 3.0000$	6.0000 r
Decagon	10	$10\sin 18^\circ = 3.090$	6.180 r
45 - sided figure	45	$45\sin 4^\circ = 3.139$	6.278 r
90 - sided figure	90	$90\sin 2^\circ = 3.141$	6.282 r
1000 - sided figure	1000	$1000\sin 0.18^\circ = 3.1416$	6.283 r

From the table we can see that as the number of sides increases, the perimeter of a regular polygon becomes approximately 6.28 times the radius. You may also recognize that the value of $(n\sin 180^\circ)/n$ comes close to the number π. We will return to this point when we discuss the circumference of a circle in section 7.5.

? EXAMPLE D (REPEATED)

Find the perimeter of a regular pentagon with radius 10, to the nearest tenth.

SOLUTION

From the table

$P = 5.8780\ r = 5.8780(10) = 58.78 = 58.8$

ANSWER : 58.8.

? EXAMPLE E

Find the apothem and area of a regular pentagon with radius 10, to the nearest tenth.

SOLUTION

In Figure 11

$\angle AOB = \frac{1}{5}(360°) = 72°$ and

$\angle AOF = \frac{1}{2}\angle AOB = \frac{1}{2}(72°) = 36°.$

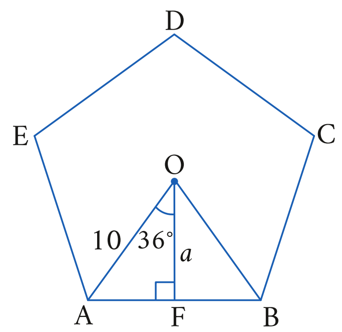


Figure 11. A regular pentagon with radius 10.

Applying trigonometry to right triangle AOF ,

$$\cos 36° = \frac{a}{10}$$
$$(10)0.8090 = \frac{a}{10}(10)$$
$$8.090 = a\ .$$

From Example D, $P = 58.78$. Therefore, by **Theorem 4**,

$A = \frac{1}{2}aP = \frac{1}{2}(8.09)(58.78) = \frac{1}{2}(475.5302) = 237.7651 = 237.8.$

ANSWER : $a = 8.1$, $P = 237.8$.

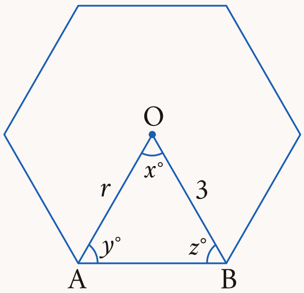
HISTORICAL NOTE

In 1936 archeologists unearthed a group of ancient Babylonian tables containing formulas for the areas of regular polygons of three, four, five, six and seven sides. There is evidence that regular polygons were commonly used in the architecture and designs of other ancient civilizations as well. A classical problem of Greek mathematics was to construct a regular polygon using just a ruler and compass. Regular polygons were usually studied in relationship to circles. As we shall see later in this chapter, the formulas for the area and perimeter of a circle can be derived from the corresponding formulas for regular polygons.

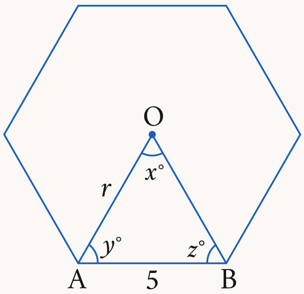
PROBLEMS

1 - 6. Find the angles x°, y°, z° and radius r of the regular polygons:

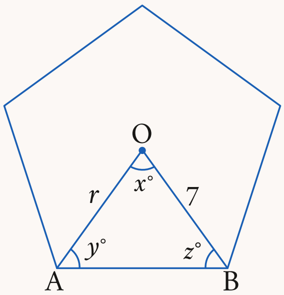
1.



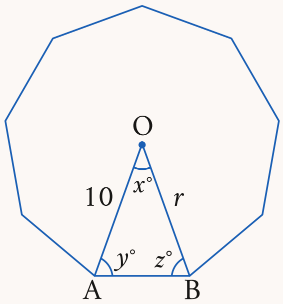
2.



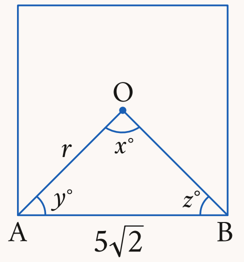
3.



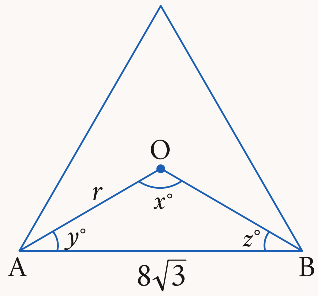
4.



5.



6.



7 - 18. Find the apothem, perimeter and area to the nearest tenth:

7. regular pentagon with side 40.

8. regular pentagon with side 16.

9. regular hexagon with side 20.

10. regular hexagon with side 16.

11. regular decagon (ten-sided figure) with side 20.

12. regular nonagon (nine-sided figure) with side 20.

13. regular pentagon with radius 20.

14. regular pentagon with radius 5.

15. regular hexagon with radius 10.

16. regular hexagon with radius 20.

17. regular decagon with radius 10.

18. regular nonagon with radius 20.

7.2 CIRCLES

The circle is one of the most frequently encountered geometric figures. Wheels, rings, phonograph records, clocks, coins are just a few examples of common objects with circular shape. The circle has many applications in the construction of machinery and in architectural and ornamental design.

To draw a circle we use an instrument called a **compass** (Figure 1). The compass consists of two arms, one ending in a sharp metal point and the other attached to a pencil. We draw the circle by rotating the pencil while the metal point is held so that it does not move. The position of the metal point is called the **center** of the circle. The distance between the center and the tip of the pencil is called the **radius** of the circle. The radius remains the same as the circle is drawn.

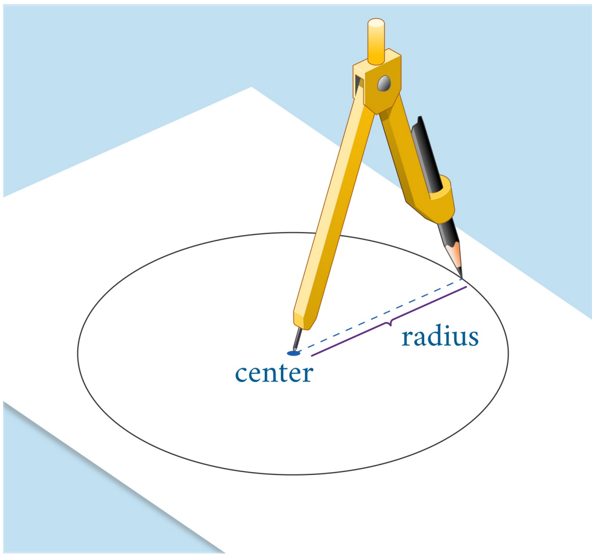


Figure 1. Using a compass to draw a circle.

The method of constructing a circle suggests the following definition:
A **circle** is a figure consisting of all points which are a given distance from a fixed point called the **center**. For example the circle in Figure 2 consists of all points which are a distance of 3 from the center O. The **radius** is the distance of any point on the circle from the center.

The circle in Figure 2 has radius 3. The term **radius** is also used to denote any of the **line segments** from a point on the circle to the center. In Figure 2, each of the line segments OA, OB and OC is a radius. It follows from the definition of circle that all radii of a circle are equal. So in Figure 2 the three radii OA, OB and OC are all equal to 3.

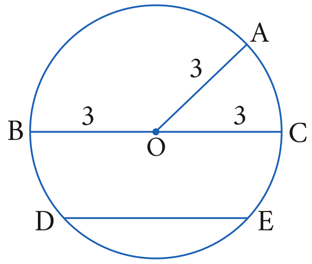
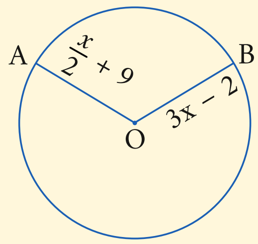


Figure 2. A circle with radius 3.

A circle is usually named for its center. The circle in Figure 2 is called **circle O**.
A **chord** is a line segment joining two points on a circle. In Figure 2, DE is a chord. A **diameter** is a chord which passes through the center. BC is a diameter. A diameter is always twice the length of a radius since it consists of two radii. Any diameter of circle O is equal to 6. All diameters of a circle are equal.

? EXAMPLE A

Find the radius and diameter:



SOLUTION

All radii are equal so

$$
\begin{aligned}
OA &= OB \\
\frac{x}{2} + 9 &= 3x - 2 \\
(2)\left(\frac{x}{2} + 9\right) &= (3x - 2)(2) \\
x + 18 &= 6x - 4 \\
22 &= 5x \\
x &= \frac{22}{5} = 4.4
\end{aligned}
$$

CHECK

OA	= OB
$\frac{x}{2} + 9$	$3x - 2$
$\frac{4.4}{2} + 9$	$3(4.4) - 2$
$2.2 + 9$	$13.2 - 2$
11.2	11.2

Therefore the radius = OA = OB = 11.2 and the diameter = 2(11.2) = 22.4.

ANSWER: radius = 11.2, diameter = 22.4.

The following three theorems show that a diameter of a circle and the perpendicular bisector of a chord in a circle are actually the same thing.

THEOREM 1

A diameter perpendicular to a chord bisects the chord.

In Figure 3, if $AB \perp CD$ then AE = EB.

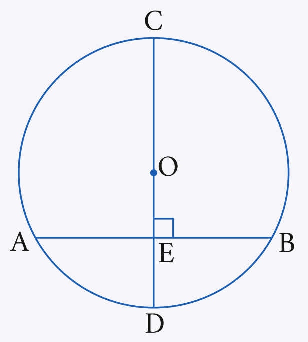


Figure 3. The diameter CD is perpendicular to chord AB.

Proof of Theorem 1:

Draw OA and OB (Figure 4). OA = OB because all radii of a circle are equal. OE = OE because of identity. Therefore $\triangle AOE \cong \triangle BOE$ by Hyp-Leg = Hyp-Leg. Hence AE = BE because they are corresponding sides of congruent triangles.

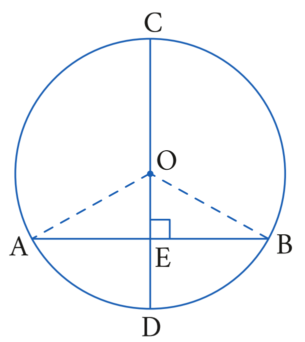
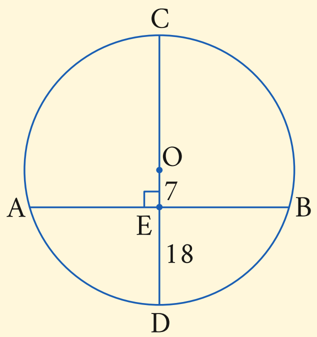


Figure 4. Draw OA and OB.

? EXAMPLE B

Find AB:



SOLUTION

Draw OA (Figure 5). OA = radius = OD = 18 + 7 = 25. $\triangle AOE$ is a right triangle and therefore we can use the Pythagorean theorem to find AE :

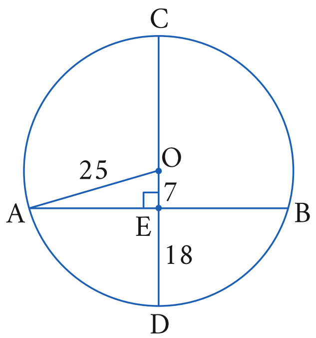


Figure 5. Draw OA.

$$AE^2 + OE^2 = OA^2$$

$$AE^2 + 7^2 = 25^2$$

$$AE^2 + 49 = 625$$

$$AE^2 = 576$$

$$AE = 24$$

By **Theorem 1**, EB = AE = 24 so AB = AE + EB = 24 + 24 = 48.

ANSWER : AB = 48.

THEOREM 2

A diameter that bisects a chord which is not a diameter is perpendicular to it.

In Figure 6, if AE = EB then AB⊥CD.

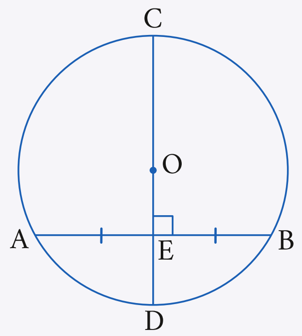


Figure 6. Diameter CD bisects chord AB.

Proof of Theorem 2:

Draw OA and OB (Figure 7). OA = OB because all radii are equal, OE = OE (identity) and AE = EB (given). Therefore △AOE ≅ △BOE by SSS=SSS. Therefore ∠AEO = ∠BEO. Since ∠AEO and ∠BEO are also supplementary we must also have ∠AEO = ∠BEO = 90°, which is what we had to prove.

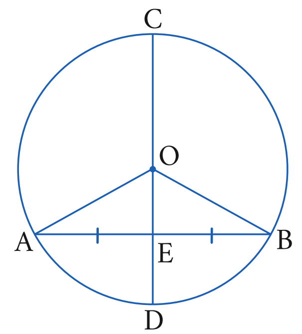
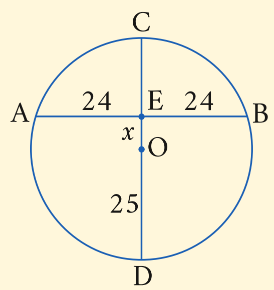


Figure 7. Draw OA and OB.

EXAMPLE C

Find x:



SOLUTION

Draw OA (Figure 8). OA = radius = OD = 25. According to **Theorem 2**, $AB \perp CD$. Therefore $\triangle AOE$ is a right triangle and we can use the Pythagorean theorem to find x:

$$OE^2 + AE^2 = OA^2$$

$$x^2 + 24^2 = 25^2$$

$$x^2 + 576 = 625$$

$$x^2 = 49$$

$$x = 7$$

ANSWER : $x = 7$.

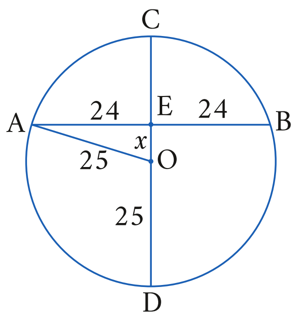


Figure 8. Draw OA.

THEOREM 3

The perpendicular bisector of a chord must pass through the center of the circle (that is, it is a diameter).

In Figure 9, if $CD \perp AB$ and $AE = EB$ then O must lie on CD.

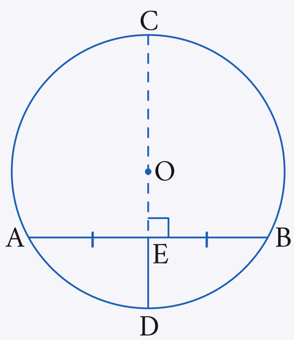


Figure 9. If CD is a perpendicular bisector of AB then CD must pass through O.

Proof of Theorem 3:

Draw a diameter FG through O perpendicular to AB at H (Figure 10). Then according to **Theorem 1** H must bisect AB . Hence H and E are the same point and FG and CD are the same line. So O lies on CD. This completes the proof.

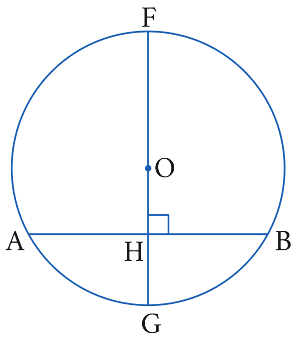
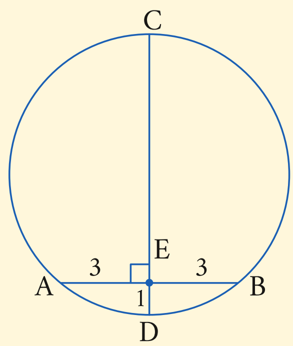


Figure 10. Draw FG through O perpendicular to AB.

? EXAMPLE D

Find the radius of the circle:



SOLUTION

According to **Theorem 3**, O must lie on CD. Draw OA (Figure 11). Let r be the radius. Then OA = OD = r and OE = r − 1. To find r we apply the Pythagorean theorem to right triangle AOE:

$$\begin{aligned} AE^2 + OE^2 &= OA^2 \\ 3^2 + (r-1)^2 &= r^2 \\ 9 + r^2 - 2r + 1 &= r^2 \\ 10 &= 2r \\ 5 &= r \end{aligned}$$

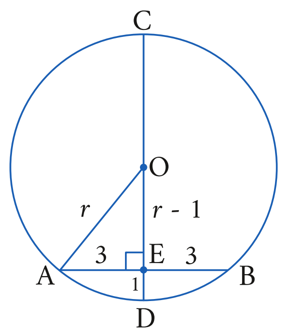
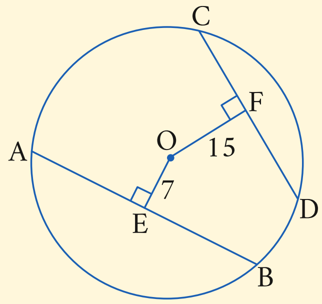


Figure 11. Draw OA and let r be the radius.

ANSWER : r = 5.

? EXAMPLE E

Find which chord, AB or CD, is larger if the radius of the circle is 25:



SOLUTION

Draw OA, OB, OC and OD(Figure 12). Each is a radius and equal to 25. We use the Pythagorean theorem, applied to right triangle AOE , to find AE:

$$\begin{aligned} AE^2 + OE^2 &= OA^2 \\ AE^2 + 7^2 &= 25^2 \\ AE^2 + 49 &= 625 \\ AE^2 &= 576 \\ AE &= 24. \end{aligned}$$

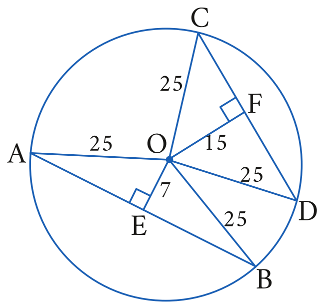


Figure 12. Draw OA, OB, OC and OD.

Since perpendicular OE bisects AB (**Theorem 1**)
BE = AE = 24 and so AB = AE + BE = 24 + 24 = 48.
Similarly, to find CF, we apply the Pythagorean theorem to right triangle COF:

$$\begin{aligned} CF^2 + OF^2 &= OC^2 \\ CF^2 + 15^2 &= 25^2 \\ CF^2 + 225 &= 625 \\ CF^2 &= 400 \\ CF &= 20 \end{aligned}$$

Again, from **Theorem 1**, we know OF bisects CD, hence DF = CF = 20 and CD = 40.

ANSWER : AB = 48, CD = 40, AB is larger than CD.

Example E suggests the following Theorem (which we state without proof):

THEOREM 4

The length of a chord is determined by its distance from the center of the circle; the closer to the center, the larger the chord.

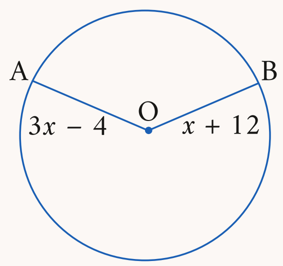
HISTORICAL NOTE

The definition of a circle and essentially all of the theorems of this and the next two sections can be found in Book III of Euclid's *Elements*.

PROBLEMS

1 - 2. Find the radius and diameter:

1.



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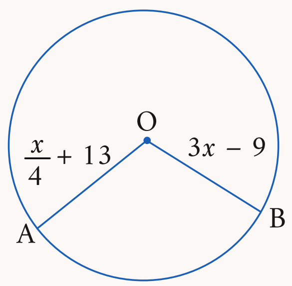
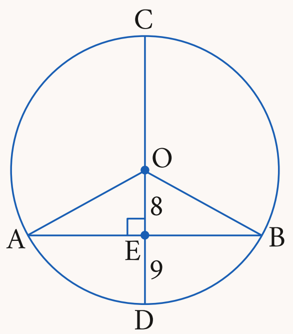


3 - 4. Find AB:

3.



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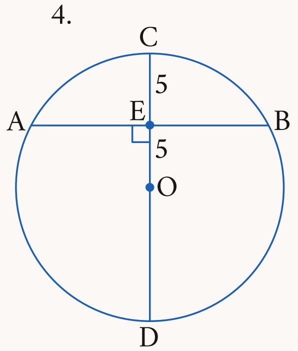
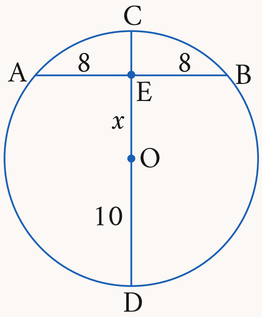


5 - 6. Find x:

5.



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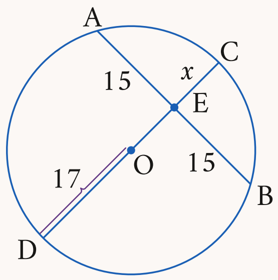
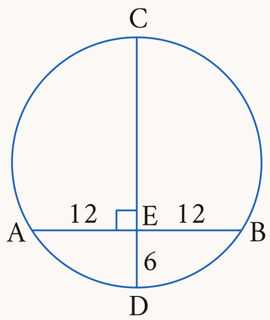


7 - 10. Find the radius and diameter:

7.



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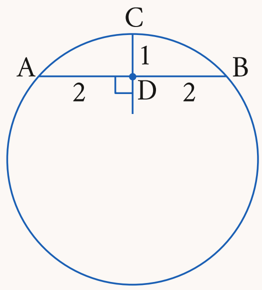

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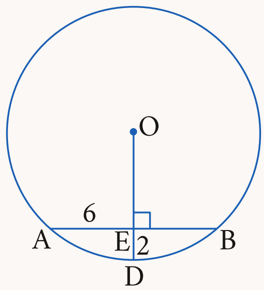
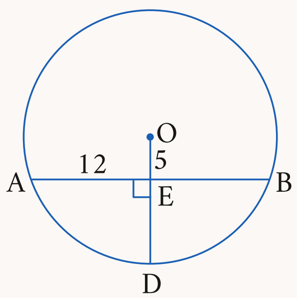


11 - 12. Find the lengths of AB and CD:

10.



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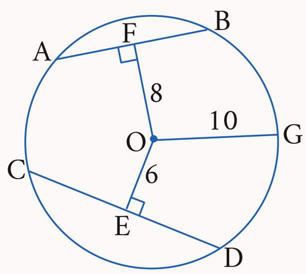

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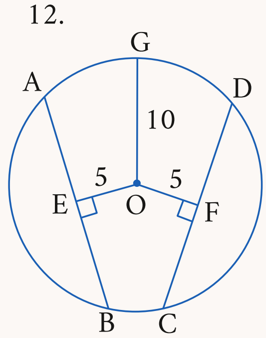


7.3 TANGENTS TO THE CIRCLE

A **tangent** to a circle is a line which intersects the circle in exactly one point. In Figure 1 line $\overleftrightarrow{AB}$ is a tangent, intersecting circle O just at point P.

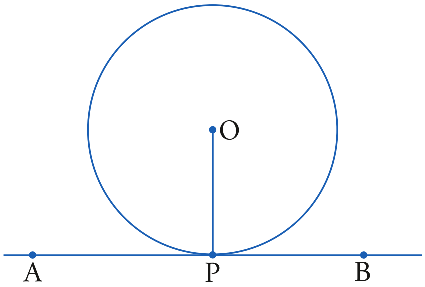


Figure 1. $\overleftrightarrow{AB}$ is tangent to circle O at point P.

A tangent has the following important property:

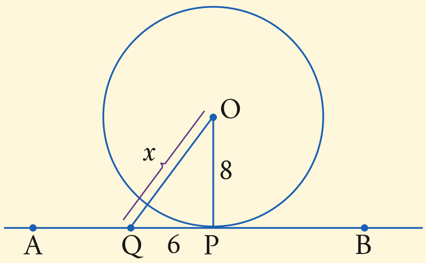
THEOREM 1

A tangent is perpendicular to the radius drawn to the point of intersection.

In Figure 1 tangent $\overleftrightarrow{AB}$ is perpendicular to radius OP at the point of intersection P.

? EXAMPLE A

Find x:



SOLUTION

According to **Theorem 1**, $\angle QPO$ is a right angle. We may therefore apply the Pythagorean theorem to right triangle QPO :

$$
\begin{aligned}
6^2 + 8^2 &= x^2 \\
36 + 64 &= x^2 \\
100 &= x^2 \\
10 &= x
\end{aligned}
$$

ANSWER : $x = 10$.

Proof of Theorem 1:
OP is the shortest line segment that can be drawn from O to line $\overleftrightarrow{AB}$. This is because if Q were another point on $\overleftrightarrow{AB}$ then OQ would be longer than radius OR = OP (Figure 2). Therefore OP$\perp\overleftrightarrow{AB}$ since the shortest line segment that can be drawn from a point to a straight line is the perpendicular (**Theorem 2**, *Section 4.6*). This completes the proof.

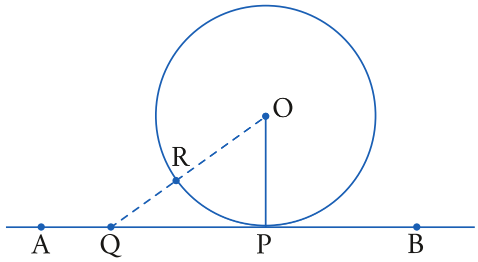


Figure 2. OP is the shortest line segment that can be drawn from O to line $\overleftrightarrow{AB}$.

The converse of **Theorem 1** is also true:

THEOREM 2

A line perpendicular to a radius at a point touching the circle must be a tangent.

In Figure 3, if OP$\perp\overleftrightarrow{AB}$ then $\overleftrightarrow{AB}$ must be a tangent; that is, P is the only point at which $\overleftrightarrow{AB}$ can touch the circle (see Figure 4).

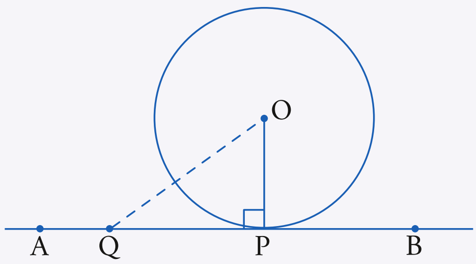


Figure 3. If OP$\perp\overleftrightarrow{AB}$ then $\overleftrightarrow{AB}$ must be a tangent.

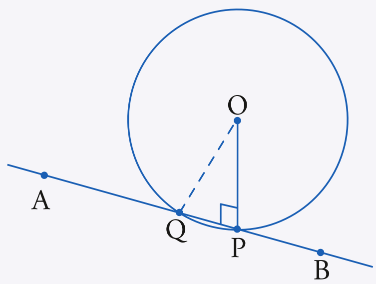
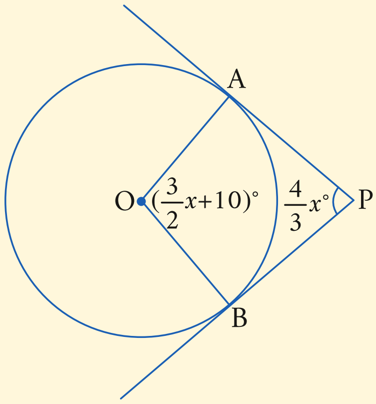


Figure 4. Theorem 2 implies that this **cannot** happen.

Proof of Theorem 2:
Suppose Q were some other point on $\overleftrightarrow{AB}$. Then OQ is the hypotenuse of right triangle OPQ (see Figure 3). According to **Theorem 1**, *section 4.6*, the hypotenuse of a right triangle is always larger than a leg. Therefore hypotenuse OQ is larger than leg OP. Since OQ is larger than the radius OP, Q cannot be on the circle. We have shown that no other point on $\overleftrightarrow{AB}$ besides P can be on the circle. Therefore $\overleftrightarrow{AB}$ is a tangent. This completes the proof.

EXAMPLE B

Find x, ∠O and ∠P:



SOLUTION

$\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangent to circle O, so by **Theorem 1**, ∠OAP = ∠OBP = 90°. The sum of the angles of quadrilateral AOBP is 360° (see Example E, *Section 1.5*), hence

$$90 + \left(\frac{3}{2}x + 10\right) + 90 + \frac{4}{3}x = 360$$

$$\frac{3}{2}x + \frac{4}{3}x + 190 = 360$$

$$\frac{3}{2}x + \frac{4}{3}x = 170$$

$$(6)\left(\frac{3}{2}x\right) + (6)\left(\frac{4}{3}x\right) = (6)(170)$$

$$9x + 8x = 1020$$

$$17x = 1020$$

$$x = 60.$$

Substituting 60 for x, we find

$$\angle O = \left(\frac{3}{2}x + 10\right)^\circ = \left(\frac{3}{2}(60) + 10\right)^\circ = (90 + 10)^\circ = 100^\circ \text{ and}$$

$$\angle P = \frac{4}{3}x^\circ = \frac{4}{3}(60^\circ) = 80^\circ.$$

CHECK: ∠A + ∠O + ∠B + ∠P = 90° + 100° + 90° + 80° = 360°.

ANSWER: x = 60, ∠O = 100°, ∠P = 80°.

If we measure line segments AP and BP in Example B we will find that they are approximately equal in length. In fact we can prove that they must be exactly equal:

THEOREM 3

Tangents drawn to a circle from an outside point are equal.

In Figure 5 if $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangents then $AP = BP$.

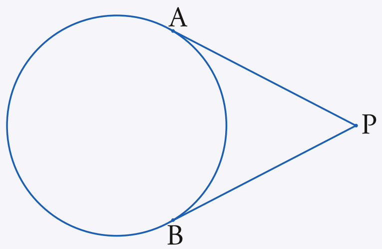


Figure 5. If $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangents then $AP = BP$.

Proof of Theorem 3:
Draw OA, OB and OP (Figure 6). $OA = OB$ (all radii are equal), $OP = OP$ (identity) and $\angle A = \angle B = 90^\circ$ (**Theorem 1**), hence $\triangle AOP \cong \triangle BOP$ by Hyp-Leg = Hyp-Leg. Therefore $AP = BP$ beause they are corresponding sides of congruent triangles.

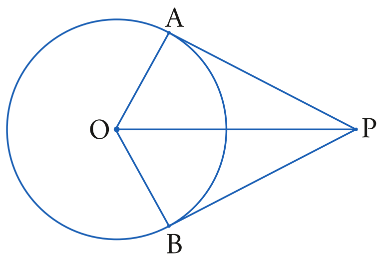
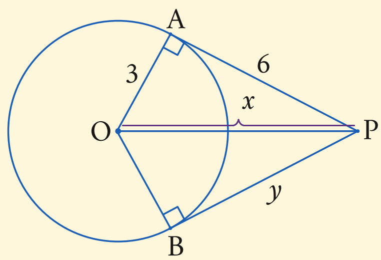


Figure 6. Draw OA, OB and OP.

EXAMPLE C

Find x and y:



SOLUTION

By the Pythagorean theorem,

$x^2 = 3^2 + 6^2$

$x^2 = 9 + 36 = 45$

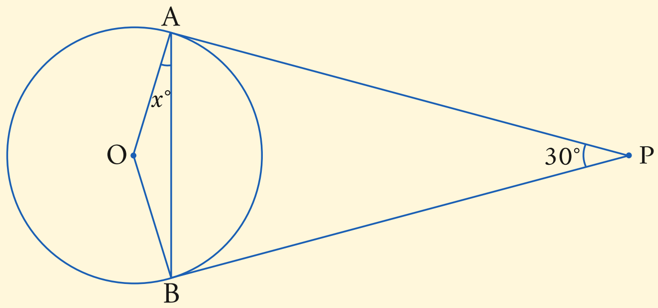
$x = \sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}.$

By **Theorem 3**, $y = BP = AP = 6$.

ANSWER : $x = 3\sqrt{5}$, $y=6$.

? EXAMPLE D

Find x:



SOLUTION

By Theorem 3, AP = BP. So $\triangle ABP$ is isosceles with $\angle PAB = \angle PBA = 75°$.
Therefore $x° = 90° - 75° = 15°$.

ANSWER : $x = 15$.

If each side of a polygon is tangent to a circle, the circle is said to be **inscribed** in the polygon and the polygon is said to be **circumscribed** about the circle. In Figure 7 circle 0 is inscribed in quadrilateral ABCD and ABCD is circumscribed about circle O.

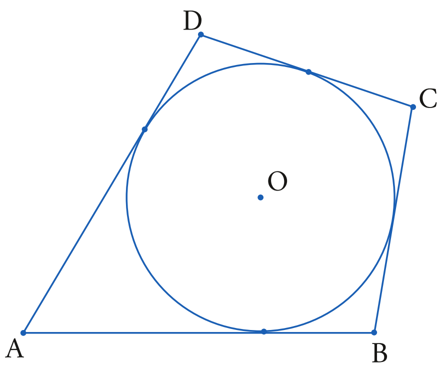
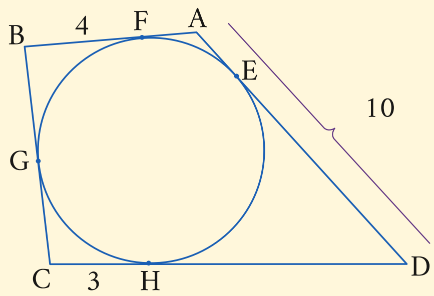


Figure 7. Circle O is inscribed in ABCD.

? EXAMPLE E

Find the perimeter of ABCD:



SOLUTION

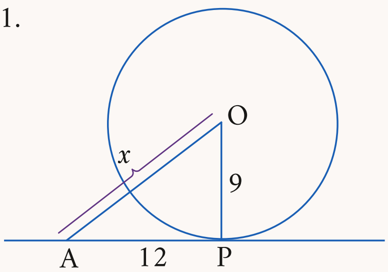
By **Theorem 3**, CG = CH = 3 and BG = BF = 4. Also DH = DE and AF = AE so DH + AF = DE + AE = 10.
Therefore the perimeter of ABCD = 3 + 3 + 4 + 4 + 10 + 10 = 34.

ANSWER : P = 34.

PROBLEMS

1 - 4. Find x:

1.



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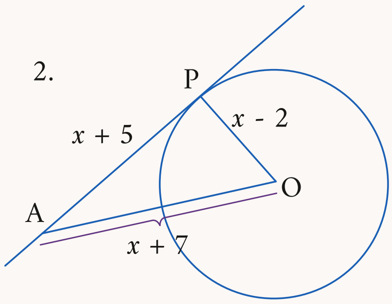

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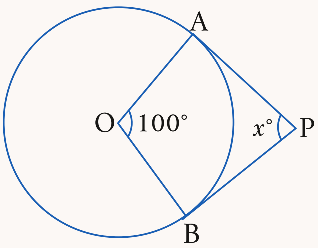

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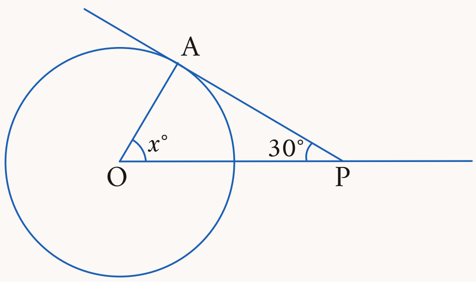
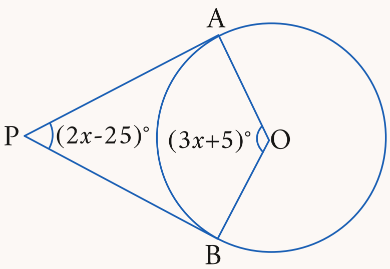
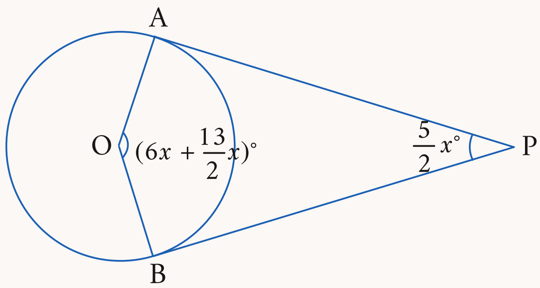


5 - 6. Find x, ∠O and ∠P :

5.

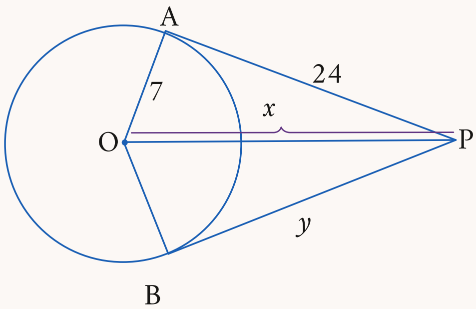


6.



7 - 8. Find x and y:

7.



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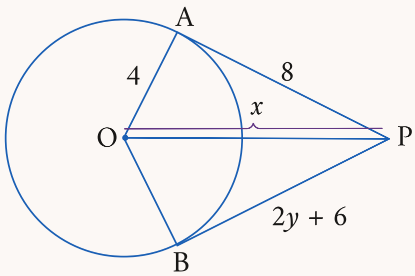
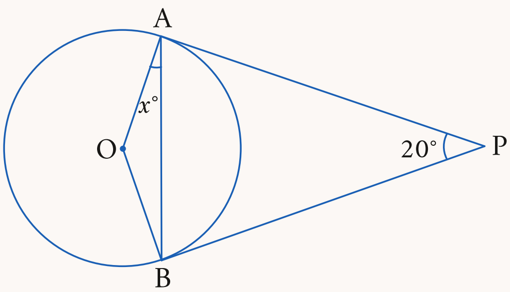


9 - 12. Find x:

9.



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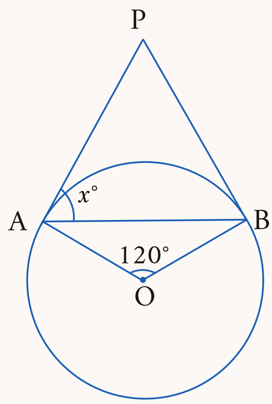

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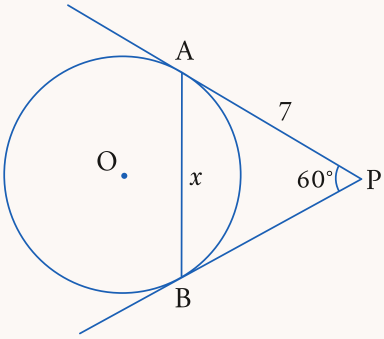

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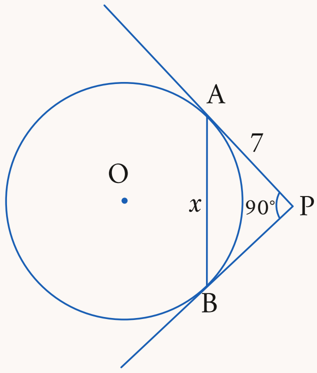
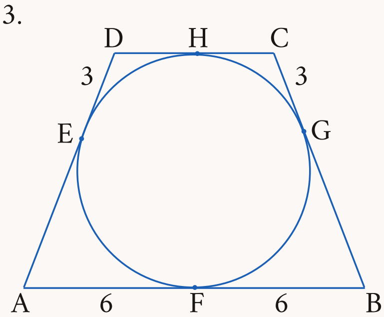


12 - 16. Find the perimeter of the polygon:

13.



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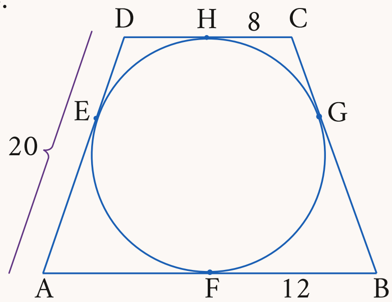

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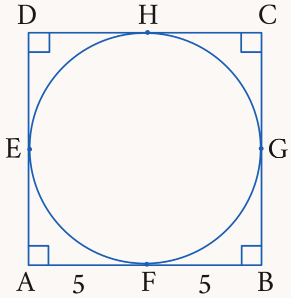

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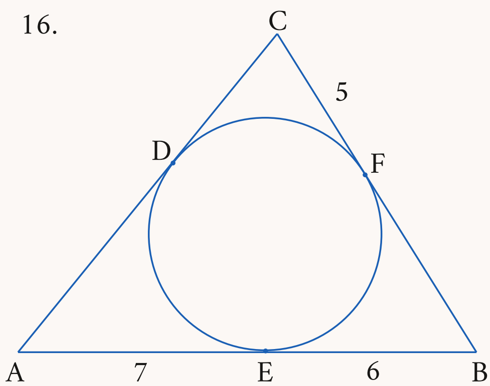


7.4 DEGREES IN AN ARC

An **arc** is a part of the circle included between two points. The symbol for the arc included between points A and B is $\overset{\frown}{AB}$. In Figure 1 there are two arcs determined by A and B. The shorter one is called the **minor arc** and the longer one is called the **major arc.** Unless otherwise indicated, $\overset{\frown}{AB}$ will always refer to the minor arc. In Figure 1 we might also write $\overset{\frown}{ACB}$ instead of $\overset{\frown}{AB}$ to indicate the major arc.

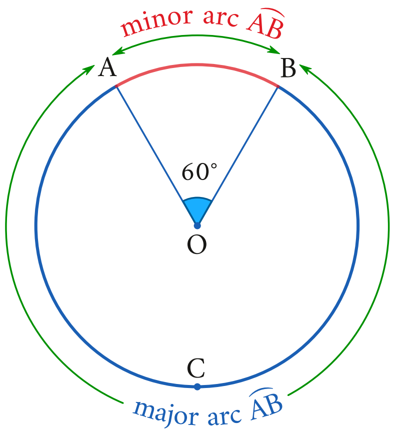


Figure 1. There are two arcs determined by A and B, the minor are and the major arc.

A **central angle** is an angle whose vertex is the center of the circle and whose sides are radii. In Figure 1, $\angle AOB$ is a central angle. $\angle AOB$ is said to **intercept** arc $\overset{\frown}{AB}$.

The number of **degrees in an arc** is defined to be the number of degrees in the central angle that intercepts the arc. In Figure 1 minor arc $\overset{\frown}{AB}$ has 60° because $\angle AOB = 60°$. We write $\overset{\frown}{AB} \overset{\circ}{=} 60°$, where the symbol $\overset{\circ}{=}$ means **equal in degrees**. The plain = symbol will be reserved for arc length, to be discussed in *Section 7.5*.

In Figure 2 $\angle AOB$ is a straight angle so $\angle AOB = 180°$ and $\overset{\frown}{ACB} \overset{\circ}{=} 180°$.

Similarly $\overset{\frown}{ADB} \overset{\circ}{=} 180°$. Each of these arcs is called a **semicircle**. The complete circle measures 360°.

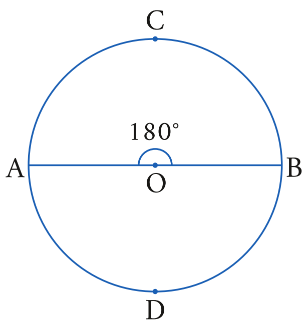
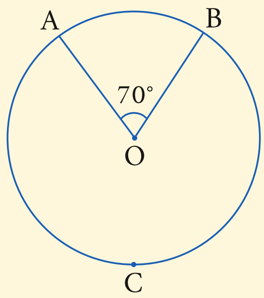


Figure 2. $\angle \overset{\frown}{AOB} = 180°$ and $\overset{\frown}{ACB} \overset{\circ}{=} 180°$.

? EXAMPLE A

Find the number of degrees in arcs $\overset{\frown}{AB}$ and $\overset{\frown}{ACB}$:



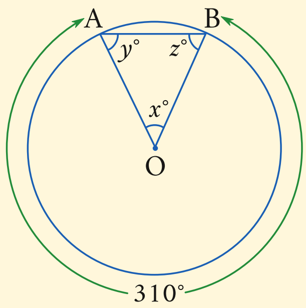
SOLUTION

$\overset{\frown}{AB} \doteq \angle AOB = 70^\circ$ and $\overset{\frown}{ACB} \doteq 360^\circ - \overset{\frown}{AB} \doteq 360^\circ - 70^\circ = 290^\circ$.

ANSWER : $\overset{\frown}{AB} \doteq 70^\circ$, $\overset{\frown}{ACB} \doteq 290^\circ$.

? EXAMPLE B

Find x, y and z:



SOLUTION

$x^\circ \doteq \overset{\frown}{AB} \doteq 360^\circ - 310^\circ = 50^\circ$, $OA = OB$ since all radii are equal. Therefore $\triangle AOB$ is isosceles with $y^\circ = z^\circ$. We have

$$\begin{aligned} x^\circ + y^\circ + z^\circ &= 180^\circ \\ 50^\circ + y^\circ + y^\circ &= 180^\circ \\ 2y^\circ &= 130^\circ \\ y^\circ &= 65^\circ \end{aligned}$$

ANSWER : $x = 50$, $y = z = 65$.

An **inscribed angle** is an angle whose vertex is on a circle and whose sides are chords of the circle. In Figure 3 $\angle ABC$ is an inscribed angle. $\angle ABC$ is said to **intercept** arc $\overset{\frown}{AC}$.

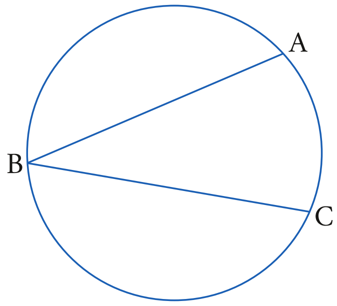


Figure 3. $\angle ABC$ is an inscribed angle.

We shall prove the following theorem:

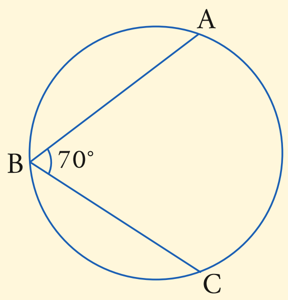
THEOREM 1

An inscribed angle $\stackrel{\circ}{=} \frac{1}{2}$ of its intercepted arc.

In Figure 3, $\angle ABC \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{AC}$.

? EXAMPLE C

Find the number of degrees in $\overset{\frown}{AC}$:



SOLUTION

$\angle ABC = 70° \stackrel{\circ}{=} \frac{1}{2}\overset{\frown}{AC}$. Therefore $\overset{\frown}{AC} \stackrel{\circ}{=} 140°$.

ANSWER: $\overset{\frown}{AC} \stackrel{\circ}{=} 140°$.

Before giving the proof of Theorem 1 let us see if we can prove the answer to Example C. Draw the diameter from B through center O (Figure 4). ∠ABC is divided by the diameter into two smaller angles, ∠ABD and ∠DBC, whose sum is 70°. Suppose ∠ABD = 30° and ∠DBC = 40° (Figure 5). AO = BO because all radii are equal. Hence △AOB is isosceles with ∠A = ∠ABD = 30°. Similarly C = ∠DBC = 40°. ∠AOD is an exterior angle of △AOB and so is equal to the sum of the remote interior angles, 30° + 30° = 60° (Theorem 2, *Section 1.5*). Similarly ∠COD = 40° + 40° = 80°. Therefore central angle ∠AOC = 60° + 80° = 140° and arc $\overset{\frown}{AC} \triangleq 140°$. This agrees with our answer to Example C.

We will now give a formal proof of **Theorem 1**, which will hold for any inscribed angle:

Proof of Theorem 1:
There are three cases according to whether the center is on, inside or outside the inscribed angle (Figures 6, 7 and 8).

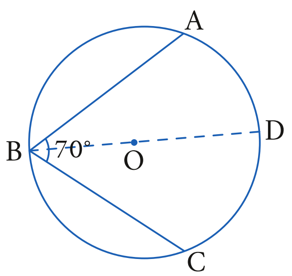


Figure 4. Draw diameter BD.

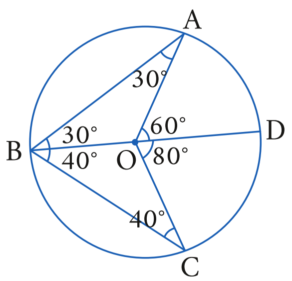


Figure 5. Suppose ∠ABD = 30° and ∠DBC = 40°.

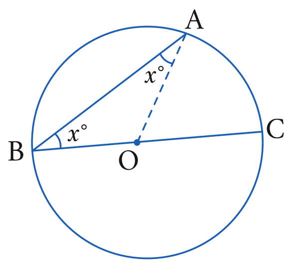


Figure 6. The center is on the inscribed angle.

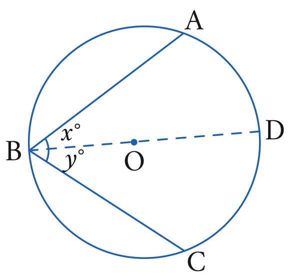


Figure 7. The center is inside the inscribed angle.

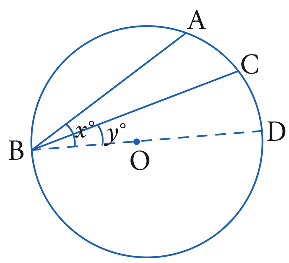


Figure 8. The center is outside the inscribed angle.

Case I.
The center is on the inscribed angle (Figure 6). Draw AO. The radii are equal so AO = BO and $\angle A = \angle B = x^\circ$. Therefore $\overset{\frown}{AC} \hat{=} \angle AOC = x^\circ + x^\circ = 2x^\circ$ and $\angle ABC = x^\circ \hat{=} \frac{1}{2}\overset{\frown}{AC}$.

Case II.
The center is inside the inscribed angle (Figure 7). Draw diameter BD from B through O.

From case I we know $\angle ABD = x^\circ \hat{=} \frac{1}{2}\overset{\frown}{AD}$ and $\angle DBC = y^\circ \hat{=} \frac{1}{2}\overset{\frown}{DC}$. Hence

$$\angle ABC = x^\circ + y^\circ \hat{=} \frac{1}{2}\overset{\frown}{AD} + \frac{1}{2}\overset{\frown}{DC} \hat{=} \frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{DC}) \hat{=} \frac{1}{2}\overset{\frown}{AC}.$$

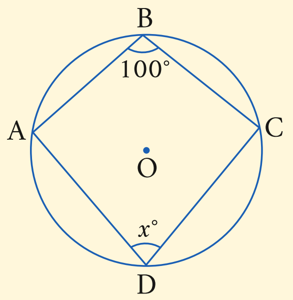
Case III.
The center is outside the inscribed angle (Figure 8). Draw diameter BD from B through O.

From case I we know $\angle ABD = x^\circ \hat{=} \frac{1}{2}\overset{\frown}{AD}$ and $CBD = y^\circ \hat{=} \frac{1}{2}\overset{\frown}{CD}$. Therefore

$$\angle ABC = x^\circ - y^\circ \hat{=} \frac{1}{2}\overset{\frown}{AD} - \frac{1}{2}\overset{\frown}{CD} \hat{=} \frac{1}{2}(\overset{\frown}{AD} - \overset{\frown}{CD}) \hat{=} \frac{1}{2}\overset{\frown}{AC}.$$

? EXAMPLE D

Find x:



SOLUTION

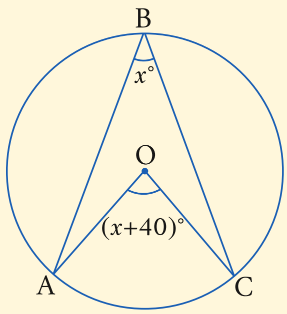
$\angle B = 100^\circ \hat{=} \frac{1}{2}\overset{\frown}{ADC}$.

Therefore $\overset{\frown}{ADC} = 200^\circ$. So $\overset{\frown}{ABC} \hat{=} 360^\circ - 200^\circ = 160^\circ$ and $x^\circ = \frac{1}{2}\overset{\frown}{ABC} = \frac{1}{2}(160^\circ) = 80^\circ$.

ANSWER : $x = 80$.

? EXAMPLE E

Find x:



SOLUTION

$\angle B = x° \circeq \frac{1}{2}\ \widehat{AC}$. Also $\widehat{AC} \circeq \angle AOC = (x + 40)°$. We have

$$\angle B \circeq \frac{1}{2}\widehat{AC}$$

$$x = \frac{1}{2}(x + 40)$$

$$2x = x + 40$$

$$x = 40$$

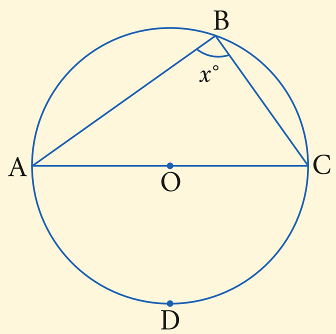
ANSWER : $x = 40$.

CHECK

$\angle B$	$\circeq \frac{1}{2}\ \widehat{AC}$
$x°$	$\frac{1}{2}\ (x + 40)°$
$40°$	$\frac{1}{2}\ (40 + 40)°$
	$\frac{1}{2}\,80°$
	$40°$

? EXAMPLE F

Find x:



SOLUTION

$\widehat{ADC}$ is a semicircle so $\widehat{ADC} \circeq 180°$. $\angle B = x° \circeq \frac{1}{2}\ \widehat{ADC} \circeq \frac{1}{2}(180°) = 90°$.

ANSWER : $x = 90$.

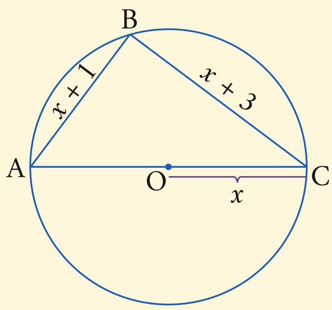
We state the result of Example F as a theorem:

THEOREM 2

An angle inscribed in a semicircle is a right angle.

EXAMPLE G

Find x:



SOLUTION

According to **Theorem 2** $\angle B = 90°$. Therefore $\triangle ABC$ is a right triangle and we can apply the Pythagorean theorem:

$$\begin{aligned} AB^2 + BC^2 &= AC^2 \\ (x+1)^2 + (x+3)^2 &= (2x)^2 \\ x^2 + 2x + 1 + x^2 + 6x + 9 &= 4x^2 \\ 2x^2 + 8x + 10 &= 4x^2 \\ 0 &= 2x^2 - 8x - 10 \\ 0 &= x^2 - 4x - 5 \\ 0 &= (x - 5)(x + 1) \end{aligned}$$

$$0 = x - 5 \qquad\qquad 0 = x + 1$$

$$x = 5 \qquad\qquad x = -1$$

We reject the answer $x = -1$ since $OC = x$ must have positive length.

ANSWER : $x = 5$.

CHECK

$$\begin{array}{r|l} AB^2 + BC^2 & = \; AC^2 \\ (x+1)^2 + (x+3)^2 & (2x)^2 \\ (5+1)^2 + (5+3)^2 & (2(5))^2 \\ 6^2 + 8^2 & 10^2 \\ 36 + 64 & 100 \\ 100 & \end{array}$$

The next four theorems are all consequences of **Theorem 1**:

THEOREM 3

Parallel lines intercept arcs equal in degrees.

In Figure 9 if AB $\|$ CD then $\overset{\frown}{AC} \circeq \overset{\frown}{BD}$.

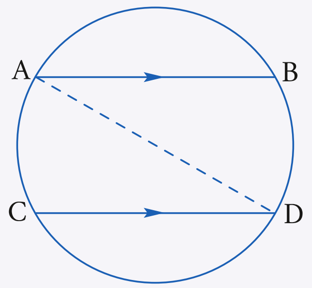
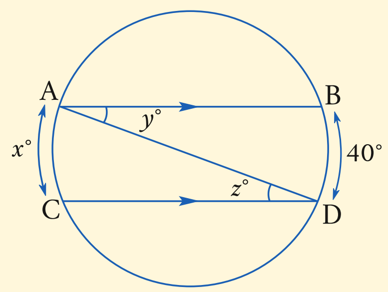


Figure 9. If AB $\|$ CD then $\overset{\frown}{AC} \circeq \overset{\frown}{BD}$.

Proof of Theorem 3:
Draw AD. Then $\angle ADC \circeq \frac{1}{2}\overset{\frown}{AC}$ and $\angle BAD \circeq \frac{1}{2}\overset{\frown}{BD}$. Also $\angle ADC = \angle BAD$ because they are alternate interior angles of parallel lines AB and CD.
Therefore $\frac{1}{2}\overset{\frown}{AC} \circeq \frac{1}{2}\overset{\frown}{BD}$ and $\overset{\frown}{AC} \circeq \overset{\frown}{BD}$.

EXAMPLE H

Find x, y and z:



SOLUTION

By **Theorem 3** $x^\circ = 40^\circ$. $y^\circ = z^\circ \circeq \frac{1}{2}\overset{\frown}{AC} \circeq \frac{1}{2}\overset{\frown}{BD} \circeq \frac{1}{2}(40^\circ) = 20^\circ$.

ANSWER: $x = 40$, $y = z = 20$.

THEOREM 4

An angle formed by a tangent and a chord is $\triangleq \frac{1}{2}$ of its intercepted arc.

In Figure 10 $\angle APC \triangleq \frac{1}{2}\overset{\frown}{PC}$ and $\angle BPC \triangleq \frac{1}{2}\overset{\frown}{PDC}$.

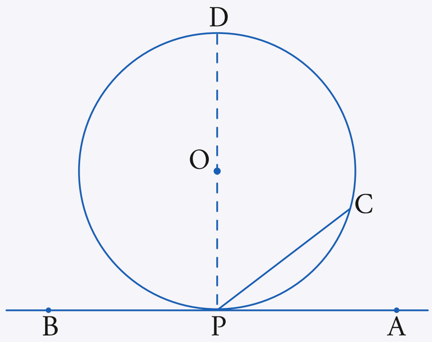


Figure 10. $\angle APC$ and $\angle BPC$ are formed by tangent $\overleftrightarrow{AB}$ and chord PC.

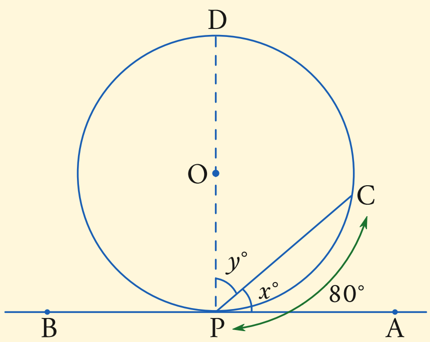
Proof of Theorem 4:
In Figure 10 draw diameter PD. Then by **Theorem 1** of *Section 7.3* $\angle APD = \angle BPD = 90°$.
By Theorem 1 of this section $\angle CPD \triangleq \frac{1}{2}\overset{\frown}{CD}$.

$$\angle APC = 90° - \angle CPD \triangleq 90° - \frac{1}{2}\overset{\frown}{CD} \triangleq 90° - \frac{1}{2}(180° - \overset{\frown}{PC}) \triangleq 90° - 90° + \frac{1}{2}\overset{\frown}{PC} \triangleq \frac{1}{2}\overset{\frown}{PC}.$$

$$\angle BPC = 90° + \angle CPD \triangleq 90° + \frac{1}{2}\overset{\frown}{CD} \triangleq \frac{1}{2}(180° + \overset{\frown}{CD}) \triangleq \frac{1}{2}\overset{\frown}{PDC}.$$

EXAMPLE I

Find x, y and $\overset{\frown}{CD}$:



SOLUTION

By **Theorem 4,**

$x^\circ \circeq \frac{1}{2}\overset{\frown}{PC} \circeq \frac{1}{2}(80^\circ) = 40^\circ.$

$y^\circ = 90^\circ - x^\circ = 90^\circ - 40^\circ = 50^\circ.$

$\overset{\frown}{CD} \circeq \frac{1}{2}180^\circ - \overset{\frown}{CP} \circeq 180^\circ - 80^\circ = 100^\circ.$

ANSWER : $x = 40$, $y = 50$, $\overset{\frown}{CD} \circeq 100^\circ$.

THEOREM 5

An angle formed by two intersecting chords is $\circeq$ to $\frac{1}{2}$ the sum of the intercepted arcs.

In Figure 11 $x^\circ \circeq \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$.

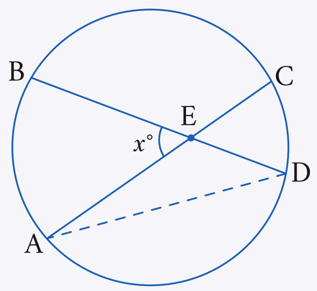


Figure 11. $x^\circ \circeq \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$.

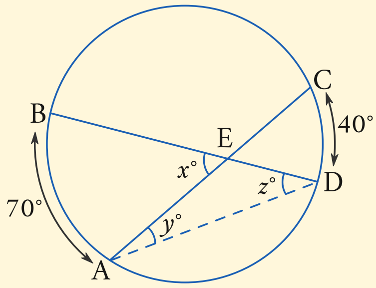
Proof of Theorem 5:

$\angle ADB \circeq \frac{1}{2}\overset{\frown}{AB}$ and $\angle CAD \circeq \frac{1}{2}\overset{\frown}{CD}$. By **Theorem 2**, *Section 1.5*, an exterior angle of a triangle is equal to the sum of the two remote interior angles.

Therefore $x^\circ = \angle ADB + \angle CAD \circeq \frac{1}{2}\overset{\frown}{AB} + \frac{1}{2}\overset{\frown}{CD} \circeq \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$.

? EXAMPLE J

Find x, y and z:



SOLUTION

By **Theorem 5**,

$$x^\circ \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD}) \overset{\circ}{=} \frac{1}{2}(70^\circ + 40^\circ) = \frac{1}{2}(110^\circ) = 55^\circ .$$

$$y^\circ \overset{\circ}{=} \frac{1}{2}\overset{\frown}{CD} = \frac{1}{2}(40^\circ) = 20^\circ.$$

$$z^\circ \overset{\circ}{=} \frac{1}{2}\overset{\frown}{AB} = \frac{1}{2}(70^\circ) = 35^\circ.$$

ANSWER : $x = 55$, $y = 20$, $z = 35$.

A line which intersects a circle in two points is called a **secant**.
In Figure 12, PC is a secant.

THEOREM 6

An angle formed outside a circle by two secants, a tangent and a secant, or two tangents is $\overset{\circ}{=} \frac{1}{2}$ the difference of the intercepted arcs.

In each of Figures 12, 13 and 14, $\angle P \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{CDE} - \overset{\frown}{AB})$.

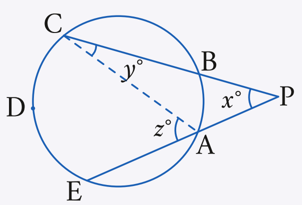


Figure 12.
∠P formed by two secants.

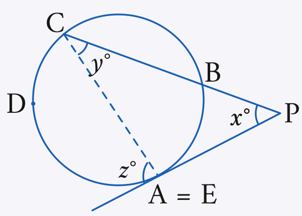


Figure 13.
∠P formed by a tangent and a secant.

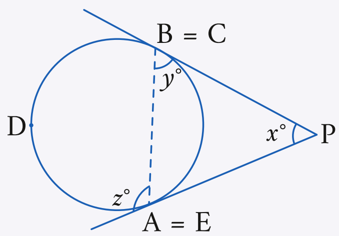


Figure 14.
∠P formed by two tangents.

Proof of Theorem 6:

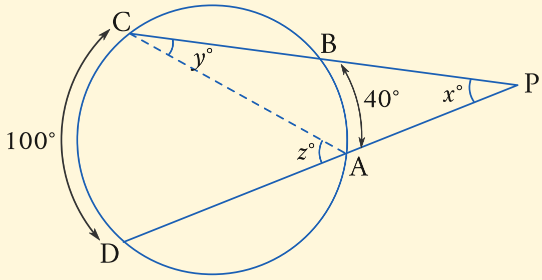
In each case $x^\circ + y^\circ = z^\circ$ (because an exterior angle of a triangle is the sum of the two remote interior angles). Therefore $x^\circ = z^\circ - y^\circ$.

Using **Theorems 1** and **4** we have

$$\angle P = x^\circ = z^\circ - y^\circ \doteq \frac{1}{2}\overset{\frown}{CDE} - \frac{1}{2}\overset{\frown}{AB} \doteq \frac{1}{2}(\overset{\frown}{CDE} - \overset{\frown}{AB}).$$

? EXAMPLE K

Find x, y and z:



SOLUTION

By **Theorem 6**,

$$x^\circ \doteq \frac{1}{2}(\overset{\frown}{CD} - \overset{\frown}{AB}) \doteq \frac{1}{2}(100^\circ - 40^\circ) = \frac{1}{2}(60^\circ) = 30^\circ.$$

By **Theorem 1**,

$$y^\circ \doteq \frac{1}{2}\overset{\frown}{AB} \doteq \frac{1}{2}(40^\circ) = 20^\circ \text{ and } z^\circ \doteq \frac{1}{2}\overset{\frown}{CD} \doteq \frac{1}{2}(100^\circ) = 50^\circ.$$

ANSWER : $x = 30$, $y = 20$, $z = 50$.

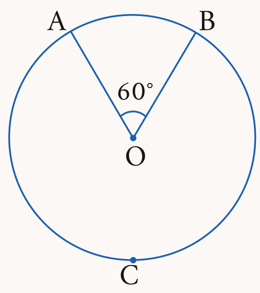
HISTORICAL NOTE

The practice of dividing the circle into 360 degrees goes back to the Greeks of the second century B.C., who in turn may have taken it over from the Babylonians. The reason for using the number 360 is not clear. It could stem from an early astronomical assumption that a year consisted of 360 days. Another explanation relies on the fact that the Babylonians used a sexagesimal or base 60 number system instead of the decimal or base 10 system that we use today. It is assumed that the Babylonians may have also used 60 as a convenient value for the radius of a circle. Since the circumference of a circle is about 6 times the radius (see next section), such a circle would consist of 360 units.

PROBLEMS

1 - 4. Find the number of degrees in arcs $\overset{\frown}{AB}$ and $\overset{\frown}{ACB}$:

1.



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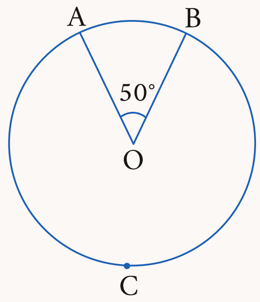

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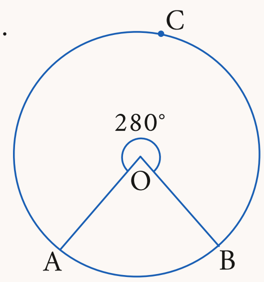
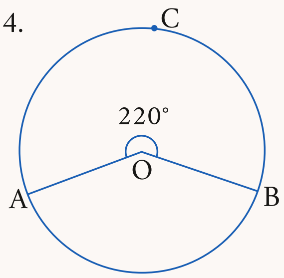


5 - 10. Find x, y and z:

4.



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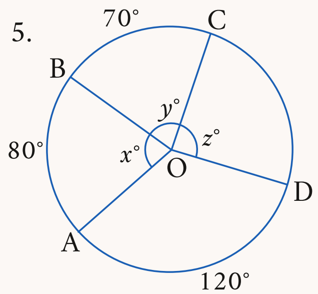

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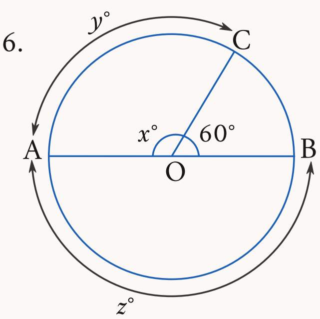

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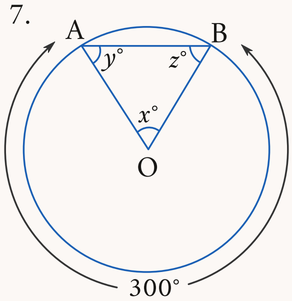

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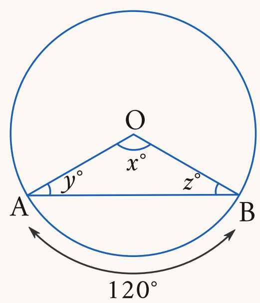

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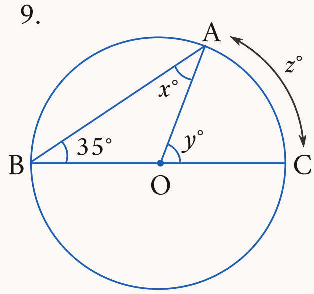
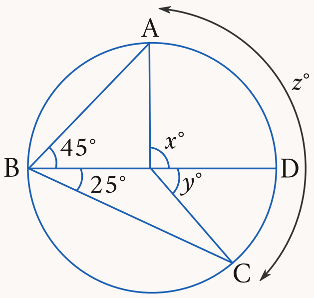


11 - 26. Find x:

10.



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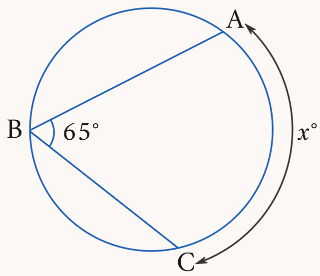

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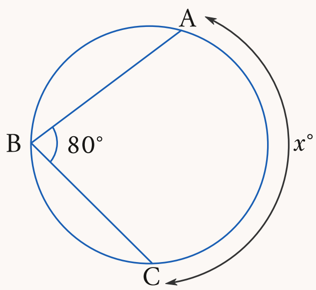

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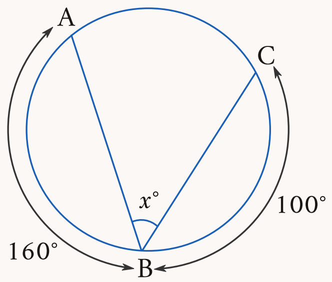

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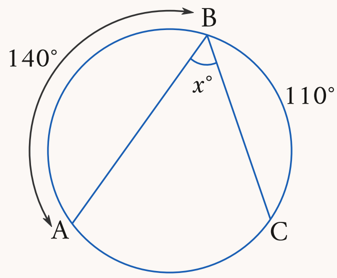

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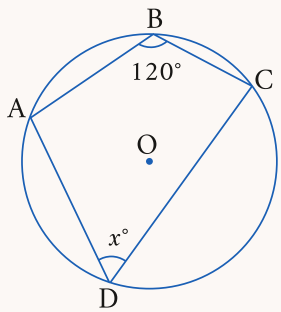

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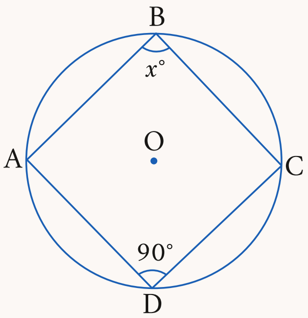

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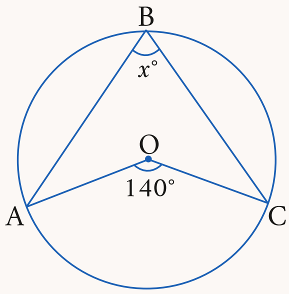

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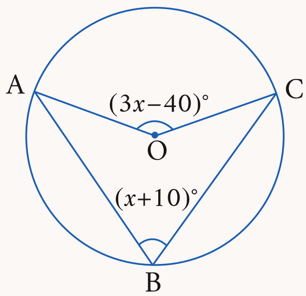

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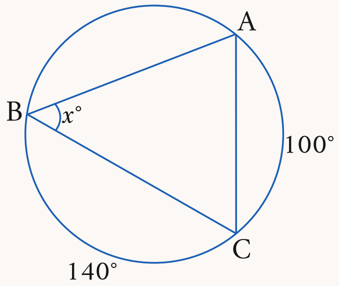

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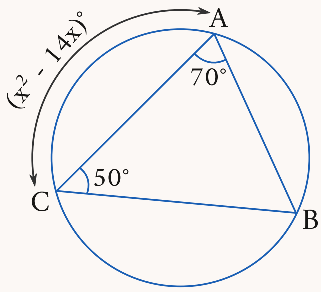

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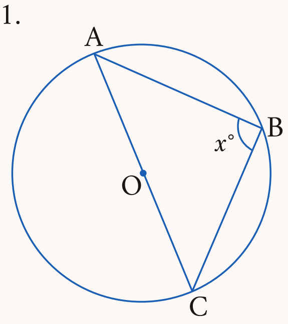

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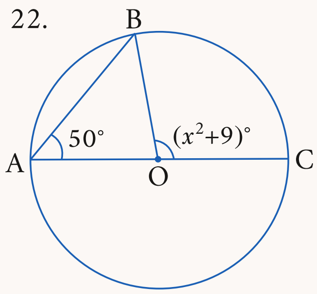

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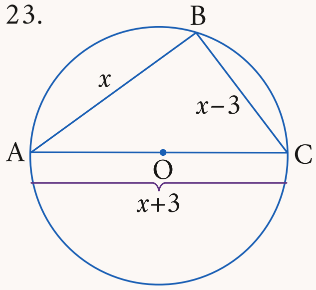

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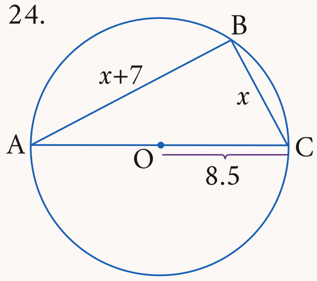

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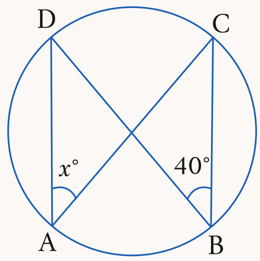

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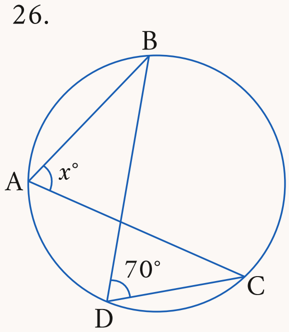
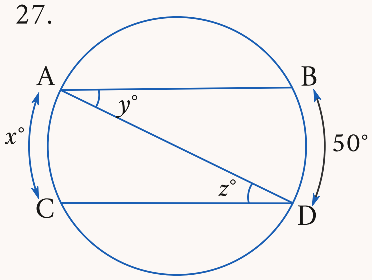


27 - 28. Find x, y and z:

27.



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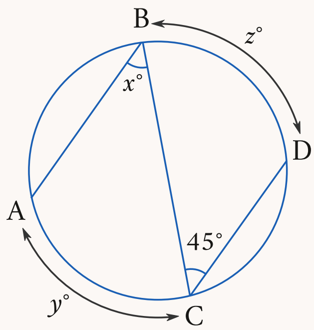
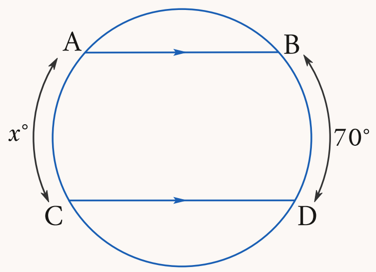


29 - 30. Find x:

29.



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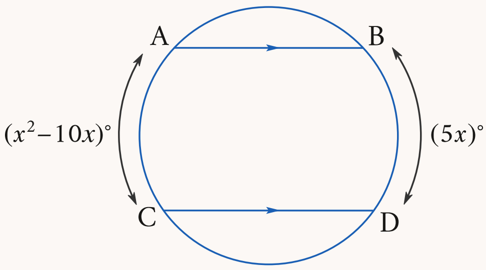
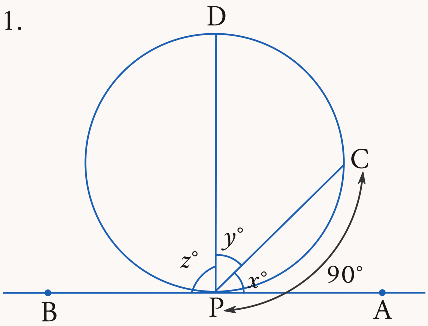


31 - 32. Find x, y and z:

31.



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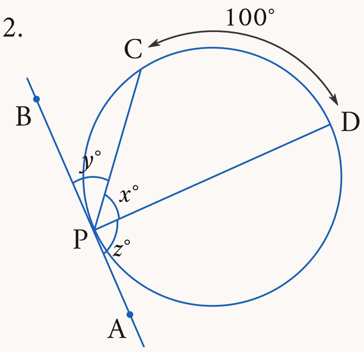
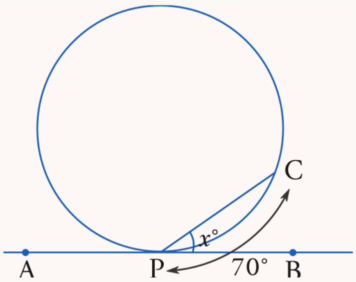


33 - 34. Find x:

33.



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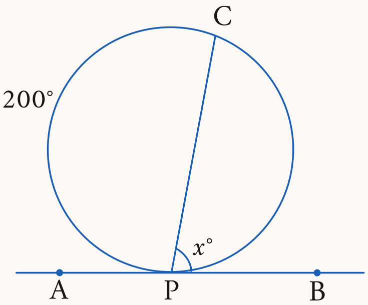
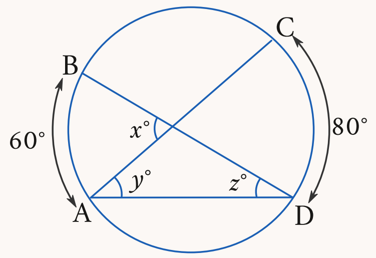


35 - 36. Find x, y and z:

35.



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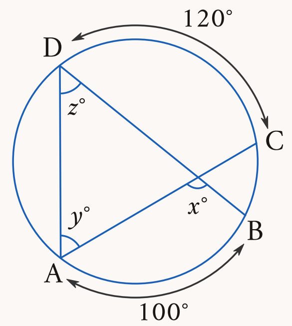
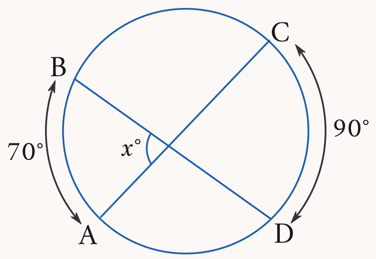


37 - 38. Find x:

37.



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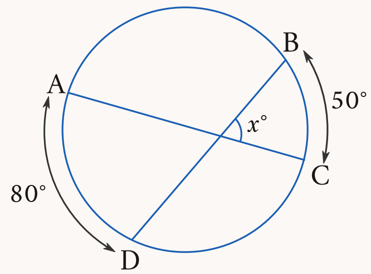
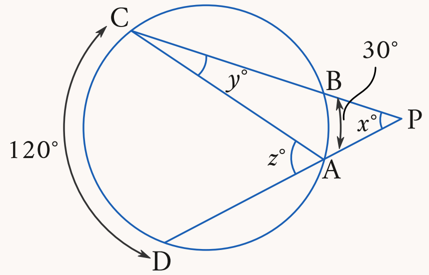


39 - 42. Find x, y and z:

39.



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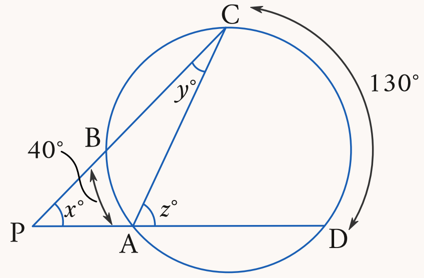

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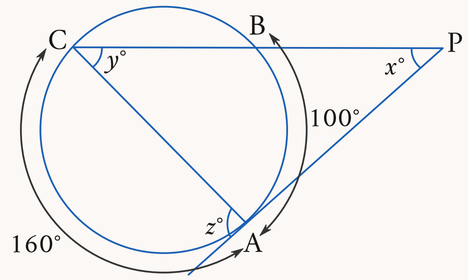

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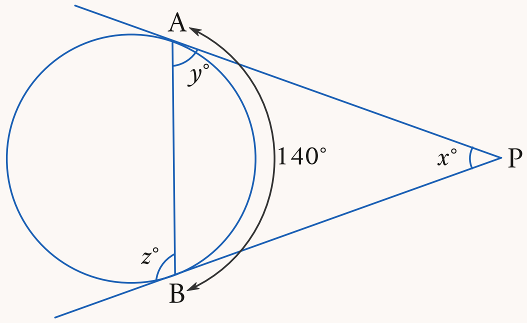
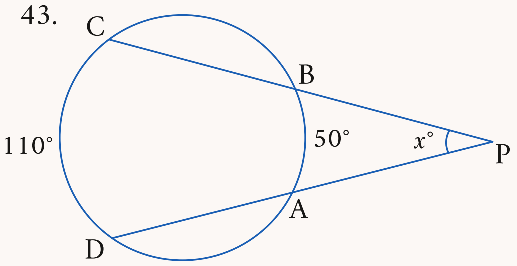


43 - 46. Find x:

43.



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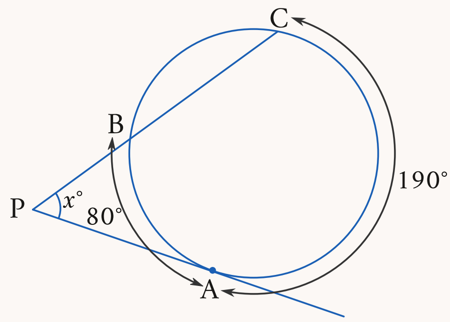

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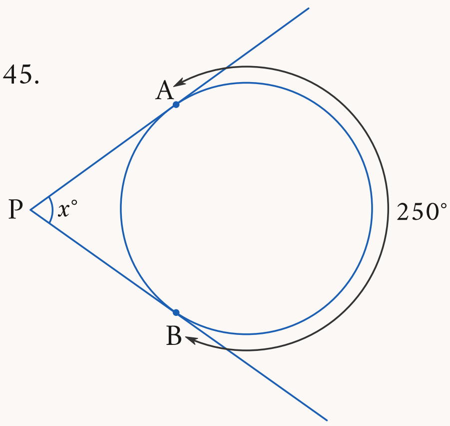

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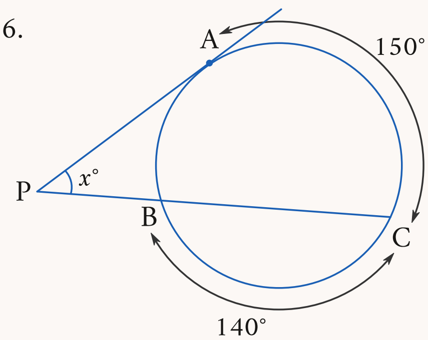


7.5 CIRCUMFERENCE OF A CIRCLE

The **circumference** of a circle is the perimeter of the circle, the length of the line obtained by cutting the circle and "straightening out the curves" (Figure 1).

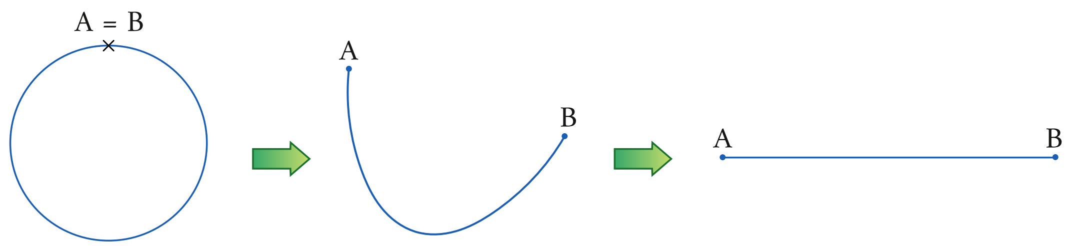


Figure 1. The circumference of a circle is the length of the line obtained by cutting the circle and straightening out the curves.

It is impractical to measure the circumference of most circular objects directly. A circular tape measure would be hard to hold in place and would become distorted as it would be bent. The object itself would be destroyed if we tried to cut it and straighten it out for measurement. Fortunately we can calculate the circumference of a circle from its radius or diameter, which are easy to measure.

An approximate value for the circumference of a circle of radius x can be obtained by calculating the perimeter of a regular hexagon of radius r inscribed in the circle (Figure 2). We see that the circumference is a little more than the perimeter of the hexagon, which is 6 times the radius or 3 times the diameter. To get a better approximation, we increase the number of sides of the inscribed regular polygon.

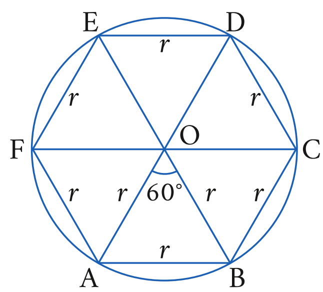


Figure 2. Regular hexagon ABCDEF is inscribed in circle O. Both have radius *r* and center *O*.

As the number of sides of a regular polygon increases, the polygon looks more and more like a circle (Figure 3). In *Section 7.1*, we calculated the perimeter of a 90-sided regular polygon to be 3.141 times the diameter or 6.282 times the radius.

The perimeter of a 1000-sided regular polygon turned out to be only slightly larger, 3.1416 times the diameter or 6.283 times the radius. It therefore seems reasonable to conclude that the circumference of a circle is about 3.14 times its diameter or 6.28 times its radius.

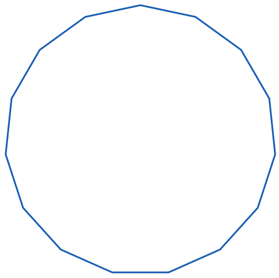


Figure 3. A regular polygon of 15 sides looks almost like a circle.

THEOREM 1

The circumference of a circle is π times its diameter or 2π times its radius, where π is approximately 3.14.

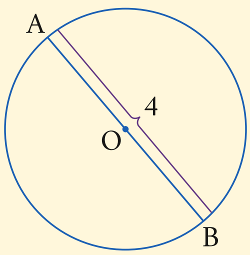
$$\boxed{C = \pi d \qquad \text{or} \qquad C = 2\pi r}$$

The symbol π (Greek letter pi) is standard notation for the number by which the diameter of a circle must be multiplied to get the circumference. Its value is usually taken to be 3.14, though 3.1416 and 22/7 are other commonly used approximations. These numbers are not exact, for like $\sqrt{2}$, it can be shown that π is an irrational number (infinite nonrepeating decimal). Its value to 50 decimal places is

3. 14159 26535 89793 23846 26433 83279 50288 41971 69399 37511.

EXAMPLE A

Find the circumference:



SOLUTION

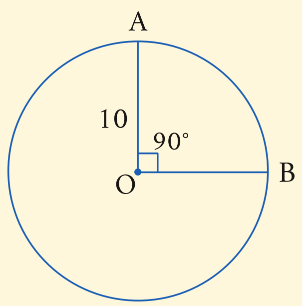
$C = \pi d = (3.14)(4) = 12.56.$

ANSWER : 12.56.

We define the **length of an arc** in the same manner as we defined circumference. We calculate it by multiplying the circumference by the appropriate fraction.

? EXAMPLE B

Find the length of arc $\overparen{AB}$:



SOLUTION

$C = 2\pi r = 3(3.14)(10) = 62.8.$

Since 90° is $\frac{1}{4}$ of 360°, $\overparen{AB}$ is $\frac{1}{4}$ of the circumference C. $\overparen{AB} = \frac{1}{4} C = \frac{1}{4}(62.8) = 15.7.$

ANSWER : 15.7.

As we stated in *Section 7.4*, the plain = symbol will be used for arc length and the $\doteq$ symbol will be used for degrees. Thus in Example B, $\overparen{AB} = 15.7$ but $\overparen{AB} \doteq 90°$.

We may also use the following formula to find arc length:

$$\text{Arc length} = \frac{\text{Degrees in Arc}}{360°} \cdot \text{Circumference}$$

or simply

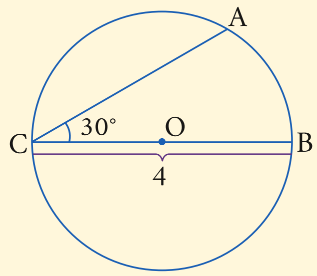
$$\boxed{L = \frac{D}{360°} \cdot C} \ .$$

Thus in Example B,

$$L = \frac{D}{360°} \cdot C \quad = \frac{90}{360°} \cdot (62.8) = \frac{1}{4}\ (62.8) = 15.7.$$

? EXAMPLE C

Find the length of arc $\overset{\frown}{AB}$:



SOLUTION

$C = \pi d = (3.14)(4) = 12.56$. $\angle ACB \doteq \frac{1}{2}\overset{\frown}{AB} \doteq 30°$.

Therefore $\overset{\frown}{AB} \doteq 60°$. Using the formula for arc length,

$$L = \frac{D}{360°} C = \frac{60}{360°} \cdot (12.56) = \frac{1}{6}(12.56) = 2.09.$$

ANSWER : 2.09.

? EXAMPLE D

Find the diameter of a circle whose circumference is 628.

SOLUTION

Letting $C = 628$ and $\pi = 3.14$ in the formula for circumference, we have

$$C = \pi d$$

$$628 = (3.14)d$$

$$\frac{628}{3.14} = \frac{3.14d}{3.14}$$

$$200 = d$$

ANSWER : diameter = 200.

Application:

The odometer and speedometer of an automobile are calibrated in accordance with the number of rotations of one of the wheels. Suppose the diameter of a tire mounted on the wheel is 2 feet. Then its circumference is $C = \pi d = (3.14)(2) = 6.28$ feet. Since 1 mile = 5280 feet, the wheel will rotate $5280 \div 6.28 = 841$ times every mile. If the size of the tires is changed for any reason the odometer and speedometer must be recalibrated.

HISTORICAL NOTE

The circumference of the earth was first accurately calculated by the Greek geographer Eratosthenes (c. 284 - 192 B.C.), who lived in Alexandria, Egypt. It was known that at noon on the day of the summer solstice the sun's rays completely lit up the wells of Syene (now called Aswan), Egypt. This indicated that the rays of the sun were perpendicular to the Earth's surface at Syene, and so, in Figure 4, $\overleftrightarrow{DS}$ passes through the earth's center O. At the same time, in Alexandria, Eratosthenes observed that the sun's rays were making an angle of 1/50 of 360° (that is, 7.2°) with the perpendicular ($\angle BAC = 7.2°$ in Figure 4). The rays of the sun are assumed to be parallel hence $\angle AOS = \angle BAC = 7.2°$ and $\overset{\frown}{AS} \doteq 7.2°$. Since the distance between Alexandria and Syene is about 500 miles (the length of $\overset{\frown}{AS}$). Eratosthenes was able to come up with a remarkably accurate figure of about (50)(500) = 25,000 miles for the circumference of the earth.

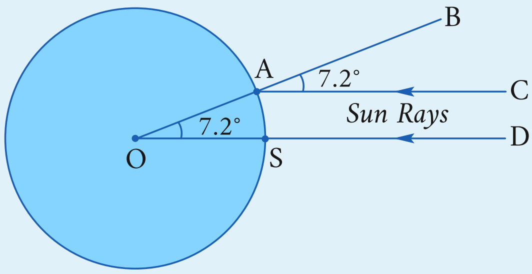


Figure 4. The sun's rays were perpendicular to the Earth's surface at S at the same time they were making an angle of 7.2° with the perpendicular at A.

Early crude estimates of the value of π were made by the Chinese ($\pi = 3$), Babylonians ($\pi = 3$ or $3\frac{1}{8}$), and Egyptians ($\pi = 3.16$). The value $\pi = 3$ is also the one assumed in the Bible (I Kings 7:23). The first accurate calculation was carried out by Archimedes (287 - 212 B.C.), the greatest mathematician of antiquity. (Archimedes was also a famous physicist and inventor. For example, he discovered the principle that a solid immersed in a liquid is buoyed up by a force equal to the weight of the fluid displaced.) In his treatise *On the Measurement of the Circle* he approximates the circumference by calculating the perimeters of inscribed and circumscribed regular polygons (Figure 5). This is similar to the method we described in the text except that Archimedes did not have accurate trigonometric tables and had to derive his own formulas. By carrying the process as far as the case of the polygon of 96 sides he found the value of π to be between $3\frac{10}{71}$ and $3\frac{1}{7}$.

(Incidentally Archimedes did not actually use the symbol π. The symbol π was not used for the ratio of the circumference to the diameter of a circle until the 18th century.)

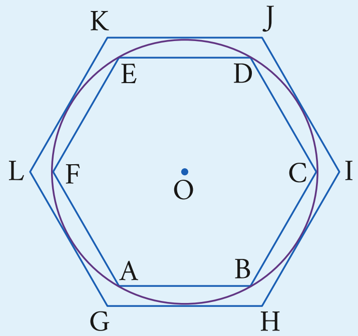


Figure 5. The circumference of circle O is greater than the perimeter of the insribed polygon ABCDEF but less than the perimeter of the circumscribed polygon GHIJKL.

The procedure of Archimedes was the beginning of a long history of increasingly accurate calculations of the value of π. Since the 17th century these calculations have involved the use of infinite series, such as

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots,$$

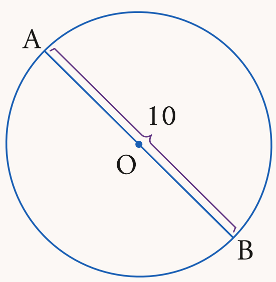
the derivation of which can be found in many calculus textbooks. Most recently, with the help of a computer, the value of π has been determined to more than a trillion decimal places.

PROBLEMS

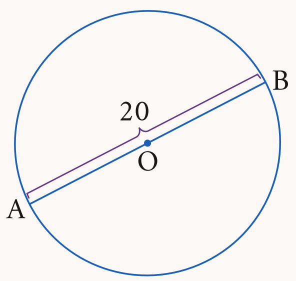
For each of the following use $\pi = 3.14$.

1 - 8. Find the circumference of each circle:

1.



2.



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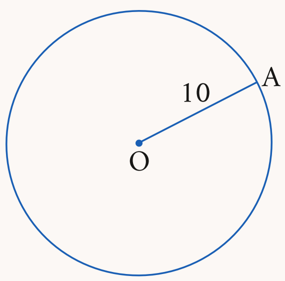

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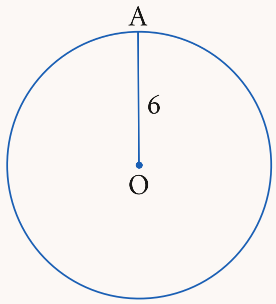

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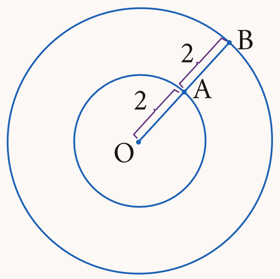

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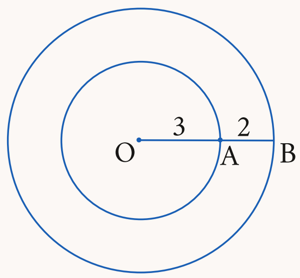

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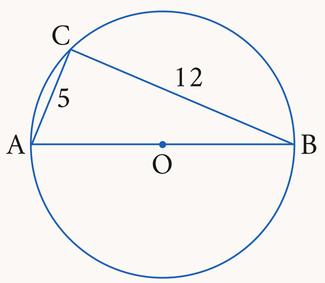

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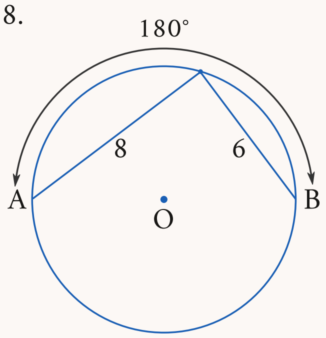
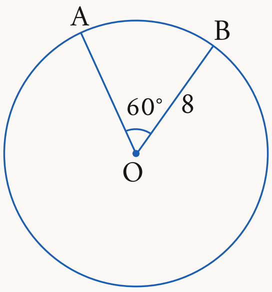


9 - 14. Find the length of arc $\overset{\frown}{AB}$:

9.



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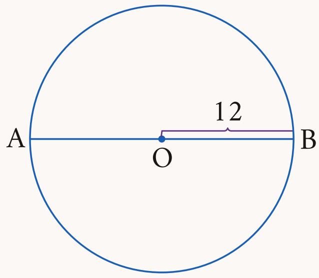

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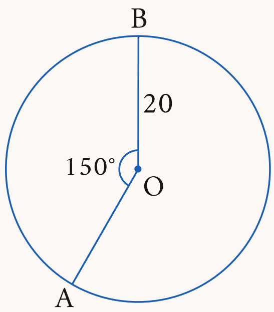

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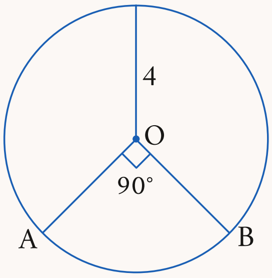

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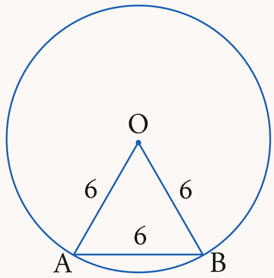

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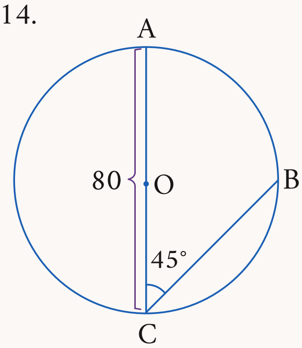
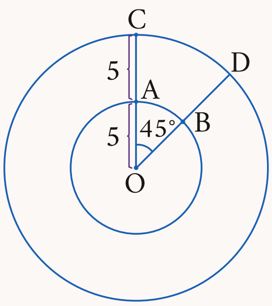


15 - 16. Find the lengths of arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$:

15.



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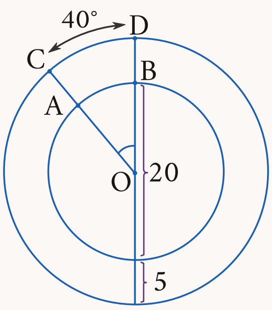
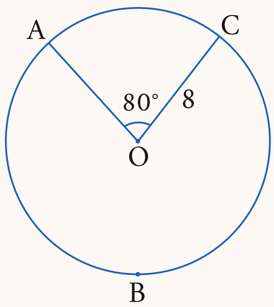


17 - 18. Find the length of major arc $\overset{\frown}{ABC}$:

17.



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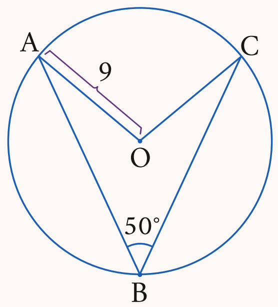


19 - 22. Find the circumference of the circle whose ...

19. diameter is 30.

20. diameter is 8.

21. radius is 10.

22. radius is 6.

23. Find the radius and diameter of the circle whose circumference is 314.

24. Find the radius and diameter of the circle whose circumference is 100 (leave answer to the nearest whole number).

25. What is the circumference of an automobile wheel whose diameter is 14 inches?

26. What is the circumference of a 12 inch phonograph record?

27. What is the diameter of the earth if its circumference is 24,830 miles?

28. What is the diameter of a quarter mile circular running track?

7.6 AREA OF A CIRCLE

In chapter 6 we defined the area of a closed figure to be the number of square units contained in the figure. To apply this definition to the circle, we will again assume a circle is a regular polygon with a large number of sides. The following formula is then obtained:

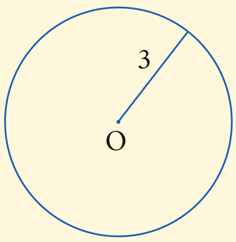
THEOREM 1

The area of circle is π times the square of its radius.

$$\boxed{A = \pi r^2}$$

EXAMPLE A

Find the area of the circle:



SOLUTION

$A = \pi r^2 = \pi(3)^2 = 9\pi = 9(3.14) = 28.26.$

ANSWER : 28.26.

***Proof of Theorem 1*:**

The area of a circle with radius r is approximately equal to the area of a regular polygon with apothem $a = r$ circumscribed about the circle (Figure 1). The approximation becomes more exact as the number of sides of the polygon becomes larger. At the same time the perimeter of the polygon approximates the circumference of the circle ($= 2\pi r$).

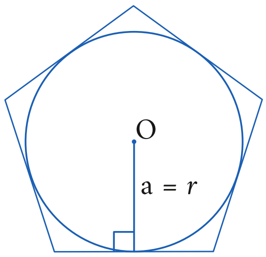


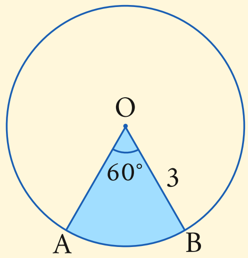
Figure 1. Regular polygon with apothem $a = r$ circumscribed about circle with radius r.

Using the formula for the area of a regular polygon (**Theorem 4**, *Section 7.1*) we have

area of circle = area of polygon = $\frac{1}{2}aP = \frac{1}{2}r(2\pi r) = \pi r^2$.

? EXAMPLE B

Find the shaded area:



SOLUTION

The shaded area OAB is $\frac{60}{360} = \frac{1}{6}$ of the total area (see Figure 2).

The area of the whole circle = $\pi r^2 = \pi(3)^2 = 9\pi = 9(3.14) = 28.26$.
Therefore the area of OAB = $\frac{1}{6}(28.26) = 4.71$.

ANSWER : 4.71.

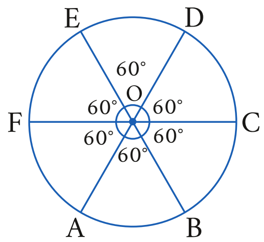


Figure 2.
A circle divided into six equal parts.

The shaded area in Example B is called a **sector** of the circle. Example B suggests the following formula for the area of a sector:

$$\text{Area of sector} = \frac{\text{Degrees in arc of sector}}{360} \cdot \text{Area of circle}$$

or simply

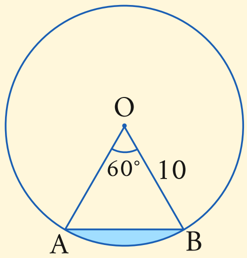
$$\boxed{A = \frac{D}{360} \cdot \pi r^2}\ .$$

Using this formula, the solution of Example B would be

$$A = \frac{D}{360}\pi r^2 = \frac{60}{360}(3.14)(3)^2 = \frac{1}{6}(3.14)(9) = \frac{1}{6}(28.26) = 4.71.$$

? EXAMPLE C

Find the shaded area:



SOLUTION

Let us first find the area of triangle OAB (Figure 3).

$\triangle OAB$ is equilateral with base b = AB = 10. Drawing in height h = OC we have that $\triangle AOC$ is a 30° - 60° - 90° triangle with AC = 5 and h = $5\sqrt{3}$. Therefore

shaded area = area of sector OAB − area of triangle OAB

$$= \frac{D}{360}\pi r^2 - \frac{1}{2}bh$$

$$= \frac{60}{360}\pi(10)^2 - \frac{1}{2}(10)(5\sqrt{3})$$

$$= \frac{1}{6}(100\pi) - \frac{1}{2}(50\sqrt{3})$$

$$= \frac{50\pi}{3} - 25\sqrt{3}$$

$$= \frac{50(3.14)}{3} - 25(1.732)$$

$$= 52.33 - 43.30 = 9.03.$$

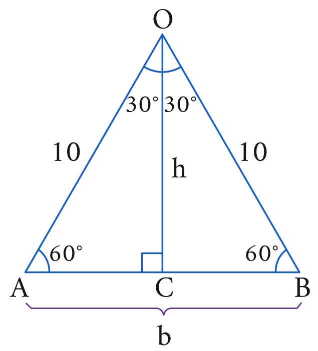


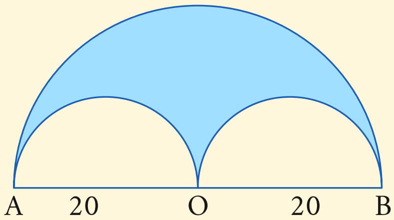
Figure 3.
Triangle OAB with base *b* and height *h*.

ANSWER : $\frac{50\pi}{3} - 25\sqrt{3}$ or 9.03.

The shaded area in Example C is called a **segment** of the circle. The area of a segment is obtained by subtracting the area of the triangle from the area of the sector.

? EXAMPLE D

Find the shaded area:



SOLUTION

The area of the large semicircle $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(20)^2 = \frac{1}{2}(400)\pi = 200\pi$.

The area of each of the smaller semicircles $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(10)^2 = \frac{1}{2}(100)\pi = 50\pi$.
Therefore

shaded area = area of large semicircle − (2)(area of small semicircles)

$= 200\pi - 2(50\pi)$

$= 200\pi - 100\pi$

$= 100\pi = 100(3.14) = 314$.

ANSWER : 100π or 314.

HISTORICAL NOTE

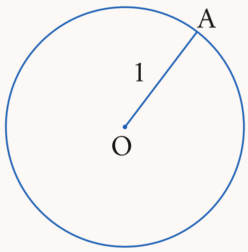
Problem 50 of the Rhind Papyrus, a mathematical treatise written by an Egyptian scribe in about 1650 B.C., states that the area of a circular field with a diameter of 9 units is the same as the area of a square with a side of 8 units. This is equivalent to using the formula $A = (\frac{8}{9}d)^2$ to find the area of a circle. If we let $d = 2r$ this becomes $A = (\frac{8}{9}d)^2 = (\frac{8}{9}\cdot 2r)^2 = (\frac{16}{9}r)^2 = \frac{256}{81}r^2$ or about $3.16r^2$. Comparing this with our modern formula $A = \pi r^2$ we find that the ancient Egyptains had a remarkably good approximation, 3.16, for the value of π.

In the same work in which he calculated the value of π, Archimedes gives a formula for the area of a circle (see **Historical Note**, *Section 7.5*). He states that the area of a circle is equal to the area of a right triangle whose base b is as long as the circumference and whose altitude h equals the radius. Letting $b = C$ and $h = r$ in the formula for the area of a triangle, we obtain $A = \frac{1}{2}bh = \frac{1}{2}Cr = \frac{1}{2}(2\pi r) = \pi r^2$, the modern formula.

PROBLEMS

1 - 6. Find the area of the circle (use π = 3.14):

1.



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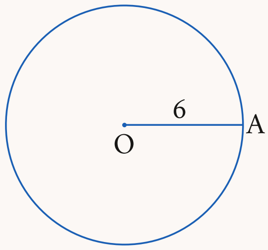

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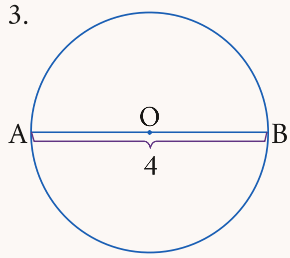

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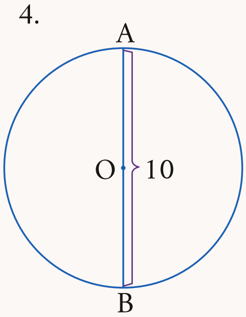

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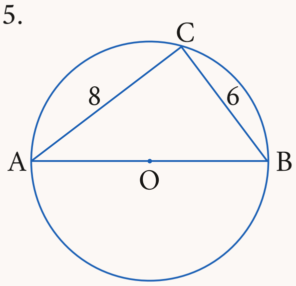

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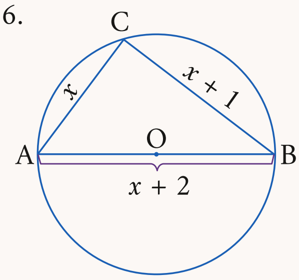


7 - 10. Find the area of the circle with ...

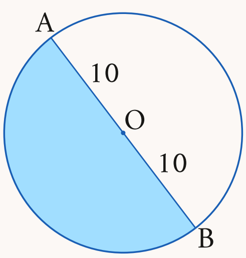
7. radius 20. 8. radius 2.5

9. diameter 12. 10. diameter 15.

(use π = 3.14)

11 - 14. Find the shaded area (use π = 3.14):

11.



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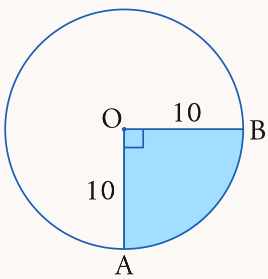

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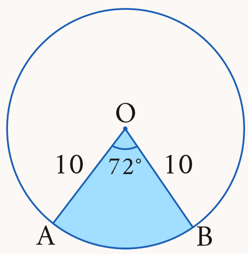

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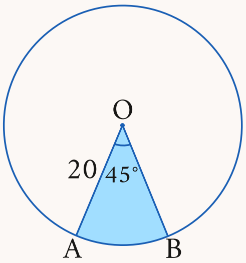
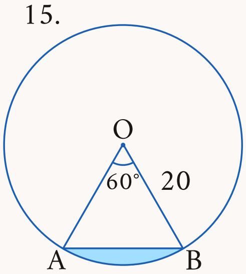


15 - 30. Find the shaded area. Answers may be left in terms of π and in radical form.

15.



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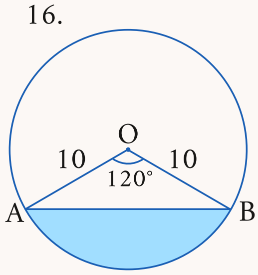

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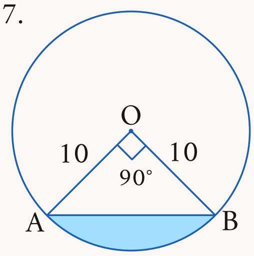

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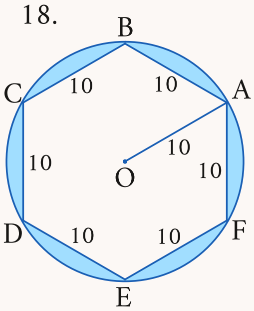

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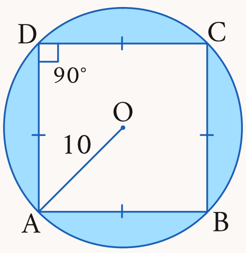

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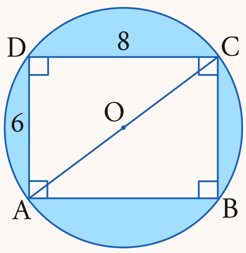

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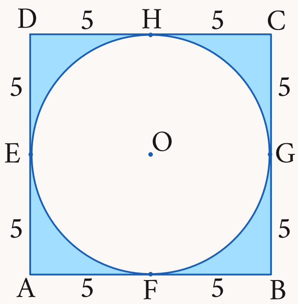

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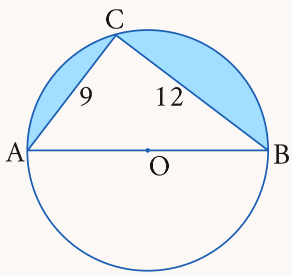

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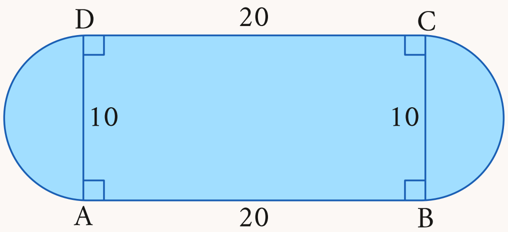

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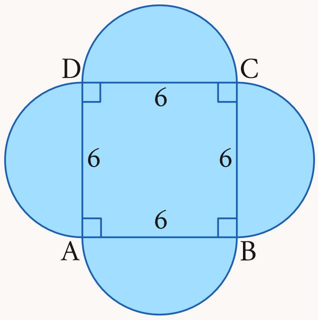

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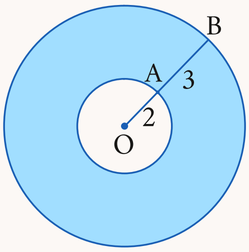

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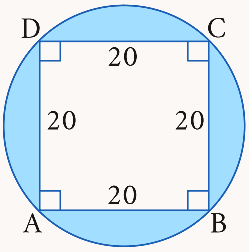

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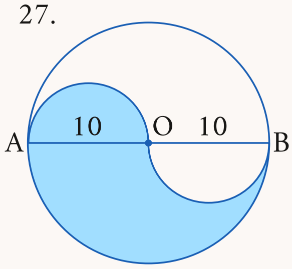

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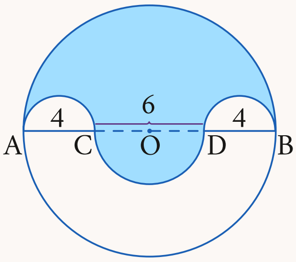

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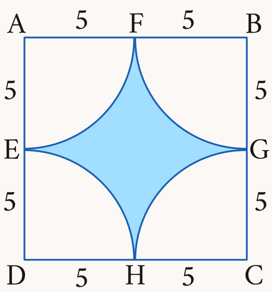

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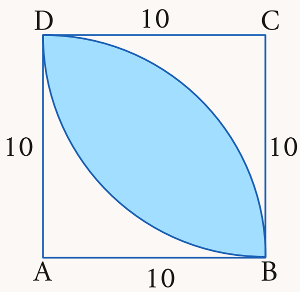

PROOF OF THE Z THEOREM

In section 1.4 we stated but did not prove the following theorem:

THEOREM 1 (THE "Z" THEOREM)

If two lines are parallel then their alternate interior angles are equal. If the alternate interior angles of two lines are equal then the lines must be parallel.

Theorem 1 consists of two statements, each one the converse of the other. We will prove the second statement first:

THEOREM 1 (SECOND PART)

If the alternate interior angles of two lines are equal then the lines must be parallel. In Figure 1, if $\angle x = \angle x'$ then $\overleftrightarrow{AB}$ must be parallel to $\overleftrightarrow{CD}$.

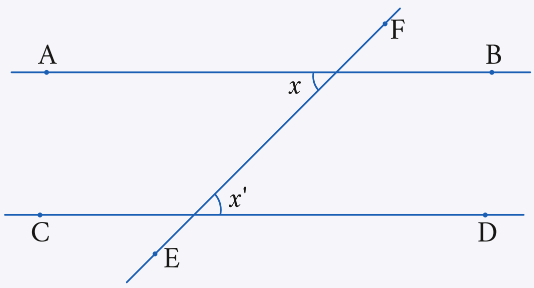

Figure 1. We will prove that if $\angle x = \angle x'$ then $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$.

Proof :
Suppose $\angle x = \angle x'$ and $\overleftrightarrow{AB}$ is **not** parallel to $\overleftrightarrow{CD}$. This means that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet at some point G, as in Figure 2, forming a triangle, $\triangle KLG$.

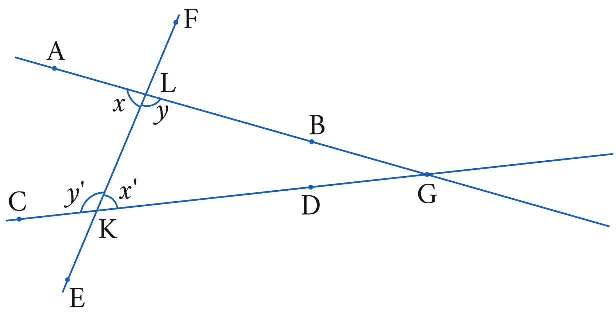

Figure 2. If $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$ then they must meet at some point G.

We know from the discussion preceding the **ASA Theorem** (Theorem 1, *Section 2.3*) that $\triangle KLG$ can be constructed from the two angles $\angle x'$ and $\angle y$ and the included side KL. Now $\angle x = \angle x'$ (by assumption) and $\angle y' = \angle y$ ($y' = 180° - \angle x' = 180° - \angle x = \angle y$). Therefore by the same construction $\angle x$ and $\angle y'$ when extended should yield a triangle congruent to $\triangle KLG$. Call this new triangle $\triangle LKH$ (see Figure 3). We now have that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are both straight lines through G and H. This is impossible since one and only one distinct straight line can be drawn through two points. Therefore our assumption that $\overleftrightarrow{AB}$ is **not** parallel to $\overleftrightarrow{CD}$ is **incorrect**, that is, $\overleftrightarrow{AB}$ must be parallel to $\overleftrightarrow{CD}$. This completes the proof.

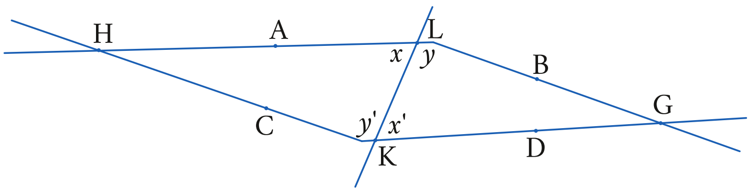

Figure 3. Since $\angle x = \angle x'$ and $\angle y' = \angle y$, $\angle x$ and $\angle y'$ when extended should also form a triangle, $\triangle LKH$.

THEOREM 1 (FIRST PART)

If two lines are parallel then their alternate interior angles are equal. In Figure 4, if $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ then $\angle x = \angle x'$.

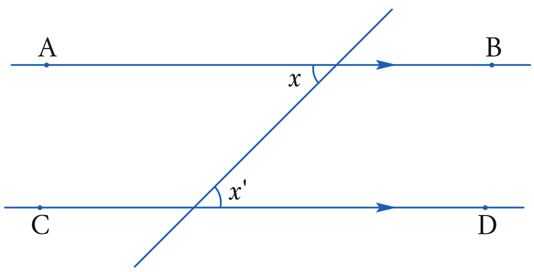

Figure 4. We will prove that if $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ then $\angle x = \angle x'$.

Proof :
Suppose $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and $\angle x \neq \angle x'$. One of the angles is larger; suppose it is $\angle x$ that is larger. Draw $\overleftrightarrow{EF}$ through P so that $\angle w = \angle x'$ as in Figure 5. $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{CD}$ because we have just proven (Theorem 1, second part) that two lines are parallel if their alternate interior angles are equal.

This contradicts the parallel postulate (*Section 1.4*) which states that through a point not on a given line (here point P and line $\overleftrightarrow{CD}$) one and only one line can be drawn parallel to the given line. Therefore ∠x must be equal to ∠x′ . This completes the proof of **Theorem 1**.

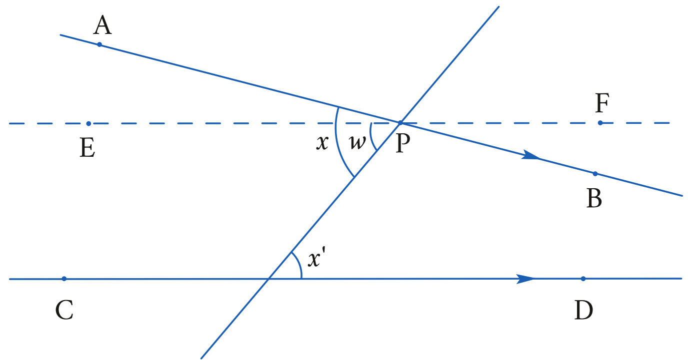

Figure 4. Draw $\overleftrightarrow{EF}$ so that ∠w = ∠x′.

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VALUES OF THE TRIGONOMETRIC FUNCTIONS

Angle	Sine	Cosine	Tangent
1°	0.0175	0.9998	0.0175
2°	0.0349	0.9994	0.0349
3°	0.0523	0.9986	0.0524
4°	0.0698	0.9976	0.0699
5°	0.0872	0.9962	0.0875
6°	0.1045	0.9945	0.1051
7°	0.1219	0.9925	0.1228
8°	0.1392	0.9903	0.1405
9°	0.1564	0.9877	0.1584
10°	0.1736	0.9848	0.1763
11°	0.1908	0.9816	0.1944
12°	0.2079	0.9781	0.2126
13°	0.2250	0.9744	0.2309
14°	0.2419	0.9703	0.2493
15°	0.2588	0.9659	0.2679
16°	0.2756	0.9613	0.2867
17°	0.2924	0.9563	0.3057
18°	0.3090	0.9511	0.3249
19°	0.3256	0.9455	0.3443
20°	0.3420	0.9397	0.3640
21°	0.3584	0.9336	0.3839
22°	0.3746	0.9272	0.4040
23°	0.3907	0.9205	0.4245
24°	0.4067	0.9135	0.4452
25°	0.4226	0.9063	0.4663
26°	0.4384	0.8988	0.4877
27°	0.4540	0.8910	0.5095
28°	0.4695	0.8829	0.5317
29°	0.4848	0.8746	0.5543
30°	0.5000	0.8660	0.5774
31°	0.5150	0.8572	0.6009
32°	0.5299	0.8480	0.6249
33°	0.5446	0.8387	0.6494
34°	0.5592	0.8290	0.6745
35°	0.5736	0.8192	0.7002
36°	0.5878	0.8090	0.7265
37°	0.6018	0.7986	0.7536
38°	0.6157	0.7880	0.7813
39°	0.6293	0.7771	0.8098
40°	0.6428	0.7660	0.8391
41°	0.6561	0.7547	0.8693
42°	0.6691	0.7431	0.9004
43°	0.6820	0.7314	0.9325
44°	0.6947	0.7193	0.9657
45°	0.7071	0.7071	1.0000
46°	0.7193	0.6947	1.0355
47°	0.7314	0.6820	1.0724
48°	0.7431	0.6691	1.1106
49°	0.7547	0.6561	1.1504
50°	0.7660	0.6428	1.1918
51°	0.7771	0.6293	1.2349
52°	0.7880	0.6157	1.2799
53°	0.7986	0.6018	1.3270
54°	0.8090	0.5878	1.3764
55°	0.8192	0.5736	1.4281
56°	0.8290	0.5592	1.4826
57°	0.8387	0.5446	1.5399
58°	0.8480	0.5299	1.6003
59°	0.8572	0.5150	1.6643
60°	0.8660	0.5000	1.7321
61°	0.8746	0.4848	1.8040
62°	0.8829	0.4695	1.8807
63°	0.8910	0.4540	1.9626
64°	0.8988	0.4384	2.0503
65°	0.9063	0.4226	2.1445
66°	0.9135	0.4067	2.2460
67°	0.9205	0.3907	2.3559
68°	0.9272	0.3746	2.4751
69°	0.9336	0.3584	2.6051
70°	0.9397	0.3420	2.7475
71°	0.9455	0.3256	2.9042
72°	0.9511	0.3090	3.0777
73°	0.9563	0.2924	3.2709
74°	0.9613	0.2756	3.4874
75°	0.9659	0.2588	3.7321
76°	0.9703	0.2419	4.0108
77°	0.9744	0.2250	4.3315
78°	0.9781	0.2079	4.7046
79°	0.9816	0.1908	5.1446
80°	0.9848	0.1736	5.6713
81°	0.9877	0.1564	6.3138
82°	0.9903	0.1392	7.1154
83°	0.9925	0.1219	8.1443
84°	0.9945	0.1045	9.5144
85°	0.9962	0.0872	11.4301
86°	0.9976	0.0698	14.3007
87°	0.9986	0.0523	19.0811
88°	0.9994	0.0349	28.6363
89°	0.9998	0.0175	57.2900
90°	1.0000	0.0000	

ANSWERS TO ODD NUMBERED PROBLEMS

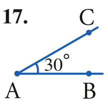
CHAPTER 1

SECTION 1.1

1. 6
3. $x = 9$, $AC = 24$.
5. 15
7. 3

SECTION 1.2

1. $\angle CBD$ or $\angle DBC$
3. $\angle AED$ or $\angle DEA$
5. $\angle ABC$ or $\angle CBA$
7. $70°$
9. $x = 130°$, $y = 50°$
11. $x = 30°$, $y = 60°$
13. $\angle A = 60°$, $\angle B = 50°$, $\angle C = 70°$
15. $\angle A = 110°$, $\angle B = 80°$, $\angle C = 70°$, $\angle D = 100°$

17.


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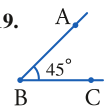

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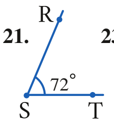
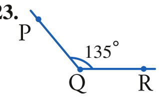


23.


25. $35°$
27. $30°$

SECTION 1.3

1. (a) $53°$ (b) $45°$ (c) $37°$ (d) $30°$
3. $15°$
5. $30°$
7. (a) $150°$ (b) $143°$ (c) $90°$ (d) $60°$
9. $30°$
11. $x=3, -3$
13. 10
15. $x = 70, y = 110, z = 70$
17. $x = 30, y = 45, z = 105$
19. $x = y = z = 90$
21. $x = 40, y = 80, z = 100$
23. 8, -8
25. 4, -5
27. $45°$

SECTION 1.4

1. $x = 50, y = z = 130$
3. $u = x = z = 120, t = v = w = y = 60$
5. 55
7. 50
9. 50
11. 55
13. 60
15. 37
17. 11
19. alternate interior:
$\angle ABD$ & $\angle CDB$ – $AB \parallel CD$; $\angle ADB$ & $\angle CBD$ – $AD \parallel BC$
21. corresponding:
$\angle BAC$ & $\angle EDC$ – $AB \parallel DE$; $\angle ABC$ & $\angle DEC$ – $AB \parallel DE$
23. interior on same side of transversal:
$\angle BAD$ & $\angle CDA$ – $AB \parallel CD$; $\angle ABC$ & $\angle DCB$ – $AB \parallel CD$
25. alternate interior:
$\angle BAC$ & $\angle DCA$ – $AB \parallel DE$; $\angle ABC$ & $\angle ECB$ – $AB \parallel DE$
27. $65°$

SECTION 1.5

1. $85°$
3. $37°$
5. $60°$
7. 30
9. 6
11. 120
13. $x = 50, y = 40, z = 50$
15. $65°$
17. 8
19. 24
21. $720°$
23. $60°$
25. $108°$

SECTION 1.6

1. 2/3
3. 6
5. $x = 1$, $AB = 2$
7. $x = 9$, $\angle ACB = 90°$
9. $\dfrac{25x + 11}{6}$, $\dfrac{37}{2}$
11. 5

CHAPTER 2

SECTION 2.1

1. AB = DE, BC = EF, AC = DF, ∠A = ∠D, ∠B = ∠E, ∠C = ∠F, x = 5, y = 6

3. AB = CD, BC = DA, AC = CA, ∠BAC = ∠DCA, ∠B = ∠D, ∠BCA = ∠DAC, x = 55, y = 35

5. △PQR ≅ △STU

7. △ABC ≅ △ABD

9. △ABD ≅ △CDB

SECTION 2.2

1. BC = 1.7, ∠B = 30°, ∠C = 90°

3. BC = 1.95, ∠B = 99°, ∠C = 41°

5. ∠B

7. ∠D

9. (1) AC, ∠A, AB of △ABC = DF, ∠D, DE of △DEF
(2) △ABC ≅ △DEF
(3) x = 65, y = 45

11. (1) AB, ∠B, BC of △ABC = EF, ∠F, FD of △EFD
(2) △ABC ≅ △EFD
(3) x = 40, y = 50

13. (1) AB, ∠B, BC of △ABC = ED, ∠D, DF of △EDF
(2) △ABC ≅ △EDF
(3) x = 8

15. (1) AB, ∠B, BC of △ABC = ED, ∠D, DF of △EDF
(2) △ABC ≅ △EDF
(3) x = 20, y = 30

17. (1) BA, ∠A, AC of △ABC = DC, ∠C, CA of△CDA
(2) △ABC ≅ △CDA
(3) x = 22

19. (1) AC, ∠ACD, CD of △ACD = BC, ∠BCD, CD of △BCD
(2) △ACD ≅ △BCD
(3) x = 50

21. (1) AD, ∠ADC, DC of △ACD = BD, ∠BDC, DC of △BCD
(2) △ACD ≅ △BCD
(3) x=2

23. (1) BC, ∠BCA, CA of △ABC = DC, ∠DCE, CE of △EDC
(2) △ABC ≅ △EDC
(3) x = 20, y = 10

25. (1) AC, ∠ACB, CB of △ABC = EC, ∠ECD, CD of △EDC
(2) △ABC ≅ △EDC
(3) x = 70

SECTION 2.3

1. BC = 1.9, AC = 2.3, ∠C= 90°

3. BC = 2.3, AC = 1.9, ∠C=90°

5. AB

7. DF

9. (1) △ABC ≅ △DEF
(2) ASA = ASA: ∠A, AB, ∠B of △ABC = ∠D, DE, ∠E of △DEF
(3) x = 5, y = 6

11. (1) △RST ≅ △UWV
(2) AAS = AAS: ∠T, ∠R, RS of △RST = ∠V, ∠U, UW of △UWV
(3) x=7,y=6

13. (1) △ABD≅△CDB
(2) ASA = ASA: ∠B, BD, ∠D of △ABD = ∠D, DB, ∠B of △CDB
(3) x = 30, y = 25

15. (1) △ABC ≅ △EDC
(2) ASA = ASA: ∠A, AC, ∠ACB of △ABC = ∠E, EC, ∠ECD of △EDC
(3) x = 11, y = 9

17. (1) △ACD≅△BCD
(2) AAS = AAS: ∠A, ∠ACD, CD of △ACD = ∠B, ∠BCD, CD of △BCD
(3) x = 5, y = 5

19. (1) △ABC ≅ △EDC
(2) ASA = ASA: ∠B, BC, ∠BCA of △ABC = ∠D, DC, ∠DCE of △EDC
(3) x = 2, y = 3

21. (1) △ABC ≅ △EDF
(2) ASA = ASA: ∠B, BC, ∠C of △ABC = ∠D, DF, ∠F of △EDF
(3) x = 2, y = 3

23. △PTB ≅ △STB, ASA = ASA: ∠PTB, TB, ∠TBP of △PTB = ∠STB, TB, ∠TBS of △STB.
SB = PB = 5, SP = SB + BP = 5 + 5 = 10.

25. △DEC ≅ △BAC, ASA = ASA: ∠E, EC, ∠ECD of △DEC = ∠A, AC, ∠ACB of △BAC. AB = ED = 7.

SECTION 2.4

1. ∠A = ∠D given, AB = DE given, ∠B = ∠E given, △ABC ≅ △DEF. ASA = ASA, AC = DF corresponding sides of ≅ △' s are = .

3. AC = EC given, ∠ACB = ∠ECD vertical ∠ 's, BC = DC given, △ABC ≅ △EDC SAS = SAS, AB = ED corresponding sides of ≅△ 's are =.

5. △ABD = ∠CDB given, BD = DB identity, ∠ADB = CBD given, △ABD ≅ △CDB. ASA = ASA, AB = CD corresponding sides of ≅△'s are =.

7. AC = BC given, ∠ACD = ∠BCD given, CD = CD identity, △ACD ≅ △BCD. SAS = SAS, ∠A = ∠B corresponding ∠ 's of ≅ △ 's are =.

9. ∠BAE = ∠DCE alternate interior ∠ 's of ∥ lines are =, AB = CD given, ∠ABE = ∠CDE alternate interior ∠ 's of ∥ lines are =, △ABE ≅ △CDE ASA = ASA, AE = CE corresponding sides of ≅△ 's are =.

11. $\angle ABC = \angle DCE$ corresponding $\angle$'s of || lines are =, $\angle A = \angle D$ given, AC = DE given, $\triangle ABC \cong \triangle DCE$ AAS = AAS, BC = CE corresponding sides of $\cong\triangle$' s are =.

13. AD = BC given, $\angle BAD = \angle ABC$ given, AB = BA identity, $\triangle ABD \cong \triangle BAC$ SAS = SAS, AC = BD corresponding sides of $\cong\triangle$'s are =.

SECTION 2.5

1. 35
3. 7
5. 45
7. x = 18, $\angle A = \angle B = 52°$, $\angle C = 76°$
9. x = 4, AB = 24, AC = BC = 21
11. x = 1, y = 4, AC = 10
13. 125

SECTION 2.6

1. $\triangle ABC \cong \triangle FDE$, SSS = SSS: AB, BC, AC of $\triangle ABC$ = FD, DE, FE of $\triangle FDE$, x = 30, y = 70, z = 80
3. $\triangle ABD \cong \triangle CDB$, SSS = SSS: AB, BD, AD of $\triangle ABD$ = CD, DB, CB of $\triangle CBD$, x = 70, y = 50, z = 60
5. $\triangle ABC \cong \triangle EDC$, SAS = SAS: AC, $\angle ACB$, CB of $\triangle ABC$ = EC, $\angle ECD$, CD of $\triangle EDC$, x = 8, y = 60, z = 56
7. $\triangle ABC \cong \triangle ADC$, ASA = ASA: $\angle BAC$, AC, $\angle ACB$ of $\triangle ABC = \angle DAC$, AC, $\angle ACD$ of $\triangle ADC$, x = 3, y = 4
9. AB = DE, BC = EF, AC = DF given, $\triangle ABC \cong \triangle DEF$. SSS = SSS, $\angle A = \angle D$ corresponding $\angle$'s of $\cong\triangle$'s are =.
11. AB = AD, BC = DC given, AC = AC identity, $\triangle ABC \cong \triangle ADC$. SSS = SSS, $\angle BAC = \angle CAD$ corresponding $\angle$'s of $\cong\triangle$'s are =.
13. AE = CE given, $\angle AEB = \angle CED$ vertical $\angle$'s are =, EB = ED given, $\triangle AEB \cong \triangle CED$ SAS = SAS, AB = CD corresponding sides of $\cong\triangle$'s are =.

SECTION 2.7

1. (1) $\triangle ABC \cong \triangle DEF$
(2) Hyp - Leg = Hyp-Leg:AB, BC of $\triangle ABC$ = DE, EF of $\triangle DEF$
(3) x = 42, y = 48
3. Triangles cannot be proven congruent.
5. Triangles cannot be proven congruent.
7. (1) $\triangle ABC \cong \triangle CDA$
(2) AAS = AAS: $\angle B$, $\angle BCA$, CA of $\triangle ABC = \angle D, \angle DAC$,AC of $\triangle CDA$
(3) x = 25, y = 20
9. (1) $\triangle ACD \cong \triangle BCD$
(2) SAS = SAS : AD, $\angle ADC$, DC of $\triangle ACD$ = BD, $\angle BDC$, DC of $\triangle BCD$
(3) x = 4
11. Triangles cannot be proven congruent.
13. Triangles cannot be proven congruent.
15. Triangles cannot be proven congruent.
17. OP = OP identity, OA = OB given, $\triangle OAP \cong \triangle OBP$ Hyp-Leg = Hyp-Leg, AP = BP corresponding sides of $\cong\triangle$'s are =.
19. AB = CD, AD = CB given, BD = DB identity, $\triangle ABD \cong \triangle CDB$, SSS = SSS, $\angle A = \angle C$ corresponding $\angle$'s of $\cong\triangle$'s are =.
21. AD = BD given, $\angle ADC = \angle BDC = 90°$ given $AB \perp CD$, CD = CD identity, $\triangle ACD \cong \triangle BCD$. SAS = SAS, $\angle A = \angle B$ corresponding $\angle$'s of $\cong\triangle$'s are =.

CHAPTER 3

SECTION 3.1

1. w = 40, y = 140, r = 4, s = 8
3. w = 35, x = 25, y = 120, z = 35
5. x = 130, y = 50, z = 130
7. x = 70, $\angle A = 70°$, $\angle B = 110°$, $\angle C = 70°$, $\angle D = 110°$
9. x = 25, y = 20, AC = 40, BD = 50
11. x = 2, AB = CD = 4 or x = 3, AB = CD = 9
13. x = 4, y = 1, AB = CD = 7, AD = BC = 3
15. x = 4, y = 2, AC = 16, BD = 12
17. x = 20, y = 10, $\angle A = 40°$, $\angle B = 140°$, $\angle C = 40$◦, $\angle D = 140°$

SECTION 3.2

1. w = 50, x = 40, y = 50, z = 50
3. x = 30, y = 60
5. x = 4, y = 4, z = 4, AC = 8, BD = 8
7. x = 40, y = 40, z = 100
9. 1
11. x = y = z = 45
13. x = 60, y = z = 120
15. x = 135, y = 100
17. w = x = 50, y = 130, z = 50
19. 5.

CHAPTER 4

SECTION 4.1

1. 1
3. 12
5. 21
7. 20
9. 6
11. 1 or 6

SECTION 4.2

1. $\triangle ABC \sim \triangle FED$
3. $\triangle ABC \sim \triangle DFE$
5. $\triangle ABC \sim \triangle DBE$
7. 6
9. 7
11. 1
13. 5
15. 6
17. $x = 6$, $y = 1.5$
19. 15
21. $x = 4.5$, $y = 1.5$, $z = 15$
23. 100 feet

SECTION 4.3

1. 5
3. 1.5
5. 4

SECTION 4.4

1. 10
3. 8
5. $\sqrt{2}$
7. $\sqrt{3}$
9. $3\sqrt{2}$
11. $x = 6$, $BC = 6$, $AC = 8$, $AB = 10$
13. $x = 17$, $PR = 8$, $QR = 15$, $PQ = 17$
15. $2\sqrt{2}$
17. $x = 3$, $AB = 16$
19. $x = 7$, $AC = 30$, $BD = 16$
21. $x = 8$, $y = 6$
23. $x = 5$, $AB = 12$, $BD = 13$
25. yes
27. no
29. no
31. 24 feet
33. no

SECTION 4.5

1. $x = 3\sqrt{3}$, $y = 6$
3. $x = 5$, $y = 5\sqrt{3}$
5. $x = \sqrt{3}$, $y = 2\sqrt{3}$
7. $x = 3$, $y = 3\sqrt{2}$
9. $x = y = 5\sqrt{2}$
11. $10\sqrt{2}$
13. $3\sqrt{2}$
15. $x = 8$, $y = 4\sqrt{3}$
17. $x = y = (5\sqrt{2})/2$
19. $x = 3$, $y = 3\sqrt{3}$
21. $x = 5\sqrt{3}$, $AB = 20$
23. $x = 3$, $y = 6$
25. $AC = 8$, $BD = 8\sqrt{3}$

SECTION 4.6

1. 4
3. $\sqrt{3}$
5. $2\sqrt{2}$

CHAPTER 5

SECTION 5.1

1. $\frac{12}{13}, \frac{5}{13}, \frac{12}{5}, \frac{5}{13}, \frac{12}{13}, \frac{5}{12}$

3. $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}, \frac{15}{17}, \frac{8}{17}, \frac{15}{8}$

5. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$

7. $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1$

9. $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}$

11. $\frac{2}{3}, \frac{\sqrt{5}}{3}, \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{3}, \frac{2}{3}, \frac{\sqrt{5}}{2}$

13. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$

15. $\frac{3}{5}, \frac{4}{3}$

17. $\frac{1}{2}, \frac{\sqrt{3}}{3}$

19. $\frac{3\sqrt{10}}{10}, \frac{\sqrt{10}}{10}$

SECTION 5.2
1. 0.1736
3. 0.1736
5. 1.0000
7. 0.3090
9. 1.1918
11. 6.4
13. 7.7
15. 11.9
17. 8.4
19. 44.8
21. 7.8
23. 20.5
25. 14.5
27. 7.3
29. 4.8
31. 42°
33. 37°
35. 56°
37. 48°
39. 4.6
41. $x = 4.6$, $y = 7.7$
43. 7.8
45. $x = 8.2$, $y = 26.5$

SECTION 5.3
1. 50.3 feet
3. 5759 feet
5. 1°
7. 18.8 feet

CHAPTER 6

SECTION 6.1
1. $A = 12$, $P = 16$
3. $A = 49$, $P = 28$
5. $A = 3$, $P = 4\sqrt{5}$
7. $A = 120$, $P = 46$
9. $A = 48$, $P = 28$
11. $A = 25\sqrt{3}$, $P = 10 + 10\sqrt{3}$
13. $A = 50$, $P = 20\sqrt{2}$
15. 4
17. 4
19. 48000 square feet
21. 296
23. 450
25. 1800 pounds

SECTION 6.2
1. $A = 240$, $P = 66$
3. $A = 36$, $P = 28$
5. $A = 96.4$, $P = 50$
7. $A = 10$, $P = 10 + 4\sqrt{2}$
9. 7
11. 4
13. $x = 8$, $y = 5$

SECTION 6.3
1. 60
3. 10
5. 11.5
7. $A = 6$, $P = 12$
9. $A = 108$, $P = 54$
11. $A = 44$, $P = 28 + 4\sqrt{5}$
13. $A = 60$, $P = 40$
15. $A = 2$, $P = 4 + 2\sqrt{2}$
17. $A = 16\sqrt{3}$, $P = 24$
19. $A = 42.0$, $P = 31.4$
21. 5
23. 4

SECTION 6.4
1. 42
3. $A = 96$, $P = 40$
5. $A = 24$, $P = 20$
7. $A = 32\sqrt{3}$, $P = 32$
9. 167.8

SECTION 6.5
1. 40
3. $A = 36$, $P = 28$
5. $A = 32$, $P = 21 + \sqrt{17}$
7. $A = 44$, $P = 32$
9. $A = 50 + 25\sqrt{3}$, $P = 40 + 10\sqrt{3}$
11. $A = 375\sqrt{3}$, $P = 95 + 5\sqrt{21}$
13. $A = 269.2$, $P = 84.9$
15. 7

CHAPTER 7

SECTION 7.1

1. $x = y = z = 60$, $r = 3$
3. $x = 72$, $y = z = 54$, $r = 7$
5. $x = 90$, $y = z = 45$, $r = 5$
7. $a = 27.5$, $P = 200$, $A = 2752.8$
9. $a = 17.3$, $P = 120$, $A = 1039.2$
11. $a = 30.8$, $P = 200$, $A = 3077.7$
13. $a = 16.2$, $P = 117.6$, $A = 951.1$
15. $a = 8.7$, $P = 60$, $A = 259.8$
17. $a = 9.5$, $P = 61.8$, $A = 293.9$

SECTION 7.2

1. $r = 20$, $d = 40$
3. 30
5. 6
7. $r = 15$, $d = 30$
9. $r = 10$, $d = 20$
11. $AB = 12$, $CD = 16$

SECTION 7.3

1. 15
3. 80
5. $x = 40$, $\angle O = 125°$, $\angle P = 55°$
7. $x = 25$, $y = 24$
9. 10
11. 7
13. 36
15. 40

SECTION 7.4

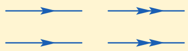
1. $\overset{\frown}{AB} \overset{\circ}{=} 60$, $\overset{\frown}{ACB} \overset{\circ}{=} 300$
3. $\overset{\frown}{AB} \overset{\circ}{=} 80°$, $\overset{\frown}{ACB} \overset{\circ}{=} 280°$
5. $x = 80$, $y = 70$, $z = 90$
7. $x = 60$, $y = 60$, $z = 60$
9. $x = 35$, $y = 70$, $z = 70$
11. 130
13. 50
15. 60
17. 70
19. 50
21. 90
23. 12
25. 40
27. $x = 50$, $y = z = 25$
29. 70
31. $x = 45$, $y = 45$, $z = 90$
33. 35
35. $x = 0.70$, $y = 40$, $z = 30$
37. 80
39. $x = 45$, $y = 15$, $z = 60$
41. $x = 30$, $y = 50$, $z = 80$
43. 30
45. 70

SECTION 7.5

1. 31.4
3. 62.8
5. 12.56, 25.12
7. 40.82
9. 8.37
11. 52.3
13. 6.28
15. $\overset{\frown}{AB} = 3.925$, $\overset{\frown}{CD} = 7.85$
17. 39.1
19. 94.2
21. 62.8
23. $r = 50$, $d = 100$
25. 43.96 inches
27. 7907.6 miles

SECTION 7.6

1. 3.14
3. 12.56
5. 78.5
7. 1256
9. 113.04
11. 157
13. 62.8
15. $(200\pi/3) - 100\sqrt{3}$
17. $25\pi - 50$
19. $100\pi - 200$
21. $100 - 25\pi$
23. $200 + 25\pi$
25. 21π
27. 50π
29. $100 - 25\pi$

A, B,	points A, B, 5	$\perp$	perpendicular, 20
$\overleftrightarrow{AB}$	line AB, 5	$\parallel$	parallel, 30
AB	line segment AB, 5		parallel lines, 30
$\overrightarrow{AB}$	ray AB, 5	$\triangle$	triangle, 45
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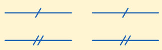	equal line segments, 5	$\sim$	similar, 141
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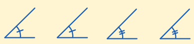	equal angles, 14	$\overset{\circ}{=}$	equal in degrees, 265
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