Intermediate Algebra and Trigonometry

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BCcampus

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[1] "Open Educational Resources," *Hewlett Foundation*, https://hewlett.org/strategy/open-educational-resources/ (accessed September 27, 2018).

Review of Operations on the Set of Real Numbers

Before we start our journey through algebra, let us review the structure of the real number system, properties of four operations, order of operations, the concept of absolute value, and set-builder and interval notation.

R1 Structure of the Set of Real Numbers

It is in human nature to group and classify objects with the same properties. For instance, items found in one's home can be classified as furniture, clothing, appliances, dinnerware, books, lighting, art pieces, plants, etc., depending on what each item is used for, what it is made of, how it works, etc. Furthermore, each of these groups could be subdivided into more specific categories (groups). For example, furniture includes tables, chairs, bookshelves, desks, etc. Sometimes an item can belong to more than one group. For example, a piece of furniture can also be a piece of art. Sometimes the groups do not have any common items (e.g. plants and appliances). Similarly to everyday life, we like to classify numbers with respect to their properties. For example, even or odd numbers, prime or composite numbers, common fractions, finite or infinite decimals, infinite repeating decimals, negative numbers, etc. In this section, we will take a closer look at commonly used groups (sets) of real numbers and the relations between those groups.

Set Notation and Frequently Used Sets of Numbers

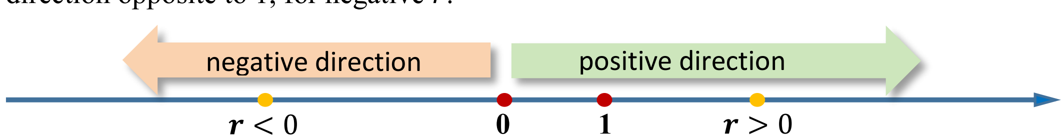
We start with terminology and notation related to sets.

Definition 1.1 ► A **set** is a collection of objects, called **elements** (or **members**) of this set.

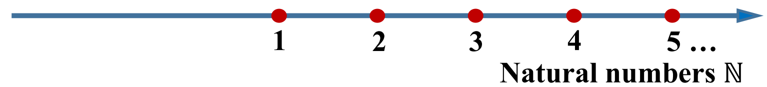
Roster Notation: A set can be given by listing its elements within the **set brackets** { } (braces).
The elements of the set are separated by commas.
To indicate that a pattern continues, we use three dots … .
Examples:
If set A consists of the numbers 1, 2, and 3, we write $A = \{1,2,3\}$.
If set B consists of all consecutive numbers, starting from 5, we write $B = \{5,6,7,8, \dots\}$.

More on Notation: To indicate that the number 2 **is an element** of set A, we write $\mathbf{2 \in A}$.
To indicate that the number 2 **is <u>not</u> an element** of set B, we write $\mathbf{2 \notin A}$.
A set with no elements, called **empty** set, is denoted by the symbol $\emptyset$ or { }.

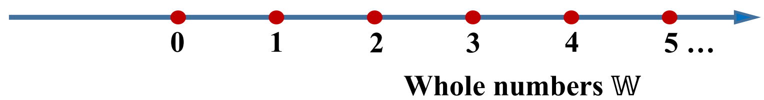
In this course we will be working with the set of **real numbers**, denoted by $\mathbb{R}$. To visualise this set, we construct a line and choose two distinct points on it, 0 and 1, to establish direction and scale. This makes it a **number line**. Each real number r can be identified with exactly one point on such a number line by choosing the endpoint of the segment of length $|r|$ that starts from 0 and follows the line in the direction of 1, for positive r, or in the direction opposite to 1, for negative r.



The set of real numbers contains several important subgroups (**subsets**) of numbers. The very first set of numbers that we began our mathematics education with is the set of counting numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, called **natural numbers** and denoted by $\mathbb{N}$.

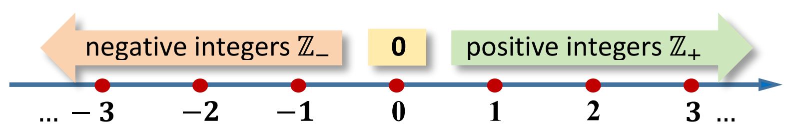


The set of natural numbers together with the number **0** creates the set of **whole numbers** $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by $\mathbb{W}$.



Notice that if we perform addition or multiplication of numbers from either of the above sets, $\mathbb{N}$ and $\mathbb{W}$, the result will still be an element of the same set. We say that the set of **natural numbers** $\mathbb{N}$ and the set of **whole numbers** $\mathbb{W}$ are both **closed** under **addition** and **multiplication**.

However, if we wish to perform subtraction of natural or whole numbers, the result may become a negative number. For example, $2 - 5 = -3 \notin \mathbb{W}$, so neither the set of whole numbers nor natural numbers is <u>not</u> closed under subtraction. To be able to perform subtraction within the same set, it is convenient to extend the set of whole numbers to include negative counting numbers. This creates the set of **integers** $\{\dots, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$, denoted by $\mathbb{Z}$.

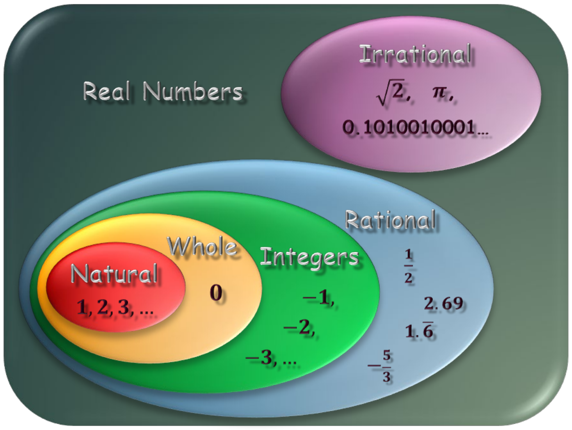


Integers $\mathbb{Z}$

Alternatively, the set of integers can be recorded using the $\pm$ sign: $\{\mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \dots\}$. The $\pm$ sign represents two numbers at once, the positive and the negative.

So the set of **integers** $\mathbb{Z}$ is **closed** under **addition**, **subtraction** and **multiplication**. What about division? To create a set that would be closed under division, we extend the set of integers by including all quotients of integers (all common fractions). This new set is called the set of **rational numbers** and denoted by $\mathbb{Q}$. Here are some examples of rational numbers: $\frac{3}{1} = \mathbf{3}$, $\frac{1}{2} = \mathbf{0.5}$, $-\frac{7}{4}$, or $\frac{4}{3} = \mathbf{1.\overline{3}}$.

Thus, the set of **rational numbers** $\mathbb{Q}$ is **closed** under **all four operations**. It is quite difficult to visualize this set on the number line as its elements are nearly everywhere. Between any two rational numbers, one can always find another rational number, simply by taking an average of the two. However, all the points corresponding to rational numbers still do not fulfill the whole number line. Actually, the number line contains a lot more unassigned points than points that are assigned to rational numbers. The remaining points correspond to numbers called **irrational** and denoted by $\mathbb{IQ}$. Here are some examples of irrational numbers: $\sqrt{\mathbf{2}}, \boldsymbol{\pi}, \boldsymbol{e}$, or $\mathbf{0.1010010001}\dots$.



By the definition, the two sets, $\mathbb{Q}$ and $\mathbb{IQ}$ fulfill the entire number line, which represents the set of **real numbers**, $\mathbb{R}$.

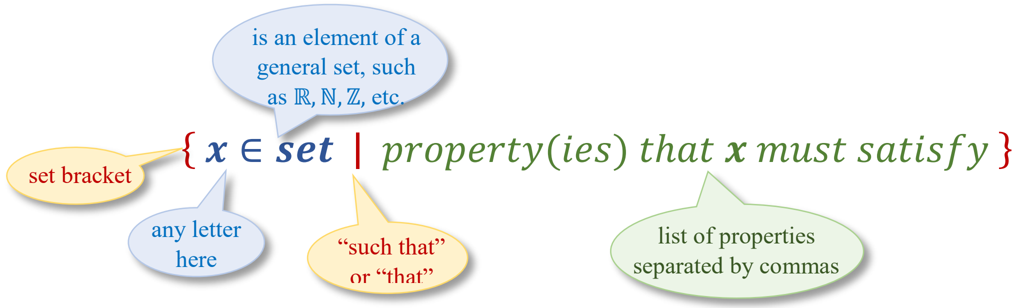
The sets $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{IQ}$, and $\mathbb{R}$ are related to each other as in the accompanying diagram. One can make the following observations:

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where $\subset$ (*read* is a **subset**) represents the operator of **inclusion of sets**;

$\mathbb{Q}$ and $\mathbb{IQ}$ are **disjoint** (they have **no common element**);

$\mathbb{Q}$ together with $\mathbb{IQ}$ create $\mathbb{R}$.

So far, we introduced six double-stroke letter signs to denote the main sets of numbers. However, there are many more sets that one might be interested in describing. Sometimes it is enough to use a subindex with the existing letter-name. For instance, the set of all positive real numbers can be denoted as $\mathbb{R}_+$ while the set of negative integers can be denoted by $\mathbb{Z}_-$. But how would one represent, for example, the set of even or odd numbers or the set of numbers divisible by 3, 4, 5, and so on? To describe numbers with a particular property, we use the **set-builder notation**. Here is the structure of set-builder notation:



For example, to describe the set of even numbers, first, we think of a property that distinguishes even numbers from other integers. This is divisibility by 2. So each even number n can be expressed as $2k$, for some integer k. Therefore, the set of even numbers could be stated as $\{n \in \mathbb{Z} \mid n = 2k, k \in \mathbb{Z}\}$ (*read*: The set of all integers n such that each n is of the form $2k$, for some integral k.)

To describe the set of rational numbers, we use the fact that any rational number can be written as a common fraction. Therefore, the set of rational numbers $\mathbb{Q}$ can be described as $\left\{x \,\middle|\, x = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0\right\}$ (*read*: The set of all real numbers x that can be expressed as a fraction $\frac{p}{q}$, for integral p and q, with $q \neq 0$.)

Convention: *If the description of a set refers to the set of real numbers, there is no need to state $x \in \mathbb{R}$ in the first part of set-builder notation. For example, we can write $\{\boldsymbol{x} \in \mathbb{R} \mid \boldsymbol{x} > \mathbf{0}\}$ or $\{\boldsymbol{x} \mid \boldsymbol{x} > \mathbf{0}\}$. Both sets represent the set of all positive real numbers, which could also be recorded as simply $\mathbb{R}_+$. However, if we work with any other major set, this set must be stated. For example, to describe all positive integers $\mathbb{Z}_+$ using set-builder notation, we write $\{\boldsymbol{x} \in \mathbb{Z} \mid \boldsymbol{x} > \mathbf{0}\}$ and $\mathbb{Z}$ is essential there.*

Example 1 ► **Listing Elements of Sets Given in Set-builder Notation**

List the elements of each set.

a. $\{n \in \mathbb{Z} \mid -2 \le n < 5\}$ **b.** $\{n \in \mathbb{N} \mid n = 5k, k \in \mathbb{N}\}$

Solution ► **a.** This is the set of integers that are at least -2 but smaller than 5. So this is $\{-2, -1, 0, 1, 2, 3, 4\}$.

b. This is the set of natural numbers that are multiples of 5. Therefore, this is the infinite set $\{5, 10, 15, 20, \dots\}$.

Example 2 ► **Writing Sets with the Aid of Set-builder Notation**

Use set-builder notation to describe each set.

a. $\{1, 4, 9, 16, 25, \dots\}$ **b.** $\{-2, 0, 2, 4, 6\}$

Solution ► **a.** First, we observe that the given set is composed of consecutive perfect square numbers, starting from 1. Since all the elements are natural numbers, we can describe this set using the set-builder notation as follows: $\{n \in \mathbb{N} \mid n = k^2, \text{for } k \in \mathbb{N}\}$.

b. This time, the given set is finite and lists all even numbers starting from -2 up to 6. Since the general set we work with is the set of integers, the corresponding set in set-builder notation can be written as $\{n \in \mathbb{Z} \mid n \text{ is even}, -2 \le n \le 6\}$, or $\{n \in \mathbb{Z} \mid -2 \le n \le 6, n = 2k, \text{for } k \in \mathbb{Z}\}$.

Observations:

* *There are many equivalent ways to describe a set using the set-builder notation.*

* *The commas used between the conditions (properties) stated after the "such that" bar play the same role as the connecting word "and".*

Rational Decimals

How can we recognize if a number in decimal notation is rational or irrational?

A **terminating** decimal (with a **finite** number of nonzero digits after the decimal dot, like 1.25 or 0.1206) can be converted to a common fraction by replacing the decimal dot with the division by the corresponding power of 10 and then simplifying the resulting fraction. For example,

$$1.25 = \frac{125}{100} = \frac{5}{4}, \quad \text{or} \quad 0.1206 = \frac{1206}{10000} = \frac{603}{5000}.$$

2 places → 2 zeros   simplify by 25   4 places → 4 zeros   simplify by 2

Therefore, any **terminating decimal is a rational number**.

One can also convert a **nonterminating** (**infinite**) decimal to a common fraction, as long as there is a **recurring** (**repeating**) sequence of digits in the decimal expansion. This can be done using the method shown in *Example 3a.* Hence, any **infinite repeating decimal is a rational number**.

Also, notice that any fraction $\frac{m}{n}$ can be converted to either a finite or infinite repeating decimal. This is because since there are only finitely many numbers occurring as remainders in the long division process when dividing by n, eventually, either a remainder becomes zero, or the sequence of remainders starts repeating.

So **a number is rational if and only if it can be represented by a finite or infinite repeating decimal**. Since the irrational numbers are defined as those that are not rational, we can conclude that **a number is irrational if and only if it can be represented as an infinite non-repeating decimal**.

Example 3 ► **Proving that an Infinite Repeating Decimal is a Rational Number**

Show that the given decimal is a rational number.

a. $0.333\ldots$ **b.** $2.3\overline{45}$

Solution ► **a.** Let $a = 0.333\ldots$. After multiplying this equation by 10, we obtain $10a = 3.333\ldots$. Since in both equations, the number after the decimal dot is exactly the same, after subtracting the equations side by side, we obtain

$$-\begin{cases} 10a = 3.\cancel{333\ldots} \\ \ \ a = 0.\cancel{333\ldots} \end{cases}$$
$$9a = 3$$

which solves to $a = \frac{3}{9} = \frac{1}{3}$. So $0.333\ldots = \frac{1}{3}$ is a rational number.

b. Let $a = 2.3\overline{45}$. The bar above 45 tells us that the sequence 45 repeats forever. To use the subtraction method as in solution to *Example 3a*, we need to create two equations involving the given number with the decimal dot moved after the repeating sequence and before the repeating sequence. This can be obtained by multiplying the equation $a = 2.3\overline{45}$ first by 1000 and then by 10, as below.

$$-\begin{cases} 1000a = 2345.\cancel{\overline{45}} \\ \ \ 10a = 23.\cancel{\overline{45}} \end{cases}$$
$$990a = 2322$$

Therefore, $a = \frac{2322}{990} = \frac{129}{55} = 2\frac{19}{55}$, which proves that $2.3\overline{45}$ is rational.

Example 4 ► **Identifying the Main Types of Numbers**

List all numbers of the set

$\left\{-10,\ -5.34,\ 0,\ 1,\ \frac{12}{3},\ 3.\overline{16},\ \frac{4}{7},\ \sqrt{2},\ -\sqrt{36},\ \sqrt{-4},\ \pi,\ 9.010010001\ldots\right\}$ that are

a. natural **b.** whole **c.** integral **d.** rational **e.** irrational

Solution

a. The only natural numbers in the given set are 1 and $\frac{12}{3} = 4$.

b. The whole numbers include the natural numbers and the number 0, so we list 0, 1 and $\frac{12}{3}$.

c. The integral numbers in the given set include the previously listed $0, 1, \frac{12}{3}$, and the negative integers -10 and $-\sqrt{36} = -6$.

d. The rational numbers in the given set include the previously listed integers $0, 1, \frac{12}{3}, -10, -\sqrt{36}$, the common fraction $\frac{4}{7}$, and the decimals -5.34 and $3.\overline{16}$.

e. The only irrational numbers in the given set are the constant π and the infinite decimal $9.010010001\ldots$.

Note: $\sqrt{-4}$ is not a real number.

R.1 Exercises

True or False? If it is false, explain why.

1. Every natural number is an integer.

2. Some rational numbers are irrational.

3. Some real numbers are integers.

4. Every integer is a rational number.

5. Every infinite decimal is irrational.

6. Every square root of an odd number is irrational.

*Use **roster notation** to list all elements of each set.*

7. The set of all positive integers less than 9

8. The set of all odd whole numbers less than 11

9. The set of all even natural numbers

10. The set of all negative integers greater than -5

11. The set of natural numbers between 3 and 9

12. The set of whole numbers divisible by 4

*Use **set-builder notation** to describe each set.*

13. $\{0, 1, 2, 3, 4, 5\}$

14. $\{4, 6, 8, 10, 12, 14\}$

15. The set of all real numbers greater than -3

16. The set of all real numbers less than 21

17. The set of all multiples of 3

18. The set of perfect square numbers up to 100

Fill in each box with one of the signs $\in, \notin, \subset, \not\subset$ or $=$ to make the statement true.

19. $-3 \ \square\ \mathbb{Z}$

20. $\{0\} \ \square\ \mathbb{W}$

21. $\mathbb{Q} \ \square\ \mathbb{Z}$

22. $0.3555\ldots \ \square\ \mathbb{IQ}$

23. $\sqrt{3} \ \square\ \mathbb{Q}$

24. $\mathbb{Z}_- \ \square\ \mathbb{Z}$

25. $\pi \square \mathbb{R}$

26. $\mathbb{N} \square \mathbb{Q}$

27. $\mathbb{Z}_+ \square \mathbb{N}$

For the given set, state the subset of ***(a)*** *natural numbers,* ***(b)*** *whole numbers,* ***(c)*** *integers,* ***(d)*** *rational numbers,* ***(e)*** *irrational numbers,* ***(f)*** *real numbers.*

28. $\left\{-1,\ 2.16,\ -\sqrt{25},\ \frac{12}{2}, -\frac{12}{5},\ 3.\overline{25},\ \sqrt{5},\ \pi,\ 3.565665666\ldots\right\}$

29. $\left\{0.999\ldots,\ -5.001,\ 0,\ 5\frac{3}{4},\ 1.4\overline{05},\ \frac{7}{8},\ \sqrt{2},\ \sqrt{16},\ \sqrt{-9},\ 9.010010001\ldots\right\}$

Show that the given decimal is a rational number.

30. $0.555\ldots$

31. $1.\overline{02}$

32. $0.1\overline{34}$

33. $2.0\overline{125}$

34. $0.25\overline{7}$

35. $5.22\overline{54}$

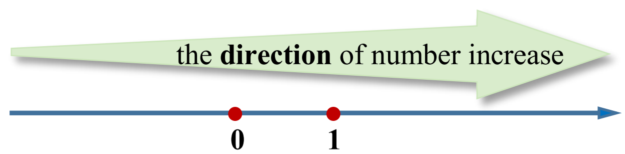
R2 Number Line and Interval Notation

As mentioned in the previous section, it is convenient to visualise the set of real numbers by identifying each number with a unique point on a number line.

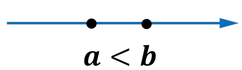
Order on the Number Line and Inequalities

Definition 2.1 ► A **number line** is a line with two distinct points chosen on it. One of these points is designated as **0** and the other point is designated as **1**.

The length of the segment from 0 to 1 represents one **unit** and provides the scale that allows to locate the rest of the numbers on the line. The **direction** from 0 to 1, marked by an **arrow** at the end of the line, indicates the **increasing order** on the number line. The numbers corresponding to the points on the line are called the **coordinates** of the points.



Note: *For simplicity, the <u>coordinates</u> of points on a number line are often identified with the points themselves.*



To compare numbers, we use **inequality signs** such as $<, \leq, >, \geq$, or $\neq$. For example, if a is **smaller than** b we write $a < b$. This tells us that the location of point a on the number line is to the left of point b. Equivalently, we could say that b is **larger than** a and write $b > a$. This means that the location of b is to the right of a.

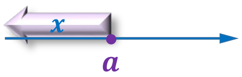
Example 1 ► **Identifying Numbers with Points on a Number Line**

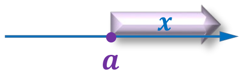
Match the numbers $-2,\ 3.5,\ \pi,\ -1.5,\ \frac{5}{2}$ with the letters on the number line:

Solution ► To match the given numbers with the letters shown on the number line, it is enough to order the numbers from the smallest to the largest. First, observe that negative numbers are smaller than positive numbers and $-2 < -1.5$. Then, observe that $\pi \approx 3.14$ is larger than $\frac{5}{2}$ but smaller than 3.5. Therefore, the numbers are ordered as follows:

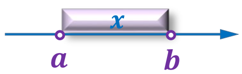
$$-2 < -1.5 < \frac{5}{2} < \pi < 3.5$$

Thus, $\boldsymbol{A = -2, B = -1.5, C = \frac{5}{2}, D = \pi}$, and $\boldsymbol{E = 3.5}$.

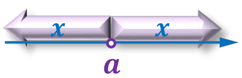




To indicate that a number $\boldsymbol{x}$ is **smaller or equal** $\boldsymbol{a}$, we write $\boldsymbol{x} \leq \boldsymbol{a}$. This tells us that the location of point $\boldsymbol{x}$ on the number line is to the left of point $\boldsymbol{a}$ or exactly at point $\boldsymbol{a}$. Similarly, if $\boldsymbol{x}$ is **larger or equal** $\boldsymbol{a}$, we write $\boldsymbol{x} \geq \boldsymbol{a}$, and we locate $\boldsymbol{x}$ to the right of point $\boldsymbol{a}$ or exactly at point $\boldsymbol{a}$.



To indicate that a number $\boldsymbol{x}$ is **between** $\boldsymbol{a}$ and $\boldsymbol{b}$, we write $\boldsymbol{a} < \boldsymbol{x} < \boldsymbol{b}$. This means that the location of point $\boldsymbol{x}$ on the number line is somewhere on the segment joining points $\boldsymbol{a}$ and $\boldsymbol{b}$, but not at $\boldsymbol{a}$ nor at $\boldsymbol{b}$. Such stream of two inequalities is referred to as a **three-part inequality**.



Finally, to state that a number $\boldsymbol{x}$ is **different than** $\boldsymbol{a}$, we write $\boldsymbol{x} \neq \boldsymbol{a}$. This means that the point $\boldsymbol{x}$ can lie anywhere on the entire number line, except at the point $\boldsymbol{a}$.

Here is a list of some English phrases that indicate the use of particular inequality signs.

English Phrases	Inequality Sign(s)
less than; smaller than	$<$
less or equal; smaller or equal; at most; no more than	$\leq$
more than; larger than; greater than;	$>$
more or equal; larger or equal; greater or equal; at least; no less than	$\geq$
different than	$\neq$
between	$< \quad <$

Example 2 ► **Using Inequality Symbols**

Write each statement as a single or a three-part inequality.

a. -7 is **less than** 5
b. $2x$ is **greater or equal** 6
c. $3x + 1$ is **between** -1 and 7
d. x is **between** 1 and 8, **including** 1 and **excluding** 8
e. $5x - 2$ is **different than** 0
f. x is **negative**

Solution ►

a. Write $-7 < 5$. *Notice:* The inequality “points” to the smaller number. This is an example of a **strong** inequality. One side is “strongly” smaller than the other side.

b. Write $2x \geq 6$. This is an example of a **weak** inequality, as it allows for equation.

c. Enclose $3x+1$ within two strong inequalities to obtain $-1<3x+1<7$. *Notice:* The word "between" indicates that the endpoints are not included.

d. Since 1 is included, the statement is $1 \le x < 8$.

e. Write $5x-2 \neq 0$.

f. Negative x means that x is smaller than zero, so the statement is $x<0$.

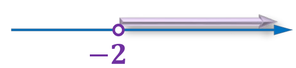
Example 3 ► Graphing Solutions to Inequalities in One Variable

Using a number line, graph all x-values that satisfy (are **solutions** of) the given inequality or inequalities:

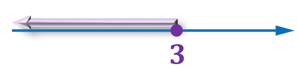
a. $x>-2$ **b.** $x \le 3$ **c.** $1 \le x < 4$

Solution ►

a. The x-values that satisfy the inequality $x>-2$ are larger than -2, so we shade the part of the number line that corresponds to numbers greater than -2. Those are all points to the right of -2, but not including -2. To indicate that the -2 is not a solution to the given inequality, we draw a hollow circle at -2.

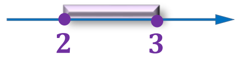


b. The x-values that satisfy the inequality $x \le 3$ are smaller than or equal to 3, so we shade the part of the number line that corresponds to the number 3 or numbers smaller than 3. Those are all points to the right of 3, including the point 3. To indicate that the 3 is a solution to the given inequality, we draw a filled-in circle at 3.



c. The x-values that satisfy the inequalities $1 \le x < 4$ are larger than or equal to 1 and at the same time smaller than 4. Thus, we shade the part of the number line that corresponds to numbers between 1 and 4, including the 1 but excluding the 4. Those are all the points that lie between 1 and 4, including the point 1 but excluding the point 4. So, we draw a segment connecting 1 with 4, with a filled-in circle at 1 and a hollow circle at 4.

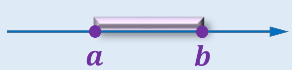
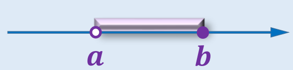
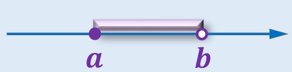
Interval Notation

As shown in the solution to *Example 3*, the graphical solutions of inequalities in one variable result in a segment of a number line (if we extend the definition of a segment to include the endpoint at infinity). To record such a solution segment algebraically, it is convenient to write it by stating its left endpoint (corresponding to the lower number) and then the right endpoint (corresponding to the higher number), using appropriate brackets that would indicate the inclusion or exclusion of the endpoint. For example, to record algebraically the segment that starts from 2 and ends on 3, including both endpoints, we write $[2,3]$. Such notation very closely depicts the graphical representation of the segment,  , and is called **interval notation**.

Interval Notation: A set of numbers satisfying a single inequality of the type $<, \leq, >$, or $\geq$ can be recorded in interval notation, as stated in the table below.

inequality	set-builder notation	graph	interval notation	comments
$x > a$	$\{x \mid x > a\}$		$(\boldsymbol{a}, \infty)$	- list the endvalues from **left to right** - to exclude the endpoint use a **round bracket (** or **)**
$x \geq a$	$\{x \mid x \geq a\}$		$[\boldsymbol{a}, \infty)$	**- infinity** sign is used with a **round bracket**, as there is no last point to include - to include the endpoint use a **square bracket [** or **]**
$x < a$	$\{x \mid x < a\}$		$(-\infty, \boldsymbol{a})$	- to indicate **negative infinity**, use the **negative sign** in front of ∞ - to indicate **positive infinity**, there is **no need** to write **a positive sign** in front of the infinity sign
$x \leq a$	$\{x \mid x \leq a\}$		$(-\infty, \boldsymbol{a}]$	- remember to list the endvalues from **left to right**; this also refers to infinity signs

Similarly, a set of numbers satisfying two inequalities resulting in a segment of solutions can be recorded in interval notation, as stated below.

inequality	set-builder notation	graph	interval notation	comments
$a < x < b$	$\{x \mid a < x < b\}$		$(\boldsymbol{a}, \boldsymbol{b})$	- we read: an **open interval** from a to b
$a \leq x \leq b$	$\{x \mid a \leq x \leq b\}$		$[\boldsymbol{a}, \boldsymbol{b}]$	- we read: a **closed interval** from a to b
$a < x \leq b$	$\{x \mid a < x \leq b\}$		$(\boldsymbol{a}, \boldsymbol{b}]$	- we read: an interval from a to b, without a but with b This is called **half-open** or **half-closed** interval.
$a \leq x < b$	$\{x \mid a \leq x < b\}$		$[\boldsymbol{a}, \boldsymbol{b})$	- we read: an interval from a to b, with a but without b This is called **half-open** or **half-closed** interval.

In addition, the set of all real numbers $\mathbb{R}$ is represented in the interval notation as $(-\infty, \infty)$.

Example 4 ► **Writing Solutions to One Variable Inequalities in Interval Notation**

Write solutions to the inequalities from *Example 3* in set-builder and interval notation.

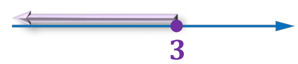
a. $x > -2$ **b.** $x \leq 3$ **c.** $1 \leq x < 4$

Solution ►

a. The solutions to the inequality $x > -2$ can be stated in set-builder notation as $\{\boldsymbol{x} | \boldsymbol{x} > -\mathbf{2}\}$. Reading the graph of this set

from **left to right**, we start from -2, without -2, and go towards infinity. So, the interval of solutions is written as $(-\mathbf{2}, \infty)$. We use the round bracket to indicate that the endpoint is not included. The infinity sign is always written with the round bracket, as infinity is a concept, not a number. So, there is no last number to include.

b. The solutions to the inequality $x \leq 3$ can be stated in set-builder notation as $\{\boldsymbol{x} | \boldsymbol{x} \leq \mathbf{3}\}$. Again, reading the graph of this set



from **left to right**, we start from $-\infty$ and go up to 3, including 3. So, the interval of solutions is written as $(-\infty, \mathbf{3}]$. We use the square bracket to indicate that the endpoint is included. As before, the infinity sign takes the round bracket. Also, we use "$-$" in front of the infinity sign to indicate negative infinity.

c. The solutions to the three-part inequality $1 \leq x < 4$ can be stated in set-builder notation as $\{\boldsymbol{x} | \mathbf{1} \leq \boldsymbol{x} < \mathbf{4}\}$. Reading the graph of this set

from **left to right**, we start from 1, including 1, and go up to 4, excluding 4. So, the interval of solutions is written as $[\mathbf{1}, \mathbf{4})$. We use the square bracket to indicate 1 and the round bracket, to exclude 4.

Absolute Value, and Distance

The **absolute value** of a number x, denoted $|x|$, can be thought of as the distance from x to 0 on a number line. Based on this interpretation, we have $|\boldsymbol{x}| = |-\boldsymbol{x}|$. This is because both numbers x and $-x$ are at the same distance from 0. For example, since both 3 and -3 are exactly three units apart from the number 0, then $|3| = |-3| = 3$.

Since distance can not be negative, we have $|\boldsymbol{x}| \geq \mathbf{0}$.

Here is a formal definition of the absolute value operator.

Definition 2.1 ► For any real number x,
$$|\boldsymbol{x}| \overset{def}{=} \begin{cases} \boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} \geq \mathbf{0} \\ -\boldsymbol{x}, & \boldsymbol{if} \quad \boldsymbol{x} < \mathbf{0} \end{cases}.$$

The above definition of absolute value indicates that for $x \geq 0$ we use the equation $|x| = x$, and for $x < 0$ we use the equation $|x| = -x$ (the absolute value of x is the opposite of x, which is a positive number).

Example 5 ► **Evaluating Absolute Value Expressions**

Evaluate.

a. $-|-4|$ **b.** $|-5| - |2|$ **c.** $|-5-(-2)|$

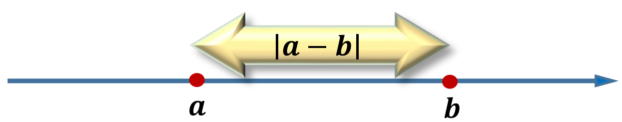
Solution ►

a. Since $|-4| = 4$ then $-|-4| = \mathbf{-4}$.

b. Since $|-5| = 5$ and $|2| = 2$ then $|-5| - |2| = 5 - 2 = \mathbf{3}$.

c. Before applying the absolute value operator, we first simplify the expression inside the absolute value sign. So we have $|-5-(-2)| = |-5+2| = |-3| = \mathbf{3}$.

On a number line, the **distance** between two points with coordinates a and b is calculated by taking the difference between the two coordinates. So, if $b > a$, the distance is $b - a$. However, if $a > b$, the distance is $a - b$. What if we don't know which value is larger, a or b? Since the distance must be positive, we can choose to calculate any of the differences and apply the absolute value on the result.



Definition 2.2 ► The **distance** $d(a, b)$ between points a and b on a number line is given by the expression $|\boldsymbol{a} - \boldsymbol{b}|$, or equivalently $|\boldsymbol{b} - \boldsymbol{a}|$.

Notice that $d(x, 0) = |x - 0| = |x|$, which is consistent with the intuitive definition of absolute value of x as the distance from x to 0 on the number line.

Example 6 ► **Finding Distance Between Two Points on a Number Line**

Find the distance between the two given points on the number line.

a. -3 and 5 **b.** x and 2

Solution ►

a. Using the distance formula for two points on a number line, we have $d(-3,5) = |-3-5| = |-8| = \mathbf{8}$. Notice that we could also calculate $|5-(-3)| = |8| = \mathbf{8}$.

b. Following the formula, we obtain $d(x, 2) = |x - 2|$. Since a is unknown, the distance between a and 2 is stated as an expression $|\boldsymbol{x} - \mathbf{2}|$ rather than a specific number.

R.2 Exercises

*Write each statement with the use of an **inequality** symbol.*

1. -6 is less than -3

2. 0 is more than -1

3. 17 is greater or equal to x

4. x is smaller or equal to 8

5. $2x + 3$ is different than zero

6. $2 - 5x$ is negative

7. x is between 2 and 5

8. $3x$ is between -5 and 7

9. $2x$ is between -2 and 6, including -2 and excluding 6

10. $x + 1$ is between -5 and 11, excluding -5 and including 11

*Graph each set of numbers on a number line and write it in **interval notation**.*

11. $\{x \mid x \geq -4\}$

12. $\{x \mid x \leq -3\}$

13. $\left\{x \middle| x < \frac{5}{2}\right\}$

14. $\left\{x \middle| x > -\frac{2}{5}\right\}$

15. $\{x \mid 0 < x < 6\}$

16. $\{x \mid -1 \leq x \leq 4\}$

17. $\{x \mid -5 \leq x < 16\}$

18. $\{x \mid -12 < x \leq 4.5\}$

Evaluate.

19. $-|-7|$

20. $|5| - |-13|$

21. $|11 - 19|$

22. $|-5 - (-9)|$

23. $-|9| - |-3|$

24. $-|-13 + 7|$

Replace each $\square$ with one of the signs $<, >, \leq, \geq, =$ to make the statement true.

25. $-7 \square -5$

26. $|-16| \square -|16|$

27. $-3 \square -|3|$

28. $x^2 \square 0$

29. $x \square |x|$

30. $|x| \square |-x|$

Find the distance between the given points.

31. $-7, -32$

32. $46, -13$

33. $-\frac{2}{3}, \frac{5}{6}$

34. $x, 0$

35. $5, y$

36. x, y

Find numbers that are 5 units apart from the given point.

37. 0

38. 3

39. a

R3 Properties and Order of Operations on Real Numbers

In algebra, we are often in need of changing an expression to a different but equivalent form. This can be observed when simplifying expressions or solving equations. To change an expression equivalently from one form to another, we use appropriate properties of operations and follow the order of operations.

Properties of Operations on Real Numbers

The four basic operations performed on real numbers are addition $(+)$, subtraction $(-)$, multiplication $(\cdot)$, and division $(\div)$. Here are the main properties of these operations:

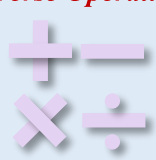
Closure: The result of an operation on real numbers is also a real number. We can say that the **set of real numbers** is **closed** under **addition**, **subtraction** and **multiplication**.

We cannot say this about division, as **division by zero is not allowed**.

Neutral Element: A real number that leaves other real numbers unchanged under a particular operation.

For example, **zero** is the **neutral element** (also called **additive identity**) of **addition**, since $\boldsymbol{a}+\mathbf{0}=\boldsymbol{a}$, and $\mathbf{0}+\boldsymbol{a}=\boldsymbol{a}$, for any real number a.
Similarly, **one** is the **neutral element** (also called **multiplicative identity**) of **multiplication**, since $\boldsymbol{a}\cdot\mathbf{1}=\boldsymbol{a}$, and $\mathbf{1}\cdot\boldsymbol{a}=\boldsymbol{a}$, for any real number a.

Inverse Operations: Operations that reverse the effect of each other. For example, **addition and subtraction** are **inverse operations**, as $\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{b}=\boldsymbol{a}$, and $\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{b}=\boldsymbol{a}$, for any real a and b.



Similarly, **multiplication and division** are **inverse operations**, as $\boldsymbol{a}\cdot\boldsymbol{b}\div\boldsymbol{b}=\boldsymbol{a}$, and $\boldsymbol{a}\cdot\boldsymbol{b}\div\boldsymbol{b}=\boldsymbol{a}$ for any real a and $b\neq 0$.

Opposites: Two quantities are **opposite** to each other if they **add to zero**. Particularly, $\boldsymbol{a}$ and $-\boldsymbol{a}$ are **opposites** (also referred to as **additive inverses**), as $\boldsymbol{a}+(-\boldsymbol{a})=\mathbf{0}$. For example, the opposite of 3 is -3, the opposite of $-\frac{3}{4}$ is $\frac{3}{4}$, the opposite of $x+1$ is $-(x+1)=-x-1$.

Reciprocals: Two quantities are **reciprocals** of each other if they **multiply to one**. Particularly, $\boldsymbol{a}$ and $\frac{\mathbf{1}}{\boldsymbol{a}}$ are **reciprocals** (also referred to as **multiplicative inverses**), since $\boldsymbol{a}\cdot\frac{\mathbf{1}}{\boldsymbol{a}}=\mathbf{1}$. For example, the reciprocal of 3 is $\frac{1}{3}$, the reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$, the reciprocal of $x+1$ is $\frac{1}{x+1}$.

Multiplication by 0: Any real quantity **multiplied by zero** becomes **zero**. Particularly, $\boldsymbol{a}\cdot\mathbf{0}=\mathbf{0}$, for any real number a.

Zero Product: If a product of two real numbers is zero, then at least one of these numbers must be zero. Particularly, for any real $\boldsymbol{a}$ and $\boldsymbol{b}$, if $\boldsymbol{a}\cdot\boldsymbol{b}=\mathbf{0}$, then $\boldsymbol{a}=\mathbf{0}$ or $\boldsymbol{b}=\mathbf{0}$.

For example, if $x(x-1)=0$, then either $x=0$ or $x-1=0$.

Commutativity: **The order of numbers does not change the value** of a particular operation. In particular, **addition** and **multiplication** is **commutative**, since

$$\boldsymbol{a+b=b+a} \text{ and } \boldsymbol{a\cdot b=b\cdot a},$$

for any real a and b. For example, $5+3=3+5$ and $5\cdot 3=3\cdot 5$.

Note: Neither subtraction nor division is commutative. See a counterexample: $5-3=2$ but $3-5=-2$, so $5-3\neq 3-5$. Similarly, $5\div 3\neq 3\div 5$.

Associativity: **Association (grouping) of numbers does not change the value** of an expression involving only one type of operation. In particular, **addition** and **multiplication** is **associative**, since

$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) \text{ and } (\boldsymbol{a}\cdot\boldsymbol{b})\cdot\boldsymbol{c}=\boldsymbol{a}\cdot(\boldsymbol{b}\cdot\boldsymbol{c}),$$

for any real a and b. For example, $(5+3)+2=5+(3+2)$ and $(5\cdot 3)\cdot 2=5\cdot(3\cdot 2)$.

Note: Neither subtraction nor division is associative. See a counterexample:
$(8-4)-2=2$ but $8-(4-2)=6$, so $(8-4)-2\neq 8-(4-2)$.
Similarly, $(8\div 4)\div 2=1$ but $8\div(4\div 2)=4$, so $(8\div 4)\div 2\neq 8\div(4\div 2)$.

Distributivity: Multiplication can be **distributed** over addition or subtraction by following the rule:

$$\boldsymbol{a}(\boldsymbol{b}\pm\boldsymbol{c})=\boldsymbol{ab}\pm\boldsymbol{ac},$$

for any real a, b and c. For example, $2(3\pm 5)=2\cdot 3\pm 2\cdot 5$, or $2(x\pm y)=2x\pm 2y$.

Note: The reverse process of distribution is known as **factoring a common factor** out. For example, $\boxed{2a}x+\boxed{2a}y=\boxed{2a}(x+y)$.

Example 1 ► **Showing Properties of Operations on Real Numbers**

Complete each statement to illustrate the indicated property.

a. $mn=$ _______ (commutativity of multiplication)
b. $5x+(7x+8)=$ __________________ (associativity of addition)
c. $5x(2-x)=$ __________________ (distributivity of multiplication)
d. $-y+$ _____ $=0$ (additive inverse)
e. $-6\cdot$ _____ $=1$ (multiplicative inverse)
f. If $7x=0$, then ____ $=0$ (zero product)

Solution ►

a. To show that multiplication is commutative, we change the order of letters, so $mn=nm$.

b. To show that addition is associative, we change the position of the bracket, so $5x+(7x+8)=(5x+7x)+8$.

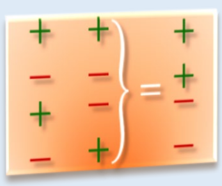
c. To show the distribution of multiplication over subtraction, we multiply $5x$ by each term of the bracket. So we have $5x(2-x)=5x\cdot 2-5x\cdot x$.

d. Additive inverse to $-y$ is its opposite, which equals to $-(-y) = y$.
So we write $-y + y = 0$.

e. Multiplicative inverse of -6 is its reciprocal, which equals to $-\frac{1}{6}$.
So we write $-6 \cdot \left(-\frac{1}{6}\right) = 1$.

f. By the zero product property, one of the factors, 7 or x, must equal to zero.
Since $7 \neq 0$, then x must equal to zero. So, we write: If $7x = 0$, then $x = 0$.

Sign Rule:



When multiplying or dividing two numbers of the **same sign**, the result is **positive**.

When multiplying or dividing two numbers of **different signs**, the result is **negative**.

This rule also applies to double signs. If the two **signs** are the **same**, they can be replaced by a **positive** sign. For example $+(+3) = 3$ and $-(-3) = 3$.
If the two **signs** are **different**, they can be replaced by a **negative** sign. For example $-(+3) = -3$ and $+(-3) = -3$.

Observation:

Since a double negative $(- -)$ *can be replaced by a positive sign* $(+)$*, the* ***opposite of an opposite*** *leaves the original quantity unchanged. For example,* $-(-2) = 2$*, and generally* $-(-a) = a$*.*
Similarly, taking the ***reciprocal of a reciprocal*** *leaves the original quantity unchanged. For example,* $\frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2$*, and generally* $\frac{1}{\frac{1}{a}} = 1 \cdot \frac{a}{1} = a$*.*

Example 2 ► **Using Properties of Operations on Real Numbers**

Use applicable properties of real numbers to simplify each expression.

a. $-\frac{-2}{-3}$ **b.** $3 + (-2) - (-7) - 11$

c. $2x(-3y)$ **d.** $3a - 2 - 5a + 4$

e. $-(2x - 5)$ **f.** $2(x^2 + 1) - 2(x - 3x^2)$

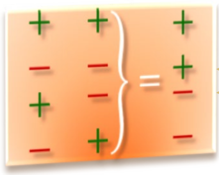
g. $-\frac{100ab}{25a}$ **h.** $\frac{2x-6}{2}$

Solution ►

a. The quotient of two negative numbers is positive, so $-\frac{-2}{-3} = -\frac{2}{3}$.

Note: *To determine the overall sign of an expression involving only multiplication and division of signed numbers, it is enough to count how many of the negative signs appear in the expression. An* ***even number of negatives*** *results in a* ***positive*** *value; an* ***odd number of negatives*** *leaves the answer* ***negative****.*

b. First, according to the sign rule, replace each double sign by a single sign. Therefore,



$$3+(-2)-(-7)-11=3-2+7-11.$$

It is convenient to treat this expression as a **sum** of signed numbers. So, it really means

$$\boxed{3}+\boxed{-2}+\boxed{7}+\boxed{-11}$$

but, for shorter notation, we tend not to write the plus signs.

Then, using the commutative property of addition, we collect all positive numbers, and all negative numbers to obtain

$$3\underbrace{-2+7}_{\substack{switch\\ addends}}-11=\underbrace{3+7}_{\substack{collect\\ positive}}\ \underbrace{-2-11}_{\substack{collect\\ negative}}=\underbrace{10-13}_{subtract}=\mathbf{-3}.$$

c. Since associativity of multiplication tells us that the order of performing multiplication does not change the outcome, there is no need to use any brackets in expressions involving only multiplication. So, the expression $2x(-3y)$ can be written as $2\cdot x\cdot(-3)\cdot y$. Here, the bracket is used only to isolate the negative number, not to prioritize any of the multiplications. Then, applying commutativity of multiplication to the middle two factors, we have

$$2\cdot\underbrace{x\cdot(-3)}_{\substack{switch\\ factors}}\cdot y=\underbrace{2\cdot(-3)}_{\substack{perform\\ multiplication}}\cdot x\cdot y=\boldsymbol{-6xy}$$

d. First, use commutativity of addition to switch the two middle addends, then factor out the a, and finally perform additions where possible.

$$3a\underbrace{-2-5a}_{\substack{switch\\ addends}}+4=\underbrace{3a-5a}_{factor\ \boldsymbol{a}\ out}-2+4=\underbrace{(3-5)}_{combine}a\ \underbrace{-2+4}_{combine}=\boldsymbol{-2a+2}$$

Note: In practice, to combine terms with the same variable, add their coefficients.

e. The expression $-(2x-5)$ represents the **opposite** to $2x-5$, which is $\boldsymbol{-2x+5}$. This expression is indeed the opposite because

$$-2x+5+2x-5=-2x\underbrace{+2x+5}_{\substack{commutativity\\ of\ addition}}-5=\underbrace{2x-2x}_{opposites}\ \underbrace{-5+5}_{opposites}=0+0=0.$$

Notice that the negative sign in front of the bracket in the expression $-(2x-5)$ can be treated as multiplication by -1. Indeed, using the distributive property of multiplication over subtraction and the sign rule, we achieve the same result

$$-1(2x-5)=-1\cdot 2x+(-1)(-5)=\boldsymbol{-2x+5}.$$

Note: *In practice, to release a bracket with a negative sign (or a negative factor) in front of it, change all the addends into opposites. For example*

$$-(2x - y + 1) = -2x + y - 1$$
$$\text{and} \quad -3(2x - y + 1) = -6x + 3y - 3$$

f. To simplify $2(x^2 + 1) - 2(x - 3x^2)$, first, we apply the distributive property of multiplication and the sign rule.

$$2(x^2 + 1) - 2(x - 3x^2) = 2x^2 + 2 - 2x + 6x^2$$

Then, using the commutative property of addition, we group the terms with the same powers of x. So, the equivalent expression is

$$2x^2 + 6x^2 - 2x + 2$$

Finally, by factoring x^2 out of the first two terms, we can add them to obtain

$$(2 + 6)x^2 - 2x + 2 = \mathbf{8x^2 - 2x + 2}.$$

Note: *In practice, to combine terms with the same powers of a variable (or variables), add their coefficients. For example*

$$\underline{\underline{2x^2}}\ \underline{\underline{-5x^2}}\ \underline{+3xy}\ \underline{-xy}\ - 3 + 2 = \underline{\underline{-3x^2}}\ \underline{+2xy} - 1.$$

g. To simplify $-\frac{100ab}{25a}$, we reduce the common factors of the numerator and denominator by following the property of the neutral element of multiplication, which is one. So,

$$-\frac{100ab}{25a} = -\frac{25 \cdot 4ab}{25a} = -\frac{25a \cdot 4b}{25a \cdot 1} = -\boxed{\frac{25a}{25a}} \cdot \frac{4b}{1} = -1 \cdot \frac{4b}{1} = \boldsymbol{-4b}.$$

This process is called **canceling** and can be recorded in short as

$$-\frac{\overset{4}{\cancel{100}}\cancel{a}b}{\cancel{25}\cancel{a}} = \boldsymbol{-4b}.$$

h. To simplify $\frac{2x-6}{2}$, factor the numerator and then remove from the fraction the factor of one by canceling the common factor of 2 in the numerator and the denominator. So, we have

$$\frac{2x - 6}{2} = \frac{\cancel{2} \cdot (2x - 6)}{\cancel{2}} = \mathbf{2x - 6}.$$

In the solution to *Example 2d* and *2f*, we used an intuitive understanding of what a "term" is. We have also shown how to combine terms with a common variable part (like terms). Here is a more formal definition of a term and of like terms.

Definition 3.1 ► A **term** is a **product** of constants (numbers), variables, or expressions. Here are examples of single terms:

$$1,\ x,\ \frac{1}{2}x^2,\ -3xy^2,\ 2(x+1),\ \frac{x+2}{x(x+1)},\ \pi\sqrt{x}.$$

Observe that the expression $2x+2$ consists of two terms connected by addition, while the equivalent expression $2(x+1)$ represents just one term, as it is a product of the number 2 and the expression $(x+1)$.

Like terms are the terms that have exactly the same variable part (the same variables or expressions raised to the same exponents). Like terms can be **combined** by adding their **coefficients** (numerical part of the term).

For example, $5x^2$ and $-2x^2$ are like, so they can be combined (added) to $3x^2$,
$(x+1)$ and $3(x+1)$ are like, so they can be combined to $4(x+1)$,
but $5x$ and $2y$ are unlike, so they cannot be combined.

Example 3 ► Combining Like Terms

Simplify each expression by combining like terms.

a. $-x^2+3y^2+x-6+2y^2-x+1$

b. $\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$

Solution ►

a. Before adding like terms, it is convenient to underline the groups of like terms by the same type of underlining. So, we have

add to zero

$$-x^2+3y^2+x-6+2y^2-x+1=\boldsymbol{-x^2+5y^2-5}$$

b. Notice that the numerical coefficients of the first two like terms in the expression

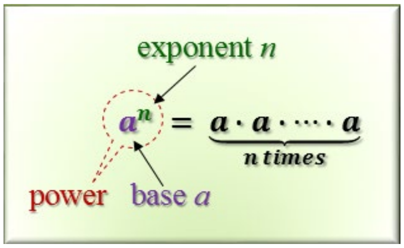
$$\frac{2}{x+1}-\frac{5}{x+1}+\sqrt{x}-\frac{\sqrt{x}}{2}$$

are 2 and -5, and of the last two like terms are 1 and $-\frac{1}{2}$. So, by adding these coefficients, we obtain

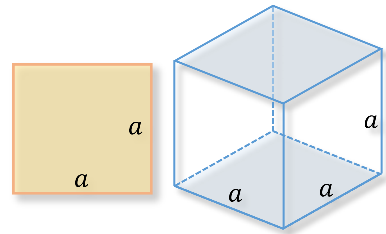
$$\boldsymbol{-\frac{3}{x+1}+\frac{1}{2}\sqrt{x}}$$

Observe that $\frac{1}{2}\sqrt{x}$ can also be written as $\frac{\sqrt{x}}{2}$. Similarly, $-\frac{3}{x+1}$, $\frac{-3}{x+1}$, or $-3\cdot\frac{1}{x+1}$ are equivalent forms of the same expression.

Exponents and Roots



Exponents are used as a shorter way of recording repeated multiplication by the same quantity. For example, to record the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, we write 2^5. The **exponent** 5 tells us how many times to multiply the **base** 2 by itself to evaluate the product, which is 32. The expression 2^5 is referred to as the 5th **power** of 2, or "2 to the 5th". In the case of exponents 2 or 3, terms "squared" or "cubed" are often used. This is because of the connection to geometric figures, a square and a cube.



The area of a square with sides of length a is expressed by a^2 (*read:* "a squared" or "the square of a") while the volume of a cube with sides of length a is expressed by a^3 (*read:* "a cubed" or "the cube of a").

If a negative number is raised to a certain exponent, a bracket must be used around the base number. For example, if we wish to multiply -3 by itself two times, we write $(-3)^2$, which equals $(-3)(-3) = 9$. The notation -3^2 would indicate that only 3 is squared, so $-3^2 = -3 \cdot 3 = -9$. This is because an **exponent refers only to the number immediately below the exponent**. Unless we use a bracket, a negative sign in front of a number is not under the influence of the exponent.

Example 4 ► **Evaluating Exponential Expressions**

Evaluate each exponential expression.

a. -3^4 **b.** $(-2)^6$
c. $(-2)^5$ **d.** $-(-2)^3$
e. $\left(-\frac{2}{3}\right)^2$ **f.** $-\left(-\frac{2}{3}\right)^5$

Solution ►

a. $-3^4 = (-1) \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \mathbf{-81}$

b. $(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = \mathbf{64}$

c. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = \mathbf{-32}$

> ***Observe:*** *Negative sign in front of a power works like multiplication by* -1.
>
> *A* ***negative*** *base raised to an* ***even exponent*** *results in a* ***positive*** *value.*
> *A* ***negative*** *base raised to an* ***odd exponent*** *results in a* ***negative*** *value.*

d. $-(-2x)^3 = -(-2x)(-2x)(-2x)$

$= -(-2)(-2)(-2)xxx = -(-2)^3x^3 = -(-8)x^3 = \mathbf{8x^3}$

e. $\left(-\frac{2}{3}\right)^2 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{(-2)^2}{3^2} = \mathbf{\frac{4}{9}}$

f. $-\left(-\frac{2}{3}\right)^5 = -\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = -\frac{(-2)^5}{3^5} = -\frac{-32}{243} = \frac{\mathbf{32}}{\mathbf{243}}$

Observe: *Exponents apply to every factor of the numerator and denominator of the base. This exponential property can be stated as*

$$\boxed{(ab)^n = a^n b^n} \quad \textit{and} \quad \boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

To reverse the process of squaring, we apply a **square root**, denoted by the **radical sign** $\sqrt{\ }$. For example, since $5 \cdot 5 = 25$, then $\sqrt{25} = 5$. Notice that $(-5)(-5) = 25$ as well, so we could also claim that $\sqrt{25} = -5$. However, we wish to define the operation of taking square root in a unique way. We choose to take the **positive** number (called **principal square root**) as the value of the square root. Therefore $\sqrt{25} = 5$, and generally

$$\boxed{\sqrt{x^2} = |x|.}$$

Since the square of any nonzero real number is positive, the square root of a negative number is not a real number. For example, we can say that $\sqrt{-16}$ **does not exist** (in the set of real numbers), as there is no real number a that would satisfy the equation $a^2 = -16$.

Example 5 ► **Evaluating Radical Expressions**

Evaluate each radical expression.

a. $\sqrt{0}$ **b.** $\sqrt{64}$

c. $-\sqrt{121}$ **d.** $\sqrt{-100}$

e. $\sqrt{\frac{1}{9}}$ **f.** $\sqrt{0.49}$

Solution ►

a. $\sqrt{0} = \mathbf{0}$, as $0 \cdot 0 = 0$

b. $\sqrt{64} = \mathbf{8}$, as $8 \cdot 8 = 64$

c. $-\sqrt{121} = \mathbf{-11}$, as we copy the negative sign and $11 \cdot 11 = 121$

d. $\sqrt{-100} = \boldsymbol{DNE}$ (*read:* doesn't exist), as no real number squared equals -100

e. $\sqrt{\frac{1}{9}} = \frac{\mathbf{1}}{\mathbf{3}}$, as $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Notice that $\frac{\sqrt{1}}{\sqrt{9}}$ also results in $\frac{1}{3}$. So, $\sqrt{\frac{1}{9}} = \frac{\sqrt{1}}{\sqrt{9}}$ and generally $\boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$ for any nonnegative real numbers a and $b \neq 0$.

f. $\sqrt{0.49} = \mathbf{0.7}$, as $0.7 \cdot 0.7 = 0.49$

Order of Operations

In algebra, similarly as in arithmetic, we like to perform various operations on numbers or on variables. To record in what order these operations should be performed, we use grouping signs, mostly brackets, but also division bars, absolute value symbols, radical symbols, etc. In an expression with many grouping signs, we perform operations in the **innermost grouping sign first**. For example, the innermost grouping sign in the expression

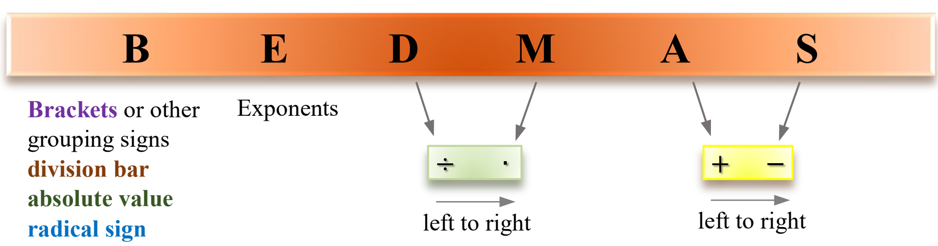
$$[4 + (3 \cdot |2 - 4|)] \div 2$$

is the absolute value sign, then the round bracket, and finally, the square bracket. So first, perform subtraction, then apply the absolute value, then multiplication, addition, and finally the division. Here are the calculations:

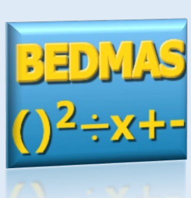
$$\begin{aligned} &[4 + (3 \cdot |2 - 4|)] \div 2 \\ &= [4 + (3 \cdot |-2|)] \div 2 \\ &= [4 + (3 \cdot 2)] \div 2 \\ &= [4 + 6] \div 2 \\ &= 10 \div 2 \\ &= \mathbf{5} \end{aligned}$$

Observe that the more operations there are to perform, the more grouping signs would need to be used. To simplify the notation, additional rules of order of operations have been created. These rules, known as BEDMAS, allow for omitting some of the grouping signs, especially brackets. For example, knowing that multiplication is performed before addition, the expression $[4 + (3 \cdot |2 - 4|)] \div 2$ can be written as $[4 + 3 \cdot |2 - 4|] \div 2$ or $\frac{4+3\cdot|2-4|}{2}$.

Let's review the BEDMAS rule.



BEDMAS Rule:



1. Perform operations in the innermost **B**racket (or other grouping sign) first.
2. Then work out **E**xponents.
3. Then perform **D**ivision and **M**ultiplication in order of their occurrence (left to right). *Notice that there is no priority between division and multiplication. However, both division and multiplication have priority before any addition or subtraction.*
4. Finally, perform **A**ddition and **S**ubtraction in order of their occurrence (left to right). *Again, there is no priority between addition and subtraction.*

Example 6 ► **Simplifying Arithmetic Expressions According to the Order of Operations**

Use the order of operations to simplify each expression.

a. $3 + 2 \cdot 6$

b. $4 \cdot 6 \div 3 - 2$

c. $12 \div 4 + 2|3 - 4|$

d. $2 \cdot 3^2 - 3(-2 + 6)$

e. $\sqrt{30 - 5} - 2\left(3 + 4 \cdot (-2)\right)^2$

f. $\frac{3-2(-3^2)}{3 \cdot \sqrt{4} - 6 \cdot 2}$

Solution

a. Out of the two operations, + and · , multiplication is performed first. So, we have

$3 + 2 \cdot 6$
$= 3 + 12$
$= \mathbf{15}$

b. There is no priority between multiplication and division, so we perform these operations in the order in which they appear, from left to right. Then we subtract. Therefore,

$4 \cdot 6 \div 3 - 2$
$= 24 \div 3 - 2$
$= 8 - 2$
$= \mathbf{6}$

c. In this expression, we have a grouping sign (the absolute value bars), so we perform the subtraction inside the absolute value first. Then, we apply the absolute value and work out the division and multiplication before the final addition. So, we obtain

$12 \div 4 + 2|3 - 4|$
$= 12 \div 4 + 2|-1|$
$= 12 \div 4 + 2 \cdot 1$
$= 3 + 2$
$= \mathbf{5}$

d. In this expression, work out the bracket first, then perform the exponent, then both multiplications, and finally the subtraction. Thus,

$2 \cdot 3^2 - 3(-2 + 6)$
$= 2 \cdot 3^2 - 3(4)$
$= 2 \cdot 9 - 3(4)$
$= 18 - 12$
$= \mathbf{6}$

e. The expression $\sqrt{30 - 5} - 2\left(3 + 4 \cdot (-2)\right)^2$ contains two grouping signs, the bracket and the radical sign. Since these grouping signs are located at separate places (they are not nested), they can be worked out simultaneously. As usual, out of the operations inside the bracket, multiplication is done before subtraction. So, we calculate

$\sqrt{30 - 5} - 2\left(3 + 4 \cdot (-2)\right)^2$
$= \sqrt{25} - 2\left(3 + (-8)\right)^2$
$= 5 - 2(-5)^2$ Work out the power first, then multiply, and finally subtract.

$= 5 - 2 \cdot 25$
$= 5 - 50$
$= \mathbf{-45}$

f. To simplify the expression $\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot 2}$, work on the numerator and the denominator before performing the division. Therefore,

$$\frac{3-2(-3^2)}{3\cdot\sqrt{4}-6\cdot 2}$$

$-3^2 = -9$

$$= \frac{3-(-18)}{3\cdot 2-6\cdot 2}$$

$$= \frac{3+18}{6-12}$$

$$= \frac{21}{-6}$$

reduce the common factor of 3

$$= -\frac{\mathbf{7}}{\mathbf{2}}$$

Example 7 ► **Simplifying Expressions with Nested Brackets**

Simplify the expression $2\{1-5[3x+2(4x-1)]\}$.

Solution ► The expression $2\{1-5[3x+2(4x-1)]\}$ contains three types of brackets: the innermost parenthesis ( ), the middle brackets [ ], and the outermost braces { }. We start with working out the innermost parenthesis first, and then after collecting like terms, we proceed with working out consecutive brackets. So, we simplify

$2\{1-5[3x+2(4x-1)]\}$ distribute 2 over the ( ) bracket
$= 2\{1-5[3x+8x-2]\}$ collect like terms before working out the [ ] bracket
$= 2\{1-5[11x-2]\}$ distribute -5 over the [ ] bracket
$= 2\{1-55x+10\}$ collect like terms before working out the { } bracket
$= 2\{-55x+11\}$ distribute 2 over the { } bracket
$= \mathbf{-110x+22}$

Evaluation of Algebraic Expressions

An **algebraic expression** consists of letters, numbers, operation signs, and grouping symbols. Here are some examples of algebraic expressions:

$$6ab, \quad x^2-y^2, \quad 3(2a+5b), \quad \frac{x-3}{3-x}, \quad 2\pi r, \quad \frac{d}{t}, \quad Prt, \quad \sqrt{x^2+y^2}$$

When a letter is used to stand for various numerical values, it is called a **variable**. For example, if t represents the number of hours needed to drive between particular towns, then t changes depending on the average speed used during the trip. So, t is a variable. Notice however, that the distance d between the two towns represents a constant number. So, even though letters in algebraic expressions usually represent variables, sometimes they may represent a **constant** value. One such constant is the letter π, which represents approximately 3.14.

Notice that algebraic expressions do not contain any comparison signs (equality or inequality, such as $=$, $\neq$, $<$, $\leq$, $>$, $\geq$), therefore, they are **not to be solved** for any variable. Algebraic expressions can only be **simplified** by implementing properties of operations (see *Example 2* and *3*) or **evaluated** for particular values of the variables. The evaluation process involves substituting given values for the variables and evaluating the resulting arithmetic expression by following the order of operations.

Advice: *To evaluate an algebraic expression for given variables, first rewrite the expression replacing each variable with **empty brackets** and then write appropriate values inside these brackets. This will help to avoid possible errors of using incorrect signs or operations.*

Example 8 ► **Evaluating Algebraic Expressions**

Evaluate each expression for $a = -2$, $b = 3$, and $c = 6$.

a. $b^2 - 4ac$ **b.** $2c \div 3a$ **c.** $\frac{|a^2-b^2|}{-a^2+\sqrt{a+c}}$

Solution ►

a. First, we replace each letter in the expression $b^2 - 4ac$ with an empty bracket. So, we write

$$(\ \ )^2 - 4(\ \ )(\ \ ).$$

Now, we fill in the brackets with the corresponding values and evaluate the resulting expression. So, we have

$$(3)^2 - 4(-2)(6) = 9 - (-48) = 9 + 48 = \mathbf{57}.$$

b. As above, we replace the letters with their corresponding values to obtain

$$2c \div 3a = 2(6) \div 3(-2).$$

Since we work only with multiplication and division here, they are to be performed in order from left to right. Therefore,

$$2(6) \div 3(-2) = 12 \div 3(-2) = 4(-2) = \mathbf{-8}.$$

c. As above, we replace the letters with their corresponding values to obtain

$$\frac{|a^2 - b^2|}{-a^2 + \sqrt{b + c}} = \frac{|(-2)^2 - (3)^2|}{-(-2)^2 + \sqrt{(3) + (6)}} = \frac{|4 - 9|}{-4 + \sqrt{9}} = \frac{|-5|}{-4 + 3} = \frac{5}{-1} = \mathbf{-5}.$$

Equivalent Expressions

Algebraic expressions that produce the same value for all allowable values of the variables are referred to as **equivalent expressions**. Notice that properties of operations allow us to rewrite algebraic expressions in a simpler but equivalent form. For example,

$$\frac{\boldsymbol{x-3}}{\boldsymbol{3-x}} = \frac{\cancel{x-3}}{-(\cancel{x-3})} = \boldsymbol{-1}$$

or

$$(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) = (x+y)x-(x+y)y = x^2+\cancel{yx}-\cancel{xy}-y^2 = \boldsymbol{x^2-y^2}.$$

To show that two expressions are **<u>not</u> equivalent**, it is enough to find a particular set of variable values for which the two expressions evaluate to a different value. For example,

$$\sqrt{x^2+y^2} \neq x+y$$

because if $x=1$ and $y=1$ then $\sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$ while $x+y=1+1=2$. Since $\sqrt{2} \neq 2$ the two expressions $\sqrt{x^2+y^2}$ and $x+y$ are not equivalent.

Example 9 ► **Determining Whether a Pair of Expressions is Equivalent**

Determine whether the given expressions are equivalent.

a. $(a+b)^2$ and a^2+b^2

b. $\frac{x^8}{x^4}$ and x^4

Solution ►

a. Suppose $a=1$ and $b=1$. Then

$$(a+b)^2 = (1+1)^2 = 2^2 = 4$$

but

$$a^2+b^2 = 1^2+1^2 = 2.$$

So the expressions $(a+b)^2$ and a^2+b^2 are not equivalent.

Using the distributive property and commutativity of multiplication, check on your own that

$$(a+b)^2 = a^2+2ab+b^2.$$

b. Using properties of exponents and then removing a factor of one, we show that

$$\frac{x^8}{x^4} = \frac{x^4 \cdot x^4}{x^4} = x^4.$$

So the two expressions are indeed equivalent.

Review of Operations on Fractions

A large part of algebra deals with performing operations on algebraic expressions by generalising the ways that these operations are performed on real numbers, particularly, on common fractions. Since operations on fractions are considered to be one of the most challenging topics in arithmetic, it is a good idea to review the rules to follow when performing these operations before we move on to other topics of algebra.

Operations on Fractions:

Simplifying

To simplify a fraction to its lowest terms, **remove** the **greatest common factor** (**GCF**) of the numerator and denominator. For example, $\frac{48}{64}=\frac{3\cdot\cancel{16}}{4\cdot\cancel{16}}=\frac{3}{4}$, and generally $\frac{a\cancel{k}}{b\cancel{k}}=\frac{a}{b}$.

This process is called ***reducing*** or ***canceling***.

Note that the reduction can be performed several times, if needed. In the above example, if we didn't notice that 16 is the greatest common factor for 48 and 64, we could reduce the fraction by dividing the numerator and denominator by any common factor (2, or 4, or 8) first, and then repeat the reduction process until there is no common factors (other than 1) anymore. For example,

$$\frac{\cancel{48}}{\cancel{64}}\underset{\div \text{ by } 2}{=}\frac{\cancel{24}}{\cancel{32}}\underset{\div \text{ by } 4}{=}\frac{\cancel{6}}{\cancel{8}}\underset{\div \text{ by } 2}{=}\frac{\mathbf{3}}{\mathbf{4}}.$$

Multiplying

To multiply fractions, we multiply their numerators and denominators. So generally,

$$\frac{\boldsymbol{a}}{\boldsymbol{b}}\cdot\frac{\boldsymbol{c}}{\boldsymbol{d}}=\frac{\boldsymbol{ac}}{\boldsymbol{bd}}.$$

However, before performing multiplication of numerators and denominators, it is a good idea to reduce first. This way, we work with smaller numbers, which makes the calculations easier. For example,

$$\frac{18}{15}\cdot\frac{25}{14}=\frac{18\cdot\cancel{25}}{\cancel{15}\cdot 14}\underset{\div \text{ by } 5}{=}\frac{\cancel{18}\cdot 5}{\cancel{3}\cdot 14}\underset{\div \text{ by } 3}{=}\frac{\cancel{6}\cdot 5}{1\cdot\cancel{14}}\underset{\div \text{ by } 2}{=}\frac{3\cdot 5}{1\cdot 7}=\frac{\mathbf{15}}{\mathbf{7}}.$$

Dividing

To divide fractions, we **multiply** the dividend (the first fraction) **by the reciprocal** of the **divisor** (the second fraction). So generally,

$$\frac{\boldsymbol{a}}{\boldsymbol{b}}\div\frac{\boldsymbol{c}}{\boldsymbol{d}}\underset{\cdot \text{ by reciprocal}}{=}\frac{\boldsymbol{a}}{\boldsymbol{b}}\cdot\frac{\boldsymbol{d}}{\boldsymbol{c}}=\frac{\boldsymbol{ad}}{\boldsymbol{bc}}.$$

For example,

$$\frac{8}{15}\div\frac{4}{5}=\frac{\cancel{8}}{\cancel{15}}\cdot\frac{\cancel{5}}{\cancel{4}}\overset{\div \text{ by } 4}{\underset{\div \text{ by } 5}{=}}\frac{2\cdot 1}{3\cdot 1}=\frac{\mathbf{2}}{\mathbf{3}}.$$

Adding or Subtracting

To add or subtract fractions, follow the steps:

1. Find the **Lowest Common Denominator** (**LCD**).
2. Extend each fraction to higher terms to obtain the desired common denominator.
3. Add or subtract the numerators, keeping the common denominator.
4. Simplify the resulting fraction, if possible.

For example, to evaluate $\frac{5}{6}+\frac{3}{4}-\frac{4}{15}$, first we find the LCD for denominators 6, 4, and 15. We can either guess that 60 is the least common multiple of 6, 4, and 15, or we can use the following method of finding LCD:

$$\begin{array}{c|ccc} 2 & 6 & 4 & 15 \\ \cdot\, 3 & 3 & 2 & 15 \\ \hline \cdot & 1 & \cdot\, 2 & \cdot\, 5 \end{array} = 60$$

- divide by a common factor of at least two numbers; for example, by 2
- write the quotients in the line below; 15 is not divisible by 2, so just copy it down
- keep dividing by common factors till all numbers become relatively prime
- the LCD is the product of all numbers listed in the letter L, so it is 60

Then, we extend the fractions so that they share the same denominator of 60, and finally perform the operations in the numerator. Therefore,

$$\frac{5}{6}+\frac{3}{4}-\frac{4}{5}=\underbrace{\frac{5\cdot 10}{6\cdot 10}+\frac{3\cdot 15}{4\cdot 15}-\frac{4\cdot 12}{5\cdot 12}}_{\substack{\textit{in practice, this step}\\ \textit{doesn't have to be written}}}=\frac{5\cdot 10+3\cdot 15-4\cdot 12}{60}=\frac{50+45-48}{60}=\mathbf{\frac{47}{60}}.$$

Example 10 ► **Evaluating Fractional Expressions**

Simplify each expression.

a. $-\frac{2}{3}-\left(-\frac{5}{12}\right)$

b. $-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$

Solution ►

a. After replacing the double negative by a positive sign, we add the two fractions, using 12 as the lowest common denominator. So, we obtain

$$-\frac{2}{3}-\left(-\frac{5}{12}\right)=-\frac{2}{3}+\frac{5}{12}=\frac{-2\cdot 4+5}{12}=\frac{-\cancel{3}}{\cancel{12}}=-\mathbf{\frac{1}{4}}.$$

b. Following the order of operations, we calculate

$$-3\left[\frac{3}{2}+\frac{5}{6}\div\left(-\frac{3}{8}\right)\right]$$

First, perform the division in the bracket by converting it to a multiplication by the reciprocal. The quotient becomes negative.

$$=-3\left[\frac{3}{2}-\frac{5}{\cancel{6}_3}\cdot\frac{\cancel{8}^4}{3}\right]$$

Reduce, before multiplying.

$$=-3\left(\frac{3}{2}-\frac{20}{9}\right)$$

Extend both fractions to higher terms using the common denominator of 18.

$$=-3\left(\frac{27-40}{18}\right)$$

Perform subtraction.

$$=-\cancel{3}\left(\frac{-13}{\cancel{18}}\right)$$

Reduce before multiplying. The product becomes positive.

$$=\mathbf{\frac{13}{6}}\ \text{or equivalently}\ \mathbf{2\frac{1}{6}}.$$

R.3 Exercises

True or False?

1. The set of integers is closed under multiplication.

2. The set of natural numbers is closed under subtraction.

3. The set of real numbers different than zero is closed under division.

4. According to the BEDMAS rule, division should be performed before multiplication.

5. For any real number $\sqrt{x^2} = x$.

6. Square root of a negative number is not a real number.

7. If the value of a square root exists, it is positive.

8. $-x^3 = (-x)^3$

9. $-x^2 = (-x)^2$

Complete each statement to illustrate the indicated property.

10. $x + (-y) =$ _________ , commutative property of addition

11. $(7 \cdot 5) \cdot 2 =$ ___________ , associative property of multiplication

12. $(3 + 8x) \cdot 2 =$ ______________ , distributive property of multiplication over addition

13. $a +$ _____ $= 0$, additive inverse

14. $-\frac{a}{b} \cdot$ _____ $= 1$, multiplicative inverse

15. $\frac{3x}{4y} \cdot$ _____ $= \frac{3x}{4y}$, multiplicative identity

16. _____ $+ (-a) = -a$, additive identity

17. $(2x - 7) \cdot$ ____ $= 0$, multiplication by zero

18. If $(x + 5)(x - 1) = 0$, then ________ $= 0$ or ________ $= 0$, zero product property

Perform operations.

19. $-\frac{2}{5} + \frac{3}{4}$

20. $\frac{5}{6} - \frac{2}{9}$

21. $\frac{5}{8} \cdot \left(-\frac{2}{3}\right) \cdot \frac{18}{15}$

22. $-3\left(-\frac{5}{9}\right)$

23. $-\frac{3}{4}(8x)$

24. $\frac{15}{16} \div \left(-\frac{9}{12}\right)$

Use order of operations to evaluate each expression.

25. $64 \div (-4) \div 2$

26. $3 + 3 \cdot 5$

27. $8 - 6(5 - 2)$

28. $20 + 4^3 \div (-8)$

29. $6\left(9 - 3\sqrt{9 - 5}\right)$

30. $-2^5 - 8 \div 4 - (-2)$

31. $-\frac{5}{6}+\left(-\frac{7}{4}\right)\div 2$

32. $\left(-\frac{3}{2}\right)\cdot\frac{1}{6}-\frac{2}{5}$

33. $-\frac{3}{2}\div\left(-\frac{4}{9}\right)-\frac{5}{4}\cdot\frac{2}{3}$

34. $-3\left(-\frac{4}{9}\right)-\frac{1}{4}\div\frac{3}{5}$

35. $2-3|3-4\cdot 6|$

36. $\frac{3|5-7|-6\cdot 4}{5\cdot 6-2|4-1|}$

Simplify each expression.

37. $-(x-y)$

38. $-2(3a-5b)$

39. $\frac{2}{3}(24x+12y-15)$

40. $\frac{3}{4}(16a-28b+12)$

41. $5x-8x+2x$

42. $3a+4b-5a+7b$

43. $5x-4x^2+7x-9x^2$

44. $8\sqrt{2}-5\sqrt{2}+\frac{1}{x}+\frac{3}{x}$

45. $2+3\sqrt{x}-6-\sqrt{x}$

46. $\frac{a-b}{b-a}$

47. $\frac{2(x-3)}{3-x}$

48. $-\frac{100ab}{75a}$

49. $-(5x)^2$

50. $\left(-\frac{2}{3}a\right)^2$

51. $5a-(4a-7)$

52. $6x+4-3(9-2x)$

53. $5x-4(2x-3)-7$

54. $8x-(-4y+7)+(9x-1)$

55. $6a-[4-3(9a-2)]$

56. $5\{x+3[4-5(2x-3)-7]\}$

57. $-2\{2+3[4x-3(5x+1)]\}$

58. $4\{[5(x-3)+5^2]-3[2(x+5)-7^2]\}$

59. $3\{[6(x+4)-3^3]-2[5(x-8)-8^2]\}$

Evaluate each algebraic expression for $a=-2$, $b=3$, and $c=2$.

60. b^2-a^2

61. $6c\div 3a$

62. $\frac{c-a}{c-b}$

63. $b^2-3(a-b)$

64. $\frac{-b+\sqrt{b^2-4ac}}{2a}$

65. $c\left(\frac{a}{b}\right)^{|a|}$

Determine whether each pair of expressions is equivalent.

66. $x^3\cdot x^2$ and x^5

67. a^2-b^2 and $(a-b)^2$

68. $\sqrt{x^2}$ and x

69. $(x^3)^2$ and x^5

Use the distributive property to calculate each value mentally.

70. $96\cdot 18+4\cdot 18$

71. $29\cdot 70+29\cdot 30$

72. $57\cdot\frac{3}{5}-7\cdot\frac{3}{5}$

73. $\frac{8}{5}\cdot 17+\frac{8}{5}\cdot 13$

Insert one pair of parentheses to make the statement true.

74. $2\cdot 3+6\div 5-3=9$

75. $9\cdot 5+2-8\cdot 3+1=22$

Attributions

Linear Equations and Inequalities

One of the main concepts in Algebra is solving equations or inequalities. This is because solutions to most application problems involve setting up and solving equations or inequalities that describe the situation presented in the problem. In this unit, we will study techniques of solving linear equations and inequalities in one variable, linear forms of absolute value equations and inequalities, and applications of these techniques in word problems.

L1 Linear Equations in One Variable

When two algebraic expressions are compared by an equal sign $(=)$, an **equation** is formed. An equation can be interpreted as a scale that balances two quantities. It can also be seen as a mathematical sentence with the verb "equals" or the verb phrase "is equal to". For example, the equation $3x - 1 = 5$ corresponds to the sentence:

One less than three times an unknown number equals five.

Unless we know the value of the unknown number (the variable x), we are unable to determine whether or not the above sentence is a true or false statement. For example, if $x = 1$, the equation $3x - 1 = 5$ becomes a false statement, as $3 \cdot 1 - 1 \neq 5$ (the "scale" is not in balance); while, if $x = 2$, the equation $3x - 1 = 5$ becomes a true statement, as $3 \cdot 2 - 1 = 5$ (the "scale" is in balance). For this reason, such sentences (equations) are called **open sentences**. Each variable value that satisfies an equation (i.e., makes it a true statement) is a **solution** (i.e., a ***root***, or a ***zero***) of the equation. An equation is **solved** by finding its **solution set**, the set of all solutions.

Attention:

* ***Equations*** *can be* ***solved*** *by finding the variable value(s) satisfying the equation.*

Example: $\overbrace{\mathbf{2x + x - 1}}^{\textit{left side}} \underset{\uparrow \atop \textit{equal sign}}{=} \overbrace{\mathbf{5}}^{\textit{right side}}$ *can be solved for x*

* ***Expressions*** *can only be* ***simplified*** *or* ***evaluated***

Example: $\mathbf{2x + x - 1}$ *can be simplified to* $3x - 1$
or evaluated for a particular x-value

Example 1 ► **Distinguishing Between Expressions and Equations**

Decide whether each of the following is an expression or an equation.

a. $4x - 16$ **b.** $4x - 16 = 0$

Solution ► **a.** $4x - 16$ is an **expression** as it does not contain any symbol of equality.

This expression can be ***evaluated*** *(for instance, if* $x = 4$*, the expression assumes the value 0), or it can be written in a different form. For example, we could* ***factor*** *it. So, we could write*

$$4x - 16 = 4(x - 4).$$

Notice that the equal symbol (=) in the above line does not indicate an equation, but rather an equivalency between the two expressions, $4x - 16$ *and* $4(x - 4)$.

b. $4x - 16 = 0$ is an **equation** as it contains an equal symbol $(=)$ that connects two sides of the equation.

To solve this equation we could factor the left-hand side expression,

$$4(x-4) = 0,$$

and then from the zero product property, we have

$$x - 4 = 0,$$

which leads us to the solution

$$x = \mathbf{4}.$$

Attention: *Even though the two algebraic forms,* $4x - 16$ *and* $4x - 16 = 0$ *are related to each other, it is important that we neither* ***voluntarily add*** *the "$= 0$" part when we want to change the form of the expression, nor* ***voluntarily drop*** *the "$= 0$" part when we solve the equation.*

Example 2 **Determining if a Given Number is a Solution to an Equation**

Determine whether the number -2 is a solution to the given equation.

a. $4x = 10 + x$

b. $x^2 - 4 = 0$

Solution

a. To determine whether -2 is a solution to the equation $4x = 10 + x$, we substitute -2 in place of the variable x and find out whether the resulting equation is a true statement. This gives us

$$4(-2) \stackrel{?}{=} 10 + (-2)$$

$$-8 \stackrel{?}{=} 8$$

Since the resulting equation is not a true statement, the number **-2 is not a solution** to the given equation.

b. After substituting -2 for x in the equation $x^2 - 4 = 0$, we obtain

$$(-2)^2 - 4 \stackrel{?}{=} 0$$

$$4 - 4 \stackrel{?}{=} 0,$$

which becomes $0 = 0$, a true statement. Therefore, the number **-2 is a solution** to the given equation.

Equations can be classified with respect to the number of solutions. There are **identities**, **conditional** equations, and **contradictions** (or ***inconsistent*** equations).

Definition 1.1 ► An **identity** is an equation that is satisfied by every real number for which the expressions on both sides of the equation are defined. Some examples of identities are

$$2x + 5x = 7x, \quad x^2 - 4 = (x+2)(x-2), \quad \text{or} \quad \frac{x}{x} = 1.$$

The solution set of the first two identities is the set of all real numbers, $\mathbb{R}$. *However, since the expression* $\frac{x}{x}$ *is undefined for* $x = 0$, *the solution set of the equation* $\frac{x}{x} = 1$ *is the set of all nonzero real numbers,* $\{\boldsymbol{x}|\boldsymbol{x} \neq \boldsymbol{0}\}$.

A **conditional** equation is an equation that is satisfied by at least one real number, but is not an identity. This is the most commonly encountered type of equation. Here are some examples of conditional equations:

$$3x - 1 = 5, \quad x^2 - 4 = 0, \quad \text{or} \quad \sqrt{x} = 2.$$

The solution set of the first equation is $\{2\}$; *of the second equation is* $\{-2,2\}$; *and of the last equation is* $\{4\}$.

A **contradiction** (an ***inconsistent*** equation) is an equation that has **no solution**. Here are some examples of contradictions:

$$5 = 1, \quad 3x - 3x = 8, \quad \text{or} \quad 0x = 1.$$

The solution set of any contradiction is the empty set, $\emptyset$.

Example 3 ► **Recognizing Conditional Equations, Identities, and Contradictions**

Determine whether the given equation is *conditional*, an *identity*, or a *contradiction*.

a. $x = x$ **b.** $x^2 = 0$ **c.** $\frac{1}{x} = 0$

Solution ►

a. This equation is satisfied by any real number. Therefore, it is an **identity**.

b. This equation is satisfied by $x = 0$, as $0^2 = 0$. However, any nonzero real number when squared becomes a positive number. So, the left side of the equation $x^2 = 0$ does not equal to zero for a nonzero x. That means that a nonzero number does not satisfy the equation. Therefore, the equation $x^2 = 0$ has exactly one solution, $x = 0$. So, the equation is **conditional**.

c. A fraction equals zero only when its numerator equals to zero. Since the numerator of $\frac{1}{x}$ does not equal to zero, then no matter what the value of x would be, the left side of the equation will never equal zero. This means that there is no x-value that would satisfy the equation $\frac{1}{x} = 0$. Therefore, the equation has no solution, which means it is a **contradiction**.

Attention: *Do not confuse the solution* $x = 0$ *to the equation in Example 3b with an empty set* $\emptyset$. *An empty set means that there is no solution.* $x = 0$ *means that there is one solution equal to zero.*

Solving Linear Equations in One Variable

In this section, we will focus on solving linear (up to the first degree) equations in one variable. Before introducing a formal definition of a linear equation, let us recall the definition of a term, a constant term, and a linear term.

Definition 1.2 ► A **term** is a **product** of numbers, letters, and possibly other algebraic expressions.

Examples of terms: $2,\ -3x,\ \frac{2}{3}(x+1),\ 5x^2y,\ -5\sqrt{x}$

A **constant term** is a number or a product of numbers.

Examples of constant terms: $2,\ -3,\ \frac{2}{3},\ 0,\ -5\pi$

A **linear term** is a product of numbers and the first power of a single variable.

Examples of linear terms: $-3x,\ \frac{2}{3}x,\ x,\ -5\pi x$

Definition 1.3 ► A **linear** equation is an equation with only **constant** or **linear terms**. A linear equation in one variable can be written in the form $\boldsymbol{Ax+B=0}$, for some real numbers A and B, and a variable x.

Here are some examples of *linear* equations: $2x+1=0, \quad 2=5, \quad 3x-7=6+2x$

Here are some examples of *nonlinear* equations: $x^2=16, \quad x+\sqrt{x}=-1, \quad 1+\frac{1}{x}=\frac{1}{x+1}$

So far, we have been finding solutions to equations mostly by guessing a value that would make the equation true. To find a methodical way of solving equations, observe the relations between equations with the same solution set. For example, equations

$$3x-1=5,\ \ 3x=6,\ \ and\ \ x=2$$

all have the same solution set $\{2\}$. While the solution to the last equation, $x=2$, is easily "seen" to be 2, the solution to the first equation, $3x-1=5$, is not readily apparent. Notice that the second equation is obtained by adding 1 to both sides of the first equation. Similarly, the last equation is obtained by dividing the second equation by 3. This suggests that to solve a linear equation, it is enough to write a sequence of simpler and simpler equations that preserve the solution set, and eventually result in an equation of the form:

$$x=constant \quad \text{or} \quad 0=constant.$$

If the resulting equation is of the form $x=constant$, the solution is this constant.
If the resulting equation is $\mathbf{0}=\mathbf{0}$, then the original equation is an **identity**, as it is true for **all real values $\boldsymbol{x}$**.
If the resulting equation is $\mathbf{0}=\boldsymbol{constant\ other\ than\ zero}$, then the original equation is a **contradiction**, as there is **no real values $\boldsymbol{x}$** that would make it true.

Definition 1.4 ► **Equivalent equations** are equations with the same solution set.

How can we transform an equation to obtain a simpler but equivalent one?

We can certainly simplify expressions on both sides of the equation, following properties of operations listed in *section R3*. Also, recall that an equation works like a scale in balance.

Therefore, adding (or subtracting) the same quantity to (from) both sides of the equation will preserve this balance. Similarly, multiplying (or dividing) both sides of the equation by a nonzero quantity will preserve the balance.

Suppose we work with an equation $A = B$, where A and B represent some algebraic expressions. In addition, suppose that C is a real number (or another expression).
Here is a summary of the basic equality operations that can be performed to produce equivalent equations:

Equality Operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) = 1+3$ can be written as $2x - 6 = 4$
Addition	$A + C = B + C$	$2x - 6 + 6 = 4 + 6$
Subtraction	$A - C = B - C$	$2x - 6 - 4 = 4 - 4$
Multiplication	$CA = CB, \quad if\ C \neq 0$	$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 10$
Division	$\frac{A}{C} = \frac{B}{C}, \quad if\ C \neq 0$	$\frac{2x}{2} = \frac{10}{2}$

Example 4 ► **Using Equality Operations to Solve Linear Equations in One Variable**

Solve each equation.

a. $4x - 12 + 3x = 3 + 5x - 2x$ **b.** $2[3(x-6) - x] = 3x - 2(5 - x)$

Solution ► **a.** First, simplify each side of the equation and then isolate the linear terms (terms containing x) on one side of the equation. Here is a sequence of equivalent equations that leads to the solution:

$$4x - 12 + 3x = 3 + 6x - 2x$$ collect like terms (1)

$$7x - 12 = 3 + 4x$$ move the x-terms to the left side by subtracting $4x$ and the constant terms to the right side by adding 12 (2)

$$7x - \cancel{12} - 4x + \cancel{12} = 3 + \cancel{4x} - \cancel{4x} + 12$$ (3)

$$7x - 4x = 3 + 12$$ collect like terms (4)

$$3x = 15$$ isolate x by dividing both sides by 3 (5)

$$\frac{\cancel{3x}}{\cancel{3}} = \frac{15}{3}$$ Simplify each side (6)

$$x = \mathbf{5}$$ (7)

Each equation is written underneath the previous one, with the **"=" symbol aligned in a column**

There is only **one "=" symbol** in each line

Let us analyze the relation between line (2) and (4).

$$7x \boxed{- 12} = 3 \boxed{+ 4x}$$ (2)

$$7x \boxed{- 4x} = 3 \boxed{+ 12}$$ (4)

By subtracting $4x$ from both sides of the equation (2), we actually 'moved' the term $+4x$ to the left side of equation (4) as $-4x$. Similarly, the addition of 12 to both sides of equation (2) caused the term -12 to 'move' to the other side as $+12$, in equation (4). This shows that the addition and subtraction property of equality allows us to change the position of a term from one side of an equation to another, by simply changing its sign. Although line (3) is helpful when explaining why we can move particular terms to another side by changing their signs, it is often cumbersome, especially when working with longer equations. So, in practice, we will avoid writing lines such as (3). Since it is important to indicate what operation is applied to the equation, we will record the operations performed in the right margin, after a slash symbol $(/)$. Here is how we could record the solution to equation (1) in a concise way.

$$4x - 12 + 3x = 3 + 6x - 2x$$

$$7x - 12 = 3 + 4x$$

$$7x - 4x = 3 + 12$$

$$3x = 15$$

$$x = \mathbf{5}$$

b. First, release all the brackets, starting from the inner-most brackets. If applicable, remember to collect like terms after releasing each bracket. Finally, isolate x by applying appropriate equality operations. Here is our solution:

$$2[3(x-6)-x] = 3x - 2(5-x) \qquad \text{release red brackets}$$

$$2[3x - 18 - x] = 3x - 10 + 2x \qquad \text{collect like terms}$$

$$2[2x - 18] = 5x - 10 \qquad \text{release the blue bracket}$$

$$4x - 36 = 5x - 10 \qquad / \; -5x, +36$$

$$-x = 26 \qquad / \; \div(-1)$$

$$x = \mathbf{-26}$$

multiplication by -1 works as well

Note: *Notice that we could choose to collect x-terms on the right side of the equation as well. This would shorten the solution by one line and save us the division by* -1. *Here is the alternative ending of the above solution.*

$$4x - 36 = 5x - 10 \qquad / \; -4x, +10$$

$$-26 = x$$

Example 5 ► **Solving Linear Equations Involving Fractions**

Solve

$$\frac{x-4}{4}+\frac{2x+1}{6}=5.$$

Solution ► First, clear the fractions and then solve the resulting equation as in *Example 4*. To clear fractions, multiply both sides of the equation by the LCD of 4 and 6, which is 12.

$$\frac{x-4}{4}+\frac{2x+1}{6}=5 \qquad / \cdot 12 \qquad (8)$$

$$\overset{3}{\cancel{12}}\left(\frac{x-4}{\cancel{4}}\right)+\overset{2}{\cancel{12}}\left(\frac{2x+1}{\cancel{6}}\right)=12\cdot 5 \qquad (9)$$

When multiplying each term by the $LCD = 12$, **simplify** it with the **denominator** before **multiplying** the result by the **numerator**.

$$3(x-4)+2(2x+1)=60 \qquad (10)$$

$$3x-12+4x+2=60 \qquad (11)$$

$$7x-10=60 \qquad (12)$$

$$7x=70 \qquad (13)$$

$$x=\frac{70}{7}=\mathbf{10} \qquad (14)$$

So the solution to the given equation is $x = \mathbf{10}$.

Note: *Notice, that if the division of 12 by 4 and then by 6 can be performed fluently in our minds, writing equation* (9) *is not necessary. One could write equation* (10) *directly after the original equation* (8). *One could think: 12 divided by 4 is 3 so I multiply the resulting 3 by the numerator* $(x-4)$. *Similarly, 12 divided by 6 is 2 so I multiply the resulting 2 by the numerator* $(2x+1)$. *It is important though that each term, including the free term 5, gets multiplied by 12.*

Also, notice that the reason we multiply equations involving fractions by LCD's is to clear the denominators of those fractions. That means that if the multiplication by an appropriate LCD is performed correctly, the resulting equation should not involve any denominators!

Example 6 ► **Solving Linear Equations Involving Decimals**

Solve $0.07x-0.03(15-x)=0.05(14)$.

Solution ► To solve this equation, it is convenient (although not necessary) to clear the decimals first. This is done by multiplying the given equation by 100.

Each **term** (product of numbers and variable expressions) needs to be multiplied by 100.

$$0.07x-0.03(15-x)=0.05(14) \qquad / \cdot 100$$

$$7x-3(15-x)=5(14)$$

$$7x-45+3x=70$$

$$10x = 70 + 45$$

$$x = \frac{115}{10} = \mathbf{11.5}$$

So the solution to the given equation is $x = \mathbf{11.5}$.

Note: *In general, if n is the highest number of decimal places to clear in an equation, we multiply it by* 10^n.

Attention: *To multiply a product AB by a number C, we multiply just one factor of this product, either A or B, but not both! For example,*

$$10 \cdot 0.3(0.5 - x) = (10 \cdot 0.3)(0.5 - x) = 3(0.5 - x) \quad \checkmark$$

or

$$10 \cdot 0.3(0.5 - x) = 0.3 \cdot [10(0.5 - x)] = 0.3(5 - 10x) \quad \checkmark$$

but

$$10 \cdot 0.3(0.5 - x) \neq (10 \cdot 0.3)[10(0.5 - x)] = 3(5 - 10x) \quad \times$$

Summary of Solving a Linear Equation in One Variable

- **Clear fractions or decimals**. Eliminate fractions by multiplying each side by the least common denominator (LCD). Eliminate decimals by multiplying by a power of 10.
- **Clear brackets** (starting from the inner-most ones) by applying the distributive property of multiplication. **Simplify** each side of the equation by **combining like terms**, as needed.
- **Collect** and **combine variable terms** on one side and free terms on the other side of the equation. Use the addition property of equality to collect all variable terms on one side of the equation and all free terms (numbers) on the other side.
- **Isolate the variable** by dividing the equation by the linear coefficient (coefficient of the variable term).

L.1 Exercises

True or False? Justify your answer.

1. The equation $5x - 1 = 9$ is equivalent to $5x - 5 = 5$.

2. The equation $x + \sqrt{x} = -1 + \sqrt{x}$ is equivalent to $x = -1$.

3. The solution set to $12x = 0$ is $\emptyset$.

4. The equation $x - 0.3x = 0.97x$ is an identity.

5. To solve $-\frac{2}{3}x = \frac{3}{5}$, we could multiply each side by the reciprocal of $-\frac{2}{3}$.

6. If a and b are real numbers, then $ax + b = 0$ has a solution.

*Decide whether each of the following is an **equation to solve** or an **expression to simplify**.*

7. $3x + 2(x - 6) - 1$

8. $3x + 2(x - 6) = 1$

9. $-5x + 19 = 3x - 5$

10. $-5x + 19 - 3x + 5$

Determine whether or not the given equation is linear.

11. $4x + 2 = x - 3$

12. $12 = x^2 + x$

13. $x + \frac{1}{x} = 1$

14. $2 = 5$

15. $\sqrt{16} = x$

16. $\sqrt{x} = 9$

Determine whether the given value is a solution of the equation.

17. $2,\ 3x - 4 = 2$

18. $-2,\ \frac{1}{x} - \frac{1}{2} = -1$

19. $6,\ \sqrt{2x + 4} = -4$

20. $-4,\ (x - 1)^2 = 25$

*Solve each equation. If applicable, tell whether the equation is an **identity** or a **contradiction**.*

21. $6x - 5 = 0$

22. $-2x + 5 = 0$

23. $-3x + 6 = 12$

24. $5x - 3 = -13$

25. $3y - 5 = 4 + 12y$

26. $9y - 4 = 14 + 15y$

27. $2(2a - 3) - 7 = 4a - 13$

28. $3(4 - 2b) = 4 - (6b - 8)$

29. $-3t + 5 = 4 - 3t$

30. $5p - 3 = 11 + 4p + p$

31. $13 - 9(2n + 3) = 4(6n + 1) - 15n$

32. $5(5n - 7) + 40 = 2n - 3(8n + 5)$

33. $3[1 - (4x - 5)] = 5(6 - 2x)$

34. $-4(3x + 7) = 2[9 - (7x + 10)]$

35. $3[5 - 3(4 - t)] - 2 = 5[3(5t - 4) + 8] - 16$

36. $6[7 - 4(8 - t)] - 13 = -5[3(5t - 4) + 8]$

37. $\frac{2}{3}(9n - 6) - 5 = \frac{2}{5}(30n - 25) - 7n$

38. $\frac{1}{2}(18 - 6n) + 5n = 10 - \frac{1}{4}(16n + 20)$

39. $\frac{8x}{3} - \frac{5x}{4} = -17$

40. $\frac{7x}{2} - \frac{5x}{10} = 5$

41. $\frac{3x-1}{4} + \frac{x+3}{6} = 3$

42. $\frac{3x+2}{7} - \frac{x+4}{5} = 2$

43. $\frac{2}{3}\left(\frac{7}{8} + 4x\right) - \frac{5}{8} = \frac{3}{8}$

44. $\frac{3}{4}\left(3x - \frac{1}{2}\right) - \frac{2}{3} = \frac{1}{3}$

45. $x - 2.3 = 0.08x + 3.5$

46. $x + 1.6 = 0.02x - 3.6$

47. $0.05x + 0.03(5000 - x) = 0.04 \cdot 5000$

48. $0.02x + 0.04 \cdot 3000 = 0.03(x + 3000)$

L2 Formulas and Applications

In the previous section, we studied how to solve linear equations. Those skills are often helpful in problem solving. However, the process of solving an application problem has many components. One of them is the ability to construct a mathematical model of the problem. This is usually done by observing the relationship between the variable quantities in the problem and writing an equation that describes this relationship.

Definition 2.1 ► An equation that represents or models a relationship between two or more quantities is called a **formula**.

To model real situations, we often use well-known formulas, such as $R \cdot T = D$, or $a^2 + b^2 = c^2$. However, sometimes we need to construct our own models.

Data Modelling

Example 1 ► **Constructing a Formula to Model a Set of Data Following a Linear Pattern**

In Santa Barbara, CA, a passenger taking a taxicab for a d-mile-long ride pays the fare of F dollars as per the table below.

distance $\boldsymbol{d}$ (in miles)	1	2	3	4	5
fare $\boldsymbol{F}$ (in dollars)	5.50	8.50	11.50	14.50	17.50

a. Write a formula that calculates fare F, in dollars, when distance driven d, in miles, is known.
b. Find the fare for a 16-mile ride by this taxi.
c. How long was the ride of a passenger who paid the fare of $29.50?

Solution ► **a.** Observe that the increase in fare when driving each additional mile after the first is constantly $3.00. This is because

$$17.5 - 14.5 = 14.5 - 11.5 = 11.5 - 8.5 = 8.5 - 5.5 = 3$$

If d represents the number of miles driven, then the number of miles after the first can be represented by $(d - 1)$. The fare for driving n miles is the cost of driving the first mile plus the cost of driving the additional miles, after the first one. So, we can write

$$\boldsymbol{fare\ F} = \begin{pmatrix} \textit{cost of the} \\ \textit{\textbf{first mile}} \end{pmatrix} + \begin{pmatrix} \textit{\textbf{cost increase}} \\ \textit{per mile} \end{pmatrix} \cdot \begin{pmatrix} \textit{\textbf{number of}} \\ \textit{additional \textbf{miles}} \end{pmatrix}$$

or symbolically,

$$F = 5.5 + 3(d - 1)$$

The above equation can be simplified to

$$F = 5.5 + 3d - 3 = 3d + 2.5.$$

Therefore, $\boldsymbol{F = 3d + 2.5}$ is the formula that models the given data.

b. Since the number of driven miles is $d = 16$, we evaluate

$$F = 3 \cdot 16 + 2.5 = 50.5$$

Therefore, the fare for a 16-mile ride is **\$50.50**.

c. This time, we are given the fare $F = 29.50$, and we are looking for the corresponding number of miles d. To find d, we substitute 31.7 for F in our formula $F = 3d + 2.5$ and then solve the resulting equation for d. We obtain

$$\begin{aligned} 29.5 &= 3d + 2.5 && /-2.5 \\ 27 &= 3d && /\div 3 \\ d &= 9 \end{aligned}$$

So, the ride was **9 miles** long.

Notice that in the solution to *Example 1c*, we could first solve the equation $F = 3d + 2.5$ for d:

$$\begin{aligned} F &= 3d + 2.5 && /-2.5 \\ F - 2.5 &= 3d && /\div 3 \\ \boldsymbol{d} &= \boldsymbol{\frac{F - 2.5}{3}}, \end{aligned}$$

and then use the resulting formula to evaluate d at $F = 29.50$.

$$d = \frac{29.5 - 2.5}{3} = \frac{27}{3} = 9.$$

The advantage of solving the formula $F = 3d + 2.5$ for the variable d first is such that the resulting formula $d = \frac{F-2.5}{3}$ makes evaluations of d for various values of F easier. For example, to find the number of miles d driven for the fare of \$35.5, we could evaluate directly using $d = \frac{35.5-2.5}{3} = \frac{33}{3} = 11$ rather than solving the equation $35.5 = 3d + 2.5$ again.

Solving Formulas for a Variable

If a formula is going to be used for repeated evaluation of a specific variable, it is convenient to rearrange this formula in such a way that the desired variable is **isolated on one side** of the equation and it does not appear on the other side. Such a formula may also be called a function.

Definition 2.2 ► A **function** is a rule for determining the value of one variable from the values of one or more other variables, in a unique way. We say that the first variable **is a function** of the other variable(s).

For example, consider the uniform motion relation between distance, rate, and time.

To evaluate rate when distance and time is given, we use the formula

$$\boldsymbol{R = \frac{D}{T}}.$$

This formula describes **rate as a function of distance and time**, as rate can be uniquely calculated for any possible input of distance and time.
To evaluate time when distance and rate is given, we use the formula

$$\boldsymbol{T = \frac{D}{R}}.$$

This formula describes **time as a function of distance and rate**, as time can be uniquely calculated for any possible input of distance and rate.
Finally, to evaluate distance when rate and time is given, we use the formula

$$\boldsymbol{D = R \cdot T}.$$

Here, the **distance is** presented as **a function of rate and time**, as it can be uniquely calculated for any possible input of rate and time.

To **solve a formula for a given variable** means to rearrange the formula so that the desired **variable equals to an expression that contains only <u>other</u> variables** but not the one that we solve for. This can be done the same way as when solving equations.

Here are some hints and guidelines to keep in mind when solving formulas:

- **Highlight** the variable of interest and solve the equation as if the other variables were just numbers (think of easy numbers), without actually performing the given operations.

  Example: To solve $mx + b = c$ for m,

  we pretend to solve, for example:

  $$m \cdot 2 + 3 = 1 \qquad /-3$$
  $$m \cdot 2 = 1 - 3 \qquad /\div 2$$
  $$m = \frac{1-3}{2}$$

  so we write:

  $$mx + b = c \qquad /-b$$
  $$mx = c - b \qquad /\div x$$
  $$\boldsymbol{m = \frac{c-b}{x}}$$

- **Reverse** (undo) **operations** to isolate the desired variable.

  Example: To solve $2L + 2W = P$ for W, first, observe the operations applied to W:

  $$W \xrightarrow{\cdot 2} 2W \xrightarrow{+2L} 2L + 2W$$

  Then, reverse these operations, starting from the last one first.

  $$W \xleftarrow[\div 2]{} 2W \xleftarrow[-2L]{} 2L + 2W$$

  So, we solve the formula as follows:

  $$2L + 2W = P \qquad /-2L$$
  $$2W = P - 2L \qquad /\div 2$$

$$\boldsymbol{W} = \frac{\boldsymbol{P - 2L}}{\boldsymbol{2}}$$

Notice that the last equation can also be written in the equivalent form $\boldsymbol{W} = \frac{\boldsymbol{P}}{\boldsymbol{2}} - \boldsymbol{L}$.

- **Keep the** desired **variable in the numerator**.

Example: To solve $R = \frac{D}{T}$ for T, we could take the reciprocal of each side of the equation to keep T in the numerator,

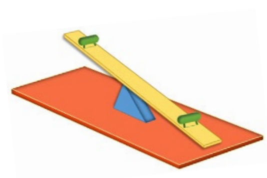
$$\frac{T}{D} = \frac{1}{R},$$

and then multiply by D to "undo" the division. Therefore, $\boldsymbol{T} = \frac{\boldsymbol{D}}{\boldsymbol{R}}$.

Observation: *Another way of solving $R = \frac{D}{T}$ for T is by multiplying both sides by T and dividing by R.*

$$\frac{T}{\cancel{R}} \cdot \cancel{R} = \frac{D}{\cancel{T}} \cdot \frac{\cancel{T}}{R}$$

This would also result in $\boldsymbol{T} = \frac{\boldsymbol{D}}{\boldsymbol{R}}$. Observe, that no matter how we solve this formula for T, the result differs from the original formula by interchanging (swapping) the variables T and R.



Note: *When working only with multiplication and division, by applying inverse operations, any factor of the numerator can be moved to the other side into the denominator, and likewise, any factor of the denominator can be moved to the other side into the numerator. Sometimes it helps to think of this movement of variables as the movement of a "teeter-totter".*

For example, the formula $\frac{bh}{2} = A$ can be solved for h by dividing by b and multiplying by 2. So, we can write directly $\boldsymbol{h} = \frac{\boldsymbol{2A}}{\boldsymbol{b}}$.

$\boldsymbol{2}$ was down so now goes up
and
$\boldsymbol{b}$ was up so now goes down

- **Keep the** desired **variable in one place.**

Example: To solve $A = P + Prt$ for P, we can factor P out,

$$A = P(1 + rt)$$

and then divide by the bracket. Thus,

$$\boldsymbol{P} = \frac{\boldsymbol{A}}{\boldsymbol{1 + rt}}.$$

Example 2 ► **Solving Formulas for a Variable**

Solve each formula for the indicated variable.

a. $a_n = a_1 + (n - 1)d$ for n

b. $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T

Solution ► **a.** To solve $a_n = a_1 + (n-1)d$ for n we use the reverse operations strategy, starting with reversing the addition, then multiplication, and finally the subtraction.

$$a_n = a_1 + (n-1)d \qquad /-a_1$$

$$a_n - a_1 = (n-1)d \qquad /\div d$$

$$\frac{a_n - a_1}{d} = n - 1 \qquad /+1$$

$$\boldsymbol{n = \frac{a_n - a_1}{d} + 1}$$

The last equation can also be written in the equivalent form $\boldsymbol{n = \frac{a_n - a_1 + d}{d}}$.

b. To solve $\frac{PV}{T} = \frac{P_0V_0}{T_0}$ for T, first, we can take the reciprocal of each side of the equation to keep T in the numerator,

$$\frac{T}{PV} = \frac{T_0}{P_0V_0},$$

and then multiply by PV to "undo" the division. So,

$$\boldsymbol{T = \frac{T_0PV}{P_0V_0}}.$$

Attention: *Taking reciprocal of each side of an equation is a good strategy only if both sides are in the form of a **single fraction**. For example, to use the reciprocal property when solving* $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ *for a, first, we perform the addition to create a single fraction,* $\frac{1}{a} = \frac{c+b}{bc}$. *Then, taking reciprocals of both sides will give us an instant result of* $a = \frac{bc}{c+b}$.

Warning! *The reciprocal of* $\frac{1}{b} + \frac{1}{c}$ *is **not** equal to* $b + c$.

Example 3 ► **Using Formulas in Application Problems**

To determine a healthy weight for a person's height, we can use the body mass index I given by the formula $\boldsymbol{I} = \frac{\boldsymbol{W}}{\boldsymbol{H^2}}$, where W represents weight, in kilograms, and H represents height, in meters. Weight is considered healthy when the index is in the range 18.5–24.9.

a. Adam is 182 cm tall and weighs 89 kg. What is his body index?

b. Barb has a body mass index of 24.5 and a height of 1.7 meters. What is her weight?

Solution ► **a.** Since the formula calls for height H is in meters, first, we convert 182 cm to 1.82 meters, and then we substitute $H = 1.82$ and $W = 89$ into the formula. So,

$$\boldsymbol{I} = \frac{89}{1.82^2} \approx \boldsymbol{26.9}$$

When rounded to one decimal place. Thus Adam is overweighted.

b. To find Barb's weight, we may want to solve the given formula for W first, and then plug in the given data. So, Barb's weight is

$$W = I\,H^2 = 24.5 \cdot 1.7^2 \approx \mathbf{70.8}\ \boldsymbol{kg}$$

Direct and Joined Variation

When two quantities vary proportionally, we say that there is a **direct variation** between them. For example, such a situation can be observed in the relation between time T and distance D covered by a car moving at a constant speed R. In particular, if $R = 60$ kph, we have

$$D = 60T$$

This relation tells us that the distance is 60 times larger than the time. Observe though that when the time doubles, the distance doubles as well. When the time triples, the distance also triples. So, the distance increases proportionally to the increase of the time. Such a **linear relation** between the two quantities is called a **direct variation**.

Definition 2.3 ► Two quantities, $\boldsymbol{x}$ and $\boldsymbol{y}$, are **directly proportional** to each other (there is a **direct variation** between them) iff there is a real constant $\boldsymbol{k \neq 0}$, such that

$$\boldsymbol{y = kx}.$$

We say that $\boldsymbol{y}$ **varies directly** as $\boldsymbol{x}$ with the **variation constant** $\boldsymbol{k}$.
(or equivalently: $\boldsymbol{y}$ is **directly proportional to** $\boldsymbol{x}$ with the **proportionality constant** $\boldsymbol{k}$.)

Example 4 ► **Solving Direct Variation Problems**

In scale drawing, the actual distance between two objects is directly proportional to the distance between the drawn objects. Suppose a kitchen room that is 4.6 meters long appears on a drawing as 2 cm long.

a. Find the direct variation equation that relates the actual distance D and the corresponding distance S on the drawing.

b. Find the actual dimensions of a 2.6 cm by 3.7 cm room on this drawing?

Solution ►

a. To find the direct variation equation that relates D and S, we need to find the variation constant k first. This can be done by substituting $D = 4.6$ and $S = 2$ into the equation $D = kS$. So, we obtain

$$4.6 = k \cdot 2 \qquad / \div 2$$

$$k = \frac{4.6}{2} = 2.3.$$

Therefore, the direct variation equation is $\boldsymbol{D = 2.3S}$.

b. To find the actual dimensions of a room that is drawn as 2.6 cm by 3.7 cm rectangle, we substitute these S-values into the above equation. This gives us

$$D = 2.3 \cdot 2.6 = 5.98 \quad \text{and} \quad D = 2.3 \cdot 3.7 = 8.51.$$

So, the actual dimensions of the room are $\mathbf{5.98}$ by $\mathbf{8.51}$ meters.

Sometimes a quantity varies directly as the n-th power of another quantity. For example, the formula $A = \pi r^2$ describes the direct variation between the area A of a circle and the square of the radius r of this circle. Here, the proportionality constant is π, while $n = 2$.

Extension: Generally, the fact that $\boldsymbol{y}$ **varies directly** as the $\boldsymbol{n}$**-th power** of $\boldsymbol{x}$ tells us that

$$\boldsymbol{y = kx^n},$$

for some nonzero constant $\boldsymbol{k}$.

Example 5 ► **Solving a Direct Variation Problem Involving the Square of a Variable**

Disregarding air resistance, the distance a body falls from rest is directly proportional to the square of the elapsed time. If a skydiver falls 24 meters in the first 2 seconds, how far will he fall in 5 seconds?

Solution ► Let d represents the distance the skydiver falls and t the time elapsed during this fall. Since d varies directly as t^2, we set the equation

$$d = kt^2$$

After substituting the data given in the problem, we find the value of k:

$$24 = k \cdot 2^2 \qquad / \div 4$$

$$k = \frac{24}{4} = 6$$

So, the direct variation equation is $d = 6t^2$. Hence, during 5 seconds the skydiver falls the distance $d = 6 \cdot 5^2 = 6 \cdot 25 = \mathbf{150}$ **meters**.

If one variable varies directly as the product of several other variables (possibly raised to some powers), we say that the first variable varies **jointly** as the other variables. For example, a joint variation can be observed in the formula for the area of a triangle, $A = \frac{1}{2}bh$, where the area A varies directly as the base b and directly as the height h of this triangle. We say that A varies jointly as b and h.

Definition 2.4 ► Variable $\boldsymbol{z}$ is **jointly proportional** to a set of variables (possibly raised to some powers) iff $\boldsymbol{z}$ is **directly proportional** to each of these variables (or equivalently: $\boldsymbol{z}$ is **directly proportional** to the product of these variables including their powers.)

Example:
z varies jointly as $\boldsymbol{x}$ and a **cube** of $\boldsymbol{y}$ iff $\boldsymbol{y = kxy^3}$, for some real constant $\boldsymbol{k \neq 0}$.

Example 6 ► **Solving Joint Variation Problems**

a. Kinetic energy is jointly proportional to the mass and the square of the velocity. Suppose a mass of 5 kilograms moving at a velocity of 4 meters per second has a kinetic energy of 40 joules.

b. Find the kinetic energy of a 3-kilogram ball moving at 6 meters per second.

c. Find the mass of an object that has 50 joules of kinetic energy when moving at 5 meters per second.

Solution ► **a.** Let E, m, and v represent respectively kinetic energy, mass, and velocity. Since E is jointly proportional to m and v^2, we set the equation

$$E = kmv^2.$$

Substituting the data given in the problem, we have

$$40 = k \cdot 5 \cdot 4^2, \qquad / \div 80$$

which gives us

$$k = \frac{40}{80} = \frac{1}{2}.$$

So, the joint variation equation is $E = \frac{1}{2}mv^2$. Hence, the kinetic energy of the 3-kilogram ball moving at 6 meters per second is $E = \frac{1}{2} \cdot 3 \cdot 6^2 = 3 \cdot 18 =$ **54 joules**.

b. First, we may want to solve the equation $E = \frac{1}{2}mv^2$ for m and then evaluate it using substitutions $E = 50$, and $v = 5$. So, we have

$$E = \frac{1}{2}mv^2 \qquad / \cdot 2$$

$$2E = mv^2 \qquad / \div v^2$$

reduce by 25

$$m = \frac{2E}{v^2} = \frac{2 \cdot \cancel{50}}{\cancel{5^2}} = 4$$

The mass of an object with the required parameters is **4 kilograms**.

L.2 Exercises

1. *When solving a formula for a particular variable, the answer can often be stated in various forms. Which of the following formulas are correct answers when solving* $A = \frac{a+b}{2}h$ *for* b*?*

   A. $b = \frac{2A-a}{h}$ **B.** $b = \frac{2A}{h} - a$ **C.** $b = \frac{2A-ah}{h}$ **D.** $b = \frac{A-ah}{\frac{1}{2}h}$

2. *Which of the following formulas are* ***not*** *correct answers when solving* $A = P + Prt$ *for* P*? Justify your answer.*

   A. $P = \frac{A}{rt}$ **B.** $P = \frac{A}{1+rt}$ **C.** $P = A - Prt$ **D.** $P = \frac{A-P}{rt}$

Solve each formula for the specified variable.

3. $I = Prt$ for r (*simple interest*)

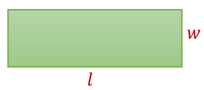
4. $C = 2\pi r$ for r (*circumference of a circle*)

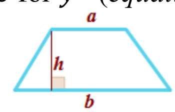
5. $E = mc^2$ for m (*mass-energy relation*)

6. $F = \frac{mv^2}{r}$ for m (*force in a circular motion*)

7. $A = \frac{(a+b)}{2}$ for b (*average*)

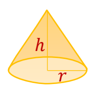
8. $Ax + By = c$ for y (*equation of a line*)

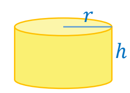




9. $P = 2l + 2w$ for l (*perimeter of a rectangle*)

10. $A = \frac{h}{2}(a + b)$ for a (*area of a trapezoid*)





11. $S = \pi rs + \pi r^2$ for π (*surface area of a cone*)

12. $S = 2\pi rh + 2\pi r^2$ for h (*surface area of a cylinder*)

13. $F = \frac{9}{5}C + 32$ for C (*Celsius to Fahrenheit*)

14. $C = \frac{9}{5}(F - 32)$ for F (*Fahrenheit to Celsius*)

15. $Q = \frac{p-q}{2}$ for p

16. $Q = \frac{p-q}{2}$ for q

17. $T = B + Bqt$ for q

18. $d = R - Rst$ for t

19. $d = R - Rst$ for R

20. $T = B + Bqt$ for B

Solve each problem.

21. On average, a passenger who drives n-kilometers in a taxicab in Abbotsford, BC, is charged C dollars as per the table below.

distance $\boldsymbol{n}$ (in kilometers)	1	2	3	4	5
cost $\boldsymbol{C}$ (in dollars)	5.10	7.00	8.90	10.80	12.70

a. Write a formula that calculates the cost C, in dollars, of driving a distance of n kilometers.
b. Find the cost for a 10-kilometer ride by this taxi.
c. If a passenger paid \$31.70, how far did he drive by this taxi?

22. On average, a passenger who drives n-kilometers in a taxicab in Vancouver, BC, is charged C dollars as per the table below.

distance $\boldsymbol{n}$ (in kilometers)	1	2	3	4	5
cost $\boldsymbol{C}$ (in dollars)	5.35	7.20	9.05	10.90	12.75

a. Write a formula that calculates the cost C, in dollars, of driving a distance of n kilometers.
b. Find the cost for a 20-kilometer ride by this taxi.
c. If a passenger paid \$22.00, how far did he drive by this taxi?

23. 

Assume that the amount of a medicine dosage for a child can be determined by the formula

$$c = \frac{ad}{a + 12},$$

where a represents the child's age, in years, and d represents the usual adult dosage, in milliliters.

a. If the adult dosage of a certain medication is 25 ml, what is the corresponding dosage for a three-year-old child?
b. Solve the formula for d.
c. Find the corresponding adult dosage, if a six-year-old child uses 5 ml of a certain medication.

24. The number of "full-time-equivalent" students, F, is often determined by the formula

$$F = \frac{n}{15},$$

where n represents the total number of credits taken by all students in a semester.
a. Suppose that in a particular institution students register for a total of 39,315 credits in one semester. What is the number of full-time-equivalent students in this institution?
b. Solve the formula for n.
c. Find the total number of credits students enroll in a semester if the number of full-time-equivalent students in this semester is 3254.

25. Suppose a cyclist can burn 530 calories in a 45-minute cycling session.

a. Write a formula that determines the number of calories C burned during two 45-minute sessions of cycling per day for d days.
b. According to this formula, how many calories would the cyclist burn in a week of cycling two 45-minute sessions per day?

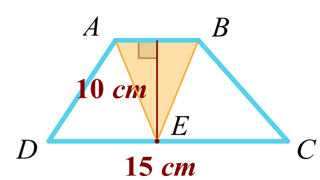
26. *Refer to information given in problem 25.*
a. On average, a person loses 1 kilogram for every 7000 calories burned. Write a formula that calculates the number of kilograms K lost in d days of cycling two 45-minute sessions per day.
b. How many kilograms could the cyclist lose in 30 days? *Round the answer to the nearest half of a kilogram.*

27. Express the width L of a rectangle in terms of its perimeter P and length W. *Here "in terms of P and W" means using an expression that involves only variables P and W.*

28. Express the area A of a circle in terms of its diameter d. *Here "in terms of d" means using an expression that involves only variable d.*

29. **a.** Solve the formula $I = Prt$ for t.
b. Using the formula from (**a**) determine how long it will take a deposit of \$125 to make the interest of \$15 when invested at 4% simple interest.

30. *Refer to information given in the accompanying figure.*
Find the area of the trapezoid $ABCD$ if its area is three times as large as the area of the shaded triangle ABE.



31. The number N of plastic bottles used each year is directly proportional to the number of people P using them.
a. Assuming that 150 people use 18,000 bottles in one year, find the variation constant and state the direct variation equation.
b. How many bottles are used each year in Vancouver, BC, which has a population of 631,490?

32. Under certain conditions, the volume V of a fixed amount of gas varies directly as its temperature T, in Kelvin degrees.
a. Assuming the gas in a hot-air balloon occupies 120 m^3 at 200 K, find the direct variation equation.
b. If the pressure of the gas remains constant, what would the volume of the gas be at 250 Kelvin degrees?

33. The recommended daily intake of fat varies directly as the number of calories consumed per day. Alice is on a 1200 diet and her healthy intake of fat is about 40 grams per day. Christopher needs about 2000 calories per day. To the nearest gram, what is his recommended daily intake of fat?

34. The distance covered by a falling object varies directly as the square of the elapsed time of the fall. A person flying in a hot-air balloon accidentally dropped a camera. Suppose the camera fell 16 meters during the first 2 seconds of the fall. If the camera hits the ground 5 seconds after it was dropped, how high was the balloon?

35. The air distance between Vancouver and Warsaw is 8,225 kilometers. The two cities are 45.2 centimeters apart on a desk globe. The air distance between Paris and Warsaw is 1367 kilometers. To the nearest millimeter, how far is Paris from Warsaw on this globe?

36. The stopping distance for a car varies directly as the square of its speed. If a car travelling 50 kilometers per hour requires 40 meters to stop, what would be the stopping distance for a car travelling 80 kilometers per hour?

37. The simple interest I varies jointly as the interest rate r and the principal P. Mother and daughter invested some money in simple interest accounts for the same period of time t. The daughter earns \$130 interest on the investment of \$2000 at 3.25%. What was the amount that the mother invested at 3.75% if she earns \$225 interest?

38. The lateral surface area (*surface area excluding the bases*) of a cylinder is jointly proportional to the height and radius of the cylinder. If a cylinder with radius 5 cm and height 8 cm has a lateral surface area of approximately 250 cm^2, what is the approximate lateral surface area of a can with a diameter of 6 cm and height of 12 cm?

39. The area of a triangle is jointly proportional to the height and the base of the triangle. If the base is increased by 50% and the height is decreased by 50%, how would the area of the triangle change?

40. The volume V of wood in a tree is directly proportional to the height h and the square of the **girth** (*circumference around the trunk*), g. Suppose the volume of a 20 meters tall tree with the girth of 1 meter is 64 cubic meters. To the nearest meter, find the height of a tree with a volume of 250 cubic meters and girth of 1.8 meters?

41. The number of barrels of oil used by a ship travelling at a constant speed is jointly proportional to the distance traveled and the square of the speed. If the ship uses 180 barrels of oil when travelling 200 miles at 40 miles per hour, find the number of barrels of oil needed for a ship that travels 300 miles at 25 miles per hour. *Round the answer to the nearest barrel.*

L3 Applications of Linear Equations

In this section, we study some strategies for solving problems with the use of linear equations, or well-known formulas. While there are many approaches to problem-solving, the following steps prove to be helpful.

Five Steps for Problem Solving
1. **Familiarize** yourself with the problem.
2. **Translate** the problem to a symbolic representation (usually an **equation** or an **inequality**).
3. **Solve** the equation(s) or the inequality(s).
4. **Check** if the answer makes sense in the original problem.
5. **State the answer** to the original problem clearly.

Here are some hints of how to **familiarize** yourself with the problem:

- **Read** the problem carefully a few times. In the first reading focus on the general setting of the problem. See if you can identify this problem as one of a motion, investment, geometry, age, mixture or solution, work, or a number problem, and draw from your experiences with these types of problems. During the second reading, focus on the specific information given in the problem, skipping unnecessary words, if possible.
- **List the information** given, including **units**, and check **what the problem asks for**.
- If applicable, **make a diagram** and label it with the given information.
- **Introduce a variable**(s) for the unknown quantity(ies). Make sure that the variable(s) is/are clearly defined (including units) by writing a "let" statement or labeling appropriate part(s) of the diagram. Choose descriptive letters for the variable(s). For example, let l be the length in centimeters, let t be the time in hours, etc.
- Express **other unknown values** in terms of the already introduced variable(s).
- Write applicable **formulas**.
- **Organize your data** in a meaningful way, for example by filling in a table associated with the applicable formula, inserting the data into an appropriate diagram, or listing them with respect to an observed pattern or rule.
- **Guess** a possible answer and check your guess. Observe the way in which the guess is checked. This may help you translate the problem into an equation.

Translation of English Phrases or Sentences to Expressions or Equations

One of the important phases of problem-solving is **translating** English words into a **symbolic representation**.

Here are the most commonly used **key words** suggesting a particular operation:

ADDITION ($+$)	SUBTRACTION ($-$)	MULTIPLICATION ($\cdot$)	DIVISION ($\div$)
sum	**difference**	**product**	**quotient**
plus	minus	multiply	divide
add	**subtract from**	times	ratio
total	**less than**	**of**	**out of**
more than	less	half of	per
increase by	decrease by	half as much as	shared
together	diminished	twice, triple	cut into
perimeter	shorter	area	

Example 1 ► **Translating English Words to an Algebraic Expression or Equation**

Translate the word description into an algebraic expression or equation.

a. The sum of half of a number and two
b. The square of a difference of two numbers
c. Triple a number, increased by five, is one less than twice the number.
d. The quotient of a number and seven diminished by the number
e. The quotient of a number and seven, diminished by the number
f. The perimeter of a rectangle is four less than its area.
g. In a package of 12 eggs, the ratio of white to brown eggs is one out of three.
h. Five percent of the area of a triangle whose base is one unit shorter than the height

Solution ► **a.** Let x represents "*a number*". Then

*The **sum of** **half of** a number **and** two* translates to $\frac{1}{2}x + 2$

Notice that the word "***sum***" indicates addition sign at the position of the word "***and***". Since addition is a binary operation (needs two inputs), we reserve space for "*half of a number*" on one side and "*two*" on the other side of the addition sign.

b. Suppose x and y are the "*two numbers*". Then

***The square of** a **difference of** two numbers* translates to $(x - y)^2$

Notice that we are squaring everything that comes after "*the square of*".

c. Let x represents "*a number*". Then

***Triple** a number, increased by five, **is** one **less than** **twice** the number.*

translates to the equation: $3x + 5 = 2x - 1$

This time, we translated a sentence that results in an equation rather than expression. Notice that the "equal" sign is used in place of the word "is". Also, remember that phrases "less than" or "subtracted from" work "backwards". For example, A **less than** B or A **subtracted from** B translates to $B - A$. However, the word "less" is used in the usual direction, from left to right. For example, A **less** B translates to $A - B$.

d. Let x represent "*a number*". Then

***The quotient of** a number **and** seven **diminished** by the number* translates to $\frac{x}{7-x}$

Notice that "**the** number" refers to the same number x.

e. Let x represent "*a number*". Then

***The quotient of** a number **and** seven, **diminished** by the number* translates to $\frac{x}{7} - x$

Here, the **comma** indicates the end of the "**quotient section**". So, we diminish the quotient rather than diminishing the seven, as in *Example 1d*.

f. Let l and w represent the length and the width of a rectangle. Then

*The **perimeter** of a rectangle **is** four **less than** its **area**.*

translates to the equation: $\mathbf{2l + 2w = lw} - 4$

Here, we use a formula for the perimeter ($\mathbf{2l + 2w}$) and for the area ($\mathbf{lw}$) of a rectangle.

g. Let w represent the number of white eggs in a package of 12 eggs. Then $(12 - w)$ represents the number of brown eggs in this package. Therefore,

*In a package of 12 eggs, **the ratio of the number of white eggs to the number of brown eggs is the same as two to three**.*

translates to the equation: $\frac{w}{12-w} = \frac{2}{3}$

Here, we expressed the unknown number of brown eggs $(12 - w)$ in terms of the number w of white eggs. Also, notice that the order of listing terms in a proportion is essential. Here, the first terms of the two ratios are written in the numerators (in blue) and the second terms (in brown) are written in the denominators.

h. Let h represent the height of a triangle. Since the base is *one unit shorter than the height,* we express it as $(h - 1)$. Using the formula $\frac{1}{2}bh$ for the area of a triangle, we translate

five percent of the area of a triangle** whose **base is one unit shorter than the height

to the expression: $\mathbf{0.05 \cdot \frac{1}{2}(h - 1)h}$

Here, we convert *five percent* to the number 0.05, as *per-cent* means *per hundred*, which tells us to divide 5 by a hundred.
Also, observe that the above word description is not a sentence, even though it contains the word "*is*". Therefore, the resulting symbolic form is an expression, not an equation. The word "*is*" relates the base and the height, which in turn allows us to substitute $(h - 1)$ in place of b, and obtain an expression in one variable.

So far, we provided some hints of how to familiarize ourself with a problem, we worked through some examples of how to translate word descriptions to a symbolic form, and we reviewed the process of solving linear equations (*see Section L1*). In the rest of this section, we will show various methods of solving commonly occurring types of problems, using representative examples.

Number Relation Problems

In number relation type of problems, we look for relations between quantities. Typically, we introduce a variable for one quantity and express the other quantities in terms of this variable following the relations given in the problem.

Example 2 ► **Solving a Number Relation Problem with Three Numbers**

The sum of three numbers is thirty-four. The second number is twice the first number, and the third number is one less than the second number. Find the three numbers.

Solution ► There are three unknown numbers such that their sum is thirty-four. This information allows us to write the equation

$$1^{\text{st}} \text{ number} + 2^{\text{nd}} \text{ number} + 3^{\text{rd}} \text{ number} = 34.$$

To solve such an equation, we wish to express all three unknown numbers in terms of one variable. Since the second number refers to the first, and the third number refers to the second, which in turn refers to the first, it is convenient to introduce a variable for the first number.
So, let n represent the first number.
The second number is twice the first, so $2n$ represents the second number.
The third number is one less than the second number, so $2n - 1$ represents the third number.

Therefore, our equation turns out to be

$$\begin{aligned} n + 2n + (2n - 1) &= 34 \\ 5n - 1 &= 34 && / +1 \\ 5n &= 35 && / \div 5 \\ n &= 7. \end{aligned}$$

Hence, the first number is **7**, the second number is $2n = 2 \cdot 7 = \mathbf{14}$, and the third number is $2n - 1 = 14 - 1 = \mathbf{13}$.

Consecutive Numbers Problems

Since **consecutive numbers** differ by one, we can represent them as $\boldsymbol{n, n+1, n+2}$, and so on.

Consecutive even or **consecutive odd** numbers differ by two, so both types of numbers can be represented by $\boldsymbol{n, n+2, n+4}$, and so on.

Notice that if the first number n is even, then $n + 2, n + 4, \ldots$ are also even; however, if the first number n is odd then $n + 2, n + 4, \ldots$ are also odd.

Example 3 ► **Solving a Consecutive Odd Integers Problem**

Find three consecutive odd integers such that three times the middle integer is five less than double the sum of the first and the third integer.

Solution ► Let the three consecutive odd numbers be called $n, n + 2$, and $n + 4$. We translate *three times the middle integer is five less than double the sum of the first and the third integer* into the equation

$$3(n + 2) = 2[n + (n + 4)] - 5$$

which gives

$$3n + 6 = 4n + 3 \qquad / -3, -3n$$

$$n = 3$$

Hence, the first number is **3**, the second number is $n + 2 = \mathbf{5}$, and the third number is $n + 4 = \mathbf{7}$.

Percent Problems

Rules to remember when solving percent problems:

$$1 = 100\% \quad \text{and} \quad \frac{\textit{is a part}}{\textit{of a whole}} = \frac{\%}{100}$$

Also, remember that

$$\textbf{\textit{percent increase}}(\textit{decrease}) = \frac{\textbf{\textit{last}} - \textbf{\textit{first}}}{\textbf{\textit{first}}} \cdot 100\%$$

Example 4 ► **Finding the Amount of Tax**

Kristin bought a new fridge for \$1712.48, including 12% of PST and GST tax. How much tax did she pay?

Solution ► Suppose the fridge costs p dollars. Then the tax paid for this fridge is 12% of p dollars, which can be represented by the expression $0.12p$. Since the total cost of the fridge including tax is \$1712.48, we set up the equation

$$p + 0.12p = 1712.48$$

which gives us

$$1.12p = 1712.48 \qquad / \div 1.12$$

$$p = 1529$$

The question calls for the amount of tax, so we calculate $0.12p = 0.12 \cdot 1529 = 183.48$.

Kristin paid \$183.48 of tax for the fridge.

Example 5 ► **Solving a Percent Increase Problem**

Susan got her hourly salary raised from \$11.50 per hour to \$12.75 per hour. To the nearest tenths of a percent, what was the percent increase in her hourly wage?

Solution ► We calculate the percent increase by following the rule $\frac{\textbf{\textit{last}}-\textbf{\textit{first}}}{\textbf{\textit{first}}} \cdot 100\%$.

So, Susan's hourly wage was increased by $\frac{12.75-11.50}{11.50} \cdot 100\% \approx \mathbf{10.9}\%$.

Investment Problems

When working with investment problems we often use the simple interest formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal (amount of money invested), r represents the interest rate, and t stands for the time in years.

Also, it is helpful to organize data in a diagram like this:

$$\textit{total amount}$$

account $\boldsymbol{I}$ account $\boldsymbol{II}$

$$\boldsymbol{interest}\ \textit{from account}\ \boldsymbol{I} + \boldsymbol{interest}\ \textit{from account}\ \boldsymbol{II} = \boldsymbol{total\ interest}$$

or

$$\boldsymbol{interest}\ \textit{from account}\ \boldsymbol{I} = \boldsymbol{interest}\ \textit{from account}\ \boldsymbol{II}$$

Example 6 ► **Solving an Investment Problem**

A student took out two student loans for a total of \$6000. One loan is at 5% annual interest and the other at 4.5% annual interest. If the total interest paid in a year is \$281, find the amount of each loan.

Solution ► To solve this problem in one equation, we would like to introduce only one variable. Suppose $\boldsymbol{x}$ is the amount of the first loan. Then the amount of the second loan is the remaining portion of the \$6000. So, it is $(6000 - \boldsymbol{x})$.

Using the simple interest formula $I = Prt$, for $t = 1$, we calculate the interest obtained from the 5% to be $0.05 \cdot \boldsymbol{x}$ and from the 4.5% account to be $0.045(6000 - \boldsymbol{x})$. Since the total interest equals to \$281, we set the equation as indicated in the diagram below.

$$\$6000$$

$\boldsymbol{x}$ at 5% $(6000 - \boldsymbol{x})$ at 4.5%

$$0.05 \cdot \boldsymbol{x} + 0.045(6000 - \boldsymbol{x}) = 281$$

For easier calculations, we may want to clear decimals by multiplying this equation by 1000.
This gives us

$$50x + 45(6000 - x) = 281000$$
$$50x + 270000 - 45x = 281000 \quad / -270000$$
$$5x = 11000 \quad / \div 5$$

and finally

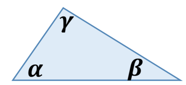
$$x = \$2200$$

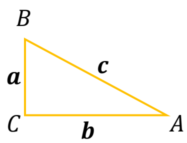
Thus, the first loan is **\$2200** and the second loan is $6000 - x = 6000 - 2200 = \$\mathbf{3800}$.

Geometry Problems

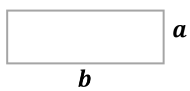
In geometry problems, we often use well-known formulas or facts that pertain to geometric figures. Here is a list of facts and formulas that are handy to know when solving various problems.

- The **sum of angles** in a triangle equals **180°**.

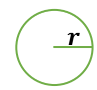


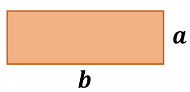


- The lengths of sides in a right-angle triangle ABC satisfy the **Pythagorean equation** $\boldsymbol{a^2 + b^2 = c^2}$, where c is the hypotenuse of the triangle.

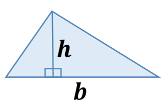


- The **perimeter of a rectangle** with sides a and b is given by the formula $\boldsymbol{2a + 2b}$.
- The **circumference** of a circle with radius r is given by the formula $\boldsymbol{2\pi r}$.

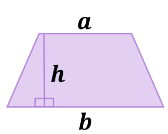




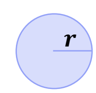
- The **area of** a **rectangle** or a **parallelogram** with base b and height h is given by the formula $\boldsymbol{bh}$.



- The **area of** a **triangle** with base b and height h is given by the formula $\boldsymbol{\frac{1}{2}bh}$.



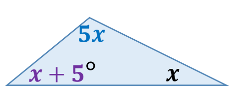
- The **area of** a **trapezoid** with bases a and b, and height h is given by the formula $\boldsymbol{\frac{1}{2}(a+b)h}$.
- The **area of** a **circle** with radius r is given by the formula πr^2.



Example 7 ► **Finding the Measure of Angles in a Triangle**

A cross section of a roof has a shape of a triangle. The largest angle of this triangle is five times as large as the smallest angle. The remaining angle is 5° greater than the smallest angle. Find the measure of each angle.

Solution ► Observe that the size of the largest and the remaining angle is compared to the size of the smallest angle. Therefore, it is convenient to introduce a variable, $\boldsymbol{x}$, for the measure of the smallest angle. Then, the expression for the measure of the largest angle, which is *five times as large as the smallest one*, is $\boldsymbol{5x}$ and the expression for the measure of the remaining angle, which is *5° greater than the smallest onet*, is $\boldsymbol{x + 5°}$. To visualize the situation, it might be helpful to draw a triangle and label the three angles.



Since the sum of angles in any triangle is equal to 180°, we set up the equation

$$x + 5x + x + 5° = 180°$$

This gives us

$$7x + 5° = 180° \qquad / -5°$$

$$7x = 175° \qquad / \div 5$$

$$x = 25°$$

So, the measure of the smallest angle is $\mathbf{25}°$,
the measure of the largest angle is $5x = 5 \cdot 25° = \mathbf{125}°$, and
the measure of the remaining angle is $x + 5° = 25° + 5° = \mathbf{30}°$.

Total Value Problems

When solving total value types of problems, it is helpful to organize the data in a table that compares the number of items and the value of these items. For example:

	item ***A***	item ***B***	total
number of items			
value of items			

Example 8 ► **Solving a Coin Problem**

The value of twenty-four coins consisting of dimes and quarters is $3.75. How many quarters are in the collection of coins?

Solution ► Suppose the number of quarters is n. Since the whole collection contains 24 coins, then the number of dimes can be represented by $24 - n$. Also, in cents, the value of n quarters is $25n$, while the value of $24 - n$ dimes is $10(24 - n)$. We can organize this information as in the table below.

	dimes	quarters	Total
number of coins	$24 - n$	n	24
value of coins (in cents)	$10(24 - n)$	$25n$	375

The value is written in cents!

Using the last row of this table, we set up the equation

$$10(24 - n) + 25n = 375$$

and then solve it for n.

$$240 - 10n + 25n = 375 \qquad / -240$$
$$15n = 135 \qquad / \div 15$$
$$n = 9$$

So, there are **9** quarters in the collection of coins.

Mixture-Solution Problems

When solving total mixture or solution problems, it is helpful to organize the data in a table that follows one of the formulas

$unit\ price \cdot number\ of\ units = total\ value$ or $percent \cdot volume = content$

	unit price ·	# of units	= value
type I			
type II			
mix			

	% ·	volume	= content
type I			
type II			
solution			

Example 9 ► **Solving a Mixture Problem**

Dark chocolate kisses costing \$13.50 per kilogram are going to be mixed with white chocolate kisses costing \$7.00 per kilogram. How many kilograms of each type of chocolate kisses should be used to obtain 30 kilograms of a mixture that costs \$10.90 per kilogram?

Solution ► In this problem, we mix two types of chocolate kisses: dark and white. Let x represent the number of kilograms of dark chocolate kisses. Since there are 30 kilograms of the mixture, we will express the number of kilograms of the white chocolate kisses as $30 - x$.

The information given in the problem can be organized as in the following table.

	unit price ·	# of units	= value (in \$)
dark kisses	13.50	x	$13.5x$
white kisses	7.00	$30 - x$	$7(30 - x)$
mix	10.90	30	327

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$13.5x + 7(30 - x) = 327$$

and then solve it for x.

$$13.5x + 210 - 7x = 327 \quad / -210$$
$$6.5x = 117 \quad / \div 6.5$$
$$x = 18$$

So, the mixture should consist of **18** kilograms of dark chocolate kisses and $30 - x = 30 - 18 = \mathbf{12}$ kilograms of white chocolate kisses.

Example 10 ► **Solving a Solution Problem**

How many milliliters of pure alcohol should be added to 80 ml of a 20% alcohol solution to make a 50% alcohol solution?

Solution ► Let x represent the volume of pure alcohol, in milliliters. The 50% solution is made by combining x ml of the pure alcohol with 80 ml of a 20% alcohol solution. So, the volume of the 50% solution can be expressed as $x + 80$.

Now, let us organize this information in the table below.

	% ·	volume	= acid
pure alcohol	1	x	x
20% solution	0.2	80	16
50% solution	0.5	$x + 80$	$0.5(x + 80)$

To complete the last column, multiply the first two columns.

Using the last column of this table, we set up the equation

$$x + 16 = 0.5(x + 80)$$

and then solve it for x.

$$x + 16 = 0.5x + 40 \qquad / -0.5x, -16$$
$$0.5x = 24 \qquad / \div 0.8$$
$$x = 12$$

So, there should be added **12** milliliters of pure alcohol.

Motion Problems

When solving motion problems, refer to the formula

$$\boldsymbol{Rate \cdot Time = Distance}$$

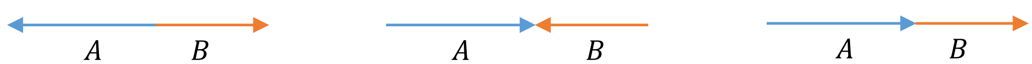
and organize date in a table like this:

	R ·	T	= D
motion I			
motion II			
total			

Some boxes in the "total" row are often left empty. For example, in motion problems, we usually do not add rates. Sometimes, the "total" row may not be used at all.

If two moving object (or two components of a motion) are analyzed, we usually encounter the following situations:

1. The two objects A and B move apart, approach each other, or move successively in the same direction (see the diagram below). In these cases, it is likely we are interested in the **total distance** covered. So, the last row in the above table will be useful to record the total values.

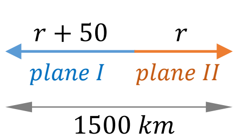


2. Both objects follow the same pathway. Then the **distances** corresponding to the two motions are the same and we may want to **equal** them. I such cases, there may not be any total values to consider, so the last row in the above table may not be used at all.

Example 11 ► **Solving a Motion Problem where Distances Add**

Two private planes take off from the same town and fly in opposite directions. The first plane is flying 50 km/h faster than the second one. In 3 hours, the planes are 1500 kilometers apart. Find the rate of each plane.

Solution ►



The rates of both planes are unknown. However, since the rate of the first plane is 50 km/h faster than the rate of the second plane, we can introduce only one variable. For example, suppose r represents the rate of the second plane. Then the rate of the first plane is represented by the expression $r + 50$.

In addition, notice that 1500 kilometers is the **total distance** covered by both planes, and 3 hours is the flight time of each plane.

Now, we can complete a table following the formula $R \cdot T = D$.

	R $\cdot$	T	$=$ D
plane I	$r + 50$	3	$3(r + 50)$
plane II	r	3	$3r$
total			1500

Notice that neither the total rate nor the total time was included here. This is because these values are not relevant to this particular problem. The equation that relates distances comes from the last column:

$$3(r + 50) + 3r = 1500$$

After solving it for r,

$$3r + 150 + 3r = 1500 \qquad / -150$$

$$6r = 1350 \qquad / \div 6$$

we obtain

$$r = 225$$

Therefore, the speed of the first plane is $r + 50 = 225 + 50 =$ **275 km/h** and the speed of the second plane is **225 km/h**.

Example 12 ► **Solving a Motion Problem where Distances are the Same**

A police officer spotted a speeding car moving at 120 km/h. Ten seconds later, the police officer starts chasing the car, travelling on a motorcycle at 140 km/h. How long does it take the police officer to catch the car?

Solution ► Let t represent the time, in minutes, needed for the police officer to catch the car. The time that the speeding car drives is 10 seconds longer than the time that the police officer drives. To match the denominations, we convert 10 seconds to $\frac{10}{60} = \frac{1}{6}$ of a minute. So, the time used by the car is $t + \frac{1}{6}$.

In addition, the rates are given in kilometers per hour, but we need to have them in kilometers per minute. So, we convert $\frac{120\ km}{1\ h} = \frac{120\ km}{60\ min} = 2\frac{km}{min}$, and similarly $\frac{140\ km}{1\ h} = \frac{140\ km}{60\ min} = \frac{7}{3}\frac{km}{min}$.

Now, we can complete a table that follows the formula $R \cdot T = D$.

	R $\cdot$	T	$=$ D
car	2	$t + \frac{1}{6}$	$2(t + \frac{1}{6})$
police	$\frac{7}{3}$	t	$\frac{7}{3}t$

Notice that this time there is no need for the "total" row.

Since distances covered by the car and the police officer are the same, we set up the equation

$$2\left(t + \frac{1}{6}\right) = \frac{7}{3}t \qquad / \cdot 3$$

To solve it for t, we may want to clear some fractions first. After multiplying by 3, we obtain

$$6\left(t+\frac{1}{6}\right)=7t$$

which becomes

$$6t+1=7t \qquad /\ -6t$$

and finally

$$1=t$$

So, the police officer needs one minute to catch this car.

Even though the above examples show a lot of ideas and methods used in solving specific types of problems, we should keep in mind that the best way to learn problem-solving is to **solve a lot of problems**. This is because every problem might present slightly different challenges than the ones that we have seen before. The more problems we solve, the more experience we gain, and with time, problem-solving becomes easier.

L.3 Exercises

Translate each word description into an algebraic expression or equation.

1. A number less seven

2. A number less than seven

3. Half of the sum of two numbers

4. Two out of all apples in the bag

5. The difference of squares of two numbers

6. The product of two consecutive numbers

7. The sum of three consecutive integers is 30.

8. Five more than a number is double the number.

9. The quotient of three times a number and 10

10. Three percent of a number decreased by a hundred

11. Three percent of a number, decreased by a hundred

12. The product of 8 more than a number and 5 less than the number

13. A number subtracted from the square of the number

14. The product of six and a number increased by twelve

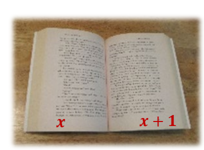
Solve each problem.

15. When the quotient of a number and 4 is added to twice the number, the result is 10 more than the number. Find the number.

16. When 25% of a number is added to 9, the result is 3 more than the number. Find the number.

17. The numbers on two adjacent safety deposit boxes add to 477. What are the numbers?

18. The sum of page numbers on two consecutive pages of a book is 543. What are the page numbers?



19. The number of international students at UFV, Abbotsford, BC, increased from 914 in the school year 2013/14 to 1708 in 2017/18. To the nearest tenths of a percent, what was the percent increase in enrollment during this time?

20. The number of domestic students at UFV, Abbotsford, BC, declined from 13762 in the school year 2013/14 to 12864 in 2017/18. To the nearest tenths of a percent, what was the percent decrease in enrollment during this time?

21. Find three consecutive odd integers such that the sum of the first, three times the second, and two times the third is 80.

22. Find three consecutive even integers such that the sum of the first, two times the second, and five times the third is 120.

23. Twice the sum of three consecutive odd integers is 210. Find the three integers.

24. Stephano paid $30,495 for a new Honda Civic. If this amount includes 7% of the sales tax, what is the cost of this car before tax?

25. After a 4% raise, the new monthly salary of a factory worker is $1924. What was the old monthly salary of this worker?

26. Jason bought a discounted fridge. The regular price of the fridge was $790.00, but he only paid $671.50. What was the percent discount?

27. The U.S. government issued about 156,000 patents in 2015. This was a decrease of about 1.7% from the number of patents issued in 2014. To the nearest hundred, how many patents were issued in 2014?

28. An investor has some funds at a 4% simple interest account and some at a 5% simple interest account. If the overall investment of $25000 gains $1134.00 of interest in one year, find the amount invested at each rate.

29. Jessie has $51,000 to invest. She plans to invest part of the money in an account paying 3% simple interest and the rest of the money into bonds paying 6.5% simple interest. How much should she invest at each rate to gain $3000 interest in a year?

30. Jan invested some money at 2.5% simple interest and twice this amount at 3.25%. Her total annual interest was $405. How much was invested at each rate?

31. Peter invested some money at 4.5% simple interest, and $2000 more than this amount at 5.25%. His total annual interest was $690. How much was invested at each rate?

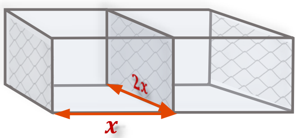
32. Daria invested $15,000 in bonds paying 6.5%. If she had some additional funds, she could invest in a saving account paying 2.5% simple interest. How much money would have to be invested at 2.5% for the average return on the two investments to be 5%?

33. Jack received a bonus payment of $12,000 and invested it in bonds paying 4.5% simple interest. If he had some additional funds, he could invest in a saving account paying 2.75% simple interest. How much additional money should he deposit in the 2.75% account so that his return on the two investments will be 4%?

34. A 126 cm long wire is cut into two pieces. Each piece is bent to form an equilateral triangle. If one triangle is twice as large as the other, how long are the sides of the triangles?

35. The measure of the smallest angle in a triangle is half the measure of the largest angle. The third angle is 15° less than the largest angle. Find the measure of each angle.

36. 35 ft of molding was used to trim a garage door. If the longer side of the door was 3 ft longer than twice the length of each of the shorter sides, then what are the dimensions of the door?



37. Billy plans to construct two adjacent rectangular outdoor cages for his rabbits. The cages would have open tops and bottoms, and share their longer side, as on the accompanying diagram. Each cage is planned to be twice as long as it is wide. If Billy has 80 ft of fencing, how large can the cages be?

38. The perimeter of a tennis court is 76 meters. The width of the court is 14 meters less than the length. Find the dimensions of the court.

39. Teresa inserted 16 coins into a vending machine to purchase a chocolate bar for \$1.25. If she used only dimes and nickels, how many of each type of coins did she use?

40. Robert used 12 coins consisting of dimes, nickels, and quarters to buy the *Vancouver Sun* for \$1.50. If he had twice as many dimes as nickels and the same many nickels as quarters, how many of each type of coins did he use?

41. A 30-kilogram mixture at \$25.28 per kilogram consists of pecans at \$27.50 per kilogram and cashews at \$23.80 per kilogram. How many kilograms of each were used to make the mixture?

42. A store owner bought 15 kilograms of peanuts for \$72. He wants to mix these peanuts with raisins costing \$7.50 per kilogram to get a mixture costing \$6 per kilogram. How many kilograms of raisins should he use?

43. Tickets to a movie theatre cost \$8.50 for an adult and \$3.50 for a child. If \$1253 were collected for selling a total of 178 tickets, how many of each type of tickets were sold?

44. Find the unit cost of a sunscreen made from 160 milliliters of lotion that cost \$1.49 per milliliter and 90 milliliters of lotion that cost \$2.49 per milliliter.

45. A tea mixture was prepared by mixing 20 kg of tea costing \$10.80 per kilogram with 30 kg of tea costing \$6.50 per pound. Find the unit cost of the tea mixture.

46. A pharmacist has 150 milliliters of a solution that contains 80% of a particular medication. How much pure water should he add to change the concentration of the medication to 25%?

47. How many grams of a 50% gold alloy must be mixed with 100 grams of an 80% gold alloy to make a 75% gold alloy?

48. A jeweller mixed 40 g of an 80% silver alloy with 60 g of a 25% silver alloy. What percent of silver contains the resulting alloy?

49. How many milliliters of water must be added to 250 ml of a 7% hydrogen peroxide solution to make a 3% hydrogen peroxide solution?

50. A car radiator contains 9 liters of a 50% antifreeze solution. How many liters need to be replaced with pure antifreeze to bring the antifreeze concentration to 80%?

51. Jessica is working with sulphuric acid solutions in a lab. She needs to dilute 50 milliliters of a 70% sulphuric acid solution to a 50% solution by mixing it with a 25% sulphuric acid solution. How many milliliters of a 25% solution should she use?

52. Two planes fly towards each other starting from two cities that are 4200 km apart. If one plane is travelling 150 km/h faster than the other and they pass each other after 2.5 hours, what is the speed of each plane?

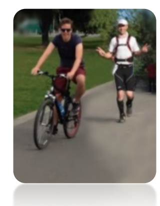
53. A plane flies at 630 km/h in still air. To the nearest minute, how long will it take the plane to travel 1000 kilometers

a. into a 90-km/h headwind?

b. with a 90-km/h tailwind?

54. At 7:00 am, Jacob left his house jogging at 10 km/h to a nearby park for his routine morning exercises. Six minutes later, his brother Andrew followed him using the same route. Running at 15 km/h, in how many minutes will Andrew catch up with his brother?

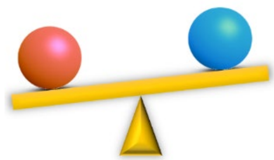
55. Tina walked at a rate of 8 km/h from home to a bike shop. She bought a bike there and rode it back home at a rate of 24 km/h. If the total time spent travelling was one hour, how far from Tina's home was the bike shop?



56. A jogger and a cyclist went to a park for their morning exercise. They start moving on a trail loop from the same point and in the same direction. On average, the cyclist travels three times as fast as the jogger. If after 20 minutes the cyclist has finished his first loop and the jogger has still 6 km to complete the loop, how long is the trail loop?

57. A 2 km long freight train is moving at 35 km/h on a straight segment of a train-track. Suppose a car is moving at 65 km/h on a parallel road, in the opposite direction. How long would it take the car to pass from the front till the end of the train?

L4 Linear Inequalities and Interval Notation



Mathematical inequalities are often used in everyday life situations. We observe speed limits on highways, minimum payments on credit card bills, maximum smartphone data usage per month, the amount of time we need to get from home to school, etc. When we think about these situations, we often refer to limits, such as "a speed limit of 100 kilometers per hour" or "a limit of 1 GB of data per month." However, we don't have to travel at exactly 100 kilometers per hour on the highway or use exactly 1 GB of data per month. The limit only establishes a boundary for what is allowable. For example, a driver travelling x kilometers per hour is obeying the speed limit of 100 kilometers per hour if $x \leq 100$ and breaking the speed limit if $x > 100$. A speed of $x = 100$ represents the boundary between obeying the speed limit and breaking it. Solving linear inequalities is closely related to solving linear equations because equality is the boundary between *greater than* and *less than*. In this section, we discuss techniques needed to solve linear inequalities and ways of presenting these solutions.

Linear Inequalities

Definition 4.1 ► A **linear inequality** is an inequality with only **constant** or **linear terms**.
A linear inequality in one variable can be written in one of the following forms:

$$\boldsymbol{Ax+B>0}, \quad \boldsymbol{Ax+B\geq 0}, \quad \boldsymbol{Ax+B<0}, \quad \boldsymbol{Ax+B\leq 0}, \quad \boldsymbol{Ax+B\neq 0},$$

for some real numbers A and B, and a variable x.

A variable value that makes an inequality true is called a **solution** to this inequality. We say that such variable value **satisfies** the inequality.

Example 1 ► **Determining if a Given Number is a Solution of an Inequality**

Determine whether each of the given values is a solution of the inequality.

a. $3x - 7 > -2$; 2, 1 **b.** $\frac{y}{2} - 6 \geq -3$; 8, 6

Solution ►

a. To check if 2 is a solution of $3x - 7 > -2$, replace x by 2 and determine whether the resulting inequality $3 \cdot 2 - 7 > -2$ is a true statement. Since $6 - 7 = -1$ is indeed larger than -2, then 2 satisfies the inequality. So 2 is a solution of $3x - 7 > -2$.

After replacing x by 1, we obtain $3 \cdot 1 - 7 > -2$, which simplifies to the false statement $-4 > -2$. This shows that 1 is not a solution of the given inequality.

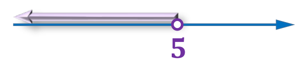
b. To check if 8 is a solution of $\frac{y}{2} - 6 \geq -3$, substitute $y = 8$. The inequality becomes $\frac{8}{2} - 6 \geq -3$, which simplifies to $-2 \geq -3$. Since this is a true statement, 8 is a solution of the given inequality.
Similarly, after substituting $y = 6$, we obtain a true statement $\frac{6}{2} - 6 \geq -3$, as the left side of this inequality equals to -3. This shows that -3 is also a solution to the original inequality.

Usually, an inequality has an infinite number of solutions. For example, one can check that the inequality

$$2x - 10 < 0$$

is satisfied by -5, 0, 1, 3, 4, 4.99, and generally by any number that is smaller than 5. So in the above example, the set of all solutions, called the **solution set**, is infinite. Generally, the solution set to a linear inequality in one variable can be stated either using **set-builder notation**, or **interval notation**. Particularly, the solution set of the above inequality could be stated as $\{x|x > 5\}$, or as $(-\infty, 5)$.

In addition, it is often beneficial to visualize solution sets of inequalities in one variable as graphs on a number line. The solution set to the above example would look like this:

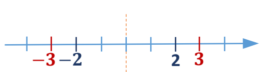


For more information about presenting solution sets of inequalities in the form of a graph or interval notation, refer to *Example 3* and the subsection on "*Interval Notation*" in *Section R2* of the *Review* chapter.

To solve an inequality means to find all the variable values that satisfy the inequality, which in turn means to find its solution set. Similarly as in the case of equations, we find these solutions by producing a sequence of simpler and simpler inequalities preserving the solution set, which eventually result in an inequality of one of the following forms:

$$x > constant, \quad x \geq constant, \quad x < constant, \quad x \leq constant, \quad x \neq constant.$$

Definition 4.2 ► **Equivalent inequalities** are inequalities with the same solution set.



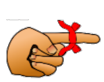
$2 < 3$ so $-2 > -3$

Figure 1

Generally, we create equivalent inequalities in the same way as we create equivalent equations, **except for *multiplying* or *dividing* an inequality by a *negative* number**. Then, we ***reverse the inequality symbol***, as illustrated in *Figure 1*.

So, if we multiply (or divide) the inequality

$$-x \geq 3$$



by -1, then we obtain an equivalent inequality

$$x \leq -3.$$

***multiplying* or *dividing* by a *negative* reverses the inequality sign**

We encourage the reader to confirm that the solution set to both of the above inequalities is $(-\infty, -3]$.

Multiplying or dividing an inequality by a positive number, leaves the inequality sign unchanged.

The table below summarizes the basic inequality operations that can be performed to produce equivalent inequalities, starting with $A < B$, where A and B are any algebraic expressions. Suppose C is a real number or another algebraic expression. Then, we have:

Inequality operation	General Rule	Example
Simplification	Write each expression in a simpler but equivalent form	$2(x-3) < 1+3$ can be written as $2x-6<4$
Addition	if $A < B$ then $A + C < B + C$	if $2x-6<4$ then $2x-6+6<4+6$
Subtraction	if $A < B$ then $A - C < B - C$	if $2x < x+4$ $2x - x < x + 4 - x$
Multiplication **when multiplying by a <u>negative</u> value, <u>reverse</u> the inequality sign**	if $C > 0$ and $A < B$ then $CA < CB$ if $C < 0$ and $A < B$ then $CA > CB$	if $2x < 10$ then $\frac{1}{2}\cdot 2x < \frac{1}{2}\cdot 10$ if $-x < -5$ then $x > 5$
Division **when dividing by a <u>negative</u> value, <u>reverse</u> the inequality sign**	if $C > 0$ and $A < B$ then $\frac{A}{C} < \frac{B}{C}$ if $C < 0$ and $A < B$ then $\frac{A}{C} > \frac{B}{C}$	if $2x < 10$ then $\frac{2x}{2} < \frac{10}{2}$ if $-2x < 10$ then $x > -5$

Example 2 ► **Using Inequality Operations to Solve Linear Inequalities in One Variable**

Solve the inequalities. Graph the solution set on a number line and state the answer in interval notation.

a. $\frac{3}{4}x + 3 > 15$ **b.** $-2(x+3) > 10$

c. $\frac{1}{2}x - 3 \le \frac{1}{4}x + 2$ **d.** $-\frac{2}{3}(2x-3) - \frac{1}{2} \ge \frac{1}{2}(5-x)$

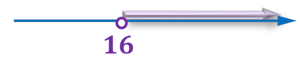
Solution ► **a.** To isolate x, we apply inverse operations in reverse order. So, first we subtract the 3, and then we multiply the inequality by the reciprocal of the leading coefficient. Thus,

$$\frac{3}{4}x + 3 > 15 \qquad / -3$$

$$\frac{3}{4}x > 12 \qquad / \cdot\frac{4}{3}$$

$$x > \frac{{}^{4}\cancel{12}\cdot 4}{\cancel{3}} = 16$$

To visualize the solution set of the inequality $x > 16$ on a number line, we graph the interval of all real numbers that are greater than 16.



Finally, we give the answer in interval notation by stating $\boldsymbol{x} \in (\mathbf{16}, \infty)$. This tells us that any x-value greater than 16 satisfies the original inequality.

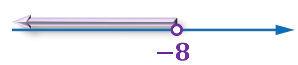
Note: The answer can be stated as $x \in (16, \infty)$, or simply as $(16, \infty)$. Both forms are correct.

b. Here, we will first simplify the left-hand side expression by expanding the bracket and then follow the steps as in *Example 2a.* Thus,

$$
\begin{aligned}
-2(x+3) &> 10 \\
-2x-6 &> 10 \qquad / -3 \\
-2x &> 16 \qquad / \div(-2) \\
x &< -8
\end{aligned}
$$

REVERSE the inequality when **dividing by a negative!**

The corresponding graph looks like this:

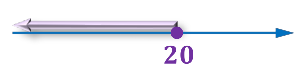


The solution set in interval notation is $(-\infty, -\mathbf{8})$.

c. To solve this inequality, we will collect and combine linear terms on the left-hand side and free terms on the right-hand side of the inequality.

$$
\begin{aligned}
\frac{1}{2}x-3 &\le \frac{1}{4}x+2 \qquad / -\frac{1}{4}x, +3 \\
\frac{1}{2}x-\frac{1}{4}x &\le 5 \\
\frac{1}{4}x &\le 5 \qquad / \cdot 4 \\
x &\le 20
\end{aligned}
$$

This can be graphed as



and stated in interval notation as $(-\infty, \mathbf{20}]$.

d. To solve this inequality, it would be beneficial to clear the fractions first. So, we will multiply the inequality by the LCD of 3 and 2, which is 6.

remember to multiply each term by 6, but only once!

$$
-\frac{2}{3}(2x-3)-\frac{1}{2} \le \frac{1}{2}(5-x) \qquad / \cdot 6
$$

$$
-\frac{2\cdot\overset{2}{\cancel{6}}}{\cancel{3}}(2x-3)-\frac{1\cdot\overset{3}{\cancel{6}}}{\cancel{2}} \le \frac{1\cdot\overset{3}{\cancel{6}}}{\cancel{2}}(5-x)
$$

$$
-4(2x-3)-3 \le 3(5-x)
$$

$$-8x + 12 - 3 \leq 15 - 3x$$

$$-8x + 9 \leq 15 - 3x \qquad / \; +8x, -15$$

At this point, we could collect linear terms on the left or on the right-hand side of the inequality. Since it is easier to work with a positive coefficient by the x-term, let us move the linear terms to the right-hand side this time.
So, we obtain

$$-6 \leq 5x \qquad / \div 5$$

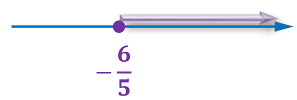
$$\frac{-6}{5} \leq x,$$

which after writing the inequality from right to left, gives us the final result

$$x \geq -\frac{6}{5}$$

$a \leq b$ can be written as $b \geq a$

The solution set can be graphed as



which means that all real numbers $x \in \left[-\frac{6}{5}, \infty\right)$ satisfy the original inequality.

Example 3 ► **Solving Special Cases of Linear Inequalities**

Solve each inequality.

a. $-2(x-3) > 5 - 2x$ **b.** $-12 + 2(3 + 4x) < 3(x - 6) + 5x$

Solution ► **a.** Solving the inequality

$$-2(x-3) > 5 - 2x$$

$$-2x + 6 > 5 - 2x \qquad / \; +2x$$

$$6 > 5,$$

ℝ

leads us to a true statement that does not depend on the variable x. This means that any real number x satisfies the inequality. Therefore, the solution set of the original inequality is equal to all real numbers $\mathbb{R}$. This could also be stated in interval notation as $(-\infty, \infty)$.

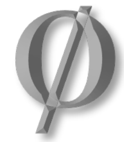
b. Solving the inequality

$$-12 + 2(3 + 4x) < 3(x - 6) + 5x$$

$$-12 + 6 + 8x < 3x - 18 + 5x$$

$$-6 + 8x < 8x - 18 \qquad / \; -8x$$

$$-6 < -18$$



leads us to a false statement that does not depend on the variable x. This means that no real number x would satisfy the inequality. Therefore, the solution set of the original inequality is an empty set $\emptyset$. We say that the inequality has **no solution**.

Three-Part Inequalities

The fact that an unknown quantity x lies between two given quantities a and b, where $a < b$, can be recorded with the use of the three-part inequality $a < x < b$. We say that x is enclosed by the values (or oscillates between the values) a and b. For example, the systolic high blood pressure p oscillates between 120 and 140 mm Hg. It is convenient to record this fact using the three-part inequality $120 < p < 140$, rather than saying that p < 140 and at the same time p > 120. The solution set of the three-part inequality $a < x < b$ or $b > x > a$ is a **bounded** interval (a, b) that can be graphed as

The hollow (open) dots indicate that thy endpoints do not belong to the solution set. Such interval is called **open**.

If the inequality symbol includes equation ($\leq$ or $\geq$), the corresponding endpoint of the interval is included in the solution set. On a graph, this is reflected as a solid (closed) dot. For example, the solution set of the three-part inequality $a \leq x < b$ is the interval $[a, b)$, which is graphed as

Such interval is called **half-open** or **half-closed**.

An interval with both endpoints included is referred to as **closed** interval. For example, $[a, b]$ is a closed interval and its graph looks like this

Any three-part inequality of the form

$$constant\ \boldsymbol{a} < (\leq)\ one\ variable\ linear\ \boldsymbol{expression} < (\leq)\ constant\ \boldsymbol{b},$$

where $a \leq b$ can be solved similarly as a single inequality, by applying inequality operations to all of the three parts. When solving such inequality, the goal is to isolate the variable in the middle part by moving all constants to the outside parts.

Example 4 ► **Solving Three-Part Inequalities**

Solve each three-part inequality. Graph the solution set on a number line and state the answer in interval notation.

a. $-2 \leq 1 - 3x \leq 3$ **b.** $-3 < \frac{2x-3}{4} \leq 6$

Solution ► **a.** To isolate x from the expression $1 - 3x$, subtract 1 first, and then divide by -3. These operations must be applied to all three parts of the inequality. So, we have

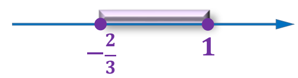
Remember to **reverse** both inequality symbols when **dividing by a negative** number!

$$-2 \le 1 - 3x \le 3 \qquad / -1$$

$$-3 \le -3x \le 2 \qquad / \div (-3)$$

$$1 \ge x \ge -\frac{2}{3}$$

The result can be graphed as



The inequality is satisfied by all $x \in \left[-\frac{2}{3}, 1\right]$.

b. To isolate x from the expression $\frac{2x-3}{4}$, we first multiply by 4, then add 3, and finally divide by 2.

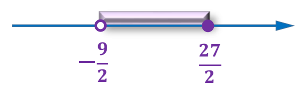
$$-3 < \frac{2x-3}{4} \le 6 \qquad / \cdot 4$$

$$-12 < 2x - 3 \le 24 \qquad / +3$$

$$-9 < 2x \le 27 \qquad / \div 2$$

$$-\frac{9}{2} < x \le \frac{27}{2}$$

The result can be graphed as



The inequality is satisfied by all $x \in \left(-\frac{9}{2}, \frac{27}{2}\right]$.

Inequalities in Application Problems

Linear inequalities are often used to solve problems in areas such as business, construction, design, science, or linear programming. The solution to a problem involving an inequality is generally an interval of real numbers. We often ask for the range of values that solve the problem.

Below is a list of common words and phrases indicating the use of particular inequality symbols.

Word expression	Interpretation
a is less (smaller) than ***b***	$a < b$
a is less than or equal to ***b***	$a \le b$
a is greater (more, bigger) than ***b***	$a > b$
a is greater than or equal to ***b***	$a \ge b$
a is at least ***b***	$a \ge b$
a is at most ***b***	$a \le b$
a is no less than ***b***	$a \ge b$
a is no more than ***b***	$a \le b$
a is exceeds ***b***	$a > b$
a is different than ***b***	$a \ne b$
x is between ***a*** and ***b***	$a < x < b$
x is between ***a*** and ***b*** inclusive	$a \le x \le b$

Example 5 ► **Translating English Words to an Inequality**

Translate the word description into an inequality and then solve it.

a. Twice a number, increased by 3 is at most 9.
b. Two diminished by five times a number is between -4 and 7

Solution ► **a.** Twice a number, increased by 3 translates to $2x+3$. Since "at most" corresponds to the symbol "$\leq$", the inequality to solve is

$$2x+3 \leq 9 \qquad / -3$$

$$2x \leq 6 \qquad / \div 2$$

$$x \leq 3$$

So, all $\boldsymbol{x} \in (-\infty, \mathbf{3}]$ satisfy the condition of the problem.

b. Two more than five times a number translates to $2-5x$. The phrase "between -4 and 7" tells us that the expression $2-5x$ is enclosed by the numbers -4 and 7, but not equal to these numbers. So, the inequality to solve is

$$-4 < 2-5x < 7 \qquad / -2$$

$$-6 < -5x < 5 \qquad / \div(-5)$$

$$\frac{6}{5} > x > -1$$

Therefore, the solution set to this problem is the interval of numbers $\left(-\mathbf{1}, \frac{\mathbf{6}}{\mathbf{5}}\right)$.

Remember: To record an interval, list its endpoints in ***increasing order*** (from the smaller to the larger number.)

Example 6 ► **Using a Linear Inequality to Compare Cellphone Charges**

A cellphone company advertises two pricing plans for day-time minutes. The first plan costs \$14.99 per month with 30 free day-time minutes and \$0.36 per minute after that. The second plan costs \$24.99 per month with 20 free day-time minutes and \$0.25 per minute after that. A customer figured that he will pay less by choosing the first plan. What could be the maximum number of day-time minutes that he predicts to use per month?

Solution ► Let n represent the number of cellphone minutes used per month. In the first plan, since the first 30 minutes are free, the number of paid-minutes can be represented by $n-30$. Hence, the total charge according to the first plan is $14.99+0.36(n-30)$. Similarly, in the second plan, the number of paid-minutes can be represented by $n-20$. Therefore the total charge according to the second plan is $24.99+0.25(n-20)$.

Since the first plan is to be cheaper, than the inequality to solve is

$$14.99+0.36(n-30) < 24.99+0.25(n-20).$$

To work with 'nicer' numbers, such as integers, we may want to eliminate the decimals by multiplying the above inequality by 100 first. Then, after removing the brackets via distribution, we obtain

$$1499 + 36n - 1080 < 2499 + 25n - 500 \qquad / \; -25n$$

$$419 + 11n < 1999 \qquad / \; -419$$

$$11n < 1580 \qquad / \; \div 11$$

$$n < \frac{1580}{11} \approx 143.6$$

So, the maximum number of day-time minutes for the first plan to be cheaper is **143**.

Example 7 ► **Finding the Test Score Range of the Missing Test**

Arek obtained 71% on a midterm test. If he wishes to bring his course mark to a B, the average of his midterm and final exam marks must be between 73% and 76%, inclusive. What range of scores on the final exam would guarantee Arek a mark of B in this course?

Solution ► Let n represent Arek's score on his final exam. Then, the average of his midterm and final exam is represented by the expression

$$\frac{71+n}{2}.$$

Since this average must be between 73% and 76% inclusive, we need to solve the three-part inequality

$$73 \le \frac{71+n}{2} \le 76 \qquad / \; \cdot 2$$

$$146 \le 71 + n \le 152 \qquad / \; -74$$

$$75 \le n \le 81$$

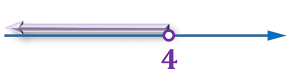
To attain a final grade of a B, Arek's score on his final exam should fall **between 75% and 81%**, inclusive.

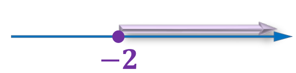
L.4 Exercises

Using interval notation, record the set of numbers represented by each graph. (Refer to the part "Interval Notation" in Section R2 of the Review chapter, if needed.)

1. 

2. 

3. 

4. 

Graph *each solution set. For each interval write the corresponding* ***inequality*** *(or inequalities), and for each inequality, write the solution set in* ***interval notation****. (Refer to the part "Interval Notation" in Section R2 of the Review chapter, if needed.)*

5. $(3, \infty)$ **6.** $(-\infty, 2]$ **7.** $[-7,5]$ **8.** $[-1, 4)$

9. $x \geq -5$ **10.** $x > 6$ **11.** $x < -2$ **12.** $x \leq 0$

13. $-4 < x < 1$ **14.** $3 \leq x < 7$ **15.** $-5 < x \leq -2$ **16.** $0 \leq x \leq 1$

Determine whether or not the given value is a solution to the inequality.

17. $4n + 15 > 6n + 20; \;\; -5$

18. $16 - 5a > 2a + 9; \;\; 1$

19. $\frac{x}{4} + 7 \geq 5; \;\; -8$

20. $6y - 7 \leq 2 - y; \;\; \frac{2}{3}$

Solve *each inequality.* ***Graph*** *the solution set and write the solution using* ***interval notation****.*

21. $2 - 3x \geq -4$

22. $4x - 6 > 12 - 10x$

23. $\frac{3}{5}x > 9$

24. $-\frac{2}{3}x \leq 12$

25. $5(x + 3) - 2(x - 4) \geq 2(x + 7)$

26. $5(y + 3) + 9 < 3(y - 2) + 6$

27. $2(3x - 4) - 4x \leq 2x + 3$

28. $7(4 - x) + 5x > 2(16 - x)$

29. $\frac{4}{5}(7x + 6) > 40$

30. $\frac{2}{3}(4x - 3) \leq 30$

31. $\frac{5}{2}(2a - 3) < \frac{1}{3}(6 - 2a)$

32. $\frac{2}{3}(3x - 1) \geq \frac{3}{2}(2x - 3)$

33. $\frac{5-2x}{2} \geq \frac{2x+1}{4}$

34. $\frac{3x-2}{-2} \geq \frac{x-4}{-5}$

35. $0.05 + 0.08x < 0.01x - 0.04(3 - 3x)$

36. $-0.2(5x + 2) > 0.4 + 1.5x$

37. $-\frac{1}{4}(p + 6) + \frac{3}{2}(2p - 5) \leq 10$

38. $\frac{3}{5}(t - 2) - \frac{1}{4}(2t - 7) \leq 3$

39. $-6 \leq 5x - 7 \leq 4$

40. $-10 < 3b - 5 < -1$

41. $2 \leq -3m - 7 \leq 4$

42. $4 < -9x + 5 < 8$

43. $-\frac{1}{2} < \frac{1}{4}x - 3 < \frac{1}{2}$

44. $-\frac{2}{3} \leq 4 - \frac{1}{4}x \leq \frac{2}{3}$

45. $-3 \leq \frac{7-3x}{2} < 5$

46. $-7 < \frac{3-2x}{3} \leq -2$

Using interval notation, state the set of numbers satisfying each description.

47. The sum of a number and 5 exceeds 12.

48. 5 times a number, decreased by 6, is smaller than -16.

49. 2 more than three times a number is at least 8.

50. Triple a number, subtracted from 5, is at most 7.

51. Half of a number increased by 3 is no more than 12.

52. Twice a number increased by 1 is different than 14.

53. Double a number is between -6 and 8.

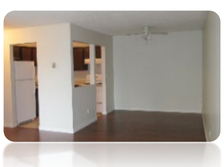
54. Half a number, decreased by 3, is between 1 and 12.

Solve each problem.

55. There are three major tests in the algebra course that Nicole takes. She already wrote the first two tests and received 79% and 89% respectively. What score must she aim for when writing her third test to keep an average test mark of 85% or higher?

56. To receive a B in a university course, the average mark needs to be between 73 and 76, inclusive. Suppose a final grade in a particular course is calculated by taking average of the four major tests, including the final exam. On the first three tests, a student obtained the following scores: 59, 71, and 86. What range of scores on the final exam will guarantee the student a B in this course?

57. A marketing company has a budget of \$1400 to run an advertisement on a particular website. The website charges \$10.50 per day to display the add and \$220 set up fee. Maximally, how many days the ad can be posted on this site?



58. Ken plans to paint a room with 340 square feet of wall area. He needs to buy some masking tape, drop sheets, and paint brushes for a total of \$32. A gallon of paint covers 100 square feet of area, and the paint is sold only in gallons. If Ken plans to stay within \$150 for the whole job what is the maximum cost per gallon of paint that he can afford?

59. Last week, the temperature in Banff, BC, ranged between 23°F and 68°F. Using the formula $F = \frac{9}{5}C + 32$, find the temperature range in degrees Celsius.

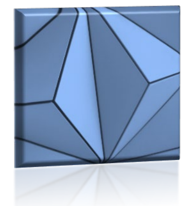
60. One day, the temperature range in Whistler, BC, was between -2°C and 18°C. Using the formula $C = \frac{5}{9}(F - 32)$, find the temperature range in degrees Fahrenheit.

61. Adon makes \$1600 a month with an additional commission of 7% of his sales. This month, Adon's earnings were higher than \$3700. How high must have been his sales?

62. Suppose a particular bank offers two chequing accounts. The first account charges \$5 per month and \$0.75 per cheque after the first 10 cheques. The second account charges \$12.50 per month with unlimited cheque writing. What is the maximum number of cheques processed for a customer who chooses the first account as the better option?

63. The Toronto Dominion Bank offers a chequing account that charges \$10.95 per month plus \$1.25 per cheque after the first 25 cheques. A competitor bank is offering an account for \$7.95 per month plus \$1.30 per cheque after the first 25 cheques. If a business chooses the first account as the cheaper option, what is the minimum number of cheques that the business predicts to write monthly?

L5 Operations on Sets and Compound Inequalities

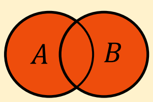


In the previous section, it was shown how to solve a three-part inequality. In this section, we will study how to solve systems of inequalities, often called **compound** inequalities, that consist of two linear inequalities joined by the words "*and*" or "*or*". Studying compound inequalities is an extension of studying three-part inequalities. For example, the three-part inequality, $2 < x \leq 5$, is in fact a system of two inequalities, $2 < x$ *and* $x \leq 5$. The solution set to this system of inequalities consists of all numbers that are larger than 2 *and* at the same time are smaller or equal to 5. However, notice that the system of the same two inequalities connected by the word "*or*", $2 < x$ *or* $x \leq 5$, is satisfied by any real number. This is because any real number is either larger than 2 *or* smaller than 5. Thus, to find solutions to compound inequalities, aside for solving each inequality individually, we need to pay attention to the joining words, "*and*" or "*or*". These words suggest particular operations on the sets of solutions.

Operations on Sets

Sets can be added, subtracted, or multiplied.

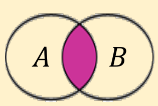
Definition 5.1 ► The result of the addition of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **union** (or sum), symbolized by $\boldsymbol{A} \cup \boldsymbol{B}$, and defined as:



$$\boldsymbol{A \cup B = \{x | x \in A \text{ or } x \in B\}}$$

This is the set of all elements that belong to either $\boldsymbol{A}$ ***or*** $\boldsymbol{B}$.

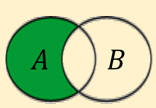
The result of the multiplication of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called an **intersection** (or product, or **common part**), symbolized by $\boldsymbol{A} \cap \boldsymbol{B}$, and defined as:



$$\boldsymbol{A \cap B = \{x | x \in A \text{ and } x \in B\}}$$

This is the set of all elements that belong to both $\boldsymbol{A}$ ***and*** $\boldsymbol{B}$.

The result of the subtraction of two sets $\boldsymbol{A}$ and $\boldsymbol{B}$ is called a **difference**, symbolized by $\boldsymbol{A} \setminus \boldsymbol{B}$, and defined as:



$$\boldsymbol{A \setminus B = \{x | x \in A \text{ and } x \notin B\}}$$

This is the set of all elements that belong to $\boldsymbol{A}$ ***and*** do not belong to $\boldsymbol{B}$.

Example 1 ► **Performing Operations on Sets**

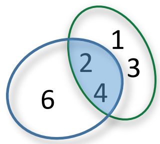
Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, and $C = \{6\}$. Find the following sets:

a. $A \cap B$ **b.** $A \cup B$

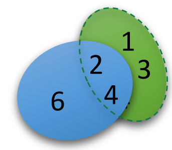
c. $A \setminus B$ **d.** $B \cup C$

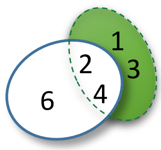
e. $B \cap C$ **f.** $A \cap C$

Solution ► **a.** The intersection of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to both sets, so $\boldsymbol{A \cap B = \{2, 4\}}$.



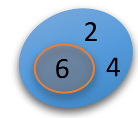
b. The union of sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{A \cup B = \{1, 2, 3, 4, 6\}}$.



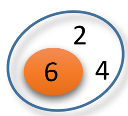


c. The difference $A \setminus B$ of sets $A = \{1, \cancel{2}, 3, \cancel{4}\}$ and $B = \{2, 4, 6\}$ consists of numbers that belong to the set A but do not belong to the set B, so $\boldsymbol{A \setminus B = \{1, 3\}}$.

d. The union of sets $B = \{2, 4, 6\}$, and $C = \{6\}$ consists of numbers that belong to at least one of the sets, so $\boldsymbol{B \cup C = \{2, 4, 6\}}$.



Notice that $B \cup C = B$. This is because C is a **subset** of B.

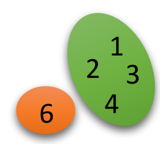


e. The intersection of sets $B = \{2, 4, 6\}$ and $C = \{6\}$ consists of numbers that belong to both sets, so $\boldsymbol{B \cap C = \{6\}}$.

Notice that $B \cap C = C$. This is because C is a **subset** of B.

f. Since the sets $A = \{1, 2, 3, 4\}$ and $C = \{6\}$ do not have any common elements, then $\boldsymbol{A \cap C} = \emptyset$.

Recall: Two sets with no common part are called **disjoint**.
Thus, the sets A and C are disjoint.

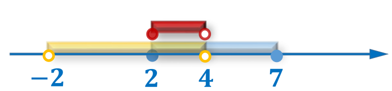


Example 2 ► **Finding Intersections and Unions of Intervals**

Write the result of each set operation as a single interval, if possible.

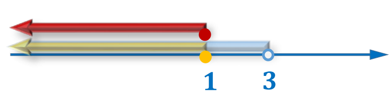
a. $(-2, 4) \cap [2, 7]$ **b.** $(-\infty, 1] \cap (-\infty, 3)$
c. $(-1, 3) \cup (1, 6]$ **d.** $(3, \infty) \cup [5, \infty)$
e. $(-\infty, 3) \cup (4, \infty)$ **f.** $(-\infty, 3) \cap (4, \infty)$
g. $(-\infty, 5) \cup (4, \infty)$ **h.** $(-\infty, 3] \cap [3, \infty)$

Solution ► **a.** The interval of points that belong to both, the interval $(-2, 4)$, in yellow, and the interval $[2, 7]$, in blue, is marked in red in the graph below.



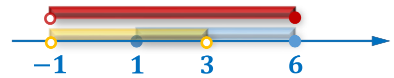
So, we write $(-2, 4) \cap [2, 7] = [2, 4)$.

b. As in problem **a.**, the common part $(-\infty, 1] \cap (-\infty, 3)$ is illustrated in red on the graph below.



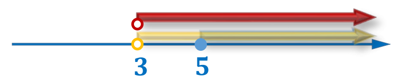
So, we write $(-\infty, 1] \cap (-\infty, 3) = (-\infty, 1]$.

c. This time, we take the union of the interval $(-1,3)$, in yellow, and the interval $(1,6]$, in blue. The result is illustrated in red on the graph below.



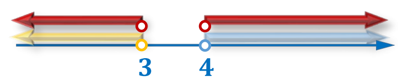
So, we write $(-1,3) \cup (1,6] = (-1,6]$.

d. The union $(3,\infty) \cup [5,\infty)$ is illustrated in red on the graph below.



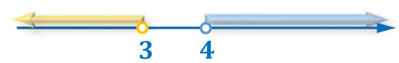
So, we write $(3,\infty) \cup [5,\infty) = (3,\infty)$.

e. As illustrated in the graph below, this time, the union $(-\infty,3) \cup (4,\infty)$ can't be written in the form of a single interval.



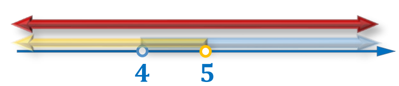
So, the expression $(-\infty,3) \cup (4,\infty)$ cannot be simplified.

f. As shown in the graph below, the interval $(-\infty,3)$ has no common part with the interval $(4,\infty)$.



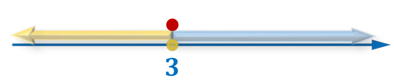
Therefore, $(-\infty,3) \cap (4,\infty) = \emptyset$

g. This time, the union $(-\infty,5) \cup (4,\infty)$ covers the entire number line.



Therefore, $(-\infty,5) \cup (4,\infty) = (-\infty,\infty)$

h. As shown in the graph below, there is only one point common to both intervals, $(-\infty,3]$ and $[3,\infty)$. This is the number 3.



Since a single number is not considered to be an interval, we should use a set rather than interval notation when recording the answer. So, $(-\infty,3] \cap [3,\infty) = \{3\}$.

Compound Inequalities

The solution set to a system of two inequalities joined by the word *and* is the intersection of solutions of each inequality in the system. For example, the solution set to the system

$$\begin{cases} x > 1 \\ x \leq 4 \end{cases}$$

is the intersection of the solution set for $x > 1$ and the solution set for $x \leq 4$. This means that the solution to the above system equals $(1, \infty) \cap (-\infty, 4] = (1, 4]$, as illustrated in the graph below.

The solution set to a system of two inequalities joined by the word *or* is the union of solutions of each inequality in the system. For example, the solution set to the system

$$x \leq 1 \text{ or } x > 4$$

is the union of the solution of $x \leq 1$ and, the solution of $x > 4$. This means that the solution to the above system equals $(-\infty, 1) \cup [4, \infty)$, as illustrated in the graph below.

Example 4 ► **Solving Compound Linear Inequalities**

Solve each compound inequality. Pay attention to the joining word *and* or *or* to find the overall solution set. Give the solution set in both interval and graph form.

a. $3x + 7 \geq 4 \quad \textit{and} \quad 2x - 5 < -1$

b. $-2x - 5 \geq 1 \quad \textit{or} \quad x - 5 \geq -3$

c. $3x - 11 < 4 \quad \textit{or} \quad 4x + 9 \geq 1$

d. $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$

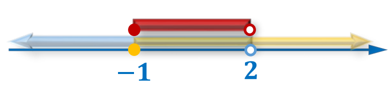
e. $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$

f. $4x - 2 < -8 \quad \textit{or} \quad 5x - 3 < 12$

Solution ► **a.** To solve this system of inequalities, first, we solve each individual inequality, keeping in mind the joining word *and*. So, we have

$$\begin{array}{cccccc} 3x + 7 \geq 4 & & \textit{and} & 2x - 5 < -1 & \\ & / -7 & & & / +5 \\ 3x \geq -3 & & \textit{and} & 2x < 4 & \\ & / \div 3 & & & / \div 2 \\ x \geq -1 & & \textit{and} & x < 2 & \end{array}$$

The joining word *and* indicates that we look for the intersection of the obtained solutions. These solutions (in yellow and blue) and their intersection (in red) are shown in the graphed below.



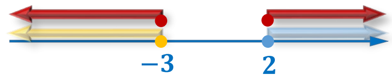
Therefore, the system of inequalities is satisfied by all $x \in [-\mathbf{1}, \mathbf{2})$.

b. As in the previous example, first, we solve each individual inequality, except this time we keep in mind the joining word *or*. So, we have

$$-2x - 5 \geq 1 \quad / +5 \qquad \textit{or} \qquad x - 5 \geq -3 \quad / +5$$

$$-2x \geq 6 \quad / \div (-2)\,! \qquad or \qquad x \geq 2$$
$$x \leq -3$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.

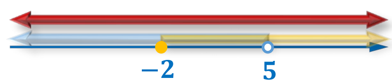


Therefore, the system of inequalities is satisfied by all $x \in (-\infty, -\mathbf{3}] \cup [\mathbf{2}, \infty)$.

c. As before, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$3x - 11 < 4 \quad / +11 \qquad or \qquad 4x + 9 \geq 1 \quad / -9$$
$$3x < 15 \quad / \div 3 \qquad or \qquad 4x \geq -8 \quad / \div 4$$
$$x < 5 \qquad or \qquad x \geq -2$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the system of inequalities is satisfied by all real numbers. The solution set equals to $\mathbb{R}$.

d. Any three-part inequality is a system of inequalities with the joining word *and*. The system $-2 < 3 - \frac{1}{4}x < \frac{1}{2}$ could be written as

$$-2 < 3 - \frac{1}{4}x \quad and \quad 3 - \frac{1}{4}x < \frac{1}{2}$$

and solved as in *Example 4a*. Alternatively, it could be solved in the three-part form, similarly as in *Section L4, Example 4*. Here is the three-part form solution.

$$-2 < 3 - \frac{1}{4}x < \frac{1}{2} \qquad / -3$$
$$-5 < -\frac{1}{4}x < \frac{1}{2} - \frac{3 \cdot 2}{2}$$
$$-5 < -\frac{1}{4}x < -\frac{5}{2} \qquad / \cdot (-4)$$

$$20 > x > \frac{5 \cdot 4}{2}$$
$$20 > x > 10$$

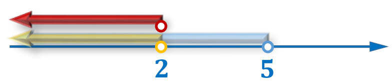
So the solution set is the interval $(10, 20)$, visualized in the graph below.

Remark: Solving a system of inequalities in three-part form has its benefits. First, the same operations are applied to all three parts, which eliminates the necessity of repeating the solving process for the second inequality. Second, the solving process of a three-part inequality produces the final interval of solutions rather than two intervals that need to be intersected to obtain the final solution set.

e. The system $\begin{cases} 4x - 7 < 1 \\ 7 - 3x > -8 \end{cases}$ consists of two inequalities joined by the word *and*. So, we solve it similarly as in *Example 4a*.

$$\begin{array}{cccccc} 4x - 7 < 1 & / +7 & \textit{and} & 7 - 3x > -8 & / -7 \\ 4x < 8 & / \div 4 & \textit{and} & -3x > -15 & / \div (-3) \\ x < 2 & & \textit{and} & x < 5 & \end{array}$$

These solutions of each individual inequality (in yellow and blue) and the intersection of these solutions (in red) are indicated in the graph below.

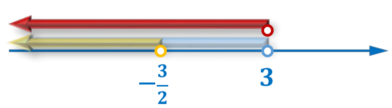


Therefore, the interval $(-\infty, 2)$ is the solution to the whole system.

f. As in *Example 4b* and *4c*, we solve each individual inequality, keeping in mind the joining word *or*. So, we have

$$\begin{array}{cccccc} 4x - 2 < -8 & / +2 & \textit{or} & 5x - 3 < 12 & / +3 \\ 4x \geq -6 & / \div 4 & \textit{or} & 5x \geq 15 & / \div 5 \\ x < -\frac{3}{2} & & \textit{or} & x < 3 & \end{array}$$

The joining word *or* indicates that we look for the union of the obtained solutions. These solutions (in yellow and blue) and their union (in red) are indicated in the graph below.



Therefore, the interval $(-\infty, 3)$ is the solution to the whole system.

Compound Inequalities in Application Problems

Compound inequalities are often used to solve problems that ask for a range of values satisfying certain conditions.

Example 5 ► **Finding the Range of Values Satisfying Conditions of a Problem**

The equation $P = 1 + \frac{d}{11}$ gives the pressure P, in atmospheres (atm), at a depth of d meters in the ocean. Atlantic cod occupy waters with pressures between 1.6 and 7 atmospheres. To the nearest meter, what is the depth range at which Atlantic cod should be searched for?

Solution ► The pressure P suitable for Atlantic cod is between 1.6 to 7 atmospheres. We record this fact in the form of the three-part inequality $1.6 \le P \le 7$. To find the corresponding depth d, in meters, we substitute $P = 1 + \frac{d}{11}$ and solve the three-part inequality for d. So, we have

$$1.6 \le 1 + \frac{d}{11} \le 7 \qquad / -1$$

$$0.6 \le \frac{d}{11} \le 6 \qquad /\cdot 11$$

$$6.6 \le d \le 66$$

Thus, Atlantic cod should be searched for between 7 and 66 meters below the surface.

Example 6 ► **Using Set Operations to Solve Applied Problems Involving Compound Inequalities**

Given the information in the table,

Film	Admissions (in millions)	Adjusted Gross Income (in millions of dollars)
Gone With the Wind	202	1825
Star Wars	178	1608
The Sound of Music	142	1286
Titanic	136	1224
Avatar	97	878

list the films that belong to each set.

a. The set of films with admissions greater than 150,000,000 *and* the adjusted gross income greater than \$1,000,000,000.

b. The set of films with admissions greater than 150,000,000 *or* the adjusted gross income greater than \$1,000,000,000.

c. The set of films with admissions smaller than 150,000,000 *and* the adjusted gross income greater than \$1,000,000,000.

Solution ► **a.** The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind* and *Star Wars*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind, Star Wars, The Sound of Music,* and *Titanic*. Therefore, the set of films satisfying both of these properties contains of ***Gone With the Wind*** and ***Star Wars***.

b. The set of films with admissions greater than 150,000,000 consists of *Gone With the Wind* and *Star Wars*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind, Star Wars, The Sound of Music,* and *Titanic*. Therefore, the set of films satisfying at least one of these properties consists of ***Gone With the Wind, Star Wars, The Sound of Music***, and ***Titanic***.

c. The set of films with admissions smaller than 180,000,000 includes *Star Wars, The Sound of Music, Titanic,* and *Avatar*. The set of films with the adjusted gross income greater than 1,000,000,000 consists of *Gone With the Wind, Star Wars, The Sound of*

Music, and *Titanic*. Therefore, the set of films satisfying both of these properties consists of ***The Sound of Music***, and ***Titanic***.

L.5 Exercises

Let $A = \{1, 2, 3, 4\}$, $B = \{1,\ 3,\ 5\}$, $C = \{5\}$. *Find each set.*

1. $A \cap B$

2. $A \cup B$

3. $B \cup C$

4. $A \setminus B$

5. $A \cap C$

6. $A \cup B \cup C$

7. $B \cap C$

8. $A \cup C$

Write the result of each set operation as a single interval, if possible.

9. $(-7, 3] \cap [1,6]$

10. $(-8, 5] \cap (-1, 13)$

11. $(0,3) \cup (1, 7]$

12. $[-7, 2] \cup (1, 10)$

13. $(-\infty, 13) \cup (1, \infty)$

14. $(-\infty, 1) \cap (2, \infty)$

15. $(-\infty, 1] \cap [1, \infty)$

16. $(-\infty, -1] \cup [1, \infty)$

17. $(-2, \infty) \cup [3, \infty)$

18. $(-2, \infty) \cap [3, \infty)$

Solve each compound inequality. Give the solution set in both interval and graph form.

19. $x + 1 > 6 \quad or \quad 1 - x > 3$

20. $-3x \geq -6 \quad and \quad -2x \leq 12$

21. $4x + 1 < 5 \quad and \quad 4x + 7 > -1$

22. $3y - 11 > 4 \quad or \quad 4y + 9 \leq 1$

23. $3x - 7 < -10 \quad and \quad 5x + 2 \leq 22$

24. $\frac{1}{4}y - 2 < -3 \quad or \quad 1 - \frac{3}{2}y \geq 4$

25. $\begin{cases} 1 - 7x \leq -41 \\ 3x + 1 \geq -8 \end{cases}$

26. $\begin{cases} 2(x + 1) < 8 \\ -2(x - 4) > -2 \end{cases}$

27. $-\frac{2}{3} \leq 3 - \frac{1}{2}a < \frac{2}{3}$

28. $-4 \leq \frac{7 - 3a}{5} \leq 4$

29. $5x + 12 > 2 \quad or \quad 7x - 1 < 13$

30. $4x - 2 > 10 \quad and \quad 8x + 2 \leq -14$

31. $7t - 1 > -1 \quad and \quad 2t - 5 \geq -10$

32. $7z - 6 > 0 \quad or \quad -\frac{1}{2}z \leq 6$

33. $\frac{5x+4}{2} \geq 7 \quad or \quad \frac{7-2x}{3} \geq 2$

34. $\frac{2x-5}{-2} \geq 2 \quad and \quad \frac{2x+1}{3} \geq 0$

35. $13 - 3x > -8 \quad and \quad 12x + 7 \geq -(1 - 10x)$

36. $1 \leq -\frac{1}{3}(4b - 27) \leq 3$

Solve each problem.

37. Two friends plan to drive between 680 and 920 kilometers per day. If they estimate that their average driving speed will be 80 km/h, how many hours per day will they be driving?

38. A substance is in a liquid state if its temperature is between its melting point and its boiling point. The melting point of phosphorus is $44°C$ and its boiling point is $280.5°C$. Using the conversion formula $C = \frac{5}{9}(F - 32)$, determine the rage of temperatures in $°F$ for which phosphorus assumes a liquid state.

39. Kevin's birthday party will cost \$400 to rent a banquet hall and an additional \$25 for every guest. If Kevin wants to keep the cost of the party between \$750 and \$1000, how many guests could he invite?

40. Michael works for \$15 per hour plus \$18 per every overtime hour after the first 40 hours per week. How many hours of overtime must he work to earn between \$800 and \$1000 per week?

41. The table below shows average tuition fees for full-time international undergraduate and graduate students during the 20018/19 academic year, by field of study.

Field of Study	Undergraduate	Graduate
Education	\$19,461	\$15,236
Humanities	\$26,175	\$13,520
Business	\$26,395	\$22,442
Mathematics	\$30,187	\$15,553
Dentistry	\$55,802	\$21,635
Nursing	\$20,354	\$13,713
Veterinary Medicine	\$60,458	\$9,088

Which fields of study belong to the following sets:

a. the set of the fields of study that cost less than \$25000 for undergraduates *or* less than \$15,000 for graduates

b. the set of the fields of study that cost less than \$25000 for undergraduates *and* less than \$15,000 for graduates

c. the set of the fields of study that cost more than \$25000 for undergraduates *or* more than \$15,000 for graduates

d. the set of the fields of study that cost more than \$25000 for undergraduates *and* more than \$15,000 for graduates

L6 Absolute Value Equations and Inequalities

The concept of **absolute value** (also called **numerical value**) was introduced in *Section R2*. Recall that when using geometrical visualisation of real numbers on a number line, the absolute value of a number x, denoted $|x|$, can be interpreted as the distance of the point x from zero. Since distance cannot be negative, the result of absolute value is always nonnegative. In addition, the distance between points x and a can be recorded as $|x-a|$ (see *Definition 2.2* in *Section R2*), which represents the nonnegative difference between the two quantities. In this section, we will take a closer look at absolute value properties, and then apply them to solve absolute value equations and inequalities.

Properties of Absolute Value

The formal definition of absolute value

$$|x| \overset{def}{=} \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0 \end{cases}$$

tells us that, when $\boldsymbol{x}$ is nonnegative, the absolute value of $\boldsymbol{x}$ is the same as $\boldsymbol{x}$, and when $\boldsymbol{x}$ is negative, the absolute value of it is the **opposite** of $\boldsymbol{x}$.

So, $|2| = 2$ and $|-2| = -(-2) = 2$. Observe that this complies with the notion of a distance from zero on a number line. Both numbers, 2 and -2 are at a distance of 2 units from zero. They are both solutions to the equation $|x| = 2$.

Since $|x|$ represents the distance of the number x from 0, which is never negative, we can claim the first absolute value property:

$\boldsymbol{|x| \geq 0}$, for any real x

Here are several other absolute value properties that allow us to simplify algebraic expressions.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ are any real numbers. Then

$\boldsymbol{|x| = 0}$ if and only if $\boldsymbol{x = 0}$

Only zero is at the distance zero from zero.

$$|-x| = |x|$$

The distance of opposite numbers from zero is the same.

$$|xy| = |x||y|$$

Absolute value of a product is the product of absolute values.

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for $y \neq 0$

Absolute value of a quotient is the quotient of absolute values.

Attention: *Absolute value doesn't 'split' over addition or subtraction! That means*

$$|\boldsymbol{x} \pm \boldsymbol{y}| \neq |\boldsymbol{x}| \pm |\boldsymbol{y}|$$

For example, $|2 + (-3)| = 1 \neq 5 = |2| + |-3|$.

Example 1 ► **Simplifying Absolute Value Expressions**

Simplify, leaving as little as possible inside each absolute value sign.

a. $|-2x|$

b. $|3x^2y|$

c. $\left|\frac{-a^2}{2b}\right|$

d. $\left|\frac{-1+x}{4}\right|$

Solution ►

a. Since absolute value can 'split' over multiplication, we have

$$|-2x| = |-2||x| = \boldsymbol{2|x|}$$

b. Using the multiplication property of absolute value and the fact that x^2 is never negative, we have

$$|3x^2y| = |3||x^2||y| = \boldsymbol{3x^2|y|}$$

c. Using properties of absolute value, we have

$$\left|\frac{-a^2}{2b}\right| = \frac{|-1||a^2|}{|2||b|} = \boldsymbol{\frac{a^2}{2|b|}}$$

c. Since absolute value does not 'split' over addition, the only simplification we can perform here is to take 4 outside of the absolute value sign. So, we have

$$\left|\frac{-1+x}{4}\right| = \boldsymbol{\frac{|x-1|}{4}} \text{ or equivalently } \boldsymbol{\frac{1}{4}|x-1|}$$

Remark: *Convince yourself that* $|x - 1|$ *is not equivalent to* $x + 1$ *by evaluating both expressions at, for example,* $x = 1$.

Absolute Value Equations

The formal definition of absolute value (see *Definition 2.1* in *Section R2*) applies not only to a single number or a variable x but also to any algebraic expression. Generally, we have

$$|expr.| \stackrel{def}{=} \begin{cases} expr., & \text{if} \quad expr. \geq 0 \\ -(expr.), & \text{if} \quad expr. < 0 \end{cases}$$

This tells us that, when an *expression* is nonnegative, the absolute value of the *expression* is the **same** as the ***expression***, and when the *expression* is negative, the absolute value of the *expression* is the **opposite** of the ***expression***.

For example, to evaluate $|x-1|$, we consider when the expression $x-1$ is nonnegative and when it is negative. Since $x-1 \geq 0$ for $x \geq 1$, we have

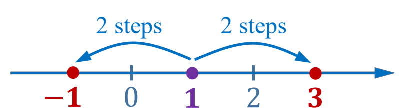
$$|x-1| = \begin{cases} x-1, & for\ x \geq 1 \\ -(x-1), & for\ x < 1 \end{cases}$$

Notice that both expressions, $x-1\ for\ x \geq 1$ and $-(x-1)\ for\ x < 1$ produce nonnegative values that represent the distance of a number x from 0.

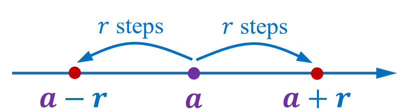
In particular,

if $x = 3$, then $|\boldsymbol{x} - \mathbf{1}| = x - 1 = 3 - 1 = \mathbf{2}$,
and
if $x = -1$, then $|\boldsymbol{x} - \mathbf{1}| = -(x-1) = -(-1-1) = -(-2) = \mathbf{2}$.

As illustrated on the number line below, both numbers, **3** and **−1** are at the distance of **2** units from **1**.



Generally, the equation $|\boldsymbol{x} - \boldsymbol{a}| = \boldsymbol{r}$ tells us that the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ is equal to $\boldsymbol{r}$. This means that $\boldsymbol{x}$ is $\boldsymbol{r}$ units away from number $\boldsymbol{a}$, in either direction.



Therefore, $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{r}$ and $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{r}$ are the solutions of the equation $|\boldsymbol{x} - \boldsymbol{a}| = \boldsymbol{r}$.

Example 1 ► **Solving Absolute Value Equations Geometrically**

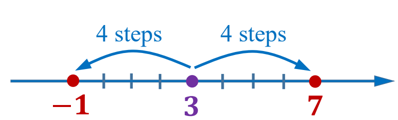
For each equation, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

a. $|x-3| = 4$

b. $|x+5| = 3$

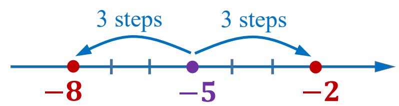
Solution ►

a. Geometrically, $|x-3|$ represents the distance between x and 3. Thus, in $|x-3| = 4$, x is a number whose distance from 3 is 4. So, $x = 3 \pm 4$, which equals either -1 or 7.



Therefore, the solution set is $\{-\mathbf{1}, \mathbf{7}\}$.

b. By rewriting $|x+5|$ as $|x-(-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x+5| = 3$, x is a number whose distance from -5 is 3. Thus, $x = -5 \pm 3$, which results in -8 or -2.



Therefore, the solution set is $\{\mathbf{-8}, \mathbf{-2}\}$.

Although the geometric interpretation of absolute value proves to be very useful in solving some of the equations, it can be handy to have an algebraic method that will allow us to solve any type of absolute value equation.

Suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.}| = \boldsymbol{r}, \quad \text{where } r > 0$$

We have two possibilities. Either the ***expression*** inside the absolute value bars is nonnegative, or it is negative. By definition of absolute value, if the ***expression*** is nonnegative, our equation becomes

$$\boldsymbol{expr.} = \boldsymbol{r}$$

If the ***expression*** is negative, then to remove the absolute value bar, we must change the sign of the ***expression***. So, our equation becomes

$$-\boldsymbol{expr.} = \boldsymbol{r},$$

which is equivalent to

$$\boldsymbol{expr.} = -\boldsymbol{r}$$

In summary, for $\boldsymbol{r} > 0$, the equation

$$|\boldsymbol{expr.}| = \boldsymbol{r}$$

is equivalent to the system of equations with the connecting word *or*.

$$\boldsymbol{expr.} = \boldsymbol{r} \quad \text{or} \quad \boldsymbol{expr.} = -\boldsymbol{r},$$

If $\boldsymbol{r} = 0$, then $|\boldsymbol{expr.}| = \mathbf{0}$ is equivalent to the equation $\boldsymbol{expr.} = \mathbf{0}$ with no absolute value.

If $\boldsymbol{r} < 0$, then $|\boldsymbol{expr.}| = \boldsymbol{r}$ has **NO SOLUTION**, as an absolute value is never negative.

Now, suppose we wish to solve an equation of the form

$$|\boldsymbol{expr.A}| = |\boldsymbol{expr.B}|$$

Since both expressions, $\boldsymbol{A}$ and $\boldsymbol{B}$, can be either nonnegative or negative, when removing absolute value bars, we have four possibilities:

$$\boldsymbol{expr.A} = \boldsymbol{expr.B} \quad \text{or} \quad \boldsymbol{expr.A} = -\boldsymbol{expr.B}$$
$$-\boldsymbol{expr.A} = -\boldsymbol{expr.B} \quad \text{or} \quad -\boldsymbol{expr.A} = \boldsymbol{expr.B}$$

However, observe that the equations in blue are equivalent. Also, the equations in green are equivalent. So, in fact, it is enough to consider just the first two possibilities.

Therefore, the equation

$$|expr.A| = |expr.B|$$

is equivalent to the system of equations with the connecting word *or*.

$$expr.A = expr.B \text{ or } expr.A = -(expr.B),$$

Example 2 ► **Solving Absolute Value Equations Algebraically**

Solve the following equations.

a. $|2 - 3x| = 7$

b. $5|x| - 3 = 12$

c. $\left|\frac{1-x}{4}\right| = 0$

d. $|6x + 5| = -4$

e. $|2x - 3| = |x + 5|$

f. $|x - 3| = |3 - x|$

Solution ►

a. To solve $|2 - 3x| = 7$, we remove the absolute value bars by changing the equation into the corresponding system of equations with no absolute value anymore. Then, we solve the resulting linear equations. So, we have

$$|2 - 3x| = 7$$

$$2 - 3x = 7 \quad \text{or} \quad 2 - 3x = -7$$

$$2 - 7 = 3x \quad \text{or} \quad 2 + 7 = 3x$$

$$x = \frac{-5}{3} \quad \text{or} \quad x = \frac{9}{3} = 3$$

Therefore, the solution set of this equation is $\left\{-\frac{5}{3}, 3\right\}$.

b. To solve $5|x| - 3 = 12$, first, we **isolate the absolute value**, and then replace the equation by the corresponding system of two linear equations.

$$5|x| - 3 = 12$$

$$5|x| = 15$$

$$|x| = 3$$

$$x = 3 \quad \text{or} \quad x = -3$$

So, the solution set of the given equation is $\{-3, 3\}$.

c. By properties of absolute value, $\left|\frac{1-x}{4}\right| = 0$ if and only if $\frac{1-x}{4} = 0$, which happens when the numerator $1 - x = 0$. So, the only solution to the given equation is $x = 1$.

d. Since an absolute value is never negative, the equation $|6x + 5| = -4$ does not have any solution.

e. To solve $|2x-3| = |x+5|$, we remove the absolute value symbols by changing the equation into the corresponding system of linear equations with no absolute value. Then, we solve the resulting equations. So, we have

$$|2x-3| = |x+5|$$

$$\begin{array}{rcl} 2x-3 = x+5 & \textit{or} & 2x-3 = -(x+5) \\ 2x-x = 5+3 & \textit{or} & 2x-3 = -x-5 \\ x = 8 & \textit{or} & 3x = -2 \\ & & x = \frac{-2}{3} \end{array}$$

Therefore, the solution set of this equation is $\left\{-\frac{\mathbf{2}}{\mathbf{3}}, \mathbf{8}\right\}$.

f. We solve $|x-3| = |3-x|$ as in *Example 2e*.

$$|x-3| = |3-x|$$

$$\begin{array}{rcl} x-3 = 3-x & \textit{or} & x-3 = -(3-x) \\ 2x = 6 & \textit{or} & x-3 = -3+x \\ x = 3 & \textit{or} & 0 = 0 \end{array}$$

Since the equation $0 = 0$ is always true, any real x-value satisfies the original equation $|x-3| = |3-x|$. So, the solution set to the original equation is $\mathbb{R}$.

Remark: *Without solving the equation in Example 2f, one could observe that the expressions $x-3$ and $3-x$ are opposite to each other and as such, they have the same absolute value. Therefore, the equation is always true.*

Summary of Solving Absolute Value Equations

Step 1 **Isolate the absolute value** expression on one side of the equation.

Step 2 **Check for special cases**, such as

$$|\boldsymbol{A}| = \mathbf{0} \leftrightarrow \boldsymbol{A} = \mathbf{0}$$
$$|\boldsymbol{A}| = -\boldsymbol{r} \rightarrow \textbf{No solution}$$

Step 2 **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations with the joining word *or*,

$$|\boldsymbol{A}| = \boldsymbol{r} \quad (\boldsymbol{r} > \mathbf{0}) \qquad\qquad |\boldsymbol{A}| = |\boldsymbol{B}|$$

$$\boldsymbol{A} = \boldsymbol{r} \;\; \textit{or} \;\; \boldsymbol{A} = -\boldsymbol{r} \qquad\qquad \boldsymbol{A} = \boldsymbol{B} \;\; \textit{or} \;\; \boldsymbol{A} = -\boldsymbol{B}$$

Step 3 **Solve** the resulting equations.

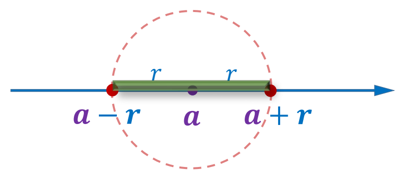
Step 4 **State the solution set** as a union of the solutions of each equation in the system.

Absolute Value Inequalities with One Absolute Value Symbol

GEOMETRIC VISUALIZATION

Suppose we wish to solve inequalities of the form $|\boldsymbol{x}-\boldsymbol{a}|<\boldsymbol{r}$ or $|\boldsymbol{x}-\boldsymbol{a}|>\boldsymbol{r}$, where $\boldsymbol{r}$ is a positive real number. Similarly as in the case of absolute value equations, we can either use a geometric interpretation with the aid of a number line, or we can rely on an algebraic procedure.

Using the geometrical visualization of $|\boldsymbol{x}-\boldsymbol{a}|$ as the distance between $\boldsymbol{x}$ and $\boldsymbol{a}$ on a number line, the inequality $|\boldsymbol{x}-\boldsymbol{a}|<\boldsymbol{r}$ tells us that the number $\boldsymbol{x}$ is less than $\boldsymbol{r}$ units from number $\boldsymbol{a}$. One could think of drawing a circle centered at $\boldsymbol{a}$, with radius $\boldsymbol{r}$. Then, the solutions of the inequality $|\boldsymbol{x}-\boldsymbol{a}|<\boldsymbol{r}$ are all the points on a number line that lie inside such a circle (*see the green segment below*).



Therefore, the solution set is the interval $(\boldsymbol{a}-\boldsymbol{r}, \boldsymbol{a}+\boldsymbol{r})$.

This result can be achieved algebraically by rewriting the absolute value inequality

$$|\boldsymbol{x}-\boldsymbol{a}|<\boldsymbol{r}$$

in an equivalent three-part inequality form

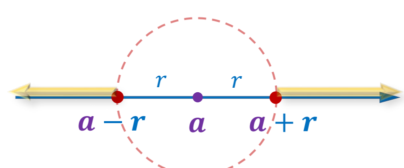
$$-\boldsymbol{r}<\boldsymbol{x}-\boldsymbol{a}<\boldsymbol{r},$$

and then solving it for $\boldsymbol{x}$

$$\boldsymbol{a}-\boldsymbol{r}<\boldsymbol{x}<\boldsymbol{a}+\boldsymbol{r},$$

which confirms that the solution set is indeed $(\boldsymbol{a}-\boldsymbol{r}, \boldsymbol{a}+\boldsymbol{r})$.

Similarly, the inequality $|\boldsymbol{x}-\boldsymbol{a}|>\boldsymbol{r}$ tells us that the number $\boldsymbol{x}$ is more than $\boldsymbol{r}$ units from number $\boldsymbol{a}$. As illustrated in the diagram below, the solutions of this inequality are all points on a number line that lie outside of the circle centered at $\boldsymbol{a}$, with radius $\boldsymbol{r}$.



Therefore, the solution set is the union $(-\infty, \boldsymbol{a}-\boldsymbol{r}) \cup (\boldsymbol{a}+\boldsymbol{r}, \infty)$.

As before, this result can be achieved algebraically by rewriting the absolute value inequality

$$|\boldsymbol{x}-\boldsymbol{a}|>\boldsymbol{r}$$

in an equivalent system of two inequalities joined by the word *or*

$$\boldsymbol{x}-\boldsymbol{a}<-\boldsymbol{r} \quad \text{or} \quad \boldsymbol{r}<\boldsymbol{x}-\boldsymbol{a},$$

and then solving it for $\boldsymbol{x}$

$$x < a - r \quad or \quad a + r < x,$$

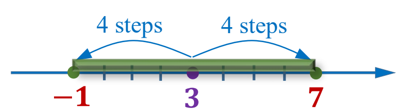
which confirms that the solution set is $(-\infty, a - r) \cup (a + r, \infty)$.

Example 3 ► **Solving Absolute Value Inequalities Geometrically**

For each inequality, state its geometric interpretation, illustrate the situation on a number line, and then find its solution set.

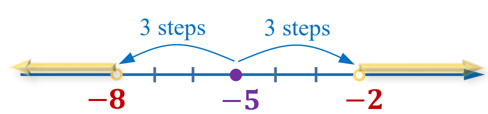
a. $|x - 3| \le 4$ **b.** $|x + 5| > 3$

Solution ► **a.** Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| \le 4$, x is a number whose distance from 3 is at most 4, in either direction.
So, $3 - 4 \le x \le 3 + 4$, which is equivalent to $-1 \le x \le 7$.



Therefore, the solution set is $[-\mathbf{1}, \mathbf{7}]$.

b. By rewriting $|x + 5|$ as $|x - (-5)|$, we can interpret this expression as the distance between x and -5. Thus, in $|x + 5| > 3$, x is a number whose distance from -5 is more than 3, in either direction. Thus, $x < -5 - 3$ or $-5 + 3 < x$, which results in $x < -8$ or $x > -2$.



Therefore, the solution set equals $(-\infty, -\mathbf{8}) \cup (-\mathbf{2}, \infty)$.

The algebraic strategy can be applied to any inequality of the form

$$|\boldsymbol{expr.}| < (\le) \boldsymbol{r}, \quad \text{or} \quad |\boldsymbol{expr.}| > (\ge) \boldsymbol{r}, \quad \text{as long as } \boldsymbol{r} > \mathbf{0}.$$

Depending on the type of inequality, we follow these rules:

$|\boldsymbol{expr.}| < \boldsymbol{r}$ $-\boldsymbol{r} < \boldsymbol{expr.} < \boldsymbol{r}$	or	$|\boldsymbol{expr.}| > \boldsymbol{r}$ $\boldsymbol{expr.} < -\boldsymbol{r}$ or $\boldsymbol{r} < \boldsymbol{expr.}$

These rules also apply to weak inequalities, such as $\le$ or $\ge$.

In the above rules, we assume that $\boldsymbol{r} > \mathbf{0}$. **What if** $\boldsymbol{r} = \mathbf{0}$ **?**

Observe that, the inequality $|\boldsymbol{expr.}| < \mathbf{0}$ is never true, so this inequality doesn't have any solution. Since $|\boldsymbol{expr.}| < \mathbf{0}$ is never true, the inequality $|\boldsymbol{expr.}| \le \mathbf{0}$ is equivalent to the equation $|\boldsymbol{expr.}| = \mathbf{0}$.

On the other hand, $|\boldsymbol{expr.}| \geq \mathbf{0}$ is always true, so the solution set equals to $\mathbb{R}$. However, since $|\boldsymbol{expr.}|$ is either positive or zero, the solution to $|\boldsymbol{expr.}| > \mathbf{0}$ consists of all real numbers except for the solutions of the equation $\boldsymbol{expr.} = \mathbf{0}$.

What if $r < 0$?

Observe that both inequalities $|\boldsymbol{expr.}| > \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \geq \boldsymbol{negative}$ are always true, so the solution set of such inequalities is equal to $\mathbb{R}$.

On the other hand, both inequalities $|\boldsymbol{expr.}| < \boldsymbol{negative}$ and $|\boldsymbol{expr.}| \leq \boldsymbol{negative}$ are never true, so such inequalities result in **NO SOLUTION**.

Example 4 ► **Solving Absolute Value Inequalities with One Absolute Value Symbol**

Solve each inequality. Give the solution set in both interval and graph form.

a. $|5x+9| \leq 4$ **b.** $|-2x-5| > 1$

e. $16 \leq |2x-3| + 9$ **f.** $1 - 2|4x-7| > -5$

Solution ► **a.** To solve $|5x+9| \leq 4$, first, we remove the absolute value symbol by rewriting the inequality in the three-part inequality, as below.

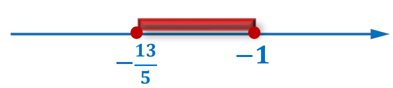
$$|5x+9| \leq 4$$

$$-4 \leq 5x + 9 \leq 4 \qquad / -9$$

$$-13 \leq 5x \leq -5 \qquad / \div 5$$

$$-\frac{13}{5} \leq x \leq -1$$

The solution is shown in the graph below.



The inequality is satisfied by all $x \in \left[-\frac{13}{5}, -\mathbf{1}\right]$.

b. As in the previous example, first, we remove the absolute value symbol by replacing the inequality with the corresponding system of inequalities, joined by the word *or*. So, we have

$$|-2x-5| > 1$$

$$-2x - 5 < -1 \quad or \quad 1 < -2x - 5 \qquad / +5$$

$$-2x < 4 \quad or \quad 6 < -2x \qquad / \div(-2)$$

$$x > -2 \quad or \quad -3 > x$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

The inequality is satisfied by all $x \in (-\infty, -\mathbf{3}) \cup (-\mathbf{2}, \infty)$.

c. To solve $16 \leq |2x-3|+9$, first, we **isolate the absolute value**, and then replace the inequality with the corresponding system of two linear equations. So, we have

$$16 \leq |2x-3|+9 \qquad /-9$$

$$7 \leq |2x-3|$$

$$2x-3 \leq -7 \quad \textit{or} \quad 7 \leq 2x-3 \qquad /+3$$

$$2x \leq -4 \quad \textit{or} \quad 2x \geq 10 \qquad /\div 2$$

$$x \leq -2 \quad \textit{or} \quad x \geq 5$$

The joining word *or* indicates that we look for the union of the obtained solutions. This union is shown in the graph below.

So, the inequality is satisfied by all $x \in (-\infty, -\mathbf{2}) \cup (\mathbf{5}, \infty)$.

d. As in the previous example, first, we **isolate the absolute value**, and then replace the inequality with the corresponding system of two inequalities.

$$1-2|4x-7| > -5 \qquad /-1$$

$$-2|4x-7| > -6 \qquad /\div(-2)$$

! reverse the signs

$$|4x-7| < 3$$

$$-3 < 4x-7 < 3 \qquad /+7$$

$$4 < 4x < 10 \qquad /\div 4$$

$$1 < x < \frac{10}{4} = \frac{5}{2}$$

So the solution set is the interval $\left(\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, visualized in the graph below.

Example 5 ► **Solving Absolute Value Inequalities in Special Cases**

Solve each inequality.

a. $\left|\frac{1}{2}x + \frac{5}{3}\right| \geq -3$

b. $|4x-7| \leq 0$

c. $|3-4x| > 0$

d. $1 - 2\left|\frac{3}{2}x - 5\right| > 3$

Solution ►

a. Since an absolute value is always bigger or equal to zero, the inequality $\left|\frac{1}{2}x + \frac{5}{3}\right| \geq -3$ is always true. Thus, it is satisfied by **any real number**. So the solution set is $\mathbb{R}$.

b. Since $|4x-7|$ is never negative, the inequality $|4x-7| \le 0$ is satisfied only by solutions to the equation $|4x-7| = 0$. So, we solve

$$|4x-7| = 0$$

$$4x - 7 = 0 \qquad / +7$$

$$4x = 7 \qquad / \div 4$$

$$x = \frac{7}{4}$$

Therefore, the inequality is satisfied only by $\boldsymbol{x = \frac{7}{4}}$.

c. Inequality $|3-4x| > 0$ is satisfied by all real x-values except for the solution to the equation $3 - 4x = 0$. Since

$$3 - 4x = 0 \qquad / +4x$$

$$3 = 4x \qquad / \div 4$$

$$\frac{3}{4} = x,$$

then the solution to the original inequality is $\left(-\infty, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \infty\right)$.

d. To solve $1 - 2\left|\frac{3}{2}x - 5\right| > 3$, first, we **isolate the absolute value**. So, we have

$$1 - 2\left|\frac{3}{2}x - 5\right| > 3 \qquad / +2\left|\frac{3}{2}x - 5\right|, -3$$

$$-2 > 2\left|\frac{3}{2}x - 5\right| \qquad / \div 2$$

$$-1 > \left|\frac{3}{2}x - 5\right|$$

Since $\left|\frac{3}{2}x - 5\right|$ is never negative, it can't be less than -1. So, there is **no solution** to the original inequality.

Summary of Solving Absolute Value Inequalities with One Absolute Value Symbol

Let $\boldsymbol{r}$ be a positive real number. To solve absolute value inequalities with one absolute value symbol, follow the steps:

- **Isolate the absolute value** expression on one side of the inequality.
- **Check for special cases**, such as

$|A| < 0 \rightarrow$ **No solution**
$|A| \le 0 \leftrightarrow$ $\boldsymbol{A = 0}$
$|A| \ge 0 \rightarrow$ **All real numbers**
$|A| > 0 \rightarrow$ **All real numbers except for solutions of $\boldsymbol{A = 0}$**
$|A| > (\ge) -\boldsymbol{r} \rightarrow$ **All real numbers**
$|A| < (\le) -\boldsymbol{r} \rightarrow$ **No solution**

- **Remove the absolute value symbol** by replacing the equation with the corresponding system of equations as below:

$$|A| < r \quad\Rightarrow\quad -r < A < r$$

$$|A| > r \quad\Rightarrow\quad A < -r \;\; or \;\; r < A$$

  This also applies to weak inequalities, such as ≤ or ≥.

- **Solve** the resulting equations.
- **State the solution set** as a union of the solutions of each equation in the system.

Applications of Absolute Value Inequalities

One of the typical applications of absolute value inequalities is in error calculations. When discussing errors in measurements, we refer to the *absolute error* or the *relative error*. For example, if M is the actual measurement of an object and x is the approximated measurement, then the ***absolute error*** is given by the formula $|x - M|$ and the ***relative error*** is calculated according to the rule $\frac{|x-M|}{M}$.

In quality control situations, the relative error often must be less than some predetermined amount. For example, suppose a machine that fills two-litre milk cartons is set for a relative error no greater than 1%. We might be interested in how much milk a two-litre carton can actually contain? What is the absolute error that this machine is allowed to make?

Since $M = 2$ litres and the relative error = 1% $= 0.01$, we are looking for all x-values that would satisfy the inequality

$$\frac{|x-2|}{2} < 0.01.$$

This is the **relative error**.

This is equivalent to

$$|x-2| < 0.02$$

This is the **absolute error**.

$$-0.02 < x - 2 < 0.02$$

$$1.98 < x < 2.02,$$

so, a two-litre carton of milk can contain any amount of milk between 1.98 and 2.02 litres. The absolute error in this situation is $0.02\ l = 20\ ml$.

Example 6 ► **Solving Absolute Value Application Problems**

A scale is considered to be accurate if it measures the weight of an object up to 0.1% of its actual weight. Suppose x represents the reading on the scale when weighing a 20 kg object. Find the set of all possible x-readings.

Solution ► The difference (*error*) between the scale reading and the actual weight can be expressed as $|x - 20|$. Since the allowable error in the scale reading with respect to the actual weight (the ***relative error***) is smaller than $0.1\% = 0.001$, then x must satisfy the inequality

$$\frac{|x-20|}{20} < 0.001$$

After solving it for x,

$$|x-20| < 0.02$$

$$-0.02 < x - 20 < 0.02$$

$$19.98 < x < 20.02$$

So, the scale readings when weighing a 20 kg object can range between **19.98** and **20.02** kg. This tells us that the scale may err up to 0.02 kg = 2 dkg when weighing this object.

L.6 Exercises

Simplify, if possible, leaving as little as possible inside the absolute value symbol.

1. $|-2x^2|$ **2.** $|3x|$ **3.** $\left|\frac{-5}{y}\right|$ **4.** $\left|\frac{3}{-y}\right|$

5. $|7x^4y^3|$ **6.** $|-3x^5y^4|$ **7.** $\left|\frac{x^2}{y}\right|$ **8.** $\left|\frac{-4x}{y^2}\right|$

9. $\left|\frac{-3x^3}{6x}\right|$ **10.** $\left|\frac{5x^2}{-25x}\right|$ **11.** $|(x-1)^2|$ **12.** $|x^2-1|$

13. *In each situation, find the* ***number of solutions*** *for the equation* $|\boldsymbol{ax} + \boldsymbol{b}| = \boldsymbol{k}$.

a. $k < 0$ **b.** $k = 0$ **c.** $k > 0$

14. *Match each absolute value equation or inequality in Column I with the graph of its solution set in Column II.*

Column I Column II

a. $|x| = 3$ **A.**

b. $|x| > 3$ **B.**

c. $|x| < 3$ **C.**

d. $|x| \geq 3$ **D.**

e. $|x| \leq 3$ **E.**

Solve each equation.

15. $|-x| = 4$

16. $|5x| = 20$

17. $|y - 3| = 8$

18. $|2y + 5| = 9$

19. $7|3x - 5| = 35$

20. $-3|2x - 7| = -12$

21. $\left|\frac{1}{2}x + 3\right| = 11$

22. $\left|\frac{2}{3}x - 1\right| = 5$

23. $|2x - 5| = -1$

24. $|7x + 11| = 0$

25. $2 + 3|a| = 8$

26. $10 - |2a - 1| = 4$

27. $\left|\frac{2x-1}{3}\right| = 5$

28. $\left|\frac{3-5x}{6}\right| = 3$

29. $|2p + 4| = |3p - 1|$

30. $|5 - q| = |q + 7|$

31. $\left|\frac{1}{2}x + 3\right| = \left|\frac{1}{5}x - 1\right|$

32. $\left|\frac{2}{3}x - 8\right| = \left|\frac{1}{6}x + 3\right|$

33. $\left|\frac{3x-6}{2}\right| = \left|\frac{5+x}{5}\right|$

34. $\left|\frac{6-5x}{4}\right| = \left|\frac{7+3x}{3}\right|$

Solve *each inequality. Give the solution set in both* ***interval*** *and* ***graph*** *form.*

35. $|x + 4| < 3$

36. $|x - 5| > 7$

37. $|x - 12| \geq 5$

38. $|x + 14| \leq 5$

39. $|5x + 3| \leq 8$

40. $|3x - 2| \geq 10$

41. $|7 - 2x| > 5$

42. $|-5x + 4| < 3$

43. $\left|\frac{1}{4}y - 6\right| \leq 24$

44. $\left|\frac{2}{5}x + 3\right| > 5$

45. $\left|\frac{3x-2}{4}\right| \geq 10$

46. $\left|\frac{2x+3}{3}\right| < 10$

47. $|-2x + 4| - 8 \geq -5$

48. $|6x - 2| + 3 < 9$

49. $7 - 2|x + 4| \geq 5$

50. $9 - 3|x - 2| < 3$

Solve each inequality.

51. $\left|\frac{2}{3}x + 4\right| \leq 0$

52. $\left|-2x + \frac{4}{5}\right| > 0$

53. $\left|\frac{6x-2}{5}\right| < -3$

54. $|-3x + 5| > -3$

55. $|-x + 4| + 5 \geq 4$

56. $|4x + 1| - 2 < -5$

Solve each problem.

57. The recommended dosage of daily intake of magnesium for a healthy adult is 370 mg with a tolerance of up to 50 mg.

a. Write an absolute value inequality that describes the recommended intake of magnesium, M, in milligrams per day.

b. Using the inequality from part (**a**), find the range of the recommended number of milligrams of magnesium intake per day.

58. The average income of an employee at a tire shop is \$36,000 per year. Patrik works in this shop, but his earnings are not within \$8000 of this average.

a. Write an absolute value inequality that describes Patrik's income, I, in this situation.

b. Solve the inequality created in part (**a**) to find the possible amounts for Patrik's annual earnings. *Assume that his earnings are positive.*

59. The recommended daily intake of calcium for females over 50 years old is 1200 mg with a tolerance of up to 100 mg. Use absolute value inequality to describe the recommended daily calcium intake C for this group of people. Then, solve the inequality to find the range of values for this daily calcium intake.

60. A police radar has a maximum allowable error of 2 km/h.

a. Use absolute value inequality to record the range of values of the actual speed s of the car if its radar reading is 63 km/h.

b. Solve the inequality in part (a) to find the minimum and maximum possible speeds for this car.

61. The speed limit on a freeway is 110 km/h. If cars are required to travel at least 60 km/h to use a freeway, what absolute value inequality describes the permitted rates r of mooving on this freeway?

62. The body temperature T of a healthy adult person is expected to be between 36.4°C and 37.6°C. Record this fact using an absolute value inequality.

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Graphs and Linear Functions

A 2-dimensional graph is a visual representation of a relationship between two variables given by an equation or an inequality. Graphs help us solve algebraic problems by analysing the geometric aspects of a problem. While equations are more suitable for precise calculations, graphs are more suitable for showing patterns and trends in a relationship. To fully utilize what graphs can offer, we must first understand the concepts and skills involved in graphing that are discussed in this chapter.

G1 System of Coordinates, Graphs of Linear Equations and the Midpoint Formula

In this section, we will review the rectangular coordinate system, graph various linear equations and inequalities, and introduce a formula for finding coordinates of the midpoint of a given segment.

The Cartesian Coordinate System

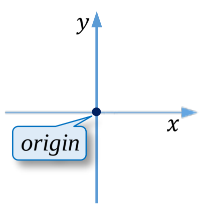


Figure 1a

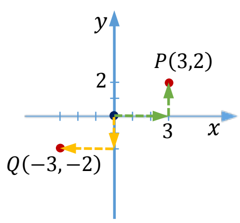


Figure 1b

A rectangular coordinate system, also called a **Cartesian coordinate system** (in honor of French mathematician, ***René Descartes)***, consists of two perpendicular number lines that cross each other at point zero, called the **origin**. Traditionally, one of these number lines, usually called the ***x*-axis**, is positioned horizontally and directed to the right (see *Figure 1a*). The other number line, usually called ***y*-axis**, is positioned vertically and directed up. Using this setting, we identify each point P of the plane with an **ordered pair** of numbers (x, y), which indicates the location of this point with respect to the origin. The first number in the ordered pair, the ***x*-coordinate**, represents the horizontal distance of the point P from the origin. The second number, the ***y*-coordinate**, represents the vertical distance of the point P from the origin. For example, to locate point $P(3,2)$, we start from the origin, go 3 steps to the right, and then two steps up. To locate point $Q(-3,-2)$, we start from the origin, go 3 steps to the left, and then two steps down (see *Figure 1b*).

Observe that the coordinates of the origin are $(0,0)$. Also, the second coordinate of any point on the x-axis as well as the first coordinate of any point on the y-axis is equal to zero. So, points on the x-axis have the form $(x, 0)$, while points on the y-axis have the form of $(0, y)$.

To **plot** (or **graph**) an ordered pair (x, y) means to place a dot at the location given by the ordered pair.

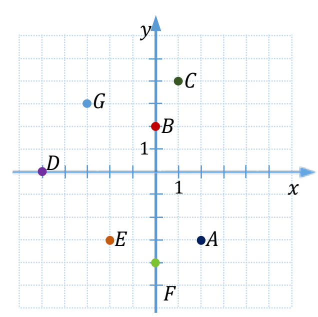
Example 1 ► **Plotting Points in a Cartesian Coordinate System**

Plot the following points:

$A(2,-3)$, $B(0,2)$, $C(1,4)$, $D(-5,0)$,
$E(-2,-3)$, $F(0,-4)$, $G(-3,3)$

Solution ►

Remember! *The order of numbers in an ordered pair is important! The **first** number represents the **horizontal** displacement and the **second** number represents the **vertical** displacement from the origin.*



Graphs of Linear Equations

A **graph of an equation** in two variables, x and y, is the set of points corresponding to **all ordered pairs** $(\boldsymbol{x}, \boldsymbol{y})$ that **satisfy** the equation (make the equation true). This means that a graph of an equation is the visual representation of the **solution set** of this equation.

To determine if a point (a, b) belongs to the graph of a given equation, we check if the equation is satisfied by $x = a$ and $y = b$.

Example 1 ► **Determining if a Point is a Solution of a Given Equation**

Determine if the points $(5,3)$ and $(-3,-2)$ are solutions of $2x - 3y = 0$.

Solution ► After substituting $x = 5$ and $y = 3$ into the equation $2x - 3y = 0$, we obtain

$$2 \cdot 5 - 3 \cdot 3 = 0$$
$$10 - 9 = 0$$
$$1 = 0,$$

which is not true. Since the coordinates of the point $(5,3)$ do not satisfy the given equation, the point $(\mathbf{5}, \mathbf{3})$ **is not a solution** of this equation.

Note: The fact that the point $(5,3)$ **does not satisfy** the given equation indicates that it **does not belong to the graph** of this equation.

However, after substituting $x = -3$ and $y = -2$ into the equation $2x - 3y = 0$, we obtain

$$2 \cdot (-3) - 3 \cdot (-2) = 0$$
$$-6 + 6 = 0$$
$$0 = 0,$$

which is true. Since the coordinates of the point $(-3,-2)$ satisfy the given equation, the point $(-\mathbf{3}, -\mathbf{2})$ **is a solution** of this equation.

Note: The fact that the point $(-3,-2)$ **satisfies** the given equation indicates that it **belongs to the graph** of this equation.

To find a solution to a given equation in two variables, we choose a particular value for one of the variables, substitute it into the equation, and then solve the resulting equation for the other variable.

For example, to find a solution to $3x + 2y = 6$, we can choose for example $x = 0$, which leads us to

$$3 \cdot 0 + 2y = 6$$
$$2y = 6$$
$$y = 3.$$

This means that the point $(0, 3)$ satisfies the equation and therefore belongs to the graph of this equation. If we choose a different x-value, for example $x = 1$, the corresponding y-value becomes

$$3 \cdot 1 + 2y = 6$$
$$2y = 3$$
$$y = \frac{3}{2}.$$

So, the point $\left(1, \frac{3}{2}\right)$ also belongs to the graph.

Since any real number could be selected for the x-value, there are infinitely many solutions to this equation. Obviously, we will not be finding all of these infinitely many ordered pairs of numbers in order to graph the solution set to an equation. Rather, based on the location of several solutions that are easy to find, we will look for a pattern and predict the location of the rest of the solutions to complete the graph.

To find more points that belong to the graph of the equation in our example, we might want to solve $3x + 2y = 6$ for y. The equation is equivalent to

$$2y = -3x + 6$$

$$y = -\frac{3}{2}x + 3$$

Observe that if we choose x-values to be multiples of 2, the calculations of y-values will be easier in this case. Here is a table of a few more (x, y) points that belong to the graph:

x	$y = -\frac{3}{2}x + 3$	(x, y)
-2	$-\frac{3}{2}(-2) + 3 = 6$	$(-2, 6)$
2	$-\frac{3}{2}(2) + 3 = 0$	$(2, 0)$
4	$-\frac{3}{2}(4) + 3 = -3$	$(4, -3)$

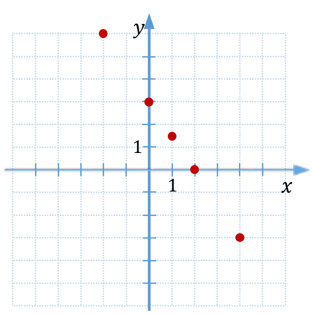


Figure 2a

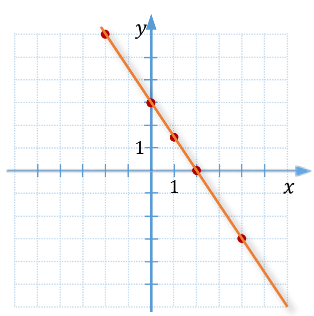


Figure 2b

After plotting the obtained solutions, $(-2, 6)$, $(0, 3)$, $\left(1, \frac{3}{2}\right)$, $(2, 0)$, $(4, -3)$, we observe that the points appear to lie on the same line (see *Figure 2a*). If all the ordered pairs that satisfy the equation $3x + 2y = 6$ were graphed, they would form the line shown in *Figure 2b*. Therefore, if we knew that the graph would turn out to be a line, it would be enough to find just two points (solutions) and draw a line passing through them.

How do we know whether or not the graph of a given equation is a line? It turns out that:

For any equation in two variables, the graph of the equation is a **line** if and only if (iff) the equation is **linear**.

So, the question is how to recognize a linear equation?

Definition 1.1 ► Any equation that can be written in the form

$$\boldsymbol{Ax + By = C}, \text{ where } A, B, C \in \mathbb{R}, \text{ and } A \text{ and } B \text{ are not both } 0,$$

is called a **linear equation** in two variables.

The form $\boldsymbol{Ax + By = C}$ is called **standard form** of a linear equation.

Example 2 ► **Graphing Linear Equations Using a Table of Values**

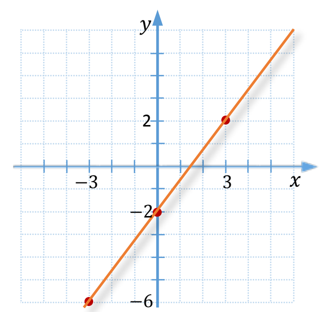
Graph $4x - 3y = 6$ using a table of values.

Solution ► Since this is a linear equation, we expect the graph to be a line. While finding two points satisfying the equation is sufficient to graph a line, it is a good idea to use a third point to guard against errors. To find several solutions, first, let us solve $4x - 3y = 6$ for y:

$$-3y = -4x + 6$$
$$y = \frac{4}{3}x - 2$$

We like to choose x-values that will make the calculations of the corresponding y-values relatively easy. For example, if x is a multiple of 3, such as -3, 0 or 3, the denominator of $\frac{4}{3}$ will be reduced. Here is the table of points satisfying the given equation and the graph of the line.

$\boldsymbol{x}$	$\boldsymbol{y = \frac{4}{3}x - 2}$	$\boldsymbol{(x, y)}$
$\boldsymbol{-3}$	$\frac{4}{3}(-3) - 2 = -6$	$(-3, -6)$
$\boldsymbol{0}$	$\frac{4}{3}(0) - 2 = -2$	$(0, -2)$
$\boldsymbol{3}$	$\frac{4}{3}(3) - 2 = 2$	$(3, 2)$



To graph a linear equation in standard form, we can develop a table of values as in *Example 2*, or we can use the x- and y-intercepts.

Definition 1.2 ► The $\boldsymbol{x}$**-intercept** is the point (if any) where the graph intersects the x-axis. So, the x-intercept has the form $(\boldsymbol{x}, \boldsymbol{0})$.

The $\boldsymbol{y}$**-intercept** is the point (if any) where the graph intersects the y-axis. So, the y-intercept has the form $(\boldsymbol{0}, \boldsymbol{y})$.

Example 3 ► **Graphing Linear Equations Using x- and y-intercepts**

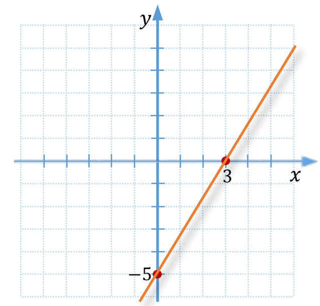
Graph $5x - 3y = 15$ by finding and plotting the x- and y-intercepts.

Solution ► To find the x-intercept, we substitute $y = 0$ into $5x - 3y = 15$, and then solve the resulting equation for y. So, we have

$$5x = 15$$
$$x = 3.$$

To find y-intercept, we substitute $x = 0$ into $5x - 3y = 15$, and then solve the resulting equation for x. So,

$$-3y = 15$$
$$y = -5.$$



Hence, we have

	x	y
x-intercept	**3**	**0**
y-intercept	**0**	**−5**

To find several points that belong to the graph of a linear equation in two variables, it was easier to solve the standard form $Ax + By = C$ for y, as follows

$$By = -Ax + C$$

$$y = -\frac{A}{B}x + \frac{C}{B}.$$

This form of a linear equation is also very friendly for graphing, as the graph can be obtained without any calculations. See *Example 4*.

Any equation $Ax + By = C$, where $B \neq 0$ can be written in the form

$$\boldsymbol{y = mx + b},$$

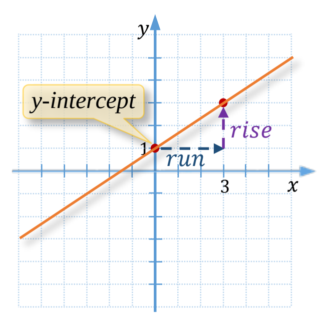
which is referred to as the **slope-intercept form** of a linear equation.
The value $\boldsymbol{m} = -\frac{\boldsymbol{A}}{\boldsymbol{B}}$ represents the **slope** of the line. Recall that $\boldsymbol{slope} = \frac{\boldsymbol{rise}}{\boldsymbol{run}}$.
The value $\boldsymbol{b}$ represents the y-intercept, so the point $(\mathbf{0}, \boldsymbol{b})$ belongs to the graph of this line.

Example 4 ► **Graphing Linear Equations Using Slope and y-intercept**

Determine the slope and y-intercept of each line and then graph it.

a. $y = \frac{2}{3}x + 1$ **b.** $5x + 2y = 8$

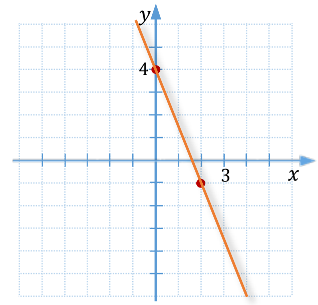
Solution ► **a.** The slope is the coefficient by x, so it is $\frac{2}{3}$.
The y-intercept equals 1.
So we plot point $(0,1)$ and then, since $\frac{2}{3} = \frac{rise}{run}$, we rise 2 units and run 3 units to find the next point that belongs to the graph.



b. To see the slope and y-intercept, we solve $5x+2y=8$ for y.

$$2y=-5x+8$$

$$y=\frac{-5}{2}x+4$$

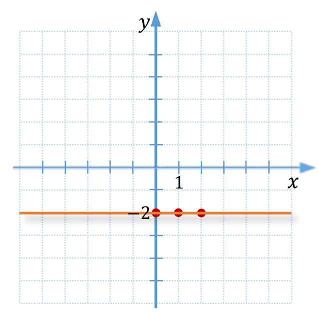


So, the slope is $\frac{-5}{2}$ and the y-intercept is 4. We start from $(0,4)$ and then run 2 units and fall 5 units (because of -5 in the numerator).

Note: Although we can *run* to the right or to the left, depending on the sign in the denominator, we usually **keep the denominator positive and always *run* forward** (to the right). If the slope is negative, we **keep the negative sign in the numerator** and either ***rise*** or ***fall***, depending on this sign. However, when finding additional points of the line, sometimes we can repeat the *run/rise* movement in either way, to the right, or to the left from one of the already known points. For example, in *Example 4a*, we could find the additional point at $(-3,-2)$ by *running* 3 units to the left and 2 units down from $(0,1)$, as the slope $\frac{2}{3}$ can also be seen as $\frac{-2}{-3}$, if needed.

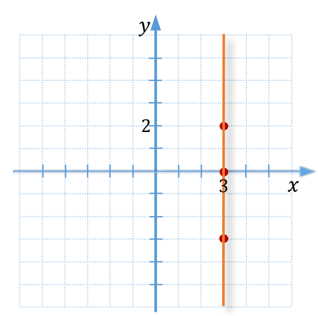
Some linear equations contain just one variable. For example, $x=3$ or $y=-2$. How would we graph such equations in the xy-plane?

Observe that $\boldsymbol{y=-2}$ can be seen as $y=0x-2$, so we can graph it as before, using the **slope** of **zero** and the y-intercept of -2. The graph consists of all points that have y-coordinates equal to -2. Those are the points of type $(x,-2)$, where x is any real number. The graph is a **horizontal line** passing through the point $(0,2)$.



Note: The horizontal line $\boldsymbol{y=0}$ is the x-axis.

The equation $\boldsymbol{x=3}$ doesn't have a slope-intercept representation, but it is satisfied by any point with x-coordinate equal to 3. So, by plotting several points of the type $(3,y)$, where y is any real number, we obtain a **vertical line** passing through the point $(3,0)$. This particular line doesn't have a y-intercept, and its ***slope*** $=\frac{\boldsymbol{rise}}{\boldsymbol{run}}$ is considered to be **undefined**. This is because the "*run*" part calculated between any two points on the line is equal to zero and we can't perform division by zero.



Note: The vertical line $\boldsymbol{x=0}$ is the y-axis.

In general, the graph of any equation of the type

$$y = b, \text{ where } b \in \mathbb{R}$$

is a **horizontal line** with the y-intercept at b. The **slope** of such line is **zero**.

The graph of any equation of the type

$$x = a, \text{ where } a \in \mathbb{R}$$

is a **vertical line** with the x-intercept at a. The **slope** of such line is **undefined**.

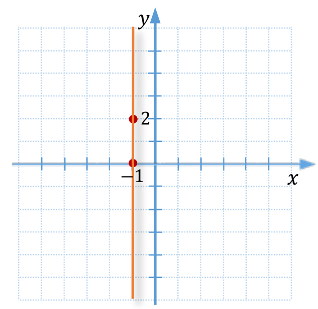
Example 5 **Graphing Special Types of Linear Equations**

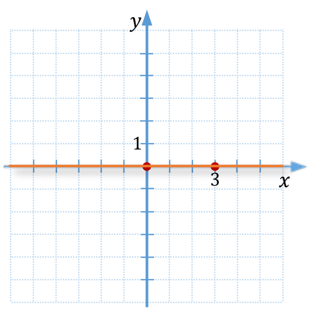
Graph each equation and state its slope.

a. $x = -1$ **b.** $y = 0$ **c.** $y = x$

Solution

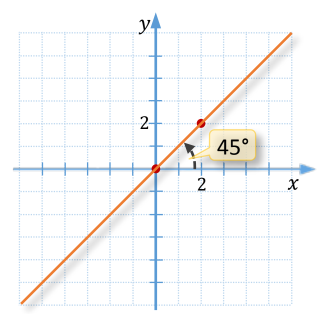
a. The solutions to the equation $x = -1$ are all pairs of the type $(-1, y)$, so after plotting points like $(-1,0)$, $(-1,2)$, etc., we observe that the graph is a **vertical line** intercepting x-axis at $x = -1$. So the **slope** of this line is **undefined**.





b. The solutions to the equation $y = 0$ are all pairs of the type $(x, 0)$, so after plotting points like $(0,0)$, $(0,3)$, etc., we observe that the graph is a **horizontal line** following the x-axis. The **slope** of this line is **zero**.

c. The solutions to the equation $y = x$ are all pairs of the type (x, x), so after plotting points like $(0,0)$, $(2,2)$, etc., we observe that the graph is a **diagonal line**, passing through the origin and making $45°$ with the x-axis. The **slope** of this line is **1**.

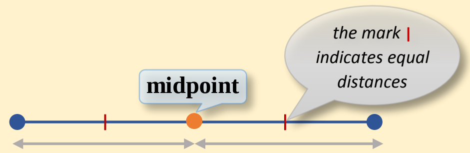


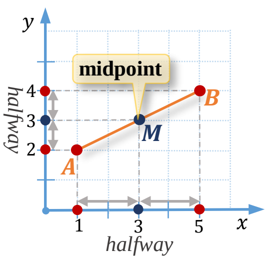
Observation: *A graph of any equation of the type* $\boldsymbol{y = mx}$ *is a line passing through the origin, as the point* $(0,0)$ *is one of the solutions.*

Midpoint Formula

To find a representative value of a list of numbers, we often calculate the average of these numbers. Particularly, to find an average of, for example, two test scores, 72 and 84, we take half of the sum of these scores. So, the average of 72 and 84 is equal to $\frac{72+84}{2} = \frac{156}{2} = 78$. Observe that 78 lies on a number line exactly halfway between 72 and 84. The idea of taking an average is employed in calculating coordinates of the midpoint of any line segment.

Definition 1.3 ► The **midpoint** of a line segment is the point of the segment that is equidistant from both ends of this segment.





Suppose $A = (x_1, y_1), B = (x_2, y_2)$, and $\boldsymbol{M}$ is the **midpoint** of the line segment $\overline{AB}$. Then the x-coordinate of M lies halfway between the two end x-values, x_1 and x_2, and the y-coordinate of M lies halfway between the two end y-values, y_1 and y_2. So, the coordinates of the midpoint are **averages** of corresponding x-, and y-coordinates:

$$\boldsymbol{M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)} \tag{1}$$

Example 6 ► **Finding Coordinates of the Midpoint**

Find the midpoint M of the line segment connecting $P = (-3,7)$ and $Q = (5,-12)$.

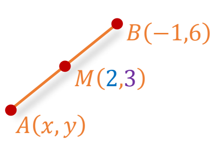
Solution ► The coordinates of the midpoint M are averages of the x- and y-coordinates of the endpoints. So,

$$M = \left(\frac{-3+5}{2}, \frac{7+(-12)}{2}\right) = \left(\mathbf{1}, -\frac{\mathbf{5}}{\mathbf{2}}\right).$$

Example 7 ► **Finding Coordinates of an Endpoint Given the Midpoint and the Other Endpoint**

Suppose segment AB has its midpoint M at $(2,3)$. Find the coordinates of point A, knowing that $B = (-1,6)$.

Solution ► Let $A = (x, y)$ and $B = (-1,6)$. Since $M = (2,3)$ is the midpoint of $\overline{AB}$, by formula (1), the following equations must hold:



$$\frac{x+(-1)}{2} = 2 \quad \text{and} \quad \frac{y+6}{2} = 3$$

Multiplying these equations by 2, we obtain

$$x + (-1) = 4 \quad \text{and} \quad y + 6 = 6,$$

which results in

$$x = 5 \quad \text{and} \quad y = 0.$$

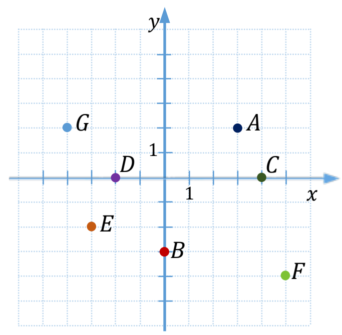
Hence, the coordinates of point A are $(\mathbf{5}, \mathbf{0})$.

G.1 Exercises

1. Plot each point in a rectangular coordinate system.

   a. $(1,2)$ **b.** $(-2,0)$ **c.** $(0,-3)$ **d.** $(4,-1)$ **e.** $(-1,-3)$

2. State the coordinates of each plotted point.



Determine if the given ordered pair is a solution of the given equation.

3. $(-2,2)$; $y = \frac{1}{2}x + 3$ **4.** $(4,-5)$; $3x + 2y = 2$ **5.** $(5,4)$; $4x - 5y = 1$

Graph each equation using the suggested table of values.

6. $y = 2x - 3$

x	y
0	
1	
2	
3	

7. $y = -\frac{1}{3}x + 2$

x	y
−3	
0	
3	
6	

8. $x + y = 3$

x	y
0	
	0
−1	
	1

9. $4x - 5y = 20$

x	y
0	
	0
2	
	−3

Graph each equation using a table of values.

10. $y = \frac{1}{3}x$ **11.** $y = \frac{1}{2}x + 2$ **12.** $6x - 3y = -9$ **13.** $6x + 2y = 8$

14. $y = \frac{2}{3}x - 1$ **15.** $y = -\frac{3}{2}x$ **16.** $3x + y = -1$ **17.** $2x = -5y$

18. $-3x = -3$ **19.** $6y - 18 = 0$ **20.** $y = -x$ **21.** $2y - 3x = 12$

*Determine the **x-** and **y-intercepts** of each line and then graph it. Find additional points, if needed.*

22. $5x + 2y = 10$ **23.** $x - 3y = 6$ **24.** $8y + 2x = -4$ **25.** $3y - 5x = 15$

26. $y = -\frac{2}{5}x - 2$ **27.** $y = \frac{1}{2}x - \frac{3}{2}$ **28.** $2x - 3y = -9$ **29.** $2x = -y$

Determine the ***slope*** *and* ***y-intercept*** *of each line and then graph it.*

30. $y = 2x - 3$

31. $y = -3x + 2$

32. $y = -\frac{4}{3}x + 1$

33. $y = \frac{2}{5}x + 3$

34. $2x + y = 6$

35. $3x + 2y = 4$

36. $-\frac{2}{3}x - y = 2$

37. $2x - 3y = 12$

38. $2x = 3y$

39. $y = \frac{3}{2}$

40. $y = x$

41. $x = 3$

Find the midpoint of each segment with the given endpoints.

42. $(-8,\ 4)$ and $(-2,-6)$

43. $(4,-3)$ and $(-1,3)$

44. $(-5,-3)$ and $(7,5)$

45. $(-7,\ 5)$ and $(-2,11)$

46. $\left(\frac{1}{2},\ \frac{1}{3}\right)$ and $\left(\frac{3}{2},-\frac{5}{3}\right)$

47. $\left(\frac{3}{5},-\frac{1}{3}\right)$ and $\left(\frac{1}{2},-\frac{5}{2}\right)$

Segment AB has the given coordinates for the endpoint A and for its midpoint M. Find the coordinates of the endpoint B.

48. $A(-3,2)$, $M(3,-2)$

49. $A(7,\ 10)$, $M(5,3)$

50. $A(5,-4)$, $M(0,6)$

51. $A(-5,-2)$, $M(-1,4)$

G2 Slope of a Line and Its Interpretation

Slope (steepness) is a very important concept that appears in many branches of mathematics as well as statistics, physics, business, and other areas. In algebra, slope is used when graphing lines or analysing linear equations or functions. In calculus, the concept of slope is used to describe the behaviour of many functions. In statistics, slope of a regression line explains the general trend in the analysed set of data. In business, slope plays an important role in linear programming. In addition, slope is often used in many practical ways, such as the slope of a road (*grade*), slope of a roof (*pitch*), slope of a ramp, etc.
In this section, we will define, calculate, and provide some interpretations of slope.

Slope

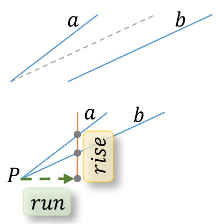


Figure 1a

Given two lines, a and b, how can we tell which one is steeper? One way to compare the steepness of these lines is to move them closer to each other, so that a point of intersection, P, can be seen, as in *Figure 1a*. Then, after running horizontally a few steps from the point P, draw a vertical line to observe how high the two lines have risen. The line that crosses this vertical line at a higher point is steeper. So, for example in *Figure 1a*, line a is steeper than line b. Observe that because we run the same horizontal distance for both lines, we could compare the steepness of the two lines just by looking at the vertical rise. However, since the *run* distance can be chosen arbitrarily, to represent the steepness of any line, we must look at the *rise* (vertical change) in respect to the *run* (horizontal change). This is where the concept of slope as a ratio of *rise* to *run* arises.

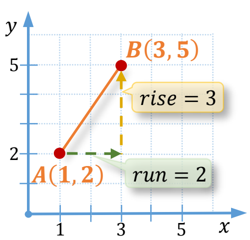


Figure 1b

To measure the slope of a line or a line segment, we choose any two distinct points of such a figure and calculate the ratio of the **vertical change** (***rise***) to the **horizontal change** (***run***) between the two points. For example, the slope between points $A(1,2)$ and $B(3,5)$ equals

$$\frac{rise}{run} = \frac{3}{2},$$

as in *Figure 1a*. If we rewrite this ratio so that the denominator is kept as one,

$$\frac{3}{2} = \frac{1.5}{1} = 1.5,$$

we can think of slope as of the **rate of change in y-values with respect to x-values**. So, a slope of 1.5 tells us that the y-value increases by 1.5 units per every increase of one unit in x-value.

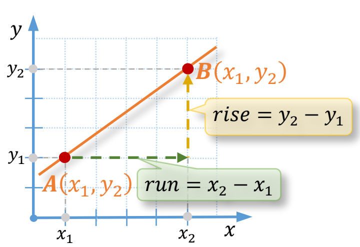


Figure 1c

Generally, the slope of a line passing through two distinct points, $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$, is the **ratio** of the change in y-values, $y_2 - y_1$, to the change in x-values, $x_2 - x_1$, as presented in *Figure 1c*. Therefore, the formula for calculating slope can be presented as

$$\frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x},$$

where the Greek letter Δ (delta) is used to denote the change in a variable.

Definition 2.1 ► Suppose a line passes through two distinct points (x_1, y_1) and (x_2, y_2).

If $x_1 \neq x_2$, then the **slope** of this line, often denoted by $\boldsymbol{m}$, is equal to

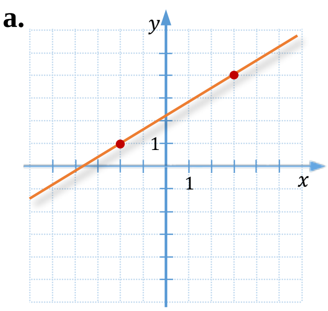
$$\boxed{m = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.}$$

If $x_1 = x_2$, then the **slope** of the line is said to be **undefined**.

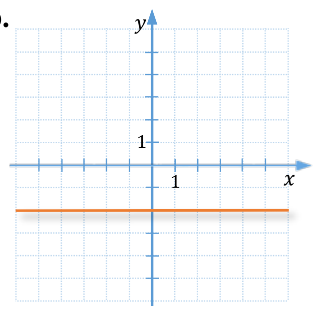
Example 1 ► Determining Slope of a Line, Given Its Graph

Determine the slope of each line.

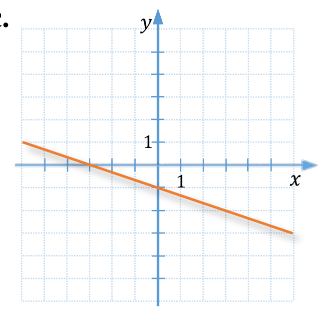
a.



b.

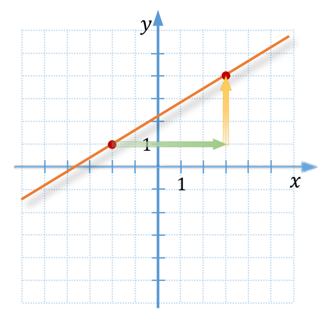


c.



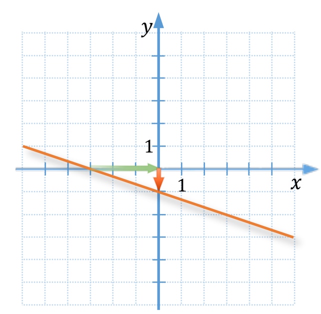
Solution ►

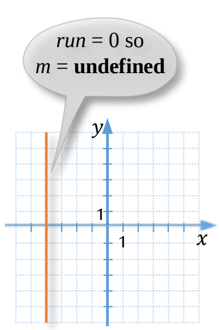
a. To read the slope we choose two distinct points with integral coefficients (often called ***lattice points***), such as the points suggested in the graph. Then, starting from the first point $(-2,1)$ we *run* 5 units and *rise* 3 units to reach the second point $(3,4)$. So, the slope of this line is $m = \frac{5}{3}$.



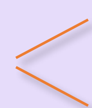
b. This is a horizontal line, so the *rise* between any two points of this line is zero. Therefore the slope of such a line is also **zero**.

c. If we refer to the lattice points $(-3,0)$ and $(0,-1)$, then the *run* is 3 and the *rise* (or rather ***fall***) is -1. Therefore the slope of this line is $m = -\frac{1}{3}$.





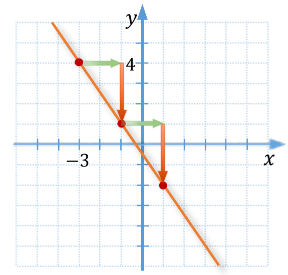
Observation:
A line that **increases** from left to right has a **positive slope**.
A line that **decreases** from left to right has a **negative slope**.
The slope of a **horizontal** line is **zero**.
The slope of a **vertical** line is **undefined**.



Example 2 ► **Graphing Lines, Given Slope and a Point**

Graph the line with slope $-\frac{3}{2}$ that passes through the point $(-3,4)$.

Solution ► First, plot the point $(-3,4)$. To find another point that belongs to this line, start at the plotted point and run 2 units, then fall 3 units. This leads us to point $(-1,1)$. For better precision, repeat the movement (two across and 3 down) to plot one more point, $(1,-2)$. Finally, draw a line connecting the plotted points.



Example 3 ► **Calculating Slope of a Line, Given Two Points**

Determine the slope of a line passing through the points $(-3,5)$ and $(7,-11)$.

Solution ► The slope of the line passing through $(-3,5)$ and $(7,-11)$ is the quotient

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-11)}{-3-7} = \frac{5+11}{-10} = -\frac{16}{10} = \mathbf{-1.6}$$

Example 4 ► **Determining Slope of a Line, Given Its Equation**

Determine the slope of a line given by the equation $2x - 5y = 7$.

Solution ► To see the slope of a line in its equation, we change the equation to its slope-intercept form, $y = mx + b$. The slope is the coefficient m. When solving $2x - 5y = 7$ for y, we obtain

$$-5y = -2x + 7$$

$$y = \frac{2}{5}x - \frac{7}{5}.$$

So, the slope of this line is equal to $\frac{2}{5}$.

Example 5 ► **Interpreting Slope as an Average Rate of Change**

The value of a particular stock has increased from \$156.60 on January 10, 2018, to \$187.48 on October 10, 2018. What is the average rate of change of the value of this stock per month for the given period of time?

Solution ► The average value of the stock has increased by $187.48 - 156.60 = 30.88$ dollars over the 9 months (from January 10 to October 10). So, the slope of the line segment connecting the the values of the stock on these two days (as marked on the above chart) equals

$$\frac{30.88}{9} \cong \mathbf{3.43}\ \$/month$$

This means that the value of the stock was increasing on average by 3.43 dollars per month between January 10, 2018, and October 10, 2018.

Observe that the change in value was actually different in each month. Sometimes the change was larger than the calculated slope, but sometimes the change was smaller or even negative. However, the **slope** of the above segment gave us the information about the **average rate of change** in the stock's value during the stated period.

Parallel and Perpendicular Lines

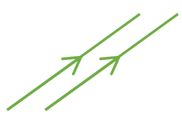


Figure 2

Since slope measures the steepness of lines, and **parallel lines** have the same steepness, then the **slopes** of **parallel lines** are **equal**.

To indicate on a diagram that lines are parallel, we draw on each line arrows pointing in the same direction (see *Figure 2*). To state in mathematical notation that two lines are parallel, we use the $\parallel$ sign.

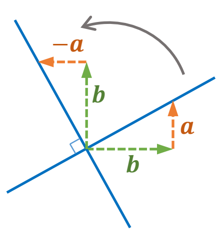


Figure 3

To see how the slopes of perpendicular lines are related, rotate a line with a given slope $\frac{a}{b}$ (where $b \neq 0$) by 90°, as in *Figure 3*. Observe that under this rotation the vertical change $\boldsymbol{a}$ becomes the horizontal change but in opposite direction ($-\boldsymbol{a}$), and the horizontal change $\boldsymbol{b}$ becomes the vertical change. So, the **slope** of the **perpendicular line** is $-\frac{b}{a}$. In other words, **slopes** of **perpendicular lines** are **opposite reciprocals**. Notice that the **product of perpendicular slopes**, $\frac{a}{b} \cdot \left(-\frac{b}{a}\right)$, is equal to $-\mathbf{1}$.

In the case of $b = 0$, the slope is undefined, so the line is vertical. After rotation by 90°, we obtain a horizontal line, with a slope of zero. So a line with a zero slope and a line with an "undefined" slope can also be considered perpendicular.

To indicate on a diagram that two lines are perpendicular, we draw a square at the intersection of the two lines, as in *Figure 3*. To state in mathematical notation that two lines are perpendicular, we use the $\perp$ sign.

In summary, if $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are **slopes** of two lines, then the lines are:

- **parallel** iff $\boldsymbol{m_1 = m_2}$, and
- **perpendicular** iff $\boldsymbol{m_1 = -\frac{1}{m_2}}$ (or equivalently, if $\boldsymbol{m_1 \cdot m_2 = -1}$)
  In addition, a **horizontal** line (with a slope of **zero**) is **perpendicular** to a **vertical** line (with **undefined** slope).

Example 6 ► **Determining Whether the Given Lines are Parallel, Perpendicular, or Neither**

For each pair of linear equations, determine whether the lines are parallel, perpendicular, or neither.

a. $3x + 5y = 7$
$5x - 3y = 4$

b. $y = x$
$2x - 2y = 5$

c. $y = 5$
$y = 5x$

Solution ►

a. As seen in *Section G1*, the slope of a line given by an equation in standard form, $Ax + By = C$, is equal to $-\frac{A}{B}$. One could confirm this by solving the equation for y and taking the coefficient by x for the slope.
Using this fact, the slope of the line $3x + 5y = 7$ is $-\frac{\mathbf{3}}{\mathbf{5}}$, and the slope of $5x - 3y = 4$ is $\frac{\mathbf{5}}{\mathbf{3}}$. Since these two slopes are opposite reciprocals of each other, the two lines are **perpendicular**.

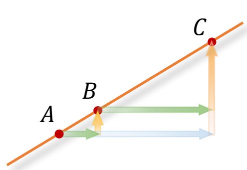
b. The slope of the line $y = x$ is $\mathbf{1}$ and the slope of $2x - 2y = 5$ is also $\frac{2}{2} = \mathbf{1}$. So, the two lines are parallel.

c. The line $y = 5$ can be seen as $y = 0x + 5$, so its slope is $\mathbf{0}$. The slope of the second line, $y = 5x$, is $\mathbf{5}$. So, the two lines are neither parallel nor perpendicular.

Collinear Points

Definition 2.2 ► Points that lie on the same line are called **collinear**.

Two points are always collinear because there is only one line passing through these points. The question is how could we check if a third point is collinear with the given two points? If we have an equation of the line passing through the first two points, we could plug in the coordinates of the third point and see if the equation is satisfied. If it is, the third point is collinear with the other two. But, can we check if points are collinear without referring to an equation of a line?



Notice that if several points lie on the same line, the slope between any pair of these points will be equal to the slope of this line. So, these slopes will be the same. One can also show that if the slopes between any two points in the group are the same, then such points lie on the same line. So, they are collinear.

Points are **collinear** iff the **slope** between each pair of points **is the same**.

Example 7 ► **Determine Whether the Given Points are Collinear**

Determine whether the points $A(-3,7)$, $B(-1,2)$, and $C = (3,-8)$ are collinear.

Solution ► Let m_{AB} represent the slope of $\overline{AB}$ and m_{BC} represent the slope of $\overline{BC}$. Since

$$m_{AB} = \frac{2-7}{-1-(-3)} = -\frac{5}{2} \quad \text{and} \quad m_{BC} = \frac{-8-2}{3-(-1)} = -\frac{10}{4} = -\frac{5}{2},$$

Then all points A, B, and C lie on the same line. Thus, they are collinear.

Example 8 ► **Finding the Missing Coordinate of a Collinear Point**

For what value of y are the points $P(2,2)$, $Q(-1,y)$, and $R(1,6)$ collinear?

Solution ► For the points P, Q, and R to be collinear, we need the slopes between any two pairs of these points to be equal. For example, the slope m_{PQ} should be equal to the slope m_{PR}. So, we solve the equation

$$m_{PQ} = m_{PR}$$

for y:

$$\frac{y-2}{-1-2} = \frac{6-2}{1-2}$$

$$\frac{y-2}{-3} = -4 \qquad /\cdot(-3)$$

$$y - 2 = 12 \qquad /+2$$

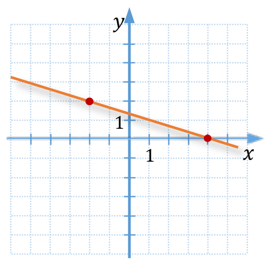
$$y = 14$$

Thus, point Q is collinear with points P and R, if $y = \mathbf{14}$.

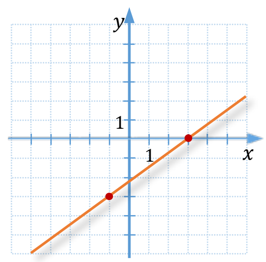
G.2 Exercises

Given the graph, find the slope of each line.

1.



2.



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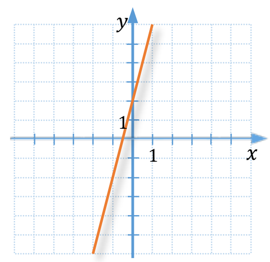

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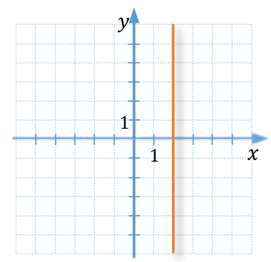


Given the equation, find the slope of each line.

5. $y = \frac{1}{2}x - 7$

6. $y = -\frac{1}{3}x + 5$

7. $4x - 5y = 2$

8. $3x + 4y = 2$

9. $x = 7$

10. $y = -\frac{3}{4}$

11. $y + x = 1$

12. $-8x - 7y = 24$

13. $-9y - 36 + 4x = 0$

Graph each line satisfying the given information.

14. passing through $(-2, -4)$ with slope $m = 4$

15. passing through $(-1, -2)$ with slope $m = -3$

16. passing through $(-3, 2)$ with slope $m = \frac{1}{2}$

17. passing through $(-3,4)$ with slope $m = -\frac{2}{5}$

18. passing through $(2, -1)$ with undefined slope

19. passing through $(2, -1)$ with slope $m = 0$

20. Which of the following formulas represent slope?

a. $m = \frac{y_1 - y_2}{x_2 - x_1}$ **b.** $m = \frac{y_1 - y_2}{x_1 - x_2}$ **c.** $m = \frac{x_2 - x_1}{y_2 - y_1}$ **d.** $m = \frac{y_2 - y_1}{x_2 - x_1}$

Find the slope of the line through each pair of points.

21. $(-2,2)$, $(4,5)$

22. $(8,7)$, $(2,-1)$

23. $(9,-4)$, $(3,-8)$

24. $(-5,2)$, $(-9,5)$

25. $(-2,3)$, $(7,-12)$

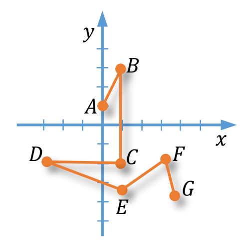
26. $(3,-1)$, $\left(-\frac{1}{2}, \frac{1}{5}\right)$

27. $(-5,2)$, $(8,2)$

28. $(-3,4)$, $(-3,10)$

29. $\left(\frac{1}{2}, 6\right)$, $\left(-\frac{2}{3}, \frac{5}{2}\right)$

30. List the line segments in the accompanying figure with respect to their slopes, from the smallest to the largest slope. List the segment with an undefined slope as last.



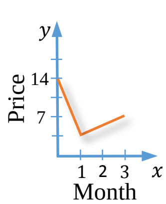
31. *Refer to the graphs of a stock price observed during a 3-month period. Match each situation in **a–d** with the most appropriate graph in **A–D**.*

a. The price rose sharply during the first month, remained stable during the second month before falling to the original price by the end of the third month.

b. The price fell sharply during the first month and then rose slowly for the next two months.

c. The price rose sharply during the first month, fell slowly during the second month before rising slowly to its highest peak during the third month.

d. The price fell slowly during the first two months before stabilizing in the third month.

A.


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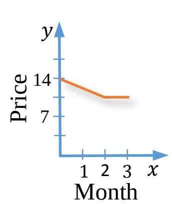
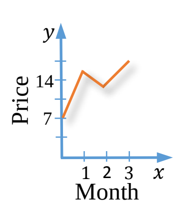
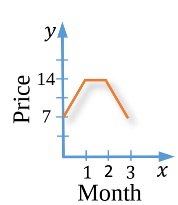
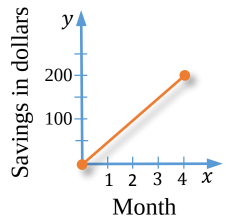


C.


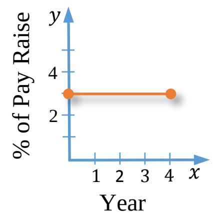
D.


Find and interpret the average rate of change illustrated in each graph.

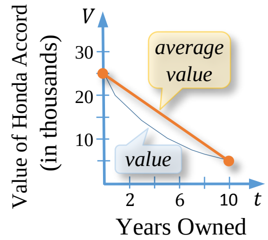
32.



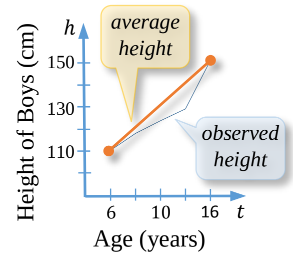
33.



34.



35.



In problems #43-46, sketch a graph depicting each situation. Assume that the roads in each problem are straight.

36. The distance that a driver is from home if he starts driving home from a town that is 30 kilometers away and he drives at a constant speed for half an hour.

37. The distance that a cyclist is from home if he bikes away from home at 30 kilometers per hour for 30 minutes and then bikes back home at 15 kilometers per hour.

38. The distance that Alice is from home if she walks 4 kilometers from home to a shoping centre, stays there for 1.5 hours, and then walks back home. Assume that Alice walks at a constant speed for 30 minutes each way.

39. The amount of water in a 500 liters outdoor pool for kids that is filled at the rate of 1500 liters per hour, left full for 4 hours, and then drained at the rate of 3000 liters per hour.

Solve each problem.

40. At 6:00 a.m. a 60,000-liter swimming pool was $\frac{1}{3}$ full and at 9:00 a.m. the pool was filled up to $\frac{3}{4}$ of its capacity. Find the rate of filling the pool with the assumption that the rate was constant.

41. Jan and Bill plan to drive to Kelowna that is 324 kilometers away. Jan noticed that during one hour they change their location from being $\frac{1}{3}$ of the way to being $\frac{2}{3}$ of the way. Assuming that they drive at a constant rate, what is their average speed of driving?

42. Suppose we see a road sign informing that a road grade is 7% for the next 1.5 kilometers. In meters, what would the expected change in elevation be 1.5 kilometers down the road?

Decide whether each pair of lines is parallel, perpendicular, or neither.

43. $y = x$
$y = -x$

44. $y = 3x - 6$
$y = -\frac{1}{3}x + 5$

45. $2x + y = 7$
$-6x - 3y = 1$

46. $x = 3$
$x = -2$

47. $3x + 4y = 3$
$3x - 4y = 5$

48. $5x - 2y = 3$
$2x - 5y = 1$

49. $y - 4x = 1$
$x + 4y = 3$

50. $y = \frac{2}{3}x - 2$
$-2x + 3y = 6$

Solve each problem.

51. Check whether or not the points $(-2, 7)$, $(1, 5)$, and $(3, 4)$ are collinear.

52. The following points, $(2, 2)$, $(-1, k)$, and $(1, 6)$ are collinear. Find the value of k.

G3 Forms of Linear Equations in Two Variables

Linear equations in two variables can take different forms. Some forms are easier to use for graphing, while others are more suitable for finding an equation of a line given two pieces of information. In this section, we will take a closer look at various forms of linear equations and their utilities.

Forms of Linear Equations

The form of a linear equation that is most useful for graphing lines is the slope-intercept form, as introduced in *Section G1*.

Definition 3.1 ► The **slope-intercept form** of the equation of a line with **slope m** and **y-intercept** $(0, b)$ is

$$y = mx + b.$$

Example 1 ► **Writing and Graphing Equation of a Line in Slope-Intercept Form**

Write the equation in slope-intercept form of the line satisfying the given conditions, and then graph this line.

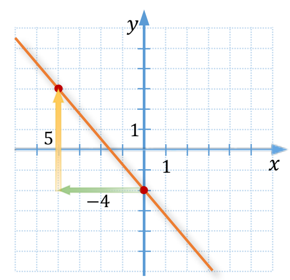
a. slope $-\frac{4}{5}$ and y-intercept $(0, -2)$

b. slope $\frac{1}{2}$ and passing through $(2, -5)$

Solution ►

a. To write this equation, we substitute $m = -\frac{4}{5}$ and $b = -2$ into the slope-intercept form. So, we obtain

$$y = -\frac{4}{5}x - 2.$$



To graph this line, we start with plotting the y-intercept $(0, -2)$. To find the second point, we follow the slope, as in *Example 2, Section G2*. According to the slope $-\frac{4}{5} = \frac{-4}{5}$, starting from $(0, -2)$, we could run 5 units to the right and 4 units down, but then we would go out of the grid. So, this time, let the negative sign in the slope be kept in the denominator, $\frac{4}{-5}$. Thus, we run 5 units to the left and 4 units up to reach the point $(0, -2)$. Then we draw the line by connecting the two points.

b. Since $m = \frac{1}{2}$, our equation has a form $y = \frac{1}{2}x + b$. To find b, we substitute point $(2, -5)$ into this equation and solve for b. So

$$-5 = \frac{1}{2}(2) + b$$

gives us

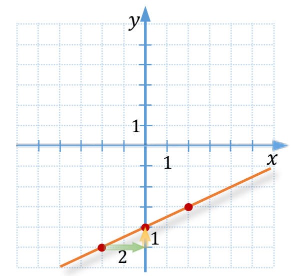
$$-5 = 1 + b$$

and finally

$$b = -6.$$

Therefore, our equation of the line is $\boldsymbol{y = \frac{1}{2}x - 6}$.

We graph it, starting by plotting the given point $(2, -5)$ and finding the second point by following the slope of $\frac{1}{2}$, as described in *Example 2, Section G2*.



The form of a linear equation that is most useful when writing equations of lines with unknown y-intercept is the slope-point form.

Definition 3.2 ► The **slope-point form** of the equation of a line with slope $\boldsymbol{m}$ and passing through the point $(\boldsymbol{x_1}, \boldsymbol{y_1})$ is

$$\boldsymbol{y - y_1 = m(x - x_1)}.$$

This form is based on the defining property of a line. A line can be defined as a set of points with a constant slope m between any two of these points. So, if (x_1, y_1) is a given (fixed) point of the line and (x, y) is any (variable) point of the line, then, since the slope is equal to m for all such points, we can write the equation

$$m = \frac{x - x_1}{y - y_1}.$$

After multiplying by the denominator, we obtain the slope-point formula, as in *Definition 3.2*.

Example 2 ► **Writing Equation of a Line Using Slope-Point Form**

Use the slope-point form to write an equation of the line satisfying the given conditions. Leave the answer in the slope-intercept form and then graph the line.

a. slope $-\frac{2}{3}$ and passing through $(1, -3)$

b. passing through points $(2, 5)$ and $(-1, -2)$

Solution ►

a. To write this equation, we plug the slope $m = -\frac{2}{3}$ and the coordinates of the point $(1, -3)$ into the slope-point form of a line. So, we obtain

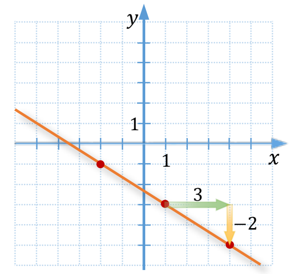
$$y - (-3) = -\frac{2}{3}(x - 1)$$

$$y + 3 = -\frac{2}{3}x + \frac{2}{3} \qquad /-3$$

$$y = -\frac{2}{3}x + \frac{2}{3} - \frac{9}{3}$$

$$\boldsymbol{y = -\frac{2}{3}x - \frac{7}{3}}$$

To graph this line, we start with plotting the point $(1,-3)$ and then apply the slope of $-\frac{2}{3}$ to find additional points that belong to the line.



b. This time the slope is not given, so we will calculate it using the given points, $(2,5)$ and $(-1,-2)$. Thus,

$$m = \frac{\Delta y}{\Delta x} = \frac{-2-5}{-1-2} = \frac{-7}{-3} = \frac{7}{3}$$

Then, using the calculated slope and one of the given points, for example $(2,5)$, we write the slope-point equation of the line

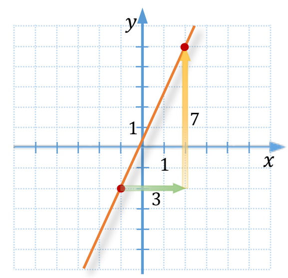
$$y - 5 = \frac{7}{3}(x-2)$$

and solve it for y:

$$y - 5 = \frac{7}{3}x - \frac{14}{3} \qquad /-5$$

$$y = \frac{7}{3}x - \frac{14}{3} + \frac{15}{3}$$

$$\boldsymbol{y = \frac{7}{3}x + \frac{1}{3}}$$



To graph this line, it is enough to connect the two given points.

One of the most popular forms of a linear equation is the standard form. This form is helpful when graphing lines based on x- and y-intercepts, as illustrated in *Example 3, Section G1*.

Definition 3.3 ► The **standard form** of a linear equation is

$$\boldsymbol{Ax + By = C},$$

Where $A, B, C \in \mathbb{R}$, A and B are not both 0, and $A \geq 0$.

When writing linear equations in standard form, the expectation is to use a **nonnegative coefficient $\boldsymbol{A}$** and **clear any fractions**, if possible. For example, to write $-x + \frac{1}{2}y = 3$ in standard form, we multiply the equation by (-2), to obtain $2x - y = -6$. In addition, we prefer to write equations in simplest form, where the greatest common factor of A, B, and C is 1. For example, we prefer to write $2x - y = -6$ rather than any multiple of this equation, such as $4x - 2y = -12$, or $6x - 3y = -18$.

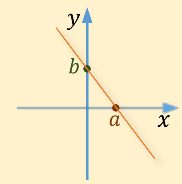
Observe that if $B \neq 0$ then the **slope** of the line given by the equation $\boldsymbol{Ax + By = C}$ is $-\frac{A}{B}$.
This is because after solving this equation for y, we obtain $\boldsymbol{y = -\frac{A}{B}x + \frac{C}{B}}$.
If $B = 0$, then the slope is **undefined**, as we are unable to divide by zero.

The form of a linear equation that is most useful when writing equations of lines based on their x- and y-intercepts is the intercept form.

Definition 3.4 ► The **intercept form** of a linear equation is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where a is the **x-intercept** and b is the **y-intercept** of the line.



We should be able to convert a linear equation from one form to another.

Example 3 ► **Converting a Linear Equation to a Different Form**

a. Write the equation $3x + 7y = 2$ in slope-intercept form.

b. Write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form.

c. Write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form.

Solution ► **a.** To write the equation $3x + 7y = 2$ in slope-intercept form, we solve it for y.

$$3x + 7y = 2 \qquad /-3x$$

$$7y = -3x + 2 \qquad /\div 7$$

$$\boldsymbol{y = -\frac{3}{7}x + \frac{2}{7}}$$

b. To write the equation $y = \frac{3}{5}x + \frac{7}{2}$ in standard form, we bring the x-term to the left side of the equation and multiply the equation by the LCD, with the appropriate sign.

$$y = \frac{3}{5}x + \frac{7}{2} \qquad /-\frac{3}{5}x$$

$$-\frac{3}{5}x + y = \frac{7}{2} \qquad /\cdot (-10)$$

$$\boldsymbol{6x - 10y = -35}$$

c. To write the equation $\frac{x}{4} - \frac{y}{3} = 1$ in standard form, we multiply it by the LCD, with the appropriate sign.

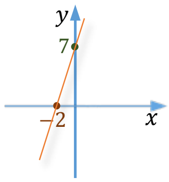
$$\frac{x}{4} - \frac{y}{3} = 1 \qquad /\cdot 12$$

$$\boldsymbol{3x - 4y = 12}$$

Example 4 ► **Writing Equation of a Line Using Intercept Form**

Write an equation of the line passing through points $(0, -2)$ and $(7, 0)$. Leave the answer in standard form.

Solution ► Since point $(0, -2)$ is the y-intercept and point $(7, 0)$ is the x-intercept of our line, to write the equation of the line we can use the intercept form with $a = -2$ and $b = 7$. So, we have

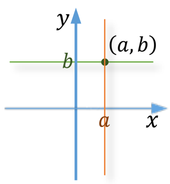


$$\frac{x}{-2} + \frac{y}{7} = 1.$$

To change this equation to standard form, we multiply it by the LCD $= -14$. Thus,

$$\mathbf{7x - 2y = -14}.$$

Equations representing horizontal or vertical lines are special cases of linear equations in standard form, and as such, they deserve special consideration.



The **horizontal line** passing through the point (a, b) has equation $\boldsymbol{y = b}$, while the **vertical line** passing through the same point has equation $\boldsymbol{x = a}$.

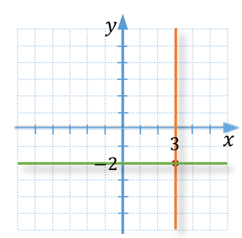
The equation of a **horizontal line**, $\boldsymbol{y = b}$, can be shown in standard form as $0x + y = b$. Observe, that the slope of such a line is $-\frac{0}{1} = 0$.

The equation of a **vertical line**, $\boldsymbol{x = a}$, can be shown in standard form as $x + 0y = a$. Observe, that the slope of such a line is $-\frac{1}{0} =$ undefined.

Example 5 ► **Writing Equations of Horizontal and Vertical Lines**

Find equations of the vertical and horizontal lines that pass through the point $(3, -2)$. Then, graph these two lines.

Solution ► Since x-coordinates of all points of the vertical line, including $(3, -2)$, are the same, then these x-coordinates must be equal to 3. So, the equation of the vertical line is $x = 3$.



Since y-coordinates of all points of a horizontal line, including $(3, -2)$, are the same, then these y-coordinates must be equal to -2. So, the equation of the horizontal line is $y = -2$.

Here is a summary of the various forms of linear equations.

Forms of Linear Equations

Equation	Description	When to Use
$\boldsymbol{y = mx + b}$	**Slope-Intercept Form** slope is $\boldsymbol{m}$ y-intercept is $(0, \boldsymbol{b})$	This form is ideal for graphing by using the y-intercept and the slope.
$\boldsymbol{y - y_1 = m(x - x_1)}$	**Slope-Point Form** slope is $\boldsymbol{m}$ the line passes through $(\boldsymbol{x_1}, \boldsymbol{y_1})$	This form is ideal for finding the equation of a line if the slope and a point on the line, or two points on the line, are known.

$\boldsymbol{Ax + By = C}$	**Standard Form** slope is $-\frac{A}{B}$, if $B \neq 0$ x-intercept is $\left(\frac{C}{A}, 0\right)$, if $A \neq 0$. y-intercept is $\left(0, \frac{C}{B}\right)$, if $B \neq 0$.	This form is useful for graphing, as the x- and y-intercepts, as well as the slope, can be easily found by dividing appropriate coefficients.
$\boldsymbol{\frac{x}{a} + \frac{y}{b} = 1}$	**Intercept Form** slope is $-\frac{b}{a}$ x-intercept is $(\boldsymbol{a}, 0)$ y-intercept is $(0, \boldsymbol{b})$	This form is ideal for graphing, using the x- and y-intercepts.
$\boldsymbol{y = b}$	**Horizontal Line** slope is 0 y-intercept is $(0, \boldsymbol{b})$	This form is used to write equations of, for example, horizontal asymptotes.
$\boldsymbol{x = a}$	**Vertical Line** slope is undefined x-intercept is $(\boldsymbol{a}, 0)$	This form is used to write equations of, for example, vertical asymptotes.

Note: Except for the equations for a horizontal or vertical line, all of the above forms of linear equations can be converted into each other via algebraic transformations.

Writing Equations of Parallel and Perpendicular Lines

Recall that the slopes of parallel lines are the same, and slopes of perpendicular lines are opposite reciprocals. See *Section G2*.

Example 6 ► **Writing Equations of Parallel Lines Passing Through a Given Point**

Find the slope-intercept form of a line parallel to $y = -2x + 5$ that passes through the point $(-4,5)$. Then, graph both lines on the same grid.

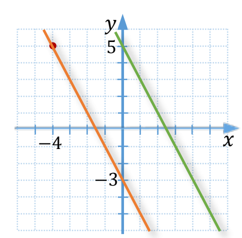
Solution ► Since the line is parallel to $y = -2x + 5$, its slope is -2. So, we plug the slope of -2 and the coordinates of the point $(-4,5)$ into the slope-point form of a linear equation.

$$y - 5 = -2(x + 4)$$

This can be simplified to the slope-intercept form, as follows:

$$y - 5 = -2x - 8$$

$$\boldsymbol{y = -2x - 3}$$



As shown in the accompanying graph, the line $y = -2x - 3$ (in orange) is parallel to the line $y = -2x + 5$ (in green) and passes through the given point $(-4,5)$.

Example 7 ► **Writing Equations of Perpendicular Lines Passing Through a Given Point**

Find the slope-intercept form of a line perpendicular to $2x - 3y = 6$ that passes through the point $(1,4)$. Then, graph both lines on the same grid.

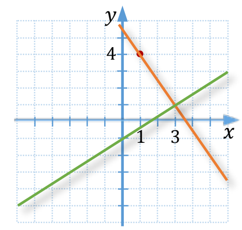
Solution ► The slope of the given line, $2x - 3y = 3$, is $\frac{2}{3}$. To find the slope of a perpendicular line, we take the opposite reciprocal of $\frac{2}{3}$, which is $-\frac{3}{2}$. Since we already know the slope and the point, we can plug these pieces of information into the slope-point formula. So, we have

$$y - 4 = -\frac{3}{2}(x - 1)$$

$$y - 4 = -\frac{3}{2}x + \frac{3}{2} \qquad /+4$$

$$y = -\frac{3}{2}x + \frac{3}{2} + \frac{8}{2}$$

$$\boldsymbol{y = -\frac{3}{2}x + \frac{11}{2}}$$



As shown in the accompanying graph, the line $2x - 3y = 6$ (in orange) is indeed perpendicular to the line $y = -\frac{3}{2}x + \frac{11}{2}$ (in green) and passes through the given point $(1,4)$.

Linear Equations in Applied Problems

Linear equations can be used to model a variety of applications in sciences, business, and other areas. Here are some examples.

Example 8 ► **Given the Rate of Change and the Initial Value, Determine the Linear Model Relating the Variables**

Lucy and Jack bought a sofa for \$1500. They have paid \$300 down and the rest is going to be paid by monthly payments of \$75 per month, till the bill is paid in full.

a. Write an equation to express the amount that is already paid off, A, in terms of the number of months, n, since their purchase.

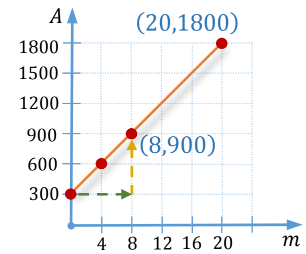
b. Graph the equation found in part **a.**

c. According to the graph, when the bill will be paid in full?

Solution ► **a.** Since each month the couple pays \$75, after n months, the amount paid off by the monthly installments is $75n$. If we add the initial payment of \$300, the equation representing the amount paid off can be written as

$$A = 75n + 300$$

b. To graph this equation, we use the slope-intercept method. Starting with the A-intercept of 300, we run 1 and rise 75, repeating this process as many times as needed to hit a lattice point on the chosen scale. As indicated in the accompanying graph, some of the points that the line passes through are $(0,300)$, $(4,600)$, $(8,900)$, and $(20,1800)$.



c. As shown in the graph, \$1800 will be paid off in 20 months.

Example 9 ► Finding a Linear Equation that Fits the Data Given by Two Ordered Pairs

In Fahrenheit scale, water freezes at 32°F and boils at 212°F. In Celsius scale, water freezes at 0°C and boils at 100°C. Write a linear equation that can be used to calculate the Celsius temperature, C, when the Fahrenheit temperature, F, is known.

Solution ►

To predict the Celsius temperature, C, knowing the Fahrenheit temperature, F, we treat the variable C as dependent on the variable F. So, we consider C as the second coordinate when setting up the ordered pairs, (F, C), of given data. The corresponding freezing temperatures give us the pair $(32,0)$ and the boiling temperatures give us the pair $(212,100)$. To find the equation of a line passing through these two points, first, we calculate the slope, and then, we use the slope-point formula. So, the slope is

$$m = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{\mathbf{5}}{\mathbf{9}},$$

and using the point $(32,0)$, the equation of the line is

$$\boldsymbol{C = \frac{5}{9}(F - 32)}$$

Example 10 ► Determining if the Given Set of Data Follows a Linear Pattern

Observe the data given in each table below. Do they follow a linear pattern? If "yes", find the slope-intercept form of an equation of the line passing through all the given points. If "not", explain why not.

a.

x	1	3	5	7	9
y	12	15	18	21	24

b.

x	5	10	15	20	25
y	15	21	26	30	35

Solution ►

a. The set of points follows a linear pattern if the slopes between consecutive pairs of these points are the same. These slopes are the ratios of increments in y-values to increments in x-values. Notice that the increases between successive x-values of the given points are constantly equal to 2. So, to check if the points follow a linear pattern, it is enough to check if the increases between successive y-values are also constant. Observe that the numbers in the list 12, 15, 18, 21, 24 steadily increase by 3. Thus, the given set of data follow a linear pattern.

To find an equation of the line passing through these points, we use the slope, which is $\frac{3}{2}$, and one of the given points, for example $(1,12)$. By plugging these pieces of information into the slope-point formula, we obtain

$$y - 12 = \frac{3}{2}(x - 1),$$

which after simplifying becomes

$$y - 12 = \frac{3}{2}x - \frac{3}{2} \qquad /+12$$

$$\boldsymbol{y = 2x + \frac{21}{2}}$$

b. Observe that the increments between consecutive x-values of the given points are constantly equal to 5, while the increments between consecutive y-values in the list 15, 21, 26, 30, 35 are 6, 5, 4, 5. So, they are not constant. Therefore, the given set of data does not follow a linear pattern.

Example 11 ► **Finding a Linear Model Relating the Number of Items Bought at a Fixed Amount**

At a local market, a farmer sells organically grown apples at \$0.50 each and pears at \$0.75 each.

a. Write a linear equation in standard form relating the number of apples, a, and pears, p, that can be bought for \$60.

b. Graph the equation from part (a).

c. Using the graph, find at least 2 points (a, p) satisfying the equation, and interpret their meanings in the context of the problem.

Solution ►

a. It costs $0.5a$ dollars to buy a apples. Similarly, it costs $0.75p$ dollars to buy p pears. Since the total charge is \$60, we have

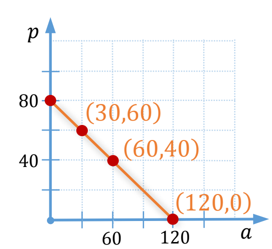
$$0.5a + 0.75p = 60$$

The coefficients can be converted into integers by multiplying the equation by a hundred. This would give us

$$50a + 75p = 6000,$$

which, after dividing by 25, turns into

$$\boldsymbol{2a + 3p = 240}.$$



b. To graph this equation, we will represent the number of apples, a, on the horizontal axis and the number of pears, p, on the vertical axis, respecting the alphabetical order of labelling the axes. Using the intercept method, we connect points $(120,0)$ and $(0,80)$.

c. Aside of the intercepts, $(120,0)$ and $(0,80)$, the graph shows us a few more points that satisfy the equation. In particular, $(\mathbf{30}, \mathbf{60})$ and $(\mathbf{60}, \mathbf{40})$ are points of the graph. If a point (a, p) of the graph has integral coefficients, it tells us that \$60 can buy a apples and p pears. For example, the point $(\mathbf{30}, \mathbf{60})$ tells us that **30 apples** and **60 pears** can be bought for **\$60**.

G.3 Exercises

*Write each equation in **standard form**.*

1. $y=-\frac{1}{2}x-7$

2. $y=\frac{1}{3}x+5$

3. $\frac{x}{5}+\frac{y}{-4}=1$

4. $y-7=\frac{3}{2}(x-3)$

5. $y-\frac{5}{2}=-\frac{2}{3}(x+6)$

6. $2y=-0.21x+0.35$

*Write each equation in **slope-intercept form**.*

7. $3y=\frac{1}{2}x-5$

8. $\frac{x}{3}+\frac{y}{5}=1$

9. $4x-5y=10$

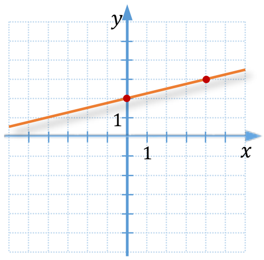
10. $3x+4y=7$

11. $y+\frac{3}{2}=\frac{2}{5}(x+2)$

12. $y-\frac{1}{2}=-\frac{2}{3}\left(x-\frac{1}{2}\right)$

*Write an equation in **slope-intercept** form of the line shown in each graph.*

13.



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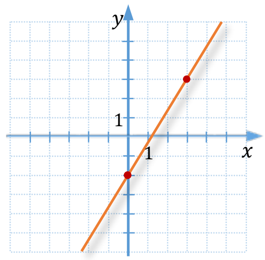

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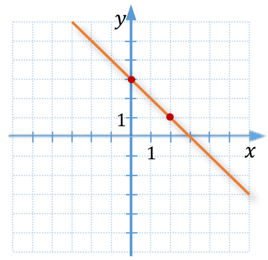

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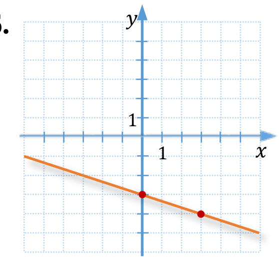


*Find an equation of the line that satisfies the given conditions. Write the equation in **slope-intercept** and **standard** form.*

17. through $(-3,2)$, with slope $m=\frac{1}{2}$

18. through $(-2,3)$, with slope $m=-4$

19. with slope $m=\frac{3}{2}$ and y-intercept at -1

20. with slope $m=-\frac{1}{5}$ and y-intercept at 2

21. through $(-1,-2)$, with y-intercept at -3

22. through $(-4,5)$, with y-intercept at $\frac{3}{2}$

23. through $(2,-1)$ and $(-4,6)$

24. through $(3,7)$ and $(-5,1)$

25. through $\left(-\frac{4}{3},-2\right)$ and $\left(\frac{4}{5},\frac{2}{3}\right)$

26. through $\left(\frac{4}{3},\frac{3}{2}\right)$ and $\left(-\frac{1}{2},\frac{4}{3}\right)$

Find an equation of the line that satisfies the given conditions.

27. through $(-5,7)$, with slope 0

28. through $(-2,-4)$, with slope 0

29. through $(-1,-2)$, with undefined slope

30. through $(-3,4)$, with undefined slope

31. through $(-3,6)$ and horizontal

32. through $\left(-\frac{5}{3},-\frac{7}{2}\right)$ and horizontal

33. through $\left(-\frac{3}{4},-\frac{3}{2}\right)$ and vertical

34. through $(5,-11)$ and vertical

*Write an equation in **standard form** for each of the lines described. In each case make a sketch of the given line and the line satisfying the conditions.*

35. through $(7,2)$ and parallel to $3x - y = 4$

36. through $(4,1)$ and parallel to $2x + 5y = 10$

37. through $(-2,3)$ and parallel to $-x + 2y = 6$

38. through $(-1,-3)$ and parallel to $-x + 3y = 12$

39. through $(-1,2)$ and parallel to $y = 3$

40. through $(-1,2)$ and parallel to $x = -3$

41. through $(6,2)$ and perpendicular to $2x - y = 5$

42. through $(0,2)$ and perpendicular to $5x + y = 15$

43. through $(-2,4)$ and perpendicular to $3x + y = 6$

44. through $(-4,-1)$ and perpendicular to $x - 3y = 9$

45. through $(3,-4)$ and perpendicular to $x = 2$

46. through $(3,-4)$ and perpendicular to $y = -3$

*For each situation, write an equation in the form **y = mx + b**, and then answer the question of the problem.*

47. Membership in the Apollo Athletic Club costs \$80, plus \$49.95 per month. Express the cost C of the membership in terms of the number of months n that the membership is good for. What is the cost of the one-year membership?

48. A cellphone plan includes 1000 anytime minutes for \$55 per month, plus a one-time activation fee of \$75. Assuming that a cellphone is included in this plan at no additional charge, express the cost C of service in terms of the number of months n of this service. How much would a one-year contract plan cost for this cellphone?

49. An air compressor can be rented for \$23 per day and a \$60 deposit. Let d represent the number of days that the compressor is rented and C represent the total charge for renting, in dollars.
a. Write an equation that relates C with d.
b. Suppose Colin rented the air compressor and paid \$198. For how long did Colin rented the conpressor?

50. A car can be rented for \$75 plus \$0.15 per kilometer. Let d represent the number of kilometers driven and C represent the cost of renting, in dollars.
a. Write an equation that relates C with d.
b. How many kilometers was the car driven if the total cost of renting is \$101.40?

Solve each problem.

51. Originally there were 8 members of a local high school Math Circle. Three years later, the Math Circle counted 25 members. Assuming that the membership continues to grow at the same rate, find an equation that represents the number N of the Math Circle members t years after.

52. Driving on a highway, Steven noticed a 152-km marker on the side of the road. Ten minutes later, he noticed a 169-km marker. Find a formula that can be used to determine the distance driven d, in kilometers, in terms of the elapsed time t, in hours.

53. The table below shows the annual tuition and fees at Oxford University for out-of-state students.

Year *y*	**2007**	**2016**
Cost *C*	\$24400	\$31600

a. Find the slope-intercept form of a line that fits the given data.
b. Interpret the slope in the context of the problem.
c. Using the line from (a), find the predicted annual tuition and fees at Oxford University in 2022.

54. The life expectancy for a person born in 1900 was 48 years, and in 2000 it was 77 years. To the nearest year, estimate the life expectancy for someone born in 1970.

55. 3 years after Stan opened his mutual funds account, the amount in the account was \$2540. Two years later, the amount in the account was \$2900. Assuming a constant average increase in \$/year, find a linear equation that represents the amount A in Stan's account t years after it was opened.

56. Connor is a car salesperson in the local auto shop. His pay consists of a base salary and a 1.5% commission on sales. One month, his sales were \$165,000, and his total pay was \$3600.
a. Write an equation in slope-intercept form that shows Conner's total monthly income I in terms of his monthly sales s.
b. Graph the equation developed in (**a**).
c. What does the I-intercept represent in the context of the problem?
d. What does the slope represent in the context of the problem?

57. A taxi driver charges \$2.50 as his base fare and a constant amount for each kilometer driven. Helen paid \$7.75 for a 3-kilometer trip.
a. Find an equation in slope-intercept form that defines the total fare $f(k)$ as a function of the number k of kilometers driven.
b. Graph the equation found in (**a**).
c. What does the slope of this graph represent in the above situation?
d. How many kilometers were driven if a passenger pays \$23.50?

G4 Linear Inequalities in Two Variables Including Systems of Inequalities

In many real-life situations, we are interested in a range of values satisfying certain conditions rather than in one specific value. For example, when exercising, we like to keep the heart rate between 120 and 140 beats per minute. The systolic blood pressure of a healthy person is usually between 100 and 120 mmHg (millimeters of mercury). Such conditions can be described using inequalities. Solving systems of inequalities has its applications in many practical business problems, such as how to allocate resources to achieve a maximum profit or a minimum cost. In this section, we study graphical solutions of linear inequalities and systems of linear inequalities.

Linear Inequalities in Two Variables

Definition 4.1 ► Any inequality that can be written as

$\boldsymbol{Ax+By<C}$, $\boldsymbol{Ax+By\le C}$, $\boldsymbol{Ax+By>C}$, $\boldsymbol{Ax+By\ge C}$, or $\boldsymbol{Ax+By\ne C}$,

where $A, B, C \in \mathbb{R}$ and A and B are not both 0, is a **linear inequality in two variables**.

To **solve** an inequality in two variables, x and y, means to **find <u>all</u> ordered pairs** $(\boldsymbol{x},\boldsymbol{y})$ satisfying the inequality.

Inequalities in two variables arise from many situations. For example, suppose that the number of full-time students, f, and part-time students, p, enrolled in upgrading courses at the University of the Fraser Valley is at most 1200. This situation can be represented by the inequality

$$f+p\le 1200.$$

Some of the solutions (f,p) of this inequality are: $(1000,200)$, $(1000,199)$, $(1000,198)$, $(600,600)$. $(550,600)$, $(1100,0)$, and many others.

The solution sets of inequalities in two variables contain infinitely many ordered pairs of numbers which, when graphed in a system of coordinates, fulfill specific regions of the coordinate plane. That is why it is more beneficial to present such solutions in the form of a graph rather than using set notation. To graph the region of points satisfying the inequality $f+p\le 1200$, we may want to solve it first for p,

$$p\le -f+1200,$$

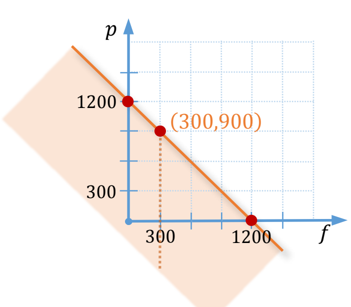


Figure 1a

and then graph the related equation, $p=-f+1200$, called the **boundary line**. Notice, that setting f to, for instance, 300 results in the inequality

$$p\le -300+1200=900.$$

So, any point with the first coordinate of 300 and the second coordinate of 900 or less satisfies the inequality (see the dotted half-line in *Figure 1a*). Generally, observe that any point with the first coordinate f and the second coordinate $-f+1200$ or less satisfies the inequality. Since the union of all half-lines that start from the boundary line and go down is the whole half-plane below the boundary line, we shade it as the solution set to the discussed inequality (see *Figure 1a*). The solution set also includes the points of the boundary line, as the inequality includes equation.

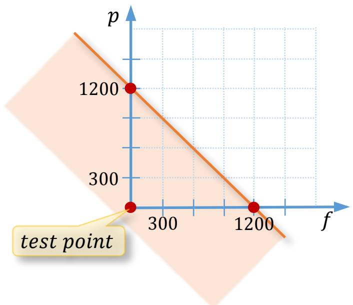


Figure 1b

The above strategy can be applied to any linear inequality in two variables. Hence, one can conclude that the solution set to a given linear inequality in two variables consists of **all points of one of the half-planes** obtained by cutting the coordinate plane by the corresponding boundary line. This fact allows us to find the solution region even faster. After graphing the boundary line, to know which half-plane to shade as the solution set, it is enough to check just one point, called a **test point**, chosen outside of the boundary line. In our example, it was enough to test for example point $(0,0)$. Since $0 \leq -0 + 1200$ is a true statement, then the point $(0,0)$ belongs to the solution set. This means that the half-plane containing this test point must be the solution set to the given inequality, so we shade it.

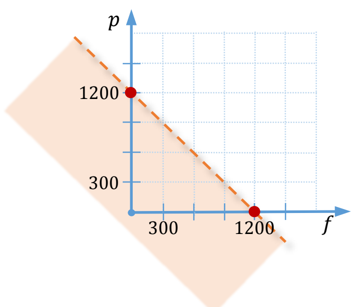


Figure 1c

The solution set of the strong inequality $p < -f + 1200$ consists of the same region as in *Figure 1b*, except for the points on the boundary line. This is because the points of the boundary line satisfy the equation $p = -f + 1200$, but not the inequality $p < -f + 1200$. To indicate this on the graph, we draw the boundary line using a dashed line (see *Figure 1c*).

In summary, to graph the solution set of a linear inequality in two variables, follow the steps:

1. Draw the graph of the corresponding **boundary line**.
   Make the line **solid** if the inequality involves $\leq$ or $\geq$.
   Make the line **dashed** if the inequality involves $<$ or $>$.

2. Choose a **test point** outside of the line and substitute the coordinates of that point into the inequality.

3. If the test point satisfies the original inequality, **shade the half-plane containing the point.**
   If the test point does not satisfy the original inequality, **shade the other half-plane** (the one that does not contain the point).

Example 1 ► **Determining if a Given Ordered Pair of Numbers is a Solution to a Given Inequality**

Determine if the points $(3,1)$ and $(2,1)$ are solutions to the inequality $5x - 2y > 8$.

Solution ► An ordered pair is a solution to the inequality $5x - 2y > 8$ if its coordinates satisfy this inequality. So, to determine whether the pair $(3,1)$ is a solution, we substitute 3 for x and 1 for y. The inequality becomes

$$5 \cdot 3 - 2 \cdot 1 > 8,$$

which simplifies to the true inequality $13 > 8$.
Thus, $(3,1)$ is a solution to $5x - 2y > 8$.

However, replacing x by 2 and y by 1 results in $5 \cdot 2 - 2 \cdot 1 > 8$, or equivalently $8 > 8$. Since 8 is not larger than 8, the point $(2,1)$ does not satisfy the inequality. Thus, $(2,1)$ is not a solution to $5x - 2y > 8$.

Example 2 ► **Graphing Linear Inequalities in Two Variables**

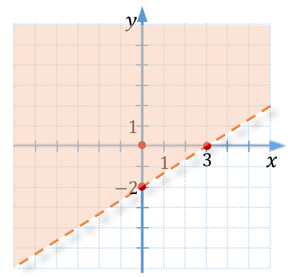
Graph the solution set of each inequality in two variables.

a. $2x - 3y < 6$ **b.** $y \le 3x - 1$

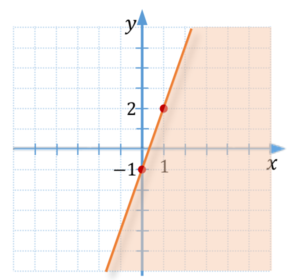
c. $x \ge -3$ **d.** $y \ne x$

Solution ►

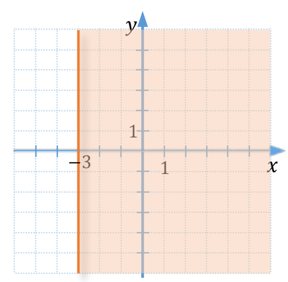
a. First, we graph the boundary line $2x - 3y = 6$, using the x- and y-intercepts: $(3,0)$ and $(0,-2)$. Since the inequality $<$ does not involve an equation, the line is marked as **dashed**, which indicates that the points on the line are not the solutions of the inequality. Then, we choose the point $(0,0)$ for the test point. Since $2 \cdot 0 - 3 \cdot 0 < 6$ is a true statement, then we shade the half-plane containing $(0,0)$ as the solution set.



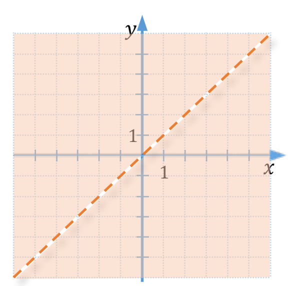
b. First, we graph the boundary line $y \le 3x - 1$, using the slope and y-intercept. Since the inequality $\le$ contains an equation, the line is marked as **solid**. This indicates that the points on the line belong to solutions of the inequality. To decide which half-plane to shade as the solution region, we observe that y is lower than or equal to the $3x - 1$, which tells us that the solution points lie below or on the boundary line. So, we shade the half-plane below the line.



c. As before, to graph $x \ge -3$, first, we graph the solid vertical line $x = -3$, and then we shade the half-plane consisting of points with x-coordinates larger or equal to -3. So the solution set is the half-plane to the right of the boundary line, including this line.



d. The solution set of the inequality $y \ne x$ consists of all points that do not satisfy the equation $y = x$. This means that we mark the boundary line as dashed and shade the rest of the points of the coordinate plane.



Systems of Linear Inequalities

Let us refer back to our original problem about the full-time and part-time students that was modelled by the inequality $f + p \leq 1200$. Since f and p represent the number of students, it is reasonable to assume that $f \geq 0$ and $p \geq 0$. This means that we are really interested in solutions to the system of inequalities

$$\begin{cases} p \leq -f + 1200 \\ f \geq 0 \\ p \geq 0 \end{cases}$$

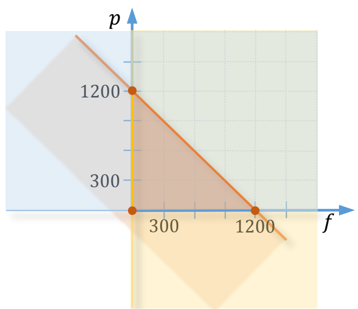


Figure 2

To find this solution set, we graph each inequality in the same coordinate system. The solutions to the first inequality are marked in orange, the second inequality, in yellow, and the third inequality, in blue (see *Figure 2*). The intersection of the three shadings, orange, yellow, and blue, results in the brown triangular region, including the border lines and the vertices. This is the overall solution set to our system of inequalities. It tells us that the coordinates of any point from the triangular region, including its boundary, could represent the actual number of full-time and part-time students enrolled in upgrading courses during the given semester.

To graph the solution set to a system of inequalities, follow the steps:

Remember that a brace indicates the "*and*" connection!

1. Using different shadings, graph the solution set to each inequality in the system, drawing the solid or dashed boundary lines, whichever applies.

2. Shade the **intersection** of the solution sets more strongly if the inequalities were connected by the word "**and**". Mark each intersection point of boundary lines with a **filled-in** circle if **both** lines are **solid**, or with a **hollow** circle if at least one of the lines is dashed.

or

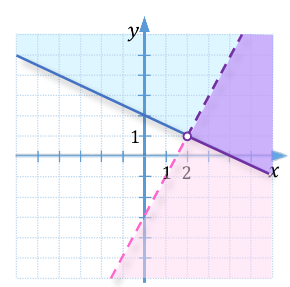
Shade the **union** of the solution sets more strongly if the inequalities were connected by the word "**or**". Mark each intersection of boundary lines with a **hollow** circle if **both** lines are **dashed**, or with a **filled-in** circle if at least one of the lines is **solid**.

Example 3 ► **Graphing Systems of Linear Inequalities in Two Variables**

Graph the solution set to each system of inequalities in two variables.

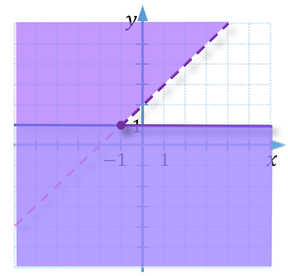
a. $\begin{cases} y < 2x - 3 \\ y \geq -\frac{1}{2}x + 1 \end{cases}$ **b.** $y > x + 2$ or $y \leq 1$

Solution ► **a.** First, we graph the solution set to $y < 2x - 3$ in pink, and the solution set to $y \geq -\frac{1}{2}x + 2$ in blue. Since both inequalities must be satisfied, the solution set of the system is the **intersection** of the solution sets of individual inequalities. So, we shade the overlapping region, in purple, indicating the solid or dashed border lines. Since the intersection of the boundary lines lies on a dashed line, it



does not satisfy one of the inequalities, so it is not a solution to the system. Therefore, we mark it with a hollow circle.

b. As before, we graph the solution set to $y > x + 2$ in pink, and the solution set to $y \le 1$ in blue. Since the two inequalities are connected with the word "or", we look for the **union** of the two solutions. So, we shade the overall region, in purple, indicating the solid or dashed border lines. Since the intersection of these lines belongs to a solid line, it satisfies one of the inequalities, so it is also a solution of this system. Therefore, we mark it by a filled-in circle.



Absolute Value Inequalities in Two Variables

As shown in *Section L6*, absolute value linear inequalities can be written as systems of linear inequalities. So we can graph their solution sets, using techniques described above.

Example 4 ► **Graphing Absolute Value Linear Inequalities in Two Variables**

Rewrite the following absolute value inequalities as systems of linear inequalities and then graph them.

a. $|x + y| < 2$ **b.** $|x + 2| \ge y$ **c.** $|x - 1| \ge 2$

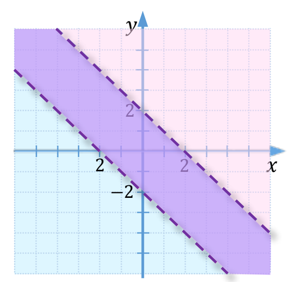
Solution ►

a. First, we rewrite the inequality $|x + y| < 2$ in the equivalent form of the system of inequalities,

$$-2 < x + y < 2.$$

The solution set to this system is the intersection of the solutions to $-2 < x + y$ and $x + y < 2$. For easier graphing, let us rewrite the last two inequalities in the explicit form

$$\begin{cases} y > -x - 2 \\ y < -x + 2 \end{cases}$$



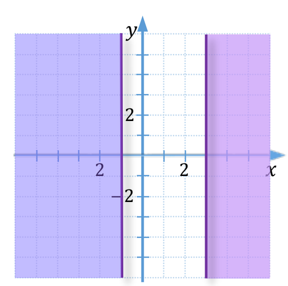
So, we graph $y > -x - 2$ in pink, $y < -x + 2$ in blue, and the final solution, in purple. Since both inequalities are strong (do not contain equation), the boundary lines are dashed.

b. We rewrite the inequality $|x - 1| \ge 2$ in the form of the system of inequalities,

$$x - 1 \ge 2 \text{ or } x - 1 \le -2,$$

or equivalently as

$$x \ge 3 \text{ or } x \le -1.$$



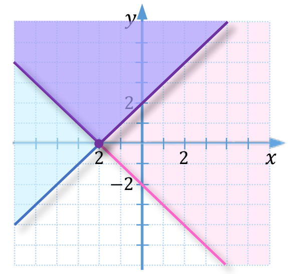
Thus, the solution set to this system is the union of the solutions to $x \ge 3$, marked in pink, and $x \le -1$, marked in blue. The overall solution to the system is marked in purple and includes the boundary lines.

c. We rewrite the inequality $|x+2| \leq y$ in the form of the system of inequalities,

$$-y \leq x+2 \leq y,$$

or equivalently as

$$y \geq -x-2 \text{ and } y \geq x+2.$$



Thus, the solution set to this system is the intersection of the solutions to $y \geq -x-2$, marked in pink, and $y \geq x+2$, marked in blue. The overall solution to the system, marked in purple, includes the border lines and the vertex.

G.4 Exercises

For each inequality, determine if the given points belong to the solution set of the inequality.

1. $y \geq -4x+3$; $(1,-1)$, $(1,0)$

2. $2x-3y<6$; $(3,0)$, $(2,-1)$

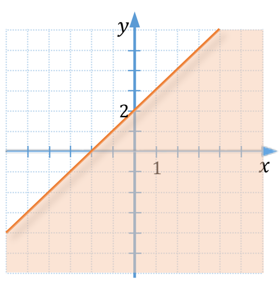
3. $y>-2$; $(0,0)$, $(-1,-1)$

4. $x \geq -2$; $(-2,1)$, $(-3,1)$

5. Match the given inequalities with the graphs of their solution sets.

a. $y \geq x+2$ **b.** $y<-x+2$ **c.** $y \leq x+2$ **d.** $y>-x+2$

I



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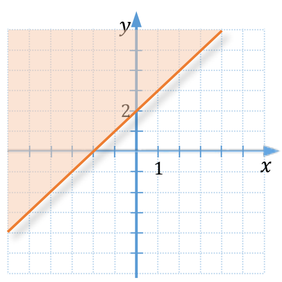

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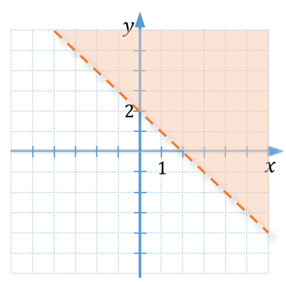

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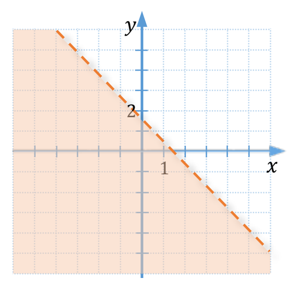


Graph each linear inequality in two variables.

6. $y \geq -\frac{1}{2}x+3$

7. $y \leq \frac{1}{3}x-2$

8. $y<2x-4$

9. $y>-x+3$

10. $y \geq -3$

11. $y<4.5$

12. $x>1$

13. $x \leq -2.5$

14. $x+3y>-3$

15. $5x-3y \leq 15$

16. $y-3x \geq 0$

17. $3x-2y<-6$

18. $3x \leq 2y$

19. $3y \neq 4x$

20. $y \neq 2$

Graph each compound inequality.

21. $\begin{cases} x + y \geq 3 \\ x - y < 4 \end{cases}$

22. $\begin{cases} x \geq -2 \\ y \leq -2x + 3 \end{cases}$

23. $\begin{cases} x - y < 2 \\ x + 2y \geq 8 \end{cases}$

24. $\begin{cases} 2x - y < 2 \\ x + 2y > 6 \end{cases}$

25. $\begin{cases} 3x + y \leq 6 \\ 3x + y \geq -3 \end{cases}$

26. $\begin{cases} y < 3 \\ x + y < 5 \end{cases}$

27. $3x + 2y > 2$ or $x \geq 2$

28. $x + y > 1$ or $x + y < 3$

29. $y \geq -1$ or $2x + y > 3$

30. $y > x + 3$ or $x > 3$

For each problem, write a system of inequalities describing the situation and then graph the solution set in the xy-plane.

31. Suppose the rates of attending a Zoo are \$30 for a regular ticket and \$20 for a student ticket. A group of tourists purchased x regular tickets and y student tickets, spending no more than \$300. Represent all possible combinations of the number of regular and student tickets purchased there by graphing appropriate region in the xy-plane.

32. Suppose a store manager bought chocolate raisins for \$8 per kilogram and chocolate candies for \$12 per kilogram. Let x be the number of kilograms of chocolate raisins and y be the number of kilograms of chocolate candies purchased by the manager. Knowing that the total cost was less than \$120, represent all possible weight combinations of the two types of candies by graphing appropriate region in the xy-plane.

G5 Concept of Function, Domain, and Range

In mathematics, we often investigate relationships between two quantities. For example, we might be interested in the average daily temperature in Abbotsford, BC, over the last few years, the amount of water wasted by a leaking tap over a certain period of time, or particular connections among a group of bloggers. The relations can be described in many different ways: in words, by a formula, through graphs or arrow diagrams, or simply by listing the ordered pairs of elements that are in the relation. A group of relations, called *functions*, will be of special importance in further studies. In this section, we will define functions, examine various ways of determining whether a relation is a function, and study related concepts such as *domain* and *range*.

Relations, Domains, and Ranges

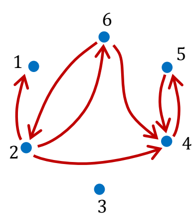


Figure 1

Consider a relation of knowing each other in a group of 6 people, represented by the arrow diagram shown in *Figure 1*. In this diagram, the points 1 through 6 represent the six people and an arrow from point x to point y tells us that the person x knows the person y. This correspondence could also be represented by listing the ordered pairs (x, y) whenever person x knows person y. So, our relation can be shown as the set of points

$$\{(2,1), (2,4), (2,6), (4,5), (5,4), (6,2), (6,4)\}$$

The x-coordinate of each pair (x, y) is called the **input**, and the y-coordinate is called the **output**.

input → (x, y) ← **output**

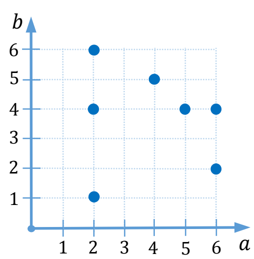


Figure 2a

The ordered pairs of numbers can be plotted in a system of coordinates, as in *Figure 2a*. The obtained graph shows that some inputs are in a relation with many outputs. For example, input 2 is in a relation with output 1, and 4, and 6. Also, the same output, 4, is assigned to many inputs. For example, the output 4 is assigned to the input 2, and 5, and 6.

The set of all the inputs of a relation is its **domain**. Thus, the domain of the above relation consists of all first coordinates

$$\{2, 4, 5, 6\}$$

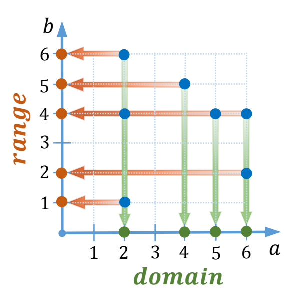


Figure 2b

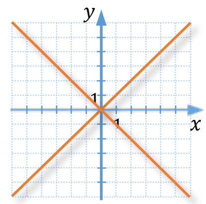
The set of all the outputs of a relation is its **range**. Thus, the range of our relation consists of all second coordinates

$$\{1, 2, 4, 5, 6\}$$

The domain and range of a relation can be seen on its graph through the **perpendicular projection** of the graph **onto the horizontal axis**, for the **domain**, and **onto the vertical axis**, for the **range**. See *Figure 2b*.

In summary, we have the following definition of a relation and its domain and range:

Definition 5.1 ► A **relation** is any **set of ordered pairs**. Such a set establishes a **correspondence** between the **input** and **output** values. In particular, any subset of a coordinate plane represents a relation.
The **domain** of a relation consists of all **inputs** (**first coordinates**).
The **range** of a relation consists of all **outputs** (**second coordinates**).



Relations can also be given by an equation or an inequality. For example, the equation

$$|y| = |x|$$

describes the set of points in the xy-plane that lie on two diagonals, $y = x$ and $y = -x$. In this case, the domain and range for this relation are both the set of real numbers because the projection of the graph onto each axis covers the entire axis.

Functions, Domains, and Ranges

Relations that have exactly one output for every input are of special importance in mathematics. This is because as long as we know the rule of a correspondence defining the relation, the output can be uniquely determined for every input. Such relations are called **functions**. For example, the linear equation $y = 2x + 1$ defines a function, as for every input x, one can calculate the corresponding y-value in a unique way. Since both the input and the output can be any real number, the domain and range of this function are both the set of real numbers.

Definition 5.2 ► A **function** is a relation that assigns **exactly one** **output** value in the **range** to each **input** value of the **domain**.

If (x, y) is an ordered pair that belongs to a function, then x can be any arbitrarily chosen input value of the domain of this function, while y must be the uniquely determined value that is assigned to x by this function. That is why x is referred to as an **independent** variable while y is referred to as the **dependent** variable (because the y-value depends on the chosen x-value).

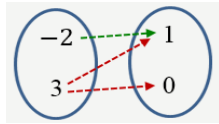
independent variable (x, y) **dependent variable**

How can we recognize if a relation is a function?

If the relation is given as a set of ordered pairs, it is enough to check if there are no two pairs with the same inputs. For example:

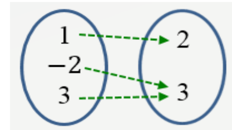
$\{(2,1), (2,4), (1,3)\}$ **relation**	$\{(2,1), (1,3), (4,1)\}$ **function**
The pairs $(2,1)$ and $(2,4)$ have the same inputs. So, there are **two y-values** assigned to the x-value 2, which makes it not a function.	There are no pairs with the same inputs, so each x-value is associated with exactly one pair and consequently with exactly one y-value. This makes it a function.

If the relation is given by a diagram, we want to check if no point from the domain is assigned to two points in the range. For example:



relation

There are **two arrows** starting from 3. So, there are two y-values assigned to 3, which makes it not a function.



function

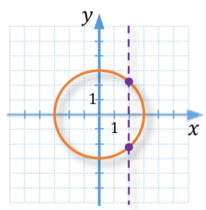
Only one arrow starts from each point of the domain, so each x-value is associated with exactly one y-value. Thus this is a function.

VERTICAL LINE TEST

If the relation is given by a graph, we use **The Vertical Line Test**:

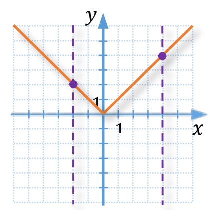
A relation is a **function** if **no vertical line** intersects the graph more than once.

For example:



relation

There is a vertical line that intersects the graph **twice**. So, there are two y-values assigned to an x-value, which makes it not a function.



function

Any vertical line intersects the graph only **once**. So, by The Vertical Line Test, this is a function.

If the relation is given by an equation, we check whether the y-value can be determined uniquely. For example:

$x^2 + y^2 = 1$

relation

Both points $(0,1)$ and $(0,-1)$ belong to the relation. So, there are **two y-values** assigned to 0, which makes it not a function.

$y = \sqrt{x}$

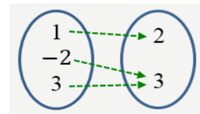
function

The y-value is uniquely defined as the square root of the x-value, for $x \geq 0$. So, this is a function.

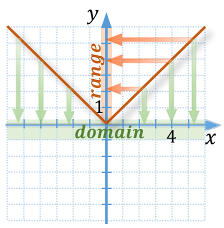
In general, to determine if a given relation is a function we analyse the relation to see whether or not it assigns two different y-values to the same x-value. If it does, it is just a relation, not a function. If it doesn't, it is a function.

Since functions are a special type of relation, the **domain and range of a function** can be determined the same way as in the case of a relation.
Let us look at domains and ranges of the above examples of functions.

The domain of the function $\{(2,1),(1,3),(4,1)\}$ is the set of the first coordinates of the ordered pairs, which is $\{1,2,4\}$. The range of this function is the set of second coordinates of the ordered pairs, which is $\{1,3\}$.

The domain of the function defined by the diagram  is the first set of points, particularly $\{1,-2,3\}$.

The range of this function is the second set of points, which is $\{2,3\}$.



The domain of the function given by the accompanying graph is the projection of the graph onto the x-axis, which is the set of all real numbers $\mathbb{R}$.
The range of this function is the projection of the graph onto the y-axis, which is the interval of points larger or equal to zero, $[0,\infty)$.

The domain of the function given by the equation $y=\sqrt{x}$ is the set of nonnegative real numbers, $[0,\infty)$, since the square root of a negative number is not real.
The range of this function is also the set of nonnegative real numbers, $[0,\infty)$, as the value of a square root is never negative.

Example 1 ► **Determining Whether a Relation is a Function and Finding Its Domain and Range**

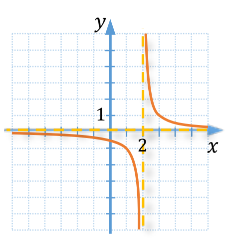
Decide whether each relation defines a function, and give the domain and range.

a. $y=\frac{1}{x-2}$ **b.** $y<2x+1$

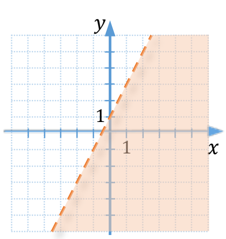
c. $x=y^2$ **d.** $y=\sqrt{2x-1}$

Solution ►

a. Since $\frac{1}{x-2}$ can be calculated uniquely for every x from its domain, the relation $y=\frac{1}{x-2}$ is a function.
The domain consists of all real numbers that make the denominator, $x-2$, different than zero. Since $x-2=0$ for $x=2$, then the domain, D, is the set of all real numbers except for 2. We write $D=\mathbb{R}\setminus\{2\}$.
Since a fraction with nonzero numerator cannot be equal to zero, the range of $y=\frac{1}{x-2}$ is the set of all real numbers except for 0. We write $range=\mathbb{R}\setminus\{0\}$.

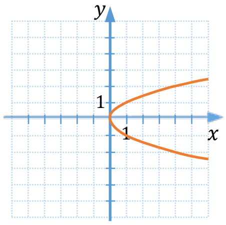


b. The inequality $y<2x+1$ is not a function as for every x-value there are many y-values that are lower than $2x+1$. Particularly, points $(0,0)$ and $(0,-1)$ satisfy the inequality and show that the y-value is not unique for $x=0$.



In general, because of the many possible y-values, no inequality defines a function.

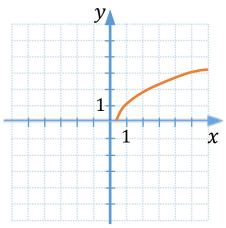
Since there are no restrictions on x-values, the domain of this relation is the set of all real numbers, $\mathbb{R}$. The range is also the set of all real numbers, $\mathbb{R}$, as observed in the accompanying graph.



c. Here, we can show two points, $(1,1)$ and $(1,-1)$, that satisfy the equation, which contradicts the requirement of a single y-value assigned to each x-value. So, this relation is not a function.

Since x is a square of a real number, it cannot be a negative number. So the domain consists of all nonnegative real numbers. We write, $D = [0, \infty)$. However, y can be any real number, so $range = \mathbb{R}$.

d. The equation $y = \sqrt{2x-1}$ represents a function, as for every x-value from the domain, the y-value can be calculated in a unique way.



The domain of this function consists of all real numbers that would make the radicand $2x - 1$ nonnegative. So, to find the domain, we solve the inequality:

$$2x - 1 \geq 0$$
$$2x \geq 1$$
$$x \geq \frac{1}{2}$$

Thus, $D = \left[\frac{1}{2}, \infty\right)$. As for the range, since the values of a square root are nonnegative, we have $range = [0, \infty)$

G.5 Exercises

Decide whether each relation defines a function, and give its ***domain*** *and* ***range****.*

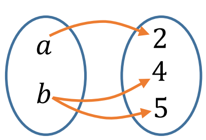
1. $\{(2,4), (0,2), (2,3)\}$

2. $\{(3,4), (1,2), (2,3)\}$

3. $\{(2,3), (3,4), (4,5), (5,2)\}$

4. $\{(1,1), (1,-1), (2,5), (2,-5)\}$

5.



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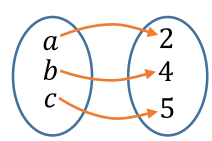

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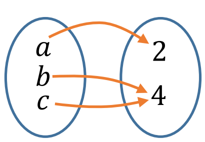

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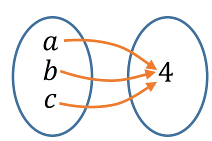


9.

x	y
0	1
0	-1
1	2
1	-2

10.

x	y
-1	4
0	2
1	0
2	-2

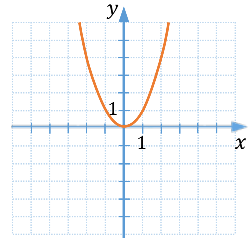
11.

x	y
3	1
6	2
9	1
12	2

12.

x	y
-2	3
-2	0
-2	-3
-2	-6

13.



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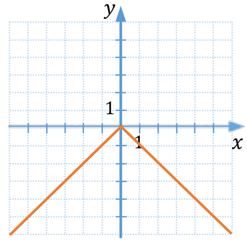

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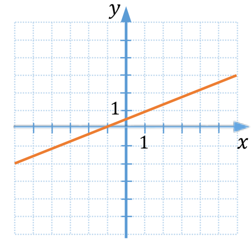

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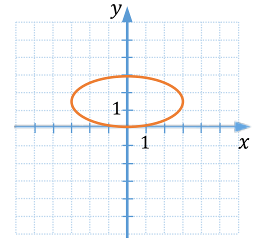

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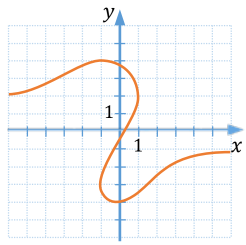

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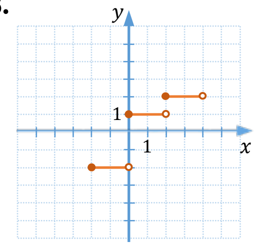

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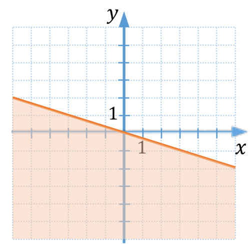

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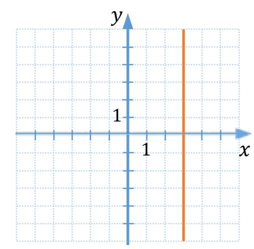


*Find the **domain** of each relation and decide whether the relation defines y as a function of x.*

21. $y = 3x + 2$

22. $y = 5 - 2x$

23. $y = |x| - 3$

24. $x = |y| + 1$

25. $y^2 = x^2$

26. $y^2 = x^4$

27. $x = y^4$

28. $y = x^3$

29. $y = -\sqrt{x}$

30. $y = \sqrt{2x - 5}$

31. $y = \frac{1}{x+5}$

32. $y = \frac{1}{2x-3}$

33. $y = \frac{x-3}{x+2}$

34. $y = \frac{1}{|2x-3|}$

35. $y \leq 2x$

36. $y - 3x \geq 0$

37. $y \neq 2$

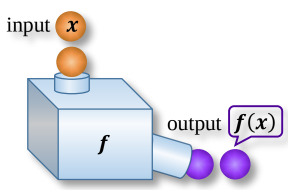
38. $x = -1$

39. $y = x^2 + 2x + 1$

40. $xy = -1$

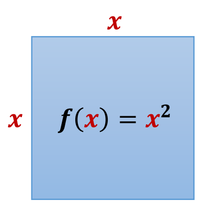
41. $x^2 + y^2 = 4$

G6 Function Notation and Evaluating Functions



A function is a correspondence that assigns a single value of the range to each value of the domain. Thus, a function can be seen as an input-output machine, where the input is taken independently from the domain, and the output is the corresponding value of the range. The rule that defines a function is often written as an equation, with the use of x and y for the independent and dependent variables, for instance, $y = 2x$ or $y = x^2$. To emphasize that y depends on x, we write $\boldsymbol{y = f(x)}$, where f is the name of the function. The expression $\boldsymbol{f(x)}$, read as "$\boldsymbol{f}$ ***of*** $\boldsymbol{x}$", represents the dependent variable assigned to the particular x. Such notation shows the dependence of the variables as well as allows for using different names for various functions. It is also handy when evaluating functions. In this section, we introduce and use *function notation*, and show how to evaluate functions at specific input-values.

Function Notation



Consider the equation $y = x^2$, which relates the length of a side of a square, x, and its area, y. In this equation, the y-value depends on the value x, and it is uniquely defined. So, we say that y is a function of x. Using function notation, we write

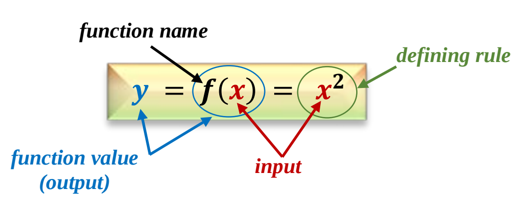
$$f(x) = x^2$$

The expression $f(x)$ is just another name for the dependent variable y, and it shouldn't be confused with a product of f and x. Even though $f(x)$ is really the same as y, we often write $f(x)$ rather than just y, because the notation $f(x)$ carries more information. Particularly, it tells us the name of the function so that it is easier to refer to the particular one when working with many functions. It also indicates the independent value for which the dependent value is calculated. For example, using function notation, we find the area of a square with a side length of 2 by evaluating $f(2) = 2^2 = 4$. So, 4 is the area of a square with a side length of 2.

The statement $f(2) = 4$ tells us that the pair $(2,4)$ belongs to function f, or equivalently, that 4 is assigned to the input of 2 by the function f. We could also say that function f attains the value 4 at 2.

If we calculate the value of function f for $x = 3$, we obtain $f(3) = 3^2 = 9$. So the pair $(3,9)$ also belongs to function f. This way, we may produce many ordered pairs that belong to f and consequently, make a graph of f.

Here is what each part of **function notation** represents:



Note: Functions are customarily denoted by a single letter, such as f, g, h, but also by abbreviations, such as sin, cos, or tan.

Function Values

Function notation is handy when evaluating functions for several input values. To evaluate a function given by an equation at a specific x-value from the domain, we substitute the x-value into the defining equation. For example, to evaluate $f(x) = \frac{1}{x-1}$ at $x = 3$, we calculate

$$f(3) = \frac{1}{3-1} = \frac{1}{2}$$

So $f(3) = \frac{1}{2}$, which tells us that when $x = 3$, the y-value is $\frac{1}{2}$, or equivalently, that the point $\left(3, \frac{1}{2}\right)$ belongs to the graph of the function f.

Notice that function f cannot be evaluated at $x = 1$, as it would make the denominator $(x-1)$ equal to zero, which is not allowed. We say that $f(1) = DNE$ (read: *Does Not Exist*). Because of this, the domain of function f, denoted D_f, is $\mathbb{R}\setminus\{1\}$.

Graphing a function usually requires evaluating it for several x-values and then plotting the obtained points. For example, evaluating $f(x) = \frac{1}{x-1}$ for $x = \frac{3}{2}, 2, 5, \frac{1}{2}, 0, -1$, gives us

$$f\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}-1} = \frac{1}{\frac{1}{2}} = 2$$

$$f(2) = \frac{1}{2-1} = \frac{1}{1} = 1$$

$$f(5) = \frac{1}{5-1} = \frac{1}{4}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}-1} = \frac{1}{-\frac{1}{2}} = -2$$

$$f(0) = \frac{1}{0-1} = -1$$

$$f(-1) = \frac{1}{-1-1} = -\frac{1}{2}$$

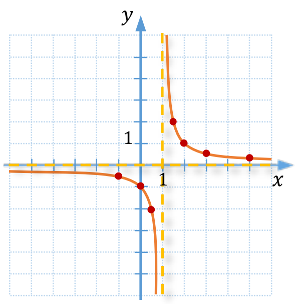


Figure 1

Thus, the points $\left(\frac{3}{2}, 2\right)$, $(2, 1)$, $\left(3, \frac{1}{2}\right)$, $\left(5, \frac{1}{4}\right)$, $\left(\frac{1}{2}, -2\right)$, $(0, -1)$, $\left(-1, -\frac{1}{2}\right)$ belong to the graph of f. After plotting them in a system of coordinates and predicting the pattern for other x-values, we produce the graph of function f, as in *Figure 1*.

Observe that the graph seems to be approaching the vertical line $x = 1$ as well as the horizontal line $y = 0$. These two lines are called ***asymptotes*** and are not a part of the graph of function f; however, they shape the graph. Asymptotes are customarily graphed by dashed lines.

Sometimes a function is given not by an equation but by a graph, a set of ordered pairs, a word description, etc. To evaluate such a function at a given input, we simply apply the function rule to the input.

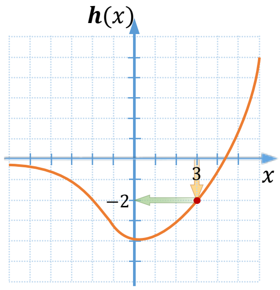


Figure 2a

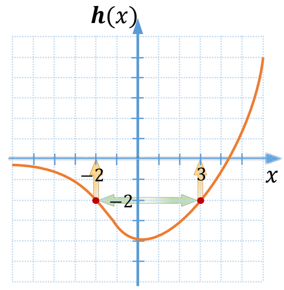


Figure 2b

For example, to find the value of function h, given by the graph in *Figure 2a*, for $x = 3$, we read the second coordinate of the intersection point of the vertical line $x = 3$ with the graph of h. Following the arrows in *Figure 2*, we conclude that $h(3) = -2$.

Notice that to find the x-value(s) for which $h(x) = -2$, we reverse the above process. This means: we read the first coordinate of the intersection point(s) of the horizontal line $y = -2$ with the graph of h. By following the reversed arrows in *Figure 2b*, we conclude that $h(x) = -2$ for $x = 3$ and for $x = -2$.

Example 1 ► **Evaluating Functions**

Evaluate each function at $x = 2$ and write the answer using function notation.

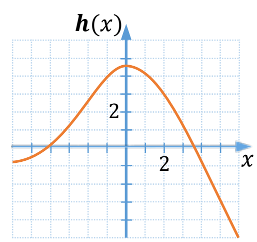
a. $f(x) = 3 - 2x$

b. function f squares the input

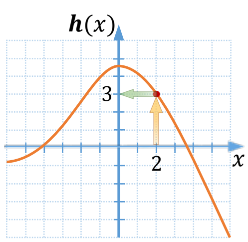
c.

x	$g(x)$
−1	2
2	5
3	−1

d.



Solution ►



a. Following the formula, we have $f(2) = 3 - 2(2) = 3 - 4 = \mathbf{-1}$

b. Following the word description, we have $f(2) = 2^2 = \mathbf{4}$

c. $g(2)$ is the value in the second column of the table that corresponds to 2 from the first column. Thus, $g(2) = \mathbf{5}$.

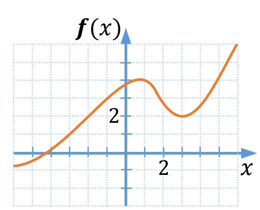
d. As shown in the graph, $h(2) = \mathbf{3}$.

Example 2 ► **Finding from a Graph the x-value for a Given $f(x)$-value**

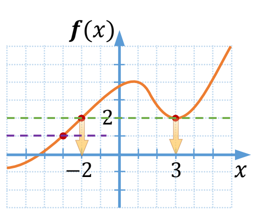
Given the graph, find all x-values for which

a. $f(x) = 1$

b. $f(x) = 2$



Solution ►



a. The purple line $y = 1$ cuts the graph at $x = -3$, so $f(x) = 1$ for $x = -3$.

b. The green line $y = 2$ cuts the graph at $x = -2$ and $x = 3$, so $f(x) = 2$ for $x \in \{-2, 3\}$.

Example 3 ► **Evaluating Functions and Expressions Involving Function Values**

Suppose $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 - 5$. Evaluate each expression.

a. $f(4)$ **b.** $g(-2)$ **c.** $g(a)$ **d.** $f(2a)$

e. $g(a-1)$ **f.** $3f(-2)$ **g.** $g(2+h)$ **h.** $f(2+h) - f(2)$

Solution ►

a. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the value 4. So,
$f(4) = \frac{1}{2}(4) - 1 = 2 - 1 = \mathbf{1}$.

b. Replace x in the equation $g(x) = x^2 - 5$ by the value -2, using parentheses around the -2. So, $g(-2) = (-2)^2 - 5 = 4 - 5 = \mathbf{-1}$.

c. Replace x in the equation $g(x) = x^2 - 5$ by the input a. So, $g(a) = \boldsymbol{a^2 - 5}$.

d. Replace x in the equation $f(x) = \frac{1}{2}x - 1$ by the input $2a$. So,
$f(2a) = \frac{1}{\cancel{2}}(\cancel{2}a) - 1 = \boldsymbol{a - 1}$.

$$\begin{aligned}&(a-1)^2\\ &= (a-1)(a-1)\\ &= a^2 - a - a + 1\\ &= a^2 - 2a + 1\end{aligned}$$

e. Replace x in the equation $g(x) = x^2 - 5$ by the input $(a-1)$, using parentheses around the input. So, $g(a-1) = (a-1)^2 - 5 = a^2 - 2a + 1 - 5 = \boldsymbol{a^2 - 2a - 4}$.

f. The expression $3f(-2)$ means three times the value of $f(-2)$, so we calculate
$3f(-2) = 3 \cdot \left(\frac{1}{\cancel{2}}(-\cancel{2}) - 1\right) = 3(-1-1) = 3(-2) = \mathbf{-6}$.

$$\begin{aligned}&(2+h)^2\\ &= (2+h)(2+h)\\ &= 4 + 2h + 2h + h^2\\ &= 4 + 4h + h^2\end{aligned}$$

g. Replace x in the equation $g(x) = x^2 - 5$ by the input $(2+h)$, using parentheses around the input. So, $g(2+h) = (2+h)^2 - 5 = 4 + 4h + h^2 - 5 = \boldsymbol{h^2 + 4h - 1}$.

h. When evaluating $f(2+h) - f(2)$, focus on evaluating $f(2+h)$ first and then, to subtract the expression $f(2)$, use a bracket just after the subtraction sign. So,

$$f(2+h) - f(2) = \underbrace{\frac{1}{2}(2+h) - 1}_{f(2+h)} - \underbrace{\left[\frac{1}{2}(2) - 1\right]}_{f(2)} = 1 + \frac{1}{2}h - 1 - [1-1] = \boldsymbol{\frac{1}{2}h}$$

Note: To perform the perfect squares in the solution to *Example 3e* and *3g*, we follow the **perfect square formula** $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$ or $\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$. One can check that this formula can be obtained as a result of applying the distributive law, often referred to as the *FOIL* method, when multiplying two binomials (see the examples in callouts in the left margin). However, we prefer to use the perfect square formula rather than the *FOIL* method, as it makes the calculation process more efficient.

Function Notation in Graphing and Application Problems

By *Definition 1.1* in *Section G1*, a linear equation is an equation of the form $Ax + By = C$. The graph of any linear equation is a line, and any nonvertical line satisfies the Vertical Line Test. Thus, any linear equation $Ax + By = C$ with $B \neq 0$ defines a linear function.

How can we write this function using function notation?

Since $y = f(x)$, we can replace the variable y in the equation $Ax + By = C$ with $f(x)$ and then solve for $f(x)$. So, we obtain

Alternatively, we can solve the original equation for y and then replace y with $f(x)$.

$$Ax + B \cdot f(x) = C \quad /-Ax$$

$$B \cdot f(x) = -Ax + C \quad /\div B$$

$$f(x) = -\frac{A}{B}x + \frac{C}{B}$$

must assume that $B \neq 0$

Definition 6.1 ► Any function that can be written in the form

$$\boldsymbol{f(x) = mx + b},$$

where m and b are real numbers, is called a **linear function**. The value $\boldsymbol{m}$ represents the **slope** of the graph, and the value $\boldsymbol{b}$ represents the $\boldsymbol{y}$**-intercept** of this function. The **domain** of any linear function is the set of all real numbers, $\mathbb{R}$.

In particular:

Definition 6.2 ► A linear function with slope $\boldsymbol{m = 0}$ takes the form

$$\boldsymbol{f(x) = b},$$

where b is a real number, and is called a **constant function**.

Note: Similarly as the domain of any linear function, the **domain** of a constant function is the set $\mathbb{R}$. However, the **range** of a constant function is the one element set $\{\boldsymbol{b}\}$, while the range of any nonconstant linear function is the set $\mathbb{R}$.

Generally, any equation in two variables, x and y, that defines a function can be written using function notation by solving the equation for y and then letting $y = f(x)$. For example, to rewrite the equation $-4x^2 + 2y = 5$ **explicitly** as a function f of x, we solve for y,

implicit form

$$2y = 4x^2 + 5$$

explicit form

$$y = 2x^2 + \tfrac{5}{2},$$

and then replace y by $f(x)$. So, $f(x) = 2x^2 + \frac{5}{2}$.

Using function notation, the graph of a function is defined as follows:

Definition 6.3 ► The graph of a function $\boldsymbol{f}$ of $\boldsymbol{x}$ is the set of ordered pairs $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))$ for every input x form the domain $\boldsymbol{D_f}$ of the function. This can be stated as

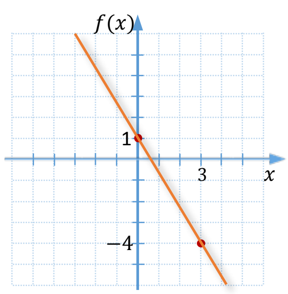
$$\boldsymbol{graph\ of\ f} = \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) \mid \boldsymbol{x} \in \boldsymbol{D_f}\}$$

Example 4 ► **Function Notation in Writing and Graphing Functions**

Each of the given equations represents a function of x. Rewrite the formula in explicit form, using function notation. Then graph this function and state its domain and range.

a. $5x + 3 \cdot f(x) = 3$ **b.** $|x| - y = -3$

Solution ► **a.** After solving the equation for $f(x)$,



$$5x + 3 \cdot f(x) = 3$$

$$3 \cdot f(x) = -5x + 3$$

$$f(x) = -\frac{5}{3}x + 1,$$

we observe that the function f is linear. So, we can graph it using the slope-intercept method. The graph confirms that the domain and range of this function are both the set of all real numbers, $\mathbb{R}$.

b. After solving the equation for y,

$$|x| - y = 2$$

$$|x| - 2 = y,$$

we obtain the function $f(x) = |x| - 2$.

If we are not sure how the graph of this function looks like, we may evaluate $f(x)$ for several x-values, plot the obtained points, and observe the pattern. For example, let $x = -2, -1, 0, 1, 2$. We fill in the table of values,

x	$\|x\| - 2 = f(x)$	$(x, f(x))$
-2	$\|-2\| - 2 = 0$	$(-2,0)$
-1	$\|-1\| - 2 = -1$	$(-1,-1)$
0	$\|0\| - 2 = -2$	$(0,-2)$
1	$\|1\| - 2 = -1$	$(1,-1)$
2	$\|2\| - 2 = 0$	$(2,0)$

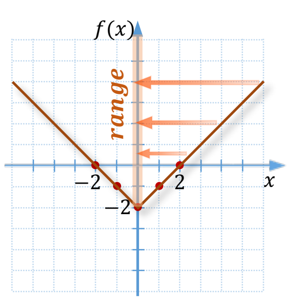


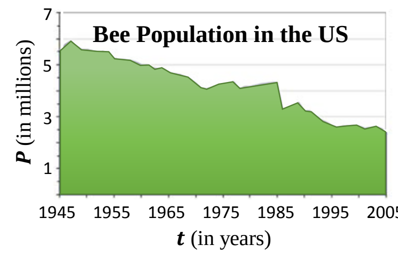
Figure 3

and plot the points listed in the third column. One may evaluate $f(x)$ for several more x-values, plot the additional points, and observe that these points form a V-shape with a vertex at $(0, -2)$. By connecting the points as in *Figure 3*, we obtain the graph of function $f(x) = |x| - 2$.

Since one can evaluate the function $f(x) = |x| - 2$ for any real x, the domain of f is the set $\mathbb{R}$. The range can be observed by projecting the graph perpendicularly onto the vertical axis. So, the range is the interval $[-2, \infty)$, as shown in *Figure 3*.

Example 5 ► **A Function in Applied Situations**

The bee population in the US was declining during the years 1945-2005, as shown in the accompanying graph.



a. Based on the graph what was the approximate value of $P(1960)$ and $P(2000)$ and what does it tell us about the bee population?

b. Estimate the average rate of change in the bee population over the years $1960 - 2000$, and interpret the result in the context of the problem.

c. Approximate the year(s) in which $P(t)$ was 4 million bees.

d. What is the general tendency of the function $P(t)$ over the years $1945 - 2005$?

e. Assuming that function P continue declining at the same rate, predict the year in which the bees in the US would become extinct.

Solution ►

a. One may read from the graph that $P(1960) \approx 5$ and $P(2000) \approx 2.6$ (see the orange line in *Figure 4a*). The first equation tells us that in 1960 there were approximately 5 million bees in the US. The second equation indicates that in the year 2000 there were approximately 2.6 million bees in the US.

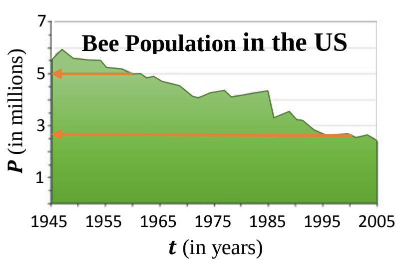


Figure 4a

b. The average rate of change is represented by the slope. Since the change in bee population over the years $1960 - 2000$ is $2.6 - 5 = -2.4$ million, and the change in time $1960 - 2000 = 40$ years, then the slope is $-\frac{2.4}{40} = -0.06$ million per year. This means that in the US, on average, 60,000 bees died each year between 1960 and 2000.

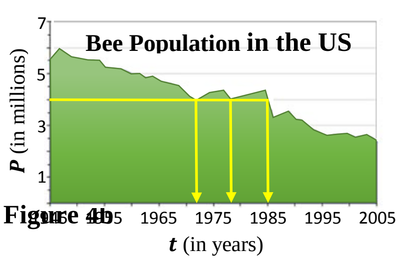


Figure 4b

c. As indicated by yellow arrows in *Figure 4b*, $P(t) = 4$ for $t \approx 1972$, $t \approx 1978$, and $t \approx 1985$.

d. The general tendency of function $P(t)$ over the years $1945 - 2005$ is declining.

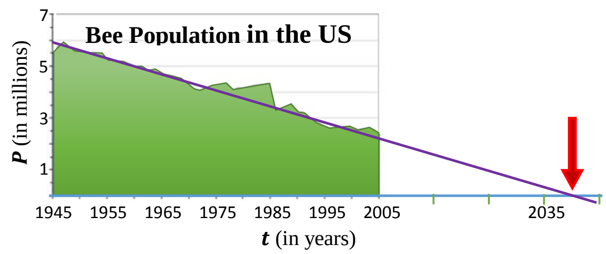
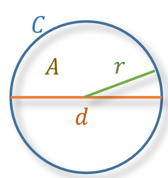


Figure 4c

e. Assuming the same declining tendency, to estimate the year in which the bee population in the US will disappear, we extend the t-axis and the approximate line of tendency (see the purple line in *Figure 4c*) to see where they intersect. After extending of the scale on the t-axis, we predict that the bee population will disappear around the year 2040.

Example 6 ► Constructing Functions



Consider a circle with area A, circumference C, radius r, and diameter d.

a. Write A as a function of r.
b. Write r as a function of d.
c. Write A as a function of d.
d. Write r as a function of C.
e. Write A as a function of C.

Solution ►

a. Using the formula for the area of a circle, $A = \pi r^2$, the function A of r is $\boldsymbol{A(r) = \pi r^2}$.

b. To express r as a function of d, we solve the formula $d = 2r$ for r. This gives us $r = \frac{d}{2}$. So, the function r of d is $\boldsymbol{r(d) = \frac{d}{2}}$.

c. To write A as a function of d, we start by connecting the formula for the area A in terms of r and the formula that expresses r in terms of d. Since

$$A = \pi r^2 \quad \text{and} \quad r = \frac{d}{2},$$

then using substitution, we have

$$A = \pi r^2 = \pi \cdot \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}.$$

Hence, our function A of d is $\boldsymbol{A(d) = \frac{1}{4}\pi d^2}$.

d. The relation between circumference C and radius r is $C = 2\pi r$. After solving this formula for r, we have $r = \frac{C}{2\pi}$. So, our function is $\boldsymbol{r(C) = \frac{C}{2\pi}}$.

e. To write A as a function of C, we use the formula $r = \frac{C}{2\pi}$ to replace r in the area formula $A = \pi r^2$ by the expression $\frac{C}{2\pi}$. This gives us

$$A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\cancel{\pi} C^2}{4\pi^{\cancel{2}}} = \frac{C^2}{4\pi}.$$

So, our function is $\boldsymbol{A(C) = \frac{C^2}{4\pi}}$.

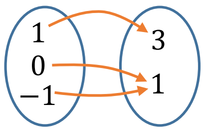
G.6 Exercises

*For each function, find **a)** $f(-1)$ and **b)** all x-values such that $f(x) = 1$.*

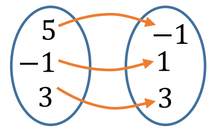
1. $\{(2,4), (-1,2), (3,1)\}$

2. $\{(-1,1), (1,2), (2,1)\}$

3.



4.



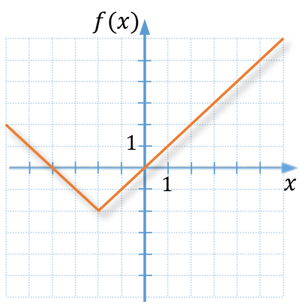
5.

x	$f(x)$
-1	4
0	2
2	1
4	-1

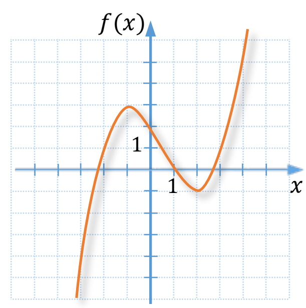
6.

x	$f(x)$
-3	1
-1	2
1	2
3	1

7.



8.



Let $f(x) = -3x + 5$ and $g(x) = -x^2 + 2x - 1$. Find the following.

9. $f(1)$

10. $g(0)$

11. $g(-1)$

12. $f(-2)$

13. $f(p)$

14. $g(a)$

15. $g(-x)$

16. $f(-x)$

17. $f(a + 1)$

18. $g(a + 2)$

19. $g(x - 1)$

20. $f(x - 2)$

21. $f(2 + h)$

22. $g(1 + h)$

23. $g(a + h)$

24. $f(a + h)$

25. $f(3) - g(3)$

26. $g(a) - f(a)$

27. $3g(x) + f(x)$

28. $f(x + h) - f(x)$

Fill in each blank.

29. The graph of the equation $2x + y = 6$ is a ____________. The point (1,___) lies on the graph of this line. Using function notation, the above equation can be written as $f(x) =$ ____________ . Since $f(1) =$ ______ , the point (___ , ___) lies on the graph of function f.

Graph each function. Give the domain and range.

30. $f(x) = -2x + 5$ **31.** $g(x) = \frac{1}{3}x + 2$ **32.** $h(x) = -3x$

33. $F(x) = x$ **34.** $G(x) = 0$ **35.** $H(x) = 2$

36. $x - h(x) = 4$ **37.** $-3x + f(x) = -5$ **38.** $2 \cdot g(x) - 2 = x$

39. $k(x) = |x - 3|$ **40.** $m(x) = 3 - |x|$ **41.** $q(x) = x^2$

42. $Q(x) = x^2 - 2x$ **43.** $p(x) = x^3 + 1$ **44.** $s(x) = \sqrt{x}$

Solve each problem.

45. A taxi driver charges \$1.50 per kilometer.

a. Complete the table by writting the charge $f(x)$ for a trip of x kilometers.

b. Find the linear function that calculates the charge $f(x) =$ ________ for a trip of x kilometers.

c. Graph $f(x)$ for the domain $\{0, 2, 4\}$.

x	$f(x)$
0	
2	
4	

46. Given the information about the linear function f, find the following:

a. $f(1)$

b. x-value such that $f(x) = -0.4$

c. slope of f

d. y-intercept of f

e. an equation for $f(x)$

x	$f(x)$
-2	3.2
-1	2.3
0	1.4
1	0.5
2	-0.4
3	-1.3

47. Suppose the cost of renting a car at Los Angeles International Airport consists of the initial fee of \$18.80 and \$24.60 per day. Let $C(d)$ represent the total cost of renting the car for d days.

a. Write a linear function that models this situation.

b. Find $C(4)$ and interpret your answer in the context of the problem.

c. Find the value of d satisfying the equation $C(d) = 191$ and interpret it in the context of this problem.

48. Suppose a house cleaning service charges \$20 per visit plus \$32 per hour.

a. Express the total charge, C, as a function of the number of hours worked, n.

b. Find $C(3)$ and interpret your answer in the context of this problem.

c. If Stacy was charged \$244 for a one-visit work, how long it took to clean her house?

49. *Refer to the given graph of function f to answer the questions below.*

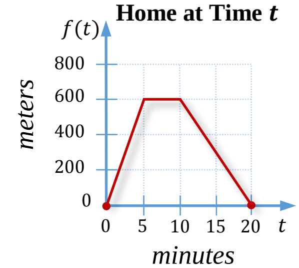
a. What is the range of possible values for the independent variable? What is the range of possible values for the independent variable?

b. For how long is the person going away from home? Coming closer to home?

c. How far away from home is the person after 10 minutes?

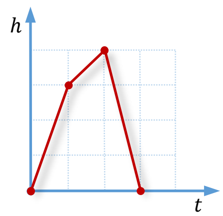
d. Call this function f. What is $f(15)$ and what does this mean in the context of the problem?

Distance of a Person from Home at Time *t*

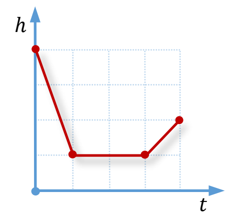


Questions 51 and 52 show graphs of the height of water in a bathtub. The t-axis represents time, and the h-axis represents height. Interpret the graph by describing the rate of change of the height of water in the bathtub.

50.



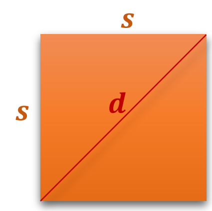
51.



52. Consider a square with area A, side s, perimeter P, and diagonal d.

a. Write A as a function of s.
b. Write s as a function of P.
c. Write A as a function of P.
d. Write A as a function of d.

(*Hint*: in part (**d**) apply the Pythagorean equation $a^2 + b^2 = c^2$, where c is the hypotenuse of a right angle triangle with arms a and b.)



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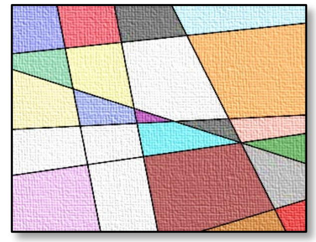
Systems of Linear Equations



As stated in *Section G1, Definition 1.1*, a linear equation in two variables is an equation of the form $Ax + By = C$, where A and B are not both zero. Such an equation has a line as its graph. Each point of this line is a solution to the equation in the sense that the coordinates of such a point satisfy the equation. So, there are infinitely many ordered pairs (x, y) satisfying the equation. Although analysis of the relation between x and y is instrumental in some problems, many application problems call for a particular, single solution. This occurs when, for example, x and y are required to satisfy an additional linear equation whose graph intersects the original line. In such a case, the solution to both equations is the point at which the lines intersect. Generally, to find unique values for **two** given **variables**, we need a system of **two equations** in these variables. In this section, we discuss several methods for solving systems of two linear equations.

E1 Systems of Linear Equations in Two Variables

Any collection of equations considered together is called a **system of equations**. For example, a system consisting of two equations, $x + y = 5$ and $4x - y = 10$, is written as

$$\begin{cases} x + y = 5 \\ 4x - y = 10 \end{cases}$$

Since the equations in the system are linear, the system is called a **linear system of equations**.

Definition 1.1 ► A **solution** of a system of two equations in two variables, x and y, is any ordered pair (x, y) satisfying both equations of the system.

A **solution set** of a system of two linear equations in two variables, x and y, is the set of all possible solutions (x, y).

Note: The two variables used in a system of two equations can be denoted by any two different letters. In such case, to construct an ordered pair, we follow an alphabetical order. For example, if the variables are p and q, the corresponding ordered pair is (p, q), as p appears in the alphabet before q. This also means that a corresponding system of coordinates has the horizontal axis denoted as p-axis and the vertical axis denoted as q-axis.

Example 1 ► **Deciding Whether an Ordered Pair Is a Solution**

Decide whether the ordered pair $(3, 2)$ is a solution of the given system.

a. $\begin{cases} x + y = 5 \\ 4x - y = 10 \end{cases}$ **b.** $\begin{cases} m + 2n = 7 \\ 3m - n = 6 \end{cases}$

Solution ► **a.** To check whether the pair $(3, 2)$ is a solution, we let $x = 3$ and $y = 2$ in both equations of the system and check whether these equations are true. Since both equations,

$$\begin{aligned} 3 + 2 &= 5 \\ 5 &= 5 \;✓ \end{aligned} \quad \text{and} \quad \begin{aligned} 4 \cdot 3 - 2 &= 10 \\ 10 &= 10, \;✓ \end{aligned}$$

are true, then the pair $(3, 2)$ is a solution to the system.

b. First, we notice that alphabetically, m is before n. So, we let $m = 3$ and $n = 2$ and substitute these values into both equations.

$$3 + 2 \cdot 2 = 7 \qquad\qquad 3 \cdot 3 - 2 = 6$$
$$7 = 7 \quad ✓ \qquad \text{but} \qquad 7 = 6 \quad ✗$$

Since the pair $(3, 2)$ is not a solution of the second equation, it is not a solution to the whole system.

Solving Systems of Linear Equations by Graphing

Solutions to a system of two linear equations are all the ordered pairs that satisfy both equations. If an ordered pair satisfies an equation, then such a pair belongs to the graph of this equation. This means that the solutions to a system of two linear equations are the points that belong to both graphs of these lines. So, to solve such system, we can graph each line and take the common points as solutions.

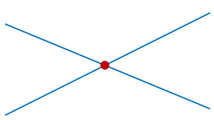


Figure 1a

How many solutions can a linear system of two equations have?

There are three possible arrangements of two lines in a plane. The lines can **intersect** each other, be **parallel**, or be the **same**.

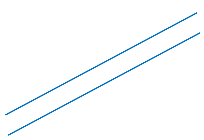


Figure 1b

1. If a system of equations corresponds to the pair of **intersecting** lines, it has exactly **one solution**. The solution set consists of the **intersection point**, as shown in *Figure 1a*.
2. If a system of equations corresponds to the pair of **parallel** lines, it has **no solutions**. The solution set is empty, as shown in *Figure 1b*.
3. If a system of equations corresponds to the pair of the **same** lines, it has **infinitely many solutions**. The solution set consists of all the **points of the line**, as shown in *Figure 1c*.

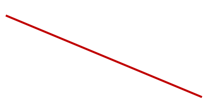
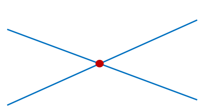


Figure 1c

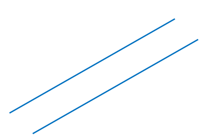
Definition 1.2 ► A linear system is called **consistent** if it has **at least one solution**. Otherwise, the system is **inconsistent**.

A linear system of two equations is called **independent** if the two lines are different. Otherwise, the system is **dependent**.

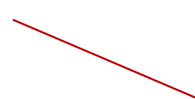
Here is the classification of systems corresponding to the following graphs:



consistent
independent



inconsistent
independent



consistent
dependent

Example 2 ► **Solving Systems of Linear Equations by Graphing**

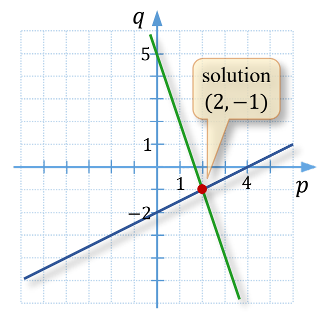
Solve each system by graphing and classify it as *consistent, inconsistent, dependent* or *independent*.

a. $\begin{cases} 3p + q = 5 \\ p - 2q = 4 \end{cases}$ **b.** $\begin{cases} 3y - 2x = 6 \\ 4x - 6y = -12 \end{cases}$ **c.** $\begin{cases} f(x) = -\frac{1}{2}x + 3 \\ 2g(x) + x = -4 \end{cases}$

Solution ► **a.** To graph the first equation, it is convenient to use the slope-intercept form,

$$q = -3p + 5.$$

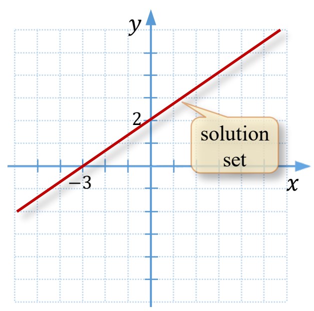
To graph the second equation, it is convenient to use the p- and q-intercepts, $(4, 0)$ and $(0, -2)$.



The first equation is graphed in green and the second – in blue. The intersection point is at $(\mathbf{2}, -\mathbf{1})$, which is the **only solution** of the system.

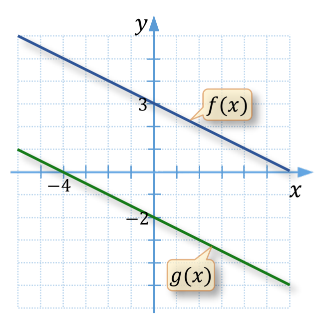
The system is **consistent**, as it has a solution, and **independent**, as the lines are different.

b. Notice that when using the x- and y-intercept method of graphing, both equations have x-intercepts equal to $(-3, 0)$ and y-intercepts equal to $(0, 2)$. So, both equations represent the same line. Therefore the solution set to this system consists of all points of the line $3y - 2x = 6$. We can record this set of points with the use of set-builder notation as $\{(\boldsymbol{x}, \boldsymbol{y}) | \mathbf{3}\boldsymbol{y} - \mathbf{2}\boldsymbol{x} = \mathbf{6}\}$, and state that the system has **infinitely many solutions**.



The system is **consistent**, as it has solutions, and **dependent**, as both lines are the same.

c. We plan to graph both functions, f and g, on the same grid. Function f is already given in the slope-intercept form, which is convenient for graphing. To graph function g, we can either use the x- and y-intercept method or solve the equation for $g(x)$ and use the slope-intercept method. So, we have



$$2g(x) = -x - 4$$

$$g(x) = -\tfrac{1}{2}x - 2$$

After graphing both functions, we observe that the two lines are parallel because they have the same slope. Having different y-intercepts, the lines do not have any common points. Therefore, the system has **no solutions**.

Such a system is **inconsistent**, as it has no solutions, and **independent**, as the lines are different.

Solving a system of equations by graphing, although useful, is not always a reliable method. For example, if the solution is an ordered pair of fractional numbers, we may have a hard time to read the exact values of these numbers from the graph. Luckily, a system of equations can be solved for exact values using algebra. Below, two algebraic methods for solving systems of two equations, referred to as substitution and elimination, are shown.

Solving Systems of Linear Equations by the Substitution Method

In the substitution method, as shown in *Example 3* below, we eliminate a variable from one equation by substituting an expression for that variable from the other equation. This method is particularly suitable for solving systems in which one variable is either already isolated or is easy enough to isolate.

Example 3 ► **Solving Systems of Linear Equations by Substitution**

Solve each system by substitution.

a. $\begin{cases} x = y + 1 \\ x + 2y = 4 \end{cases}$ **b.** $\begin{cases} 3a - 2b = 6 \\ 6a + 4b = -20 \end{cases}$

Solution ►

a. Since x is already isolated in the first equation, $x = y + 1$, we replace x by $y + 1$ in the second equation, $x = y + 1$. Thus,

$$(y + 1) + 2y = 4 \qquad /-1$$

which after solving for y, gives us

$$\begin{aligned} 3y &= 3 \qquad /\div 3 \\ \boldsymbol{y} &= \mathbf{1} \end{aligned}$$

Then, we substitute the value $y = 1$ back into the first equation, and solve for x. This gives us $\boldsymbol{x} = 1 + 1 = \mathbf{2}$.

One can check that $x = 2$ and $y = 1$ satisfy both of the original equations. So, the solution set of this system is $\{(\mathbf{2}, \mathbf{1})\}$.

b. To use the substitution method, we need to solve one of the equations for one of the variables, whichever is easier. Out of the coefficients by the variables, -2 seems to be the easiest coefficient to work with. So, let us solve the first equation, $3a - 2b = 6$, for b.

$$\begin{aligned} 3a - 2b &= 6 \qquad /-3a \\ -2b &= -3a + 6 \qquad /\div(-2) \\ b &= \tfrac{3}{2}a - 3 \end{aligned}$$

substitution equation

Then, substitute the expression $\frac{3}{2}a - 3$ to the second equation, $6a + 4b = -20$, for b. So, we obtain

$$6b + 4\left(\tfrac{3}{2}a - 3\right) = -20,$$

which can be solved for a:

$$\begin{aligned} 6a + 6a - 12 &= -20 \qquad /+12 \\ 12a &= -8 \qquad /\div 12 \end{aligned}$$

equation in one variable

$$a = -\frac{8}{12}$$
$$\boldsymbol{a} = -\frac{\mathbf{2}}{\mathbf{3}}$$

Then, we plug $a = -\frac{2}{3}$ back into the substitution equation $b = \frac{3}{2}a - 3$ to find the b-value. This gives us

$$\boldsymbol{b} = \frac{3}{2}\left(-\frac{2}{3}\right) - 3 = -1 - 3 = -\mathbf{4}$$

To check that the values $a = -\frac{2}{3}$ and $b = -4$ satisfy both equations of the system, we substitute them into each equation, and simplify each side. Since both equations,

$$3\left(-\frac{2}{3}\right) - 2(-4) = 6 \qquad\qquad 6\left(-\frac{2}{3}\right) + 4(-4) = -20$$
$$-2 + 8 = 6 \qquad \text{and} \qquad -4 - 16 = -20$$
$$6 = 6 \checkmark \qquad\qquad -20 = -20, \checkmark$$

are satisfied, the solution set of this system is $\left\{\left(-\frac{\mathbf{2}}{\mathbf{3}}, -\mathbf{4}\right)\right\}$.

Summary of Solving Systems of Linear Equations by Substitution

Step 1	**Solve one of the equations for one of the variables.** Choose to solve for the variable with the easiest coefficient to work with. The obtained equation will be referred to as the **substitution equation**.
Step 2	**Plug the substitution equation into the other equation.** The result should be an equation with just one variable.
Step 3	**Solve** the resulting equation to find the value of the variable.
Step 4	**Find the value of the other variable** by substituting the result from Step 3 into the substitution equation from Step 1.
Step 5	**Check** if the variable values satisfy both of the original equations. Then **state the solution set** by listing the ordered pair(s) of numbers.

Solving Systems of Linear Equations by the Elimination Method

Another algebraic method, the **elimination method**, involves combining the two equations in a system so that one variable is eliminated. This is done using the addition property of equations.

Recall: If $\boldsymbol{a} = \boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{d}$, then $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$.

Example 6 ► **Solving Systems of Linear Equations by Elimination**

Solve each system by elimination.

a. $\begin{cases} r + 2s = 3 \\ 3r - 2s = 5 \end{cases}$ **b.** $\begin{cases} 2x + 3y = 6 \\ 3x + 5y = -2 \end{cases}$

Solution ► **a.** Notice that the equations contain opposite terms, $2s$ and $-2s$. Therefore, if we add these equations, side by side, the s-variable will be eliminated. So, we obtain

$$+\begin{cases} r + 2s = 3 \\ 3r - 2s = 5 \end{cases}$$
$$\overline{\quad 4r = 8 \quad}$$
$$r = 2$$

Now, since the r-value is already known, we can substitute it to one of the equations of the system to find the s-value. Using the first equation, we obtain

$$2 + 2s = 3$$
$$2s = 1$$
$$s = \frac{1}{2}$$

One can check that the values $r = 2$ and $s = \frac{1}{2}$ make both equations of the original system true. Therefore, the pair $(\mathbf{2}, \frac{\mathbf{1}}{\mathbf{2}})$ is the solution of this system. We say that the solution set is $\left\{(\mathbf{2}, \frac{\mathbf{1}}{\mathbf{2}})\right\}$.

b. First, we choose which variable to eliminate. Suppose we plan to remove the x-variable. To do this, we need to transform the equations in such a way that the coefficients in the x-terms become opposite. This can be achieved by multiplying, for example, the first equation by 3 and the second equation, by -2.

$$\begin{cases} 2x + 3y = 6 & /\cdot 3 \\ 3x + 5y = -2 & /\cdot (-2) \end{cases}$$

$$+\begin{cases} 6x + 9y = 18 \\ -6x - 10y = 4 \end{cases}$$

Then, we add the two equations, side by side,

$$-y = 22 \qquad /\cdot (-1)$$

and solve the resulting equation for y,

$$y = -22.$$

To find the x-value, we substitute $y = -22$ to one of the original equations. Using the first equation, we obtain

$$2x + 3(-22) = 6$$
$$2x - 66 = 6 \qquad /+66$$
$$2x = 72 \qquad /\div 2$$
$$x = 36$$

One can check that the values $x = 36$ and $y = -22$ make both equations of the original system true. Therefore, the solution of this system is the pair $(\mathbf{36}, \mathbf{-22})$. We say that the solution set is $\{(\mathbf{36}, \mathbf{-22})\}$.

Summary of Solving Systems of Linear Equations by Elimination

- **Write both equations in standard form $Ax + By = C$.** Keep A and B as integers by clearing any fractions, if needed.
- **Choose a variable to eliminate.**
- **Make the chosen variable's terms opposites** by multiplying one or both equations by appropriate numbers if necessary.
- **Eliminate a variable by adding the respective sides of the equations** and then solve for the remaining variable.
- **Find the value of the other variable** by substituting the result from Step 4 into either of the original equations and solve for the other variable.
- **Check** if the variable values satisfy both of the original equations. Then **state the solution set** by listing the ordered pair(s) of numbers.

Comparing Methods of Solving Systems of Equations

When deciding which method to use, consider the suggestions in the table below.

Method	Strengths	Weaknesses
Graphical	• **Visualization.** The solutions can be "seen" and approximated.	• **Inaccuracy.** When solutions involve numbers that are not integers, they can only be approximated. • **Grid limitations.** Solutions may not appear on the part of the graph drawn.
Substitution	• **Exact solutions.** • Most convenient to use when a variable has a **coefficient of 1**.	• **Computations.** Often requires extensive computations with fractions.
Elimination	• **Exact solutions.** • Most convenient to use when all **coefficients** by variables are **different than 1**.	• **Preparation.** The method requires that the coefficients by one of the variables are opposite.

Solving Systems of Linear Equations in Special Cases

As it was shown in solving linear systems of equations by graphing, some systems have no solution or infinitely many solutions. The next example demonstrates how to solve such systems algebraically.

Example 7 ► Solving Inconsistent or Dependent Systems of Linear Equations

Solve each system algebraically.

a. $\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$ **b.** $\begin{cases} 2x - y = 3 \\ 6x - 3y = 9 \end{cases}$

Solution ►

a. When trying to eliminate one of the variables, we might want to multiply the first equation by 2. This, however, causes both variables to be eliminated, resulting in

$$+\begin{cases} 2x + 6y = 8 \\ -2x - 6y = 3 \end{cases}$$
$$0 = 11,$$

parallel lines

which is *never true*. This means that there is no ordered pair (x, y) that would make this equation true. Therefore, there is **no solution** to this system. The solution set is $\emptyset$. The system is **inconsistent**, so the equations must describe **parallel lines**.

b. When trying to eliminate one of the variables, we might want to multiply the first equation by 3. This, however, causes both variables to be eliminated and we obtain

$$+\begin{cases} 6x - 3y = 9 \\ 6x - 3y = 9 \end{cases}$$
$$0 = 0,$$

same line

which is *always true*. This means that any x-value together with its corresponding y-value satisfy the system. Therefore, there are **infinitely many solutions** to this system. These solutions are all points of one of the equations. Therefore, the solution set can be recorded in set-builder notation, as

Read: the set of all ordered pairs (x, y), such that $2x - y = 3$

$$\{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{2x - y = 3}\}$$

Since the equations of the system are equivalent, they represent the same line. So, the system is **dependent**.

Summary of Special Cases of Linear Systems

If both variables are eliminated when solving a linear system of two equations, then the solution sets are determined as follows.

Case 1 If the resulting statement is **true**, there are **infinitely many solutions**. The system is ***consistent***, and the equations are ***dependent***.

Case 2 If the resulting statement is **false**, there is **no solution**. The system is ***inconsistent***, and the equations are ***independent***.

Another way of determining whether a system of two linear equations is inconsistent or dependent is by examining slopes and y-intercepts in the two equations.

Example 8 ► **Using Slope-Intercept Form to Determine the Number of Solutions and the Type of System**

For each system, determine the number of solutions and classify the system without actually solving it.

a. $\begin{cases} \frac{1}{2}x = \frac{1}{8}y + \frac{1}{4} \\ 4x - y = -2 \end{cases}$

b. $\begin{cases} 2x + 5y = 6 \\ 0.4x + y = 1.2 \end{cases}$

Solution ►

a. First, let us clear the fractions in the first equation by multiplying it by 8,

$$\begin{cases} 4x = y + 2 & /-2 \\ 4x - y = -2\,' & /+y, +2 \end{cases}$$

and then solve each equation for y.

parallel lines

$$\begin{cases} 4x - 2 = y \\ 4x + 2 = y \end{cases}$$

Then, observe that the slopes in both equations are the same and equal to 4, but the y-intercepts are different, -2 and 2. The same slopes tell us that the corresponding lines are **parallel** while different y-intercepts tell us that the two lines are **different**. So, the system has **no solution**, which means it is **inconsistent**, and the lines are **independent**.

b. We will start by solving each equation for y. So, we have

$$\begin{cases} 2x + 5y = 6 & /-2x \\ 0.4x + y = 1.2 & /-0.4x \end{cases}$$

$$\begin{cases} 5y = -2x + 6 & /\div 5 \\ y = -0.4x + 1.2 \end{cases}$$

same line

$$\begin{cases} y = -\frac{2}{5}x + \frac{6}{5} \\ y = -0.4x + 1.2 \end{cases}$$

Notice that $-\frac{2}{5} = -0.4$ and $\frac{6}{5} = 1.2$. Since the resulting equations have the same slopes and the same y-intercepts, they represent the same line. Therefore, the system has **infinitely many solutions**, which means it is **consistent**, and the lines are **dependent**.

E.1 Exercises

Decide whether the given ordered pair is a solution of the given system.

12. $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$; $(5, 2)$

13. $\begin{cases} x + y = 1 \\ 2x - 3y = -8 \end{cases}$; $(-1, 2)$

14. $\begin{cases} p + 3q = 1 \\ 5p - q = -9 \end{cases}$; $(-2, 1)$

15. $\begin{cases} 2a + b = 3 \\ a - 2b = -9 \end{cases}$; $(-1, 5)$

Solve each system of equations ***graphically****. Then, classify the system as* ***consistent*** *or* ***inconsistent*** *and* ***dependent*** *or* ***independent****.*

16. $\begin{cases} 3x + y = 5 \\ x - 2y = 4 \end{cases}$

17. $\begin{cases} 3x + 4y = 8 \\ x + 2y = 6 \end{cases}$

18. $\begin{cases} f(x) = x - 1 \\ g(x) = -2x + 5 \end{cases}$

19. $\begin{cases} f(x) = -\frac{1}{4}x + 1 \\ g(x) = \frac{1}{2}x - 2 \end{cases}$

20. $\begin{cases} y - x = 5 \\ 2x - 2y = 10 \end{cases}$

21. $\begin{cases} 6x - 2y = 2 \\ 9x - 3y = -1 \end{cases}$

22. $\begin{cases} y = 3 - x \\ 2x + 2y = 6 \end{cases}$

23. $\begin{cases} 2x - 3y = 6 \\ 3y - 2x = -6 \end{cases}$

24. $\begin{cases} 2u + v = 3 \\ 2u = v + 7 \end{cases}$

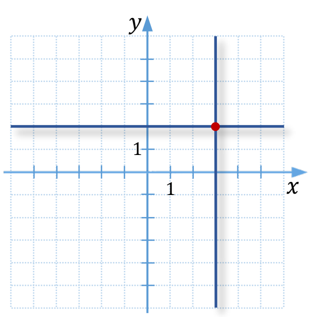
25. $\begin{cases} 2b = 6 - a \\ 3a - 2b = 6 \end{cases}$

26. $\begin{cases} f(x) = 2 \\ x = -3 \end{cases}$

27. $\begin{cases} f(x) = x \\ g(x) = -1.5 \end{cases}$

28. Classify each system **I** to **VI** as *consistent* or *inconsistent* and the equations as *dependent* or *independent*. Then, match it with the corresponding system of equations **A** to **F**.

I



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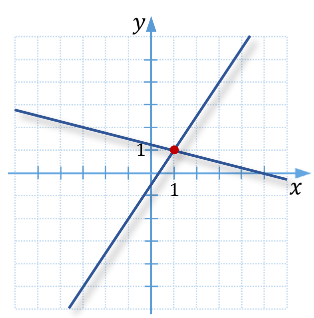
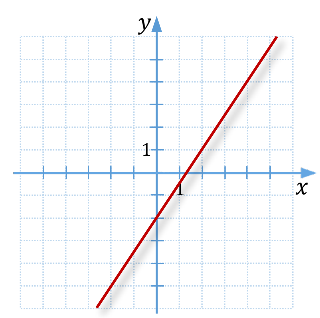
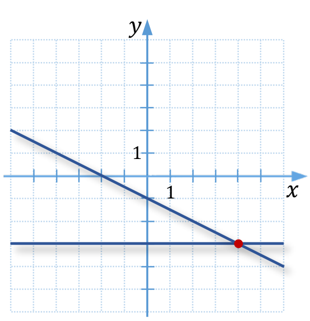


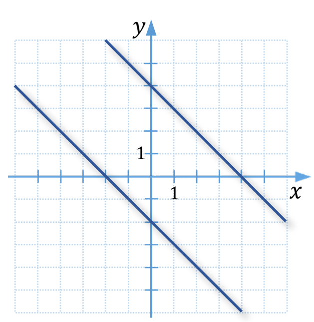
III



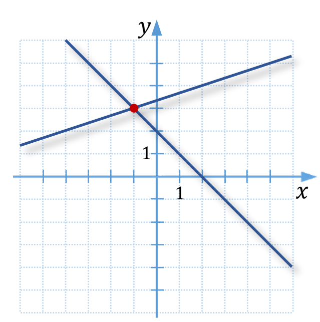
IV



V



VI



A $\begin{cases} 3y - x = 10 \\ x = -y + 2 \end{cases}$ **B** $\begin{cases} 9x - 6y = 12 \\ y = \frac{3}{2}x - 2 \end{cases}$ **C** $\begin{cases} 2y - 3x = -1 \\ x + 4y = 5 \end{cases}$

D $\begin{cases} x + y = 4 \\ y = -x - 2 \end{cases}$ **E** $\begin{cases} \frac{1}{2}x + y = -1 \\ y = -3 \end{cases}$ **F** $\begin{cases} x = 3 \\ y = 2 \end{cases}$

*Solve each system by **substitution**. If the system describes **parallel lines** or the **same line**, say so.*

29. $\begin{cases} y = 2x + 1 \\ 3x - 4y = 1 \end{cases}$ **30.** $\begin{cases} 5x - 6y = 23 \\ x = 6 - 3y \end{cases}$ **31.** $\begin{cases} x + 2y = 3 \\ 2x + y = 5 \end{cases}$

32. $\begin{cases} 2y = 1 - 4x \\ 2x + y = 0 \end{cases}$ **33.** $\begin{cases} y = 4 - 2x \\ y + 2x = 6 \end{cases}$ **34.** $\begin{cases} y - 2x = 3 \\ 4x - 2y = -6 \end{cases}$

35. $\begin{cases} 4s - 2t = 18 \\ 3s + 5t = 20 \end{cases}$ **36.** $\begin{cases} 4p + 2q = 8 \\ 5p - 7q = 1 \end{cases}$ **37.** $\begin{cases} \frac{x}{2} + \frac{y}{2} = 5 \\ \frac{3x}{2} - \frac{2y}{3} = 2 \end{cases}$

38. $\begin{cases} \frac{x}{4} + \frac{y}{3} = 0 \\ \frac{x}{8} - \frac{y}{6} = 2 \end{cases}$ **39.** $\begin{cases} 1.5a - 0.5b = 8.5 \\ 3a + 1.5b = 6 \end{cases}$ **40.** $\begin{cases} 0.3u - 2.4v = -2.1 \\ 0.04u + 0.03v = 0.7 \end{cases}$

*Solve each system by **elimination**. If the system describes **parallel lines** or the **same line**, say so.*

41. $\begin{cases} x + y = 20 \\ x - y = 4 \end{cases}$ **42.** $\begin{cases} 6x + 5y = -7 \\ -6x - 11y = 1 \end{cases}$ **43.** $\begin{cases} x - y = 5 \\ 3x + 2y = 10 \end{cases}$

44. $\begin{cases} x - 4y = -3 \\ -3x + 5y = 2 \end{cases}$ **45.** $\begin{cases} 2x + 3y = 1 \\ 3x - 5y = -8 \end{cases}$ **46.** $\begin{cases} -2x + 5y = 14 \\ 7x + 6y = -2 \end{cases}$

47. $\begin{cases} 2x + 3y = 1 \\ 4x + 6y = 2 \end{cases}$ **48.** $\begin{cases} 6x - 10y = -4 \\ 5y - 3x = 7 \end{cases}$ **49.** $\begin{cases} 0.3x - 0.2y = 4 \\ 0.2x + 0.3y = 1 \end{cases}$

50. $\begin{cases} \frac{2}{3}x + \frac{1}{7}y = -11 \\ \frac{1}{7}x - \frac{1}{3}y = -10 \end{cases}$ **51.** $\begin{cases} 3a + 2b = 3 \\ 9a - 8b = -2 \end{cases}$ **52.** $\begin{cases} 5m - 9n = 7 \\ 7n - 3m = -5 \end{cases}$

53. Can a linear system of two equations has exactly two solutions? Justify your answer.

*Write each equation in **slope-intercept form** and then tell how many solutions the system has. Do not actually solve.*

54. $\begin{cases} -x + 2y = 8 \\ 4x - 8y = 1 \end{cases}$ **55.** $\begin{cases} 6x = -9y + 3 \\ 2x = -3y + 1 \end{cases}$ **56.** $\begin{cases} y - x = 6 \\ x + y = 6 \end{cases}$

Solve each system by the method of your choice.

57. $\begin{cases} 3x + y = -7 \\ x - y = -5 \end{cases}$ **58.** $\begin{cases} 3x - 2y = 0 \\ 9x + 8y = 7 \end{cases}$ **59.** $\begin{cases} 3x - 5y = 7 \\ 2x + 3y = 30 \end{cases}$

60. $\begin{cases} 2x + 3y = 10 \\ -3x + y = 18 \end{cases}$ **61.** $\begin{cases} \frac{1}{6}x + \frac{1}{3}y = 8 \\ \frac{1}{4}x + \frac{1}{2}y = 30 \end{cases}$ **62.** $\begin{cases} \frac{1}{2}x - \frac{1}{8}y = -\frac{1}{2} \\ 4x - y = -2 \end{cases}$

63. $\begin{cases} a+4b=2 \\ 5a-b=3 \end{cases}$

64. $\begin{cases} 3a-b=7 \\ 2a+2b=5 \end{cases}$

65. $\begin{cases} 6\cdot f(x)=2x \\ -7x+15\cdot g(x)=10 \end{cases}$

Solve the system of linear equations. Assume that $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *represent nonzero constants.*

66. $\begin{cases} x+ay=1 \\ 2x+2ay=4 \end{cases}$

67. $\begin{cases} -ax+y=4 \\ ax+y=4 \end{cases}$

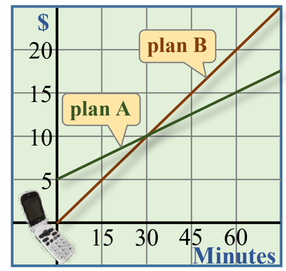
68. $\begin{cases} -ax+y=2 \\ ax+y=4 \end{cases}$

69. $\begin{cases} ax+by=2 \\ -ax+2by=1 \end{cases}$

70. $\begin{cases} 2ax-y=3 \\ y=5ax \end{cases}$

71. $\begin{cases} 3ax+2y=1 \\ -ax+y=2 \end{cases}$

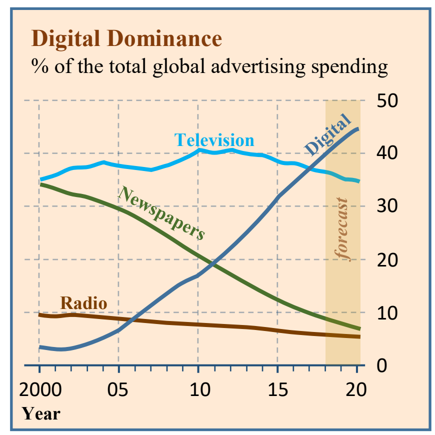
Refer to the accompanying graph to answer questions 72-73.



72. According to the graph, for how many long-distance minutes the charge would be the same in plan A as in plan B? Give an ordered pair of the form (minutes, dollars) to represent this situation.

73. For what range of long-distance minutes would plan B be cheaper?

Refer to the accompanying graph to answer questions 74-77.



74. According to the predictions shown in the graph, what percent of the global advertising spending will be allocated to digital advertising in 2020? Give an ordered pair of the form (year, percent) to represent this information.

75. Estimate the year in which spending on digital advertising matches spending on television advertising.

76. Since when did spending on digital advertising exceed spending on advertising in newspapers? What was this spending as a percentage of global advertising spending at that time?

77. Since when did spending on digital advertising exceed spending on radio advertising? What was this spending as a percentage of global advertising spending at that time?

E2 Applications of Systems of Linear Equations in Two Variables

Systems of equations are frequently used in solving applied problems. Although many problems with two unknowns can be solved with the use of a single equation with one variable, it is often easier to translate the information given in an application problem with two unknowns into two equations in two variables.

Here are some guidelines to follow when solving applied problems with two variables.

Solving Applied Problems with the Use of System of Equations

- **Read** the problem, several times if necessary. When reading, watch the given information and what the problem asks for. Recognize the type of problem, such as geometry, total value, motion, solution, percent, investment, etc.
- **Assign variables** for the unknown quantities. Use meaningful letters, if possible.
- **Organize** the given information. Draw appropriate tables or diagrams; list relevant formulas.
- **Write equations** by following a relevant formula(s) or a common sense pattern.
- **Solve** the system of equations.
- **Check** if the solution is reasonable in the context of the problem.
- **State the answer** to the problem.

Below we show examples of several types of applied problems that can be solved with the aid of systems of equations.

Number Relations Problems

Example 1 ► **Finding Unknown Numbers**

The difference between twice a number and a second number is 3. The sum of the two numbers is 18. Find the two numbers.

Solution ► Let a be the first number and b be the second number. The first sentence of the problem translates into the equation

$$2a - b = 3.$$

The second sentence translates to

$$a + b = 18.$$

Now, we can solve the system of the above equations, using the elimination method. Since

$$+\begin{cases} 2a - b = 3 \\ a + b = 18 \end{cases}$$

$$3a = 21 \quad / \div 3$$

$$a = \mathbf{7},$$

then $b = 18 - a = 18 - 7 = \mathbf{11}$. Therefore, the two numbers are **7** and **11**.

Observation: *A single equation in two variables gives us infinitely many solutions. For example, some of the solutions of the equation* $a+b=16$ *are* $(0,16),(1,15),(2,14)$, *and so on. Generally, any ordered pair of the type* $(a,16-a)$ *is the solution to this equation. So, when working with two variables, to find a specific solution we are in need of a second equation (not equivalent to the first) that relates these variables. This is why problems with two unknowns are solved with the use of systems of two equations.*

Geometry Problems

When working with geometry problems, we often use formulas for perimeter, area, or volume of basic figures. Sometimes, we rely on particular properties or theorems, such as *the sum of angles in a triangle is* $180°$ or *the ratios of corresponding sides of similar triangles are equal*.

Example 2 ► **Finding Dimensions of a Rectangle**

Pat plans a rectangular vegetable garden. The width of the rectangle is to be 5 meters shorter than the length. If the perimeter is planned to be 34 meters, what will the dimensions of the garden be?

Solution ►

The problem refers to the perimeter of a rectangular garden. Suppose L and W represent the length and width of the rectangle. Then the perimeter is represented by the expression $2L+2W$. Since the perimeter of the garden should equal 34 meters, we set up the first equation

$$2L+2W=34 \qquad (1)$$

The second equation comes directly from translating the second sentence of the problem, which tells us that the width is to be 5 meters shorter than the length. So, we write

$$W=L-5 \qquad (2)$$

Now, we can solve the system of the above equations, using the substitution method. After substituting equation (2) into equation (1), we obtain

$$2L+2(L-5)=34$$

$$2L+2L-10=34 \qquad /+10$$

$$4L=44 \qquad /\div 4$$

$$L=\mathbf{11}$$

So, $W=L-5=11-5=\mathbf{6}$.

Therefore, the garden is **11 meters** long and **6 meters** wide.

Number-Value Problems

Problems that refer to the number of different types of items and the value of these items are often solved by setting two equations. Either one equation compares the number of items, and the other compares the value of these items, like in coin types of problems, or both equations compare the values of different arrangements of these items.

Example 3 ► **Finding the Number of Each Type of Items**

A restoration company purchased 45 paintbrushes, some at \$7.99 each and some at \$9.49 each. If the total charge before tax was \$379.05, how many of each type were purchased?

Solution ► Let x represent the number of brushes at \$7.99 each, and let y represent the number of brushes at \$9.49 each. Then the value of x brushes at \$7.99 each is $7.99x$. Similarly, the value of y brushes at \$9.49 each is $9.49y$. To organize the given information, we suggest to create and complete the following table.

	brushes at \$7.99 each +	brushes at \$9.49 each	= Total
number of brushes	x	y	45
value of brushes (in \$)	$7.99x$	$9.49y$	379.05

Since we work with two variables, we need two different equations in these variables. The first equation comes from comparing the number of brushes, as in the middle row. The second equation comes from comparing the values of these brushes, as in the last row. So, we have the system

$$\begin{cases} x + y = 45 \\ 7.99x + 9.49y = 379.05 \end{cases}$$

to solve. This can be solved by substitution. From the first equation, we have $y = 45 - x$, which after substituting to the second equation gives us

$$7.99x + 9.49(45 - x) = 379.05 \qquad /\cdot 100$$

$$799x + 949(45 - x) = 37905$$

$$799x + 42705 - 949x = 37905 \qquad /-43875$$

$$-150x = -4800 \qquad /\div(-125)$$

$$x = \mathbf{32}$$

Then, $y = 45 - x = 45 - 32 = \mathbf{13}$.

Therefore, the restoration company purchased **32** brushes at \$7.99 each and **13** brushes at \$9.49 each.

Example 4 ► **Finding the Unit Cost of Each Type of Items**

The cost of 48 ft of red oak and 72 ft of fibreboard is \$271.20. At the same prices, 32 ft of red oak and 60 ft of fibreboard cost \$200. Find the unit price of red oak and fibreboard.

Solution ► Let r be the unit price of red oak and let f be the unit price of fibreboard. Then the value of 48 ft of red oak is represented by $48r$, and the value of 72 ft of fibreboard is represented by $72f$. Using the total cost of \$271.20, we write the first equation

$$48r + 72f = 271.20$$

Similarly, using the total cost of \$200 for 32 ft of red oak and 60 ft of fibreboard, we write the second equation

$$32r + 60f = 200$$

We will solve the system of the above equations via the elimination method.

$$\begin{cases} 48r + 72f = 271.20 \\ 32r + 60f = 200 \end{cases} \qquad \begin{matrix} /\cdot(-2) \\ /\cdot 3 \end{matrix}$$

$$+\begin{cases} -96r - 144f = -542.40 \\ 96r + 180f = 600 \end{cases}$$

$$36f = 57.40 \qquad /\div 36$$

$$f = \mathbf{1.60}$$

After substituting $f = 1.60$ into the second equation, we obtain

$$32r + 60 \cdot 1.60 = 200$$

$$32r + 96 = 200 \qquad /-96$$

$$32r = 104 \qquad /\div 32$$

$$r = \mathbf{3.25}$$

So, red oak costs **3.25 \$/ft**, and fibreboard costs **1.60 \$/ft**.

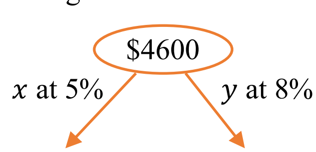
Investment Problems

Investment problems usually involve calculation of simple annual interest according to the formula $\boldsymbol{I = Prt}$, where I represents the amount of interest, P represents the principal which is the amount of investment, r represents the annual interest rate in decimal form, and t represents the time in years.

Example 5 ► **Finding the Amount of Each Loan**

Marven takes two student loans totaling \$4600 to cover the cost of his yearly tuition. One loan is at 5% simple interest, and the other is at 8% simple interest. If his total annual interest charge is \$278, find the amount of each loan.

Solution ► Suppose the amount of loan taken at 5% is x, and the amount of loan taken at 8% is y. The situation can be visualized by the diagram



one year interest: $0.05x + 0.08y = 278$

So, the system of equations to solve is

$$\begin{cases} x + y = 4600 \\ 0.05x + 0.08y = 278 \end{cases}$$

Using the substitution method, we solve the first equation for x,

$$x = 4600 - y, \qquad (*)$$

substitute it into the second equation,

decimal elimination is optional

$$0.05(4600 - y) + 0.08y = 278, \qquad /\cdot 100$$

and after elimination of decimals via multiplication by 100,

$$5(4600 - y) + 8y = 27800,$$

solve it for y:

$$23000 - 5y + 8y = 27800 \qquad /-23000$$

$$3y = 4800$$

$$y = \mathbf{1600}$$

Then, after plugging in the y-value to the substitution equation $(*)$, we obtain

$$x = 4600 - 1600 = \mathbf{3000}$$

So, the amount of loan taken at 5% is \$**3000**, and the amount of loan taken at 8% is \$**1600**.

Mixture – Solution Problems

In mixture or solution problems, we typically mix two or more mixtures or solutions with different concentrations of a particular substance that we will refer to as the content. For example, if we are interested in the salt concentration in salty water, the salt is referred to as the content. When solving mixture problems, it is helpful to organize data in a table such as the one shown below.

	% ·	volume =	content
type I			
type II			
mixture/solution			

Example 6 ► **Solving a Mixture Problem**

Olivia wants to prepare 3 kg mixture of nuts and dried fruits that contains 30% of cranberries by mixing a blend that is 10% cranberry with a blend that is 40% cranberry. How much of each type of blend should she use to obtain the desired mixture?

Solution ► Suppose x is the amount of the 10% blend and y is the amount of the 40% blend. We complete the table

	% $\cdot$	volume =	cranberries
10% blend	0.1	x	$0.1x$
40% blend	0.4	y	$0.4y$
30% blend	0.3	3	0.9

The last column is completed by multiplying the "%" and "volume" columns.

The first equation comes from combining the weight of blends as shown in the "volume" column. The second equation comes from combining the weight of cranberries, as indicated in the last column. So, we solve

$$\begin{cases} x + y = 3 \\ 0.1x + 0.4y = 0.9 \end{cases}$$

Using the substitution method, we solve the first equation for x,

$$x = 3 - y, \qquad (*)$$

and substitute it into the second equation and solve it for y:

$$0.1(3 - y) + 0.4y = 0.9 \qquad / \cdot 10$$

$$3 - y + 4y = 9$$

$$3 + 3y = 9 \qquad / -3$$

$$3y = 6 \qquad / \div 3$$

$$y = \mathbf{2}$$

Then, using the substitution equation $(*)$, we find the value of x:

$$x = 3 - 2 = \mathbf{1}$$

So, to obtain the desired blend, Olivia should mix **1 kg** of 10% and **2 kg** of 40% blend.

Motion Problems

In motion problems, we follow the formula $\boldsymbol{Rate \cdot Time = Distance}$. Drawing a diagram and completing a table based on the $\boldsymbol{R \cdot T = D}$ formula is usually helpful. In some motion problems, in addition to the rate of the moving object itself, we need to consider the rate of a moving medium such as water current or wind. The overall rate of a moving object is typically either the sum or the difference between the object's own rate and the rate of the moving medium.

Example 7 ► **Finding Rates in a Motion Problem**

A motorcycle travels 280 km in the same time that a car travels 245 km. If the motorcycle moves 14 kilometers per hour faster than the car, find the speed of each vehicle.

Solution ► Using meaningful letters, let m and c represent the speed of the motorcycle and the car, respectively. Since the speed of the motorcycle, m, is 14 km/h faster than the speed of the car, c, we can write the first equation:

$$m = c + 14$$

The second equation comes from comparing the travel time of each vehicle, as indicated in the table below.

	R ·	T	$=$ D
motorcycle	m	$\frac{280}{m}$	280
car	c	$\frac{245}{c}$	245

To find the expression for time, we follow the formula $T = \frac{D}{R}$, which comes from solving $R \cdot T = D$ for T.

So, we need to solve the system

$$\begin{cases} m = c + 14 \\ \frac{280}{m} = \frac{245}{c} \end{cases}$$

Cross-multiplication can **only** be applied to a **proportion** (an equation with a single fraction on each side.)

Notice that multiplication by the **LCD** would give the same result.

After substituting the first equation into the second, we obtain

$$\frac{280}{c + 14} = \frac{245}{c},$$

which can be solved by cross-multiplying

$$280c = 245(c + 14)$$

$$280c = 245c + 3430 \qquad /-245c$$

$$35c = 3430 \qquad /\div 35$$

$$c = \mathbf{98}$$

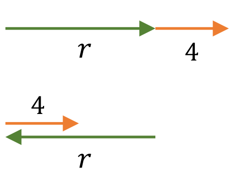
Then, we use this value to find $m = c + 14 = 98 + 14 = \mathbf{112}$.

So, the speed of the motorcycle is **112 km/h**, and the speed of the car is **98 km/h**.

Example 8 **Solving a Motion Problem with a Current**

A motorboat trip upstream a river takes 6 hours, while the return trip takes only 2 hours. Assuming the constant current of 4 km/h, find the speed of the boat in still water.

Solution



Let r be the speed of the boat in still water.
Then the speed of the boat moving downstream is 4 mph faster because of the current going in the same direction as the boat. So, it is represented by $r + 4$.
The speed of the boat moving against the current is 4 mph slower. So, it is represented by $r - 4$.
Also, let d represent the distance covered by the boat going in one direction.

To organize the information, we can complete the table below.

	R ·	T	$=$ D
downstream	$r + 4$	2	d
upstream	$r - 4$	6	d

The two equations come from following the formula $R \cdot T = D$, as indicated in each row.

$$\begin{cases}(r+4)\cdot 2 = d\\(r-4)\cdot 6 = d\end{cases}$$

Since left sides of both equations represent the same distance d, we can equal them and solve for r:

$$\begin{aligned}(r+4)\cdot 2 &= (r-4)\cdot 6\\ 2r+8 &= 6r-24 && /-2r,+24\\ 32 &= 4r && /\div 4\\ \mathbf{8} &= r\end{aligned}$$

So, the speed of the boat in still water was **8 km/h**.

E.2 Exercises

1. If a barrel contains 50 liters of 16% alcohol wine, what is the volume of pure alcohol there?

2. If \$3000 is invested into bonds paying 4.5% simple annual interest, how much interest is expected in a year?

3. If a kilogram of baked chicken breast costs \$9.25, how much would n kilograms of this meat cost?

4. A ticket to a concert costs \$18. Write an expression representing the revenue from selling n such tickets.

5. A canoe that moves at r km/h in still water encounters c km/h current. Write an expression that would represent the speed of this canoe moving **a.** with the current (*downstream*), **b.** against the current **(**upstream).

6. A helicopter moving at r km/h in still air encounters a wind that blows at w km/h. Write an expression representing the rate of the helicopter travelling **a.** with the wind (*tailwind*), **b.** against the wind (*headwind*).

Solve each problem using two variables.

7. The larger out of two complementary angles is 6° more than three times the smaller one. Find the measure of each angle. (*Recall:* complementary angles add to 90°)

8. The larger out of two supplementary angles is 5° more than four times the smaller one. Find the measure of each angle. (*Recall:* supplementary angles add to 180°)

9. The sum of the height and the base of a triangle is 231 centimeters. The height is half the base. Find the base and height.

10. A tennis court is 13 meters longer than it is wide. What are the dimensions of the court if its perimeter is 72 meters.

11. A marathon is a run that covers about 42 km. In 2017, Mary Keitany won the "Women Only" marathon in London. During this marathon, at a particular moment, she was five times as far from the start of the course as she was from its end. At that time, how many kilometers had she already run?

12. Jane asked her students to find the two numbers that she had in mind. She told them that the smaller number is 2 more than one-third of the larger number and that three times the larger number is 1 less than eight times the smaller one. Find the numbers.

13. During the 2014 Winter Olympics in Sochi, Russia won a total of 33 medals. There were 4 more gold medals than bronze ones. If the number of silver medals was the average of the number of gold and the number of bronze medals, how many of each type of medal did Russia earn?

14. A 156 cm long piece of wire is cut into two pieces. Then, each piece is bent to make an equilateral triangle. If the side of one triangle is twice as long as the side of the other triangle, how should the wire be cut?

15. A bistro cafe sells espresso and cappuccino cups of coffee. One day, the cafe has sold 452 cups of coffee. How many cups of each type of coffee the shop has sold that day if the number of cappuccino cups was 3 times as large as the number of espresso cups of coffee sold?

16. During an outdoor festival, a retail booth was selling solid-colour scarfs for \$8.75 each and printed scarfs for \$11.49 each. If \$478.60 was collected for selling 50 scarfs, how many of each type were sold?

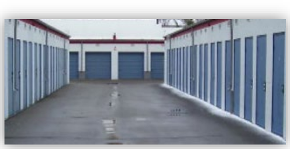
17. The Mission Folk Music Festival attracts local community every summer since 1988. In recent years, a one-day admission to this festival costs \$45 for an adult ticket and \$35 for a youth ticket. If the total of \$24,395 were collected from the sale of 605 tickets, how many of each type of tickets were sold?

18. Ellena's total GED score in Mathematics and Science was 328. If she scored 24 points higher in Mathematics than in Science, what were her GED scores in each subject?

19. One day, at his food stand, Tom sold 12 egg salad sandwiches and 18 meat sandwiches, totaling \$101.70. The next day he sold 23 egg salad sandwiches and 9 meat sandwiches, totaling \$93.18. How much did each type of sandwich cost?

20. At lunchtime, a group of conference members ordered three cappuccinos and four expressos for a total of \$20.07. Another group ordered two cappuccinos and three expressos for a total of \$14.43. How much did each type of coffee cost?

21. New York City and Paris are one of the most expensive cities to live in. Based on the average weekly cost of living in each city (not including accommodation), 2 weeks in New York and 3 weeks in Paris cost \$1852, while 4 weeks in New York and 2 weeks in Paris cost \$2344. Find the average weekly cost of living in each city?



22. One of the local storage facilities rents two types of storage lockers, a small one with 180 ft^2 in area, and a large one with the area of 600 ft^2. In total, the facility has 42 storage lockers that provide 15,120 ft^2 of the overall storage area. How many of each type of storage lockers does the facility have?

23. Ryan took two student loans for a total of \$4800. One of these loans was borrowed at 3.25% simple interest and the other one at 2.75%. If after one year Ryan's overall interest charge for both of the loans was \$143.50, what was the amount of each loan?

24. An investor made two investments totaling \$36,000. In one year, these investments generated \$1650 in simple interest. If the interest rate for the two investments were 5% and 3.75%, how much was invested at each rate?

25. A stockbroker invested some money in a low-risk fund and twice as much in a high-risk fund. In a year, the low-risk fund earned 3.7%, and the high-risk fund lost 8.2%. If the two investments resulted in the overall loss of \$111.20, how much was invested in each fund?

26. Patricia's bank offers her two types of investments, one at 4.5% and the other one at 6.25% simple interest. Patricia invested \$1500 more at 6.25% than at 4.5%. How much was invested at each rate if the total interest accumulated after one year was \$462.25?

27. How many liters of 4% brine and 20% brine should be mixed to obtain 12 liters of 8% brine?

28. Sam has \$11 in dimes and quarters. If he counted 71 coins in all, how many of each type of coin are there?

29. Cottage cheese contains 12% of protein and 6% of carbs while vanilla yogurt is 5% protein and 15% carbs. How many grams of each product should be used to serve a meal that contains 10 grams of protein and 10 grams of carbs?

30. Kidney beans contain 24% protein while lima beans contain just 8% protein. How many dekagrams of each type of bean should be used to prepare 60 dekagrams of a bean-mix that is 12% protein?

31. Cezary purchased a shirt costing \$42.75 with a \$50 bill. The cashier gave him the change in quarters and loonies. If the change consisted of 14 coins, how many of each kind were there?

32. When travelling with the current, a speedboat covers 24 km in half an hour. It takes 40 minutes for the boat to cover the same distance against the current. Find the rate of the boat in still water and the rate of the current.

33. A houseboat travelling with the current went 45 km in 3 hours. It took 2 hours longer to travel the same distance against the current. Find the rate of the houseboat in still water and the rate of the current.

34. When flying with the wind, a passenger plane covers a distance of 1760 km in 2 hours. When flying against the same wind, the plane covers 2400 km in 3 hours. Determine the rate of the plane in still air and the rate of the wind.

35. Flying with a tailwind, a pilot of a small plane could cover the distance of 1500 km between two cities in 5 hours. Flying with the headwind of the same intensity, he would need 6 hours to cover the same distance. Find the rate of the plane in still air and the rate of the wind.

36. Robert kayaked 10 km downstream a river in 2 hours. When returning, Robert could kayak only 6 km in the same amount of time. What was his rate of kayaking in still water and what was the rate of the current?

37. A small private plane flying into a wind covered 1080 km in 4 hours. When flying back, with a tailwind of the same intensity, the plane needed only 3 hours to cover the same distance. What are the rate of the plane in still air and the rate of the wind?

38. Flight time against a headwind for a trip of 2300 kilometers is 4 hours. If the headwind were half as great, the same flight would take 10 minutes less time. Find the rate of the wind and the rate of the plane in still air.

39. Teresa was late for her conference presentation after driving at an average speed of 60 km/h. If she had driven 4 km/h faster, her travelling time would be half an hour shorter. How far was the conference?

40. Two vehicles leave a gas station at the same time and travel in the same direction. One travels at 96 km/h and the other at 108 km/h. The drivers of the two vehicles can communicate with each other with a short-distance radio device as long as they are within 10 km range. When will they lose this contact?

41. The windshield fluid tank in Izabella's car contains 5 L of 7% antifreeze. To the nearest tenths of a litre, how much of this mixture should be drained and replaced with pure antifreeze so that the mixture becomes 20% antifreeze?

42. Mike has two gallons of stain that is 10% brown and 90% neutral and a gallon of pure brown stain. To stain a deck, he needs 2 gallons of a stain that is 40% brown and 60% neutral. How much of each type of stain should he use to prepare the desired mix?

Attributions

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Polynomials and Polynomial Functions

One of the simplest types of algebraic expressions are polynomials. They are formed only by addition and multiplication of variables and constants. Since both addition and multiplication produce unique values for any given inputs, polynomials are in fact functions. One of the simplest polynomial functions are linear functions, such as $P(x) = 2x + 1$, or quadratic functions, such as $Q(x) = x^2 + x - 6$. Due to the comparably simple form, polynomial functions appear in a wide variety of areas of mathematics and science, economics, business, and many other areas of life. Polynomial functions are often used to model various natural phenomena, such as the shape of a mountain, the distribution of temperature, the trajectory of projectiles, etc. The shape and properties of polynomial functions are helpful when constructing such structures as roller coasters or bridges, solving optimization problems, or even analysing stock market prices.

In this chapter, we will introduce polynomial terminology, perform operations on polynomials, and evaluate and compose polynomial functions.

P1 Addition and Subtraction of Polynomials

Terminology of Polynomials

Recall that products of constants, variables, or expressions are called **terms** (see *Section R3, Definition 3.1*). **Terms** that are **products** of only **numbers** and **variables** are called **monomials**. Examples of monomials are $-2x$, xy^2, $\frac{2}{3}x^3$, etc.

Definition 1.1 ► A **polynomial** is a sum of monomials.

A **polynomial** in a single variable is the sum of terms of the form ax^n, where a is a **numerical coefficient**, x is the variable, and n is a whole number.

An ***n*-th degree polynomial** in x-variable has the form

$$\boldsymbol{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0},$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$.

Note: A polynomial can always be considered as a sum of monomial terms even though there are negative signs when writing it.
For example, polynomial $x^2 - 3x - 1$ can be seen as the sum of signed terms

$$x^2 + -3x + -1$$

Definition 1.2 ► The **degree of a monomial** is the sum of exponents of all its variables.

For example, the degree of $5x^3y$ is 4, as the sum of the exponent of x^3, which is 3 and the exponent of y, which is 1. To record this fact, we write $\deg(5x^3y) = 4$.

The **degree of a polynomial** is the highest degree out of all its terms.

For example, the degree of $2x^2y^3 + 3x^4 - 5x^3y + 7$ is 5 because $\deg(2x^2y^3) = 5$ and the degrees of the remaining terms are not greater than 5.

Polynomials that are sums of two terms, such as $x^2 - 1$, are called **binomials**.
Polynomials that are sums of three terms, such as $x^2 + 5x - 6$ are called **trinomials**.

The **leading term** of a polynomial is the highest degree term.
The **leading coefficient** is the numerical coefficient of the leading term.
So, the leading term of the polynomial $1 - x - x^2$ is $-x^2$, even though it is not the first term. The leading coefficient of the above polynomial is -1, as $-x^2$ can be seen as $(-1)x^2$.

A first degree term is often referred to as a **linear term**. A second degree term can be referred to as a **quadratic term**. A zero degree term is often called a **constant** or a **free term**.

Below are the parts of an n-th degree polynomial in a single variable x:

leading coefficient

$$\underbrace{a_n x^n}_{\substack{\text{leading}\\ \text{term}}} + a_{n-1}x^{n-1} + \cdots + \underbrace{a_2 x^2}_{\substack{\text{quadratic}\\ \text{term}}} + \underbrace{a_1 x}_{\substack{\text{linear}\\ \text{term}}} + \underbrace{a_0}_{\substack{\text{constant}\\ (\text{free})\\ \text{term}}}$$

Note: Single variable polynomials are usually arranged in descending powers of the variable. Polynomials in more than one variable are arranged in decreasing degrees of terms. If two terms are of the same degree, they are arranged with respect to the descending powers of the variable that appers first in alphabetical order.

For example, polynomial $x^2 + x - 3x^4 - 1$ is customarily arranged as follows

$$-3x^4 + x^2 + x - 1,$$

while polynomial $3x^3y^2 + 2y^6 - y^2 + 4 - x^2y^3 + 2xy$ is usually arranged as below.

$$\underbrace{2y^6}_{\substack{6\text{th}\\ \text{degree}\\ \text{term}}} \underbrace{+3x^3y^2 - x^2y^3}_{\substack{5\text{th degree terms}\\ \text{arranged}\\ \text{with respect to } x}} \underbrace{+2xy - y^2}_{\substack{2\text{nd degree}\\ \text{terms arranged}\\ \text{with respect to } x}} \underbrace{+4}_{\substack{\text{zero}\\ \text{degree}\\ \text{term}}}$$

Example 1 ► **Writing Polynomials in Descending Order and Identifying Parts of a Polynomial**

Suppose $P = x - 6x^3 - x^6 + 4x^4 + 2$ and $Q = 2y - 3xyz - 5x^2 + xy^2$. For each polynomial:

a. Write the polynomial in descending order.
b. State the degree of the polynomial and the number of its terms.
c. Identify the leading term, the leading coefficient, the coefficient of the linear term, the coefficient of the quadratic term, and the free term of the polynomial.

Solution ► **a.** After arranging the terms in descending powers of x, polynomial P becomes

$$\boldsymbol{-x^6 + 4x^4 - 6x^3 + x + 2},$$

while polynomial Q becomes

$$\boldsymbol{xy^2 - 3xyz - 5x^2 + 2y}.$$

Notice that the first two terms, xy^2 and $-3xyz$, are both of the same degree. So, to decide which one should be written first, we look at powers of x. Since these powers are again the same, we look at powers of y. This time, the power of y in xy^2 is higher than the power of y in $-3xyz$. So, the term xy^2 should be written first.

b. The polynomial P has **5 terms**. The highest power of x in P is 6, so the **degree** of the polynomial P is **6**.
The polynomial Q has **4 terms**. The highest degree terms in Q are xy^2 and $-3xyz$, both third degree. So the **degree** of the polynomial Q is **3**.

$x = 1 \cdot x$
$-x = (-1)x$

c. The leading term of the polynomial $P = -x^6 + 4x^4 - 6x^3 + x - 2$ is $-x^6$, so the **leading coefficient** equals $-\mathbf{1}$.
The linear term of P is x, so the **coefficient of the linear term** equals **1**.
P doesn't have any quadratic term so the coefficient of the quadratic term equals **0**.
The **free term** of P equals $-\mathbf{2}$.

The leading term of the polynomial $Q = xy^2 - 3xyz - 5x^2 + 2y$ is xy^2, so the **leading coefficient** is equal to **1**.
The linear term of Q is $2y$, so the **coefficient of the linear term** equals **2**.
The quadratic term of Q is $-5x^2$, so the **coefficient of the quadratic term** equals $-\mathbf{5}$.
The polynomial Q does not have a free term, so the **free term** equals **0**.

Example 2 ► **Classifying Polynomials**

Describe each polynomial as a *constant*, *linear*, *quadratic*, or n*-th degree* polynomial. Decide whether it is a *monomial*, *binomial*, or *trinomial*, if applicable.

a.	$x^2 - 9$	**b.**	$-3x^7y$
c.	$x^2 + 2x - 15$	**d.**	π
e.	$4x^5 - x^3 + x - 7$	**f.**	$x^4 + 1$

Solution ►

a. $x^2 - 9$ is a second degree polynomial with two terms, so it is a **quadratic binomial**.

b. $-3x^7y$ is an **8-th degree monomial**.

c. $x^2 + 2x - 15$ is a second degree polynomial with three terms, so it is a **quadratic trinomial**.

d. π is a 0-degree term, so it is a **constant monomial**.

e. $4x^5 - x^3 + x - 7$ is a **5-th degree polynomial**.

f. $x^4 + 1$ is a **4-th degree binomial**.

Polynomials as Functions and Evaluation of Polynomials

Each term of a polynomial in one variable is a product of a number and a power of the variable. The polynomial itself is either one term or a sum of several terms. Since taking a power of a given value, multiplying, and adding given values produce unique answers, polynomials are also functions. While f, g, or h are the most commonly used letters to represent functions, other letters can also be used. To represent polynomial functions, we customarily use capital letters, such as P, Q, R, etc.

Any polynomial function P of degree n, has the form

$$\boldsymbol{P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$, and $n \in \mathbb{W}$.

Since polynomials are functions, they can be evaluated for different x-values.

Example 3 ► **Evaluating Polynomials**

Given $P(x) = 3x^3 - x^2 + 4$, evaluate the following expressions:

a. $P(0)$ **b.** $P(-1)$
c. $2 \cdot P(1)$ **d.** $P(a)$

Solution ►

a. $P(0) = 3 \cdot 0^3 - 0^2 + 4 = \mathbf{4}$

b. $P(-1) = 3 \cdot (-1)^3 - (-1)^2 + 4 = 3 \cdot (-1) - 1 + 4 = -3 - 1 + 4 = \mathbf{0}$

When evaluating at negative x-values, it is essential to use brackets in place of the variable before substituting the desired value.

c. $2 \cdot P(1) = 2 \cdot \underbrace{(3 \cdot 1^3 - 1^2 + 4)}_{this\ is\ P(1)} = 2 \cdot (3 - 1 + 4) = 2 \cdot 6 = \mathbf{12}$

d. To find the value of $P(a)$, we replace the variable x in $P(x)$ with a. So, this time the final answer,

$$P(a) = \mathbf{3a^3 - a^2 + 4},$$

is an expression in terms of a rather than a specific number.

Since polynomials can be evaluated at any real x-value, then the **domain** (*see Section G3, Definition 5.1*) of any polynomial is the set $\mathbb{R}$ of all real numbers.

Addition and Subtraction of Polynomials

Recall that terms with the same variable part are referred to as **like terms** (see *Section R3, Definition 3.1*). Like terms can be **combined** by adding their coefficients. For example,

$$\underbrace{2x^2y - 5x^2y = (2-5)x^2y}_{\substack{by\ distributive\ property \\ (factoring)}} = -3x^2y$$

In practice, this step is not necessary to write.

Unlike terms, such as $2x^2$ and $3x$, **cannot be combined**.

Example 4 ► Simplifying Polynomial Expressions

Simplify each polynomial expression.

a. $5x - 4x^2 + 2x + 7x^2$ **b.** $8p - (2 - 3p) + (3p - 6)$

Solution ►

a. To simplify $5x - 4x^2 + 2x + 7x^2$, we combine like terms, starting from the highest degree terms. It is suggested to underline the groups of like terms, using different type of underlining for each group, so that it is easier to see all the like terms and not to miss any of them. So,

$$\underline{5x}\ \underline{-4x^2}\ \underline{+2x}\ \underline{+7x^2} = \mathbf{3x^2} + \mathbf{7x}$$

Remember that the sign in front of a term belongs to this term.

b. To simplify $8p - (2 - 3p) + (3p - 6)$, first we remove the brackets using the distributive property of multiplication and then we combine like terms. So, we have

$-(2 - 3p)$
$= (-1)(2 - 3p)$

$$8p - (2 - 3p) + (3p - 6)$$
$$= \underline{8p} - 2 \underline{+3p} \underline{+3p} - 6$$
$$= \mathbf{11p - 8}$$

Example 5 ► Adding or Subtracting Polynomials

Perform the indicated operations.

a. $(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$
b. $(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$
c. $[9p - (3p - 2)] - [4p - (3 - 7p) + p]$

Solution ►

a. To add polynomials, combine their like terms. So,

remove any bracket preceeded by a "+" sign

$$(6a^5 - 4a^3 + 3a - 1) + (2a^4 + a^2 - 5a + 9)$$
$$= 6a^5 \underline{-4a^3} \underline{+3a} \underline{-1} + 2a^4 \underline{+a^3} \underline{-5a} \underline{+9}$$
$$= \mathbf{6a^5 + 2a^4 - 3a^3 - 8a + 8}$$

b. To subtract a polynomial, add its opposite. In practice, remove any bracket preceeded by a negative sign by reversing the signs of all the terms of the polynomial inside the bracket. So,

$$(4y^3 - 3y^2 + y + 6) - (y^3 + 3y - 2)$$
$$= \underline{4y^3} - 3y^2 \underline{+y} \underline{+6} \underline{-y^3} \underline{-3y} \underline{+2}$$
$$= \mathbf{3y^3 - 3y^2 - 2y + 8}$$

To remove a bracket preceeded by a "−" sign, reverse each sign inside the bracket.

c. First, perform the operations within the square brackets and then subtract the resulting polynomials. So,

$$[9p-(3p-2)]-[4p-(3-7p)+p]$$
$$=[9p-3p+2]-[4p-3+7p+p]$$
$$=[6p+2]-[12p-3]$$
$$=6p+2-12p+3$$
$$=\boldsymbol{-6p+5}$$

collect like terms before removing the next set of brackets

Addition and Subtraction of Polynomial Functions

Similarly as for polynomials, addition and subtraction can also be defined for general functions.

Definition 1.3 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **sum function** $\boldsymbol{f+g}$ is defined as

$$(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})$$

and the **difference function** $\boldsymbol{f-g}$ is defined as

$$(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the sum or difference function is the intersection $\boldsymbol{D_f\cap D_g}$ of the domains of the two functions.

A frequently used application of a sum or difference of polynomial functions comes from the business area. The fact that profit P equals revenue R minus cost C can be recorded using function notation as

$$P(x)=(R-C)(x)=R(x)-C(x),$$

where x is the number of items produced and sold. Then, if $R(x)=6.5x$ and $C(x)=3.5x+900$, the profit function becomes

$$\boldsymbol{P}(\boldsymbol{x})=R(x)-C(x)=6.5x-(3.5x+900)=6.5x-3.5x-900=\boldsymbol{3x-900}.$$

Example 6 ► **Adding or Subtracting Polynomial Functions**

Suppose $P(x)=x^2-6x+4$ and $Q(x)=2x^2-1$. Find the following:

a. $(P+Q)(x)$ and $(P+Q)(2)$
b. $(P-Q)(x)$ and $(P-Q)(-1)$
c. $(P+Q)(k)$
d. $(P-Q)(2a)$

Solution ► **a.** Using the definition of the sum of functions, we have

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{x}) = P(x) + Q(x) = \underbrace{x^2 - 6x + 4}_{P(x)} + \underbrace{2x^2 - 1}_{Q(x)} = \boldsymbol{3x^2 - 6x + 3}$$

Therefore, $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{2}) = 3 \cdot 2^2 - 6 \cdot 2 + 3 = 12 - 12 + 3 = \boldsymbol{3}$.

Alternatively, $(P+Q)(2)$ can be calculated without refering to the function $(P+Q)(x)$, as shown below.

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{2}) = P(2) + Q(2) = \underbrace{2^2 - 6 \cdot 2 + 4}_{P(2)} + \underbrace{2 \cdot 2^2 - 1}_{Q(2)}$$

$$= 4 - 12 + 4 + 8 - 1 = \boldsymbol{3}.$$

b. Using the definition of the difference of functions, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{x}) = P(x) - Q(x) = \underbrace{x^2 - 6x + 4}_{P(x)} - \underbrace{(2x^2 - 1)}_{Q(x)}$$

$$= x^2 - 6x + 4 - 2x^2 + 1 = \boldsymbol{-x^2 - 6x + 5}$$

To evaluate $(P-Q)(-1)$, we will take advantage of the difference function calculated above. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{-1}) = -(-1)^2 - 6(-1) + 5 = -1 + 6 + 5 = \boldsymbol{10}.$$

c. By *Definition 1.3*,

$$(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = P(k) + Q(k) = k^2 - 6k + 4 + 2k^2 - 1 = \boldsymbol{3k^2 - 6k + 3}$$

Alternatively, we could use the sum function already calculated in the solution to *Example 6a*. Then, the result is instant: $(\boldsymbol{P}+\boldsymbol{Q})(\boldsymbol{k}) = \boldsymbol{3k^2 - 6k + 3}$.

d. To find $(P-Q)(2a)$, we will use the difference function calculated in the solution to *Example 6b*. So, we have

$$(\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{2a}) = -(2a)^2 - 6(2a) + 5 = \boldsymbol{-4a^2 - 12a + 5}.$$

P.1 Exercises

Determine whether the expression is a monomial.

1. $-\pi x^3 y^2$ **2.** $5x^{-4}$ **3.** $5\sqrt{x}$ **4.** $\sqrt{2}x^4$

Identify the degree and coefficient.

5. xy^3 **6.** $-x^2 y$ **7.** $\sqrt{2}xy$ **8.** $-3\pi x^2 y^5$

Arrange each polynomial in descending order of powers of the variable. Then, identify the degree and the leading coefficient of the polynomial.

9. $5 - x + 3x^2 - \frac{2}{5}x^3$

10. $7x + 4x^4 - \frac{4}{3}x^3$

11. $8x^4 + 2x^3 - 3x + x^5$

12. $4y^3 - 8y^5 + y^7$

13. $q^2 + 3q^4 - 2q + 1$

14. $3m^2 - m^4 + 2m^3$

State the degree of each polynomial and identify it as a monomial, binomial, trinomial, or n-th degree polynomial if $n > 2$.

15. $7n - 5$

16. $4z^2 - 11z + 2$

17. 25

18. $-6p^4q + 3p^3q^2 - 2pq^3 - p^4$

19. $-mn^6$

20. $16k^2 - 9p^2$

Let $P(x) = -2x^2 + x - 5$ and $Q(x) = 2x - 3$. Evaluate each expression.

21. $P(-1)$

22. $P(0)$

23. $2P(1)$

24. $P(a)$

25. $Q(-1)$

26. $Q(5)$

27. $Q(a)$

28. $Q(3a)$

29. $3Q(-2)$

30. $3P(a)$

31. $3Q(a)$

32. $Q(a+1)$

Simplify each polynomial expression.

33. $5x + 4y - 6x + 9y$

34. $4x^2 + 2x - 6x^2 - 6$

35. $6xy + 4x - 2xy - x$

36. $3x^2y + 5xy^2 - 3x^2y - xy^2$

37. $9p^3 + p^2 - 3p^3 + p - 4p^2 + 2$

38. $n^4 - 2n^3 + n^2 - 3n^4 + n^3$

39. $4 - (2 + 3m) + 6m + 9$

40. $2a - (5\mathrm{a} - 3) - (7a - 2)$

41. $6 + 3x - (2x + 1) - (2x + 9)$

42. $4y - 8 - (-3 + y) - (11y + 5)$

Perform the indicated operations.

43. $(x^2 - 5y^2 - 9z^2) + (-6x^2 + 9y^2 - 2z^2)$

44. $(7x^2y - 3xy^2 + 4xy) + (-2x^2y - xy^2 + xy)$

45. $(-3x^2 + 2x - 9) - (x^2 + 5x - 4)$

46. $(8y^2 - 4y^3 - 3y) - (3y^2 - 9y - 7y^3)$

47. $(3r^6 + 5) + (-7r^2 + 2r^6 - r^5)$

48. $(5x^{2a} - 3x^a + 2) + (-x^{2a} + 2x^a - 6)$

49. $(-5a^4 + 8a^2 - 9) - (6a^3 - a^2 + 2)$

50. $(3x^{3a} - x^a + 7) - (-2x^{3a} + 5x^{2a} - 1)$

51. $(10xy - 4x^2y^2 - 3y^3) - (-9x^2y^2 + 4y^3 - 7xy)$

52. Subtract $(-4x + 2z^2 + 3m)$ from the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$.

53. Subtract the sum of $(2z^2 - 3x + m)$ and $(z^2 - 2m)$ from $(-4x + 2z^2 + 3m)$.

54. $[2p - (3p - 6)] - \big[\big(5p - (8 - 9p)\big) + 4p\big]$

55. $-[3z^2 + 5z - (2z^2 - 6z)] + \big[\big(8z^2 - (5z - z^2)\big) + 2z^2\big]$

56. $5k - (5k - [2k - (4k - 8k)]) + 11k - (9k - 12k)$

For each pair of functions, find ***a)*** $(f+g)(x)$ *and* ***b)*** $(f-g)(x)$.

57. $f(x) = 5x - 6, \quad g(x) = -2 + 3x$

58. $f(x) = x^2 + 7x - 2, \quad g(x) = 6x + 5$

59. $f(x) = 3x^2 - 5x, \quad g(x) = -5x^2 + 2x + 1$

60. $f(x) = 2x^n - 3x - 1, \quad g(x) = 5x^n + x - 6$

61. $f(x) = 2x^{2n} - 3x^n + 3, \quad g(x) = -8x^{2n} + x^n - 4$

Let $\boldsymbol{P(x) = x^2 - 4}$, $\boldsymbol{Q(x) = 2x + 5}$, *and* $\boldsymbol{R(x) = x - 2}$. *Find each of the following.*

62. $(P+R)(-1)$

63. $(P-Q)(-2)$

64. $(Q-R)(3)$

65. $(R-Q)(0)$

66. $(R-Q)(k)$

67. $(P+Q)(a)$

68. $(Q-R)(a+1)$

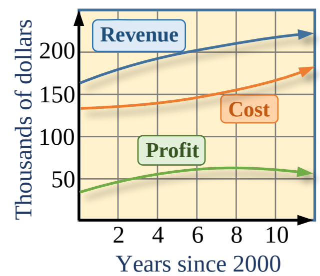
69. $(P+R)(2k)$

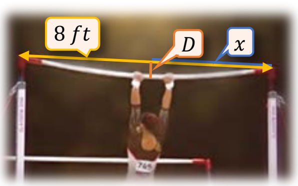
Solve each problem.

70. Suppose that during the years 2000-2012 the revenue R and the cost C of a particular business are modelled by the polynomials

$R(t) = -0.296t^2 + 9.72t + 164$ and $C(t) = 0.154t^2 + 2.15t + 135$,

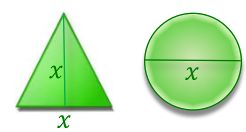
where t represents the number of years since 2000 and both $R(t)$ and $C(t)$ are in thousands of dollars. Write a polynomial that models the profit $P(t)$ of this business during the years 2000-2012.





71. Suppose that the deflection D of an 8 feet-long gymnastic bar can be approximated by the polynomial function $D(x) = 0.037x^4 - 0.59x^3 + 2.35x^2$, where x is the distance in feet from one end of the bar and D is in centimeters. To the nearest tenths of a centimeter, determine the maximum deflection for this bar, assuming that it occurs at the middle of the bar.

72. Write a polynomial that can be used to calculate the sum of areas of a triangle with the base and height of length x and a circle with diameter x. Determine the total area of the two shapes for $x = 5$ centimeters. *Round the answer to the nearest centimeter square.*



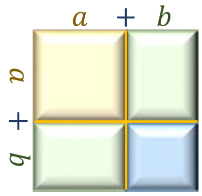
73. Suppose the cost in dollars of sewing n dresses is given by $C(n) = 32n + 1500$. If the dresses can be sold for \$56 each, complete the following.

a. Write a function $R(n)$ that gives the revenue for selling n dresses.

b. Write a formula $P(n)$ for the profit. *Recall that profit is defined as the difference between revenue and cost.*

c. Evaluate $P(100)$ and interpret the answer.

P2 Multiplication of Polynomials



As shown in the previous section, addition and subtraction of polynomials results in another polynomial. This means that the **set of polynomials** is **closed under** the operation of **addition** and **subtraction**. In this section, we will show that the set of polynomials is also closed under the operation of **multiplication**, meaning that a product of polynomials is also a polynomial.

Properties of Exponents

Since multiplication of polynomials involves multiplication of powers, let us review properties of exponents first.

Recall:

power, exponent, base

$$\boldsymbol{a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \textit{ times}}}$$

For example, $x^4 = x \cdot x \cdot x \cdot x$ and we read it "x to the fourth power" or shorter "x to the fourth". If $n = 2$, the power x^2 is customarily read "x squared". If $n = 3$, the power x^3 is often read "x cubed".

Let $a \in \mathbb{R}$, and $m, n \in \mathbb{W}$. The table below shows basic exponential rules with some examples justifying each rule.

Power Rules for Exponents

General Rule	Description	Example
$\boldsymbol{a^m \cdot a^n = a^{m+n}}$	To **multiply powers** of the same bases, keep the base and **add** the **exponents**.	$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x^{2+3} = x^5$
$\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$	To **divide powers** of the same bases, keep the base and **subtract** the **exponents**.	$\frac{x^5}{x^2} = \frac{(x \cdot x \cdot x \cdot \cancel{x} \cdot \cancel{x})}{(\cancel{x} \cdot \cancel{x})} = x^{5-2} = x^3$
$\boldsymbol{(a^m)^n = a^{mn}}$	To raise a **power to a power, multiply the exponents.**	$(x^2)^3 = (x \cdot x)(x \cdot x)(x \cdot x) = x^{2 \cdot 3} = x^6$
$\boldsymbol{(ab)^n = a^n b^n}$	To raise a **product to a power**, raise **each factor to that power**.	$(2x)^3 = 2^3 x^3$
$\boldsymbol{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$	To raise a **quotient to a power**, raise the **numerator and** the **denominator to that power**.	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2}$
$\boldsymbol{a^0 = 1}$ for $\boldsymbol{a \neq 0}$ $\boldsymbol{0^0}$ is undefined	A **nonzero number** raised **to the power of zero** equals **one**.	$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1$

Example 1 ► **Simplifying Exponential Expressions**

Simplify.

a. $(-3xy^2)^4$ **b.** $(5p^3q)(-2pq^2)$

c. $\left(\frac{-2x^5}{x^2y}\right)^3$ **d.** $x^{2a}x^a$

Solution ►

a. To simplify $(-3xy^2)^4$, we apply the fourth power to each factor in the bracket. So,

$$(-3xy^2)^4 = \underbrace{(-3)^4}_{\substack{\text{even power}\\ \text{of a negative}\\ \text{is a positive}}} \cdot x^4 \cdot \underbrace{(y^2)^4}_{\substack{\text{multiply}\\ \text{exponents}}} = \mathbf{3^4x^4y^8}$$

$p = p^1$

b. To simplify $(5p^3q)(-2pq^2)$, we multiply numbers, powers of p, and powers of q. So,

$$(5p^3q)(-2pq^2) = (-2)\cdot 5 \cdot \underbrace{p^3 \cdot p}_{\substack{\text{add}\\ \text{exponents}}} \cdot \underbrace{q \cdot q^2}_{\substack{\text{add}\\ \text{exponents}}} = \mathbf{-10p^4q^3}$$

c. To simplify $\left(\frac{-2x^5}{x^2y}\right)^3$, first we reduce the common factors and then we raise every factor of the numerator and denominator to the third power. So, we obtain

$$\left(\frac{-2x^{\cancel{5}\,3}}{\cancel{x^2}y}\right)^3 = \left(\frac{-2x^3}{y}\right)^3 = \frac{(-2)^3(x^3)^3}{y^3} = \frac{\mathbf{-8x^9}}{\mathbf{y^3}}$$

d. When multiplying powers with the same bases, we add exponents, so $x^{2a}x^a = \mathbf{x^{3a}}$

Multiplication of Polynomials

Multiplication of polynomials involves finding products of monomials. To multiply monomials, we use the commutative property of multiplication and the product rule of powers.

Example 2 ► **Multiplying Monomials**

Find each product.

a. $(3x^4)(5x^3)$ **b.** $(5b)(-2a^2b^3)$ **c.** $-4x^2(3xy)(-x^2y)$

Solution ►

To find the product of monomials, find the following:

- the final **sign**,
- the **number**,
- the **power**.

a. $(3x^4)(5x^3) = 3 \cdot \underbrace{x^4 \cdot 5}_{\substack{\text{commutative}\\ \text{property}}} \cdot x^3 = 3 \cdot 5 \cdot \underbrace{x^4 \cdot x^3}_{\substack{\text{product}\\ \text{rule of powers}}} = \mathbf{15x^7}$

The intermediate steps are not necessary to write. The final answer is immediate if we follow the order: **sign**, **number**, **power** of each variable.

b. $(5b)(-2a^2b^3) = 5(-2)a^2bb^3 = \mathbf{-10a^2b^4}$

c. $-4x^2(3xy)(-x^2y) = \underbrace{(-4)\cdot 3 \cdot (-1)}_{\substack{\text{multiply}\\ \text{coefficients}}} \underbrace{x^2xx^2\,yy}_{\substack{\text{apply product}\\ \text{rule of powers}}} = \mathbf{12x^5y^2}$

To multiply polynomials by a monomial, we use the distributive property of multiplication.

Example 3 ► **Multiplying Polynomials by a Monomial**

Find each product.

a. $-2x(3x^2 - x + 7)$ **b.** $(5b - ab^3)(3ab^2)$

Solution ► **a.** To find the product $-2x(3x^2 - x + 7)$, we distribute the monomial $-2x$ to each term inside the bracket. So, we have

$$-2x(3x^2 - x + 7) = \underbrace{-2x(3x^2) - 2x(-x) - 2x(7)}_{\textit{this step can be done mentally}} = \boldsymbol{-6x^3 + 2x^2 - 14x}$$

b. $$(5b - ab^3)(3ab^2) = \underbrace{5b(3ab^2) - ab^3(3ab^2)}_{\textit{this step can be done mentally}}$$

$$= 15ab^3 - 3a^2b^5 = \boldsymbol{-3a^2b^5 + 15ab^3}$$

arranged in decreasing order of powers

When multiplying polynomials by polynomials we **multiply each term of the first polynomial by each term of the second polynomial**. This process can be illustrated with finding areas of a rectangle whose sides represent each polynomial. For example, we multiply $(2x + 3)(x^2 - 3x + 1)$ as shown below

	x^2	$-3x$	$+1$
$2x$	$2x^3$	$-6x^2$	$2x$
$+3$	$3x^2$	$-9x$	3

So, $$\begin{aligned}(2x + 3)(x^2 - 3x + 1) = 2x^3 &- 6x^2 + 2x \\ &+ 3x^2 - 9x + 3 \\ \hline = \boldsymbol{2x^3} &\boldsymbol{- 3x^2 - 7x + 3}\end{aligned}$$

line up like terms to combine them

Example 4 ► **Multiplying Polynomials by Polynomials**

Find each product.

a. $(3y^2 - 4y - 2)(5y - 7)$ **b.** $4a^2(2a - 3)(3a^2 + a - 1)$

Solution ► **a.** To find the product $(3y^2 - 4y - 2)(5y - 7)$, we can distribute the terms of the second bracket over the first bracket and then collect the like terms. So, we have

$$\begin{aligned}(3y^2 - 4y - 2)(5y - 7) &= 15y^3 - 20y^2 - 10y \\ &\qquad\; - 21y^2 + 28y + 14 \\ \hline &= \boldsymbol{15y^3 - 41y^2 + 18y + 14}\end{aligned}$$

b. To find the product $4a^2(2a - 3)(3a^2 + a - 1)$, we will multiply the two brackets first, and then multiply the resulting product by $4a^2$. So,

$$4a^2(2a-3)(3a^2+a-1) = 4a^2\underbrace{\left(6a^3 \underline{+\,2a^2}\ \underline{\underline{-2a}}\ \underline{-\,9a^2}\ \underline{\underline{-3a}} + 3\right)}_{\textit{collect like terms before removing the bracket}}$$

$$= 4a^2(6a^3 - 7a^2 - 5a + 3) = \mathbf{24a^5 - 28a^4 - 20a^3 + 12a^2}$$

In multiplication of binomials, it might be convenient to keep track of the multiplying terms by following the **FOIL** mnemonic, which stands for multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials. Here is how it works:

First Last

$$(2x-3)(x+5) = 2x^2 \underbrace{+10x - 3x}_{\substack{\textit{the } \textbf{sum} \textit{ of the} \\ \textbf{Outer} \textit{ and } \textbf{Inner} \\ \textit{terms becomes} \\ \textit{the } \textbf{middle} \textit{ term}}} - 15 = \mathbf{2x^2 + 7x - 15}$$

Inner

Outer

Example 5 ► **Using the FOIL Method in Binomial Multiplication**

Find each product.

a. $(x+3)(x-4)$ **b.** $(5x-6)(2x+3)$

Solution ► **a.** To find the product $(x+3)(x-4)$, we may follow the **FOIL** method

x^2 -12

$$(x+3)(x-4) = \mathbf{x^2 - x - 12}$$

$3x$

$-4x$

To find the **linear (middle)** term try to add the inner and outer products mentally.

b. Observe that the linear term of the product $(5x-6)(2x+3)$ is equal to the sum of $-12x$ and $15x$, which is $3x$. So, we have

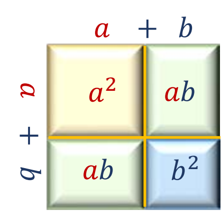
$$(5x-6)(2x+3) = \mathbf{10x^2 + 3x - 18}$$

Special Products

Suppose we want to find the product $(a+b)(a+b)$. This can be done via the FOIL method

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}) = a^2 + ab + ab + b^2 = \boldsymbol{a^2 + 2ab + b^2},$$

or via the geometric visualization:



Observe that since the products of the inner and outer terms are both equal to ab, we can use a shortcut and write the middle term of the final product as $2ab$. We encorage the reader to come up with similar observations regarding the product $(a-b)(a-b)$. This regularity in multiplication of a binomial by itself leads us to the **perfect square formula**:

$$\boldsymbol{(a \pm b)^2 = a^2 \pm 2ab + b^2}$$

In the above notation, the " $\pm$ " sign is used to record two formulas at once, the perfect square of a sum and the perfect square of a difference. It tells us to either use a " $+$ " in both places, or a " $-$ " in both places. The $\boldsymbol{a}$ and $\boldsymbol{b}$ can actually represent any expression. For example, to expand $(2x-y^2)^2$, we can apply the perfect square formula by treating $2x$ as a and y^2 as b. Here is the calculation.

$$(2x-y^2)^2 = (2x)^2 - 2(2x)y^2 + (y^2)^2 = \boldsymbol{2x^2 - 4xy^2 + y^4}$$

Conjugate binomials have the **same first terms** and **opposite second terms**.

Another interesting pattern can be observed when multiplying two **conjugate** brackets, such as $(a+b)$ and $(a-b)$. Using the FOIL method,

$$\boldsymbol{(a+b)(a-b)} = a^2 + \cancel{ab} - \cancel{ab} - b^2 = \boldsymbol{a^2 - b^2},$$

we observe, that the products of the inner and outer terms are opposite. So, they add to zero and the product of conjugate brackets becomes the difference of squares of the two terms. This regularity in multiplication of conjugate brackets leads us to the **difference of squares formula**.

$$\boldsymbol{(a+b)(a-b) = a^2 - b^2}$$

Again, $\boldsymbol{a}$ and $\boldsymbol{b}$ can represent any expression. For example, to find the product $(3x+0.1y^2)(3x-0.1y^2)$, we can apply the difference of squares formula by treating $3x$ as a and $0.1y^2$ as b. Here is the calculation.

$$\boldsymbol{(3x+0.1y^3)(3x-0.1y^3)} = (3x)^2 - (0.1y^3)^2 = \boldsymbol{9x^2 - 0.01y^6}$$

We encourage to use the above formulas whenever applicable, as it allows for more efficient calculations and helps to observe patterns useful in future factoring.

Example 6 ► **Useing Special Product Formulas in Polynomial Multiplication**

Find each product. Apply special products formulas, if applicable.

a. $(5x+3y)^2$ **b.** $(x+y-5)(x+y+5)$

Solution ► **a.** Applying the perfect square formula, we have

$$(5x+3y)^2 = (5x)^2 + 2(5x)3y + (3y)^2 = \boldsymbol{25x^2 + 30xy + 9y^2}$$

b. The product $(x+y-5)(x+y+5)$ can be found by multiplying each term of the first polynomial by each term of the second polynomial, using the distributive property. However, we can find the product $(x+y-5)(x+y+5)$ in a more efficient way by applying the difference of squares formula. Treating the expression $x+y$ as the first term a and the 5 as the second term b in the formula $(a+b)(a-b)=a^2-b^2$, we obtain

$$(x+y-5)(x+y+5)=(x+y)^2-5^2$$
$$=\underbrace{\boldsymbol{x^2+2xy+y^2}}_{\substack{\text{here we apply}\\ \text{the perfect square}\\ \text{formula}}}\ \boldsymbol{-\,25}$$

Caution: *The perfect square formula shows that* $(\boldsymbol{a}+\boldsymbol{b})^{\mathbf{2}} \neq \boldsymbol{a}^{\mathbf{2}}+\boldsymbol{b}^{\mathbf{2}}$.
The difference of squares formula shows that $(\boldsymbol{a}-\boldsymbol{b})^{\mathbf{2}} \neq \boldsymbol{a}^{\mathbf{2}}-\boldsymbol{b}^{\mathbf{2}}$.
More generally, $(\boldsymbol{a}\pm\boldsymbol{b})^{\boldsymbol{n}} \neq \boldsymbol{a}^{\boldsymbol{n}}\pm\boldsymbol{b}^{\boldsymbol{n}}$ *for any natural* $n\neq 1$.

Product Functions

The operation of multiplication can be defined not only for polynomials but also for general functions.

Definition 2.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **product function**, denoted $\boldsymbol{f\cdot g}$ or $\boldsymbol{fg}$, is defined as

$$(\boldsymbol{f\cdot g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x}).$$

The **domain** of the product function is the intersection $\boldsymbol{D_f}\cap\boldsymbol{D_g}$ of the domains of the two functions.

Example 7 ► **Multiplying Polynomial Functions**

Suppose $P(x)=x^2-4x$ and $Q(x)=3x+2$. Find the following:

a. $(PQ)(x)$, $(PQ)(-2)$, and $P(-2)Q(-2)$
b. $(QQ)(x)$ and $(QQ)(1)$
c. $2(PQ)(k)$

Solution ► **a.** Using the definition of the product function, we have

$$(PQ)(x)=P(x)\cdot Q(x)=(x^2-4x)(3x+2)=3x^3+2x^2-12x^2-8x$$
$$=\boldsymbol{3x^3-10x^2-8x}$$

To find $(PQ)(-2)$, we substitute $x=-2$ to the above polynomial function. So,

$$(PQ)(-2)=3(-2)^3-10(-2)^2-8(-2)=3\cdot(-8)-10\cdot 4+16$$
$$=-24-40+16=\boldsymbol{-48}$$

To find $P(-2)Q(-2)$, we calculate
$$P(-2)Q(-2)=\left((-2)^2-4(-2)\right)(3(-2)+2)=(4+8)(-6+2)=12\cdot(-4)$$
$$=\boldsymbol{-48}$$

Observe that both expressions result in the same value. This was to expect, as by the definition, $(PQ)(-2) = P(-2) \cdot Q(-2)$.

b. Using the definition of the product function as well as the perfect square formula, we have

$$(QQ)(x) = Q(x) \cdot Q(x) = [Q(x)]^2 = (3x+2)^2 = \mathbf{9x^2 + 12x + 4}$$

Therefore, $(QQ)(1) = 9 \cdot 1^2 + 12 \cdot 1 + 4 = 9 + 12 + 4 = \mathbf{25}$.

c. Since $(PQ)(x) = 3x^3 - 10x^2 - 8x$, as shown in the solution to *Example 7a*, then $(PQ)(k) = 3k^3 - 10k^2 - 8k$. Therefore,

$$2(PQ)(k) = 2[3k^3 - 10k^2 - 8k] = \mathbf{6k^3 - 20k^2 - 16k}$$

P.2 Exercises

1. Decide whether each expression has been simplified correctly. If not, correct it.

a. $x^2 \cdot x^4 = x^8$ **b.** $-2x^2 = 4x^2$ **c.** $(5x)^3 = 5^3x^3$

d. $-\left(\frac{x}{5}\right)^2 = -\frac{x^2}{25}$ **e.** $(a^2)^3 = a^5$ **f.** $4^5 \cdot 4^2 = 16^7$

g. $\frac{6^5}{3^2} = 2^3$ **h.** $xy^0 = 1$ **i.** $(-x^2y)^3 = -x^6y^3$

Simplify each expression.

2. $3x^2 \cdot 5x^3$ **3.** $-2y^3 \cdot 4y^5$ **4.** $3x^3(-5x^4)$

5. $2x^2y^5(7xy^3)$ **6.** $(6t^4s)(-3t^3s^5)$ **7.** $(-3x^2y)^3$

8. $\frac{12x^3y}{4xy^2}$ **9.** $\frac{15x^5y^2}{-3x^2y^4}$ **10.** $(-2x^5y^3)^2$

11. $\left(\frac{4a^2}{b}\right)^3$ **12.** $\left(\frac{-3m^4}{n^3}\right)^2$ **13.** $\left(\frac{-5p^2q}{pq^4}\right)^3$

14. $3a^2(-5a^5)(-2a)^0$ **15.** $-3a^3b(-4a^2b^4)(ab)^0$ **16.** $\frac{(-2p)^2pq^3}{6p^2q^4}$

17. $\frac{(-8xy)^2y^3}{4x^5y^4}$ **18.** $\left(\frac{-3x^4y^6}{18x^6y^7}\right)^3$ **19.** $((-2x^3y)^2)^3$

20. $((-a^2b^4)^3)^5$ **21.** x^nx^{n-1} **22.** $3a^{2n}a^{1-n}$

23. $(5^a)^{2b}$ **24.** $(-7^{3x})^{4y}$ **25.** $\frac{-12x^{a+1}}{6x^{a-1}}$

26. $\frac{25x^{a+b}}{-5x^{a-b}}$ **27.** $\left(x^{a+b}\right)^{a-b}$ **28.** $(x^2y)^n$

Find each product.

29. $8x^2y^3(-2x^5y)$ **30.** $5a^3b^5(-3a^2b^4)$ **31.** $2x(-3x+5)$

32. $4y(1-6y)$ **33.** $-3x^4y(4x-3y)$ **34.** $-6a^3b(2b+5a)$

35. $5k^2(3k^2-2k+4)$ **36.** $6p^3(2p^2+5p-3)$ **37.** $(x+6)(x-5)$

38. $(x-7)(x+3)$ **39.** $(2x+3)(3x-2)$ **40.** $3p(5p+1)(3p+2)$

41. $2u^2(u-3)(3u+5)$ **42.** $(2t+3)(t^2-4t-2)$ **43.** $(2x-3)(3x^2+x-5)$

44. $(a^2-2b^2)(a^2-3b^2)$ **45.** $(2m^2-n^2)(3m^2-5n^2)$ **46.** $(x+5)(x-5)$

47. $(a+2b)(a-2b)$ **48.** $(x+4)(x+4)$ **49.** $(a-2b)(a-2b)$

50. $(x-4)(x^2+4x+16)$ **51.** $(y+3)(y^2-3y+9)$

52. $(x^2+x-2)(x^2-2x+3)$ **53.** $(2x^2+y^2-2xy)(x^2-2y^2-xy)$

True or False? If it is false, show a counterexample by choosing values for a and b that would not satisfy the equation.

54. $(a+b)^2=a^2+b^2$ **55.** $a^2-b^2=(a-b)(a+b)$ **56.** $(a-b)^2=a^2+b^2$

57. $(a+b)^2=a^2+2ab+b^2$ **58.** $(a-b)^2=a^2+ab+b^2$ **59.** $(a-b)^3=a^3-b^3$

Find each product. Use the ***difference of squares*** *or the* ***perfect square*** *formula, if applicable.*

60. $(2p+3)(2p-3)$ **61.** $(5x-4)(5x+4)$ **62.** $\left(b-\frac{1}{3}\right)\left(b+\frac{1}{3}\right)$

63. $\left(\frac{1}{2}x-3y\right)\left(\frac{1}{2}x+3y\right)$ **64.** $(2xy+5y^3)(2xy-5y^3)$ **65.** $(x^2+7y^3)(x^2-7y^3)$

66. $(1.1x+0.5y)(1.1x-0.5y)$ **67.** $(0.8a+0.2b)(0.8a+0.2b)$ **68.** $(x+6)^2$

69. $(x-3)^2$ **70.** $(4x+3y)^2$ **71.** $(5x-6y)^2$

72. $\left(3a+\frac{1}{2}\right)^2$ **73.** $\left(2n-\frac{1}{3}\right)^2$ **74.** $(a^3b^2-1)^2$

75. $(x^4y^2+3)^2$ **76.** $(3a^2+4b^3)^2$ **77.** $(2x^2-3y^3)^2$

78. $3y(5xy^3+2)(5xy^3-2)$ **79.** $2a(2a^2+5ab)(2a^2+5ab)$ **80.** $3x(x^2y-xy^3)^2$

81. $(-xy+x^2)(xy+x^2)$ **82.** $(4p^2+3pq)(-3pq+4p^2)$ **83.** $(x+1)(x-1)(x^2+1)$

84. $(2x-y)(2x+y)(4x^2+y^2)$ **85.** $(a-b)(a+b)(a^2-b^2)$ **86.** $(a+b+1)(a+b-1)$

87. $(2x+3y-5)(2x+3y+5)$ **88.** $(3m+2n)(3m-2n)(9m^2-4n^2)$

89. $\left((2k-3)+h\right)^2$ **90.** $\left((4x+y)-5\right)^2$

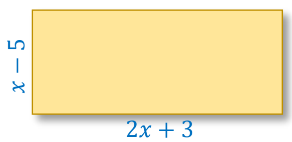
91. $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}+y^{2b}\right)$ **92.** $\left(x^a+y^b\right)\left(x^a-y^b\right)\left(x^{2a}-y^{2b}\right)$

Use the difference of squares formula, $(a+b)(a-b)=a^2-b^2$, to find each product.

93. $101 \cdot 99$ **94.** $198 \cdot 202$ **95.** $505 \cdot 495$

Find the area of each figure. Express it as a polynomial in descending powers of the variable x.

96.



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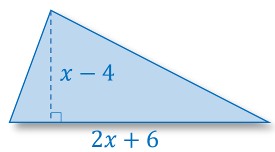

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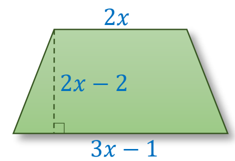


*For each pair of functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, find the **product** function $(\boldsymbol{fg})(\boldsymbol{x})$.*

99. $f(x)=5x-6,\ \ g(x)=-2+3x$

100. $f(x)=x^2+7x-2,\ \ g(x)=6x+5$

101. $f(x)=3x^2-5x,\ \ g(x)=9+x-x^2$

102. $f(x)=x^n-4,\ \ g(x)=x^n+1$

Let $\boldsymbol{P(x)=x^2-4}$, $\boldsymbol{Q(x)=2x}$, and $\boldsymbol{R(x)=x-2}$. Find each of the following.

103. $(PR)(x)$ **104.** $(PQ)(x)$ **105.** $(PQ)(a)$

106. $(PR)(-1)$ **107.** $(PQ)(3)$ **108.** $(PR)(0)$

109. $(QR)(x)$ **110.** $(QR)\left(\frac{1}{2}\right)$ **111.** $(QR)(a+1)$

112. $P(a-1)$ **113.** $P(2a+3)$ **114.** $P(1+h)-P(1)$

Solve each problem.

115. Squares with x centimeters long sides are cut out from each corner of a rectangular piece of cardboard measuring 50 cm by 70 cm. Then the flaps of the remaining cardboard are folded up to construct a box. Find the volume $V(x)$ of the box in terms of the length x.

116. A rectangular flower-bed has a perimeter of 60 meters. If the rectangle is w meters wide, write a polynomial that can be used to determine the area $A(w)$ of the flower-bed in terms of w.

P3 Division of Polynomials

In this section we will discuss dividing polynomials. The result of division of polynomials is not always a polynomial. For example, $x+1$ divided by x becomes

$$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x},$$

which is not a polynomial. Thus, the set of polynomials is not closed under the operation of division. However, we can perform division with remainders, mirroring the algorithm of division of natural numbers. We begin with dividing a polynomial by a monomial and then by another polynomial.

Division of Polynomials by Monomials

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial, and then simplify each quotient. In other words, we use the reverse process of addition of fractions, as illustrated below.

$$\boldsymbol{\frac{a+b}{d} = \frac{a}{d} + \frac{b}{d}}$$

Example 1 ► **Dividing Polynomials by Monomials**

Divide and simplify.

a. $(6x^3 + 15x^2 - 2x) \div (3x)$ **b.** $\frac{xy^2 - 8x^2y + 6x^3y^2}{-2xy^2}$

Solution ►

a. $(6x^3 + 15x^2 - 2x) \div (3x) = \frac{6x^3+15x^2-2x}{3x} = \frac{\overset{2}{\cancel{6}}\overset{2}{\cancel{x^3}}}{\cancel{3}\cancel{x}} + \frac{\overset{5}{\cancel{15}}x^2}{\cancel{3}\cancel{x}} - \frac{2\cancel{x}}{3\cancel{x}} = \mathbf{2x^2 + 5x - \frac{2}{3}}$

b. $\frac{xy^2-8x^2y+6x^3y^2}{-2xy^2} = -\frac{\cancel{xy^2}}{2\cancel{xy^2}} + \frac{\overset{4}{\cancel{8}}x^2y}{\cancel{2}xy^2} - \frac{\overset{3}{\cancel{6}}\overset{2}{\cancel{x^3}}\cancel{y^2}}{\cancel{2}\cancel{x}\cancel{y^2}} = \mathbf{-\frac{1}{2} + \frac{4x}{y} - 3x^2}$

Division of Polynomials by Polynomials

To divide a polynomial by another polynomial, we follow an algorithm similar to the long division algorithm used in arithmetic. For example, observe the steps taken in the long division algorithm when dividing 158 by 13 and the corresponding steps when dividing $x^2 + 5x + 8$ by $x + 3$.

Step 1: Place the dividend under the long division symbol and the divisor in front of this symbol.

$$13\,\overline{)\,158} \qquad\qquad \underbrace{x+3}_{divisor}\,\overline{)\,\underbrace{x^2+5x+8}_{dividend}}$$

Remember: Both polynomials should be written in **decreasing order of powers**. Also, any **missing terms** after the leading term should be displayed with a **zero coefficient**. This will ensure that the terms in each column are of the same degree.

Step 2: Divide the first term of the dividend by the first term of the divisor and record the quotient above the division symbol.

$$\begin{array}{r} 1 \\ \underbrace{13}\,\big)\,\overline{\underbrace{15}8} \end{array} \qquad\qquad \begin{array}{r} \overbrace{\quad x \qquad\qquad}^{quatient} \\ \underbrace{x}+3\,\big)\,\overline{\underbrace{x^2}+5x+8} \end{array}$$

Step 3: Multiply the quotient from *Step 2* by the divisor and write the product under the dividend, lining up the columns with the same degree terms.

$$\begin{array}{r} 1 \\ 13\,\big)\,\overline{158} \\ 13 \end{array} \qquad\qquad \begin{array}{l} \qquad\quad\; x \\ x+3\,\big)\,\overline{x^2+5x+8} \\ \qquad\quad\; x^2+3x \end{array}$$

Step 4: Underline and subtract by adding opposite terms in each column. We suggest to record the new sign in a circle, so that it is clear what is being added.

$$\begin{array}{r} 1 \\ 13\,\big)\,\overline{158} \\ -\underline{13} \\ 2 \end{array} \qquad\qquad \begin{array}{r} x \qquad\quad \\ x+3\,\big)\,\overline{x^2+5x+8} \\ -\underline{(x^2 \overset{\ominus}{+} 3x)} \quad \\ 2x \qquad\quad \end{array}$$

Step 5: Drop the next term (or digit) and repeat the algorithm until the degree of the remainder is lower than the degree of the divisor.

$$\begin{array}{r} 12 \\ 13\,\big)\,\overline{158} \\ -\underline{13}\; \\ 28 \\ -\underline{26} \\ 2 \end{array} \qquad\qquad \begin{array}{r} x \;+\; 2 \qquad \\ x+3\,\big)\,\overline{x^2+5x+8} \\ -\underline{(x^2 \overset{\ominus}{+} 3x)} \qquad \\ 2x+8 \\ -\underline{(2x \overset{\ominus}{+} 6)} \\ 2 \leftarrow remainder \end{array}$$

In the example of long division of numbers, we have $158 = 13 \cdot 12 + 2$.

So, the quotient can be written as $\frac{158}{13} = \mathbf{12} + \frac{\mathbf{2}}{\mathbf{13}}$.

In the example of long division of polynomials, we have

$$x^2 + 5x + 8 = (x + 3) \cdot (x \;\; + \;\; 2) + 2.$$

So, the quotient can be written as $\frac{x^2+5x+8}{x+3} = \boldsymbol{x + 2} + \frac{\mathbf{2}}{\boldsymbol{x+3}}$.

Generally, if P, D, Q, and R are polynomials, such that $P(x) = D(x) \cdot Q(x) + R(x)$, then the ratio of polynomials P and D can be written as

$$\boxed{\frac{\boldsymbol{P(x)}}{\boldsymbol{D(x)}} = \boldsymbol{Q(x)} + \frac{\boldsymbol{R(x)}}{\boldsymbol{D(x)}},}$$

where $Q(x)$ is the quotient polynomial, and $R(x)$ is the remainder from the division of $P(x)$ by the divisor $D(x)$.

Observe: The degree of the remainder must be lower than the degree of the divisor, as otherwise, we could apply the division algorithm one more time.

Example 2 ► **Dividing Polynomials by Polynomials**

Divide.

a. $(3x^3 - 2x^2 + 5) \div (x^2 - 3)$ **b.** $\frac{2p^3+2p+3p^2}{5+2p}$

Solution ► **a.** When writing the polynomials in the long division format, we use a zero placeholder term in place of the missing linear terms in both, the dividend and the divisor. So, we have

$$
\begin{array}{r}
\mathbf{3x \;-\; 2} \\
x^2 + 0x - 3 \;\overline{)\; 3x^3 \;-\; 2x^2 \;+\; 0x + 5} \\
-\,\underline{(\,3x^3 \;+\; 0x^2 \;\overset{\ominus}{+}\; 9x)} \\
-2x^2 \;-\; 9x + 5 \\
-\,\underline{(-2x^2 \;-\; 0x \;\overset{\ominus}{+}\; 6)} \\
-\;\mathbf{9x - 1}
\end{array}
$$

Thus, $(3x^3 - 2x^2 + 5) \div (x^2 - 3) = 3x - 2 + \frac{-9x-1}{x^2-3} = \mathbf{3x - 2 - \frac{9x+1}{x^2-3}}$.

b. To perform this division, we arrange both polynomials in decreasing order of powers, and replace the constant term in the dividend with a zero. So, we have

$$
\begin{array}{r}
\boldsymbol{p^2 - \;p \;+\; \frac{7}{2}} \\
2p + 5 \;\overline{)\; 2p^3 + \;3p^2 + 2p \;+\; 0} \\
-\,\underline{(\,2p^3 \overset{\ominus}{+} 5p^2)} \\
-2p^2 + 2p \\
-\,\underline{(-2p^2 \overset{\oplus}{-} 5p)} \\
7p + 0 \\
-\,\underline{\left(7p \overset{\ominus}{+} \frac{35}{2}\right)} \\
-\,\boldsymbol{\frac{35}{2}}
\end{array}
$$

Thus, $\frac{2p^3+2p+3p^2}{5+2p} = p^2 - p + \frac{7}{2} + \frac{-\frac{35}{2}}{2p+5} = \boldsymbol{p^2 - p + \frac{7}{2} - \frac{35}{4p+10}}$.

Observe in the above answer that $\frac{-\frac{35}{2}}{2p+5}$ is written in a simpler form, $-\frac{35}{4p+10}$. This is because $\frac{-\frac{35}{2}}{2p+5} = -\frac{35}{2} \cdot \frac{1}{2p+5} = -\frac{35}{4p+10}$.

Quotient Functions

Similarly as in the case of polynomials, we can define quotients of functions.

Definition 3.1 ► Suppose f and g are functions of x with the corresponding domains D_f and D_g.

Then the **quotient function**, denoted $\boldsymbol{\frac{f}{g}}$, is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The **domain** of the quotient function is the intersection of the domains of the two functions, D_f and D_g, excluding the x-values for which $g(x) = 0$. So,

$$D_{\frac{f}{g}} = D_f \cap D_g \setminus \{x | g(x) = 0\}$$

Example 3 ► **Dividing Polynomial Functions**

Suppose $P(x) = 2x^2 - x - 6$ and $Q(x) = x - 2$. Find the following:

a. $\left(\frac{P}{Q}\right)(x)$ and $\left(\frac{P}{Q}\right)(2a)$,

b. $\left(\frac{P}{Q}\right)(-3)$ and $\left(\frac{P}{Q}\right)(2)$,

c. domain of $\frac{P}{Q}$.

Notice that this equation holds only for $x \neq 2$.

Solution ►

a. By *Definition 3.1*, $\left(\frac{P}{Q}\right)(x) = \frac{P(x)}{Q(x)} = \frac{2x^2-x-6}{x-2} = \frac{(2x+3)(x-2)}{x-2} = \mathbf{2x + 3}$

So, $\left(\frac{P}{Q}\right)(-3) = 2(-3) + 3 = \mathbf{-3}$. One can verify that the same value is found by evaluating $\frac{P(-3)}{Q(-3)}$.

b. Since the equation $\frac{(2x+3)(x-2)}{x-2} = 2x + 3$ is true only for $x \neq 2$, the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ cannot be used to evaluate $\left(\frac{P}{Q}\right)(2)$. However, by *Definition 3.1*, we have

$\left(\frac{P}{Q}\right)(2)$ is undefined, so 2 is not in the domain of $\frac{P}{Q}$

$$\left(\frac{P}{Q}\right)(2) = \frac{P(2)}{Q(2)} = \frac{2(2)^2 - (2) - 6}{(2) - 2} = \frac{8 - 2 - 6}{0} = \frac{0}{0} = \boldsymbol{undefined}$$

To evaluate $\left(\frac{P}{Q}\right)(2a)$, we first notice that if $a \neq 1$, then $2a \neq 2$. So, we can use the simplified formula $\left(\frac{P}{Q}\right)(x) = 2x + 3$ and evaluate $\left(\frac{P}{Q}\right)(2a) = 2(2a) + 3 = \mathbf{4a + 3}$.

c. The domain of any polynomial is the set of all real numbers. So, the domain of $\frac{P}{Q}$ is the set of all real numbers except for the x-values for which the denominator $Q(x) =$

$x - 2$ is equal to zero. Since the solution to the equation $x - 2 = 0$ is $x = 2$, then the value 2 must be excluded from the set of all real numbers. Therefore, $\boldsymbol{D}_{\frac{P}{Q}} = \mathbb{R} \setminus \{\mathbf{2}\}$.

P.3 Exercises

1. *True or False?* The quotient in a division of a six-degree polynomial by a second-degree polynomial is a third-degree polynomial. Justify your answer.

2. *True or False?* The remainder in a division of a polynomial by a second-degree polynomial is a first-degree polynomial. Justify your answer.

Divide.

3. $\frac{20x^3-15x^2+5x}{5x}$
4. $\frac{27y^4+18y^2-9y}{9y}$
5. $\frac{8x^2y^2-24xy}{4xy}$
6. $\frac{5c^3d+10c^2d^2-15cd^3}{5cd}$
7. $\frac{9a^5-15a^4+12a^3}{-3a^2}$
8. $\frac{20x^3y^2+44x^2y^3-24x^2y}{-4x^2y}$
9. $\frac{64x^3-72x^2+12x}{8x^3}$
10. $\frac{4m^2n^2-21mn^3+18mn^2}{14m^2n^3}$
11. $\frac{12ab^2c+10a^2bc+18abc^2}{6a^2bc}$

Divide.

12. $(x^2 + 3x - 18) \div (x + 6)$
13. $(3y^2 + 17y + 10) \div (3y + 2)$
14. $(x^2 - 11x + 16) \div (x + 8)$
15. $(t^2 - 7t - 9) \div (t - 3)$
16. $\frac{6y^3-y^2-10}{3y+4}$
17. $\frac{4a^3+6a^2+14}{2a+4}$
18. $\frac{4x^3+8x^2-11x+3}{4x+1}$
19. $\frac{10z^3-26z^2+17z-13}{5z-3}$
20. $\frac{2x^3+4x^2-x+2}{x^2+2x-1}$
21. $\frac{3x^3-2x^2+5x-4}{x^2-x+3}$
22. $\frac{4k^4+6k^3+3k-1}{2k^2+1}$
23. $\frac{9k^4+12k^3-4k-1}{3k^2-1}$
24. $\frac{2p^3+7p^2+9p+3}{2p+2}$
25. $\frac{5t^2+19t+7}{4t+12}$
26. $\frac{x^4-4x^3+5x^2-3x+2}{x^2+3}$
27. $\frac{p^3-1}{p-1}$
28. $\frac{x^3+1}{x+1}$
29. $\frac{y^4+16}{y+2}$
30. $\frac{x^5-32}{x-2}$

For each pair of polynomials, $P(x)$ and $D(x)$, find such polynomials $Q(x)$ and $R(x)$ that $\boldsymbol{P(x) = Q(x) \cdot D(x) + R(x)}$.

31. $P(x) = 4x^3 - 4x^2 + 13x - 2$ and $D(x) = 2x - 1$
32. $P(x) = 3x^3 - 2x^2 + 3x - 5$ and $D(x) = 3x - 2$

*For each pair of functions, $\boldsymbol{f}$ and $\boldsymbol{g}$, find the quotient function $\left(\frac{f}{g}\right)(\boldsymbol{x})$ and state its **domain**.*

33. $f(x) = 6x^2 - 4x, \;\; g(x) = 2x$

34. $f(x) = 6x^2 + 9x, \;\; g(x) = -3x$

35. $f(x) = x^2 - 36, \;\; g(x) = x + 6$

36. $f(x) = x^2 - 25, \;\; g(x) = x - 5$

37. $f(x) = 2x^2 - x - 3, \;\; g(x) = 2x - 3$

38. $f(x) = 3x^2 + x - 4, \;\; g(x) = 3x + 4$

39. $f(x) = 8x^3 + 125, \;\; g(x) = 2x + 5$

40. $f(x) = 64x^3 - 27, \;\; g(x) = 4x - 3$

Let $\boldsymbol{P(x) = x^2 - 4}$, $\boldsymbol{Q(x) = 2x}$, *and* $\boldsymbol{R(x) = x - 2}$. *Find each of the following. If the value can't be evaluated, say DNE (does not exist).*

41. $\left(\frac{R}{Q}\right)(x)$

42. $\left(\frac{P}{R}\right)(x)$

43. $\left(\frac{R}{P}\right)(x)$

44. $\left(\frac{R}{Q}\right)(2)$

45. $\left(\frac{R}{Q}\right)(0)$

46. $\left(\frac{P}{R}\right)(3)$

47. $\left(\frac{R}{P}\right)(-2)$

48. $\left(\frac{R}{P}\right)(2)$

49. $\left(\frac{P}{R}\right)(a)$, for $a \neq 2$

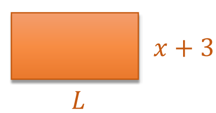
50. $\left(\frac{R}{Q}\right)\left(\frac{3}{2}\right)$

51. $\frac{1}{2}\left(\frac{Q}{R}\right)(x)$

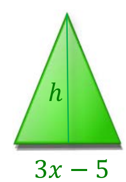
52. $\left(\frac{Q}{R}\right)(a - 1)$

Solve each problem.

53. The area A of a rectangle is $3x^2 + 7x - 6$ and its width W is $x + 3$.
a. Find a polynomial that represents the length L of the rectangle.
b. Find the length of the rectangle if the width is 7 meters.

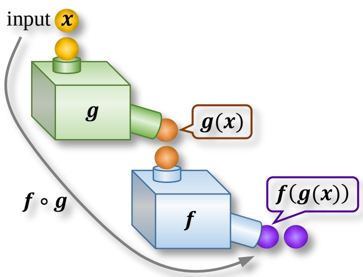


54. The area A of a triangle is $6x^2 - 13x + 5$. Find the height h of the triangle whose base is $3x - 5$. What is the height of such a triangle if its base is 7 centimeters?



P4 Composition of Functions and Graphs of Basic Polynomial Functions

In the last three sections, we have created new functions with the use of function operations such as addition, subtraction, multiplication, and division. In this section, we will introduce one more function operation, called **composition** of functions. This operation will allow us to represent situations in which some quantity depends on a variable that, in turn, depends on another variable. For instance, the number of employees hired by a firm may depend on the firm's profit, which may in turn depend on the number of items the firm produces. In this situation, we might be interested in the number of employees hired by a firm as a function of the number of items the firm produces. This illustrates a **composite function**. It is obtained by composing the number of employees with respect to the profit and the profit with respect to the number of items produced.



In the second part of this section, we will examine graphs of basic polynomial functions, such as constant, linear, quadratic, and cubic functions.

Composition of Functions

Consider women's shoe size scales in U.S., Italy, and Great Britain.

U.S.		Italy		Britain
4		32		2
5	$\xrightarrow{g}$	34	$\xrightarrow{f}$	3
6		36		4
7		38		5
8		40		6

h

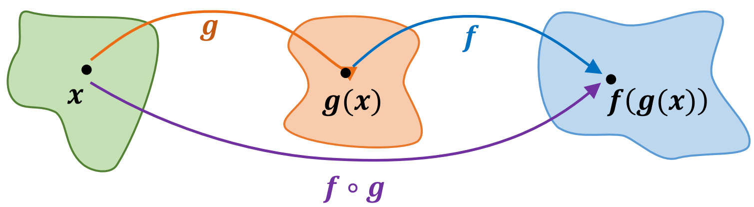
The reader is encouraged to confirm that the function $g(x) = 2x + 24$ gives the women's shoe size in Italy for any given U.S. shoe size x. For example, a U.S. shoe size 7 corresponds to a shoe size of $g(7) = 2 \cdot 7 + 24 = 38$ in Italy. Similarly, the function $f(x) = \frac{1}{2}x - 14$ gives the women's shoe size in Britain for any given Italian shoe size x. For example, an Italian shoe size 38 corresponds to a shoe size of $f(38) = \frac{1}{2} \cdot 38 - 14 = 5$ in Italy. So, by converting the U.S. shoe size first to the corresponding Italian size and then to the Great Britain size, we create a third function, h, that converts the U.S. shoe size directly to the Great Britain size. We say, that function h is the **composition of functions $\boldsymbol{g}$** and $\boldsymbol{f}$. Can we find a formula for this composite function? By observing the corresponding data for U.S. and Great Britain shoe sizes, we can conclude that $h(x) = x - 2$. This formula can also be derived with the use of algebra.

Since the U.S. shoe size x corresponds to the Italian shoe size $g(x)$, which in turn corresponds to the Britain shoe size $f(g(x))$, the composition function $h(x)$ is given by the formula

$$\boldsymbol{h(x)} = \boldsymbol{f(g(x))} = f(2x + 24) = \frac{1}{2}(2x + 24) - 14 = x + 12 - 14 = \boldsymbol{x - 2},$$

which confirms our earlier observation.

Generally, the composition of functions f and g, denoted $f \circ g$, is the function that acts on the input x by mapping it by the function g to the value $g(x)$, which in turn becomes the input for function f, to be mapped to the value $f(g(x))$. See the diagram below.



Definition 4.1 ► If f and g are functions, then the **composite function** $\boldsymbol{f \circ g}$, or **composition** of f and g, is defined by

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})),$$

for all x in the domain of g, such that $g(x)$ is in the domain of f.

Note: We read $f(g(x))$ as "f of g of x".

Example 1 ► **Evaluating a Composite Function**

Suppose $f(x) = x^2$ and $g(x) = 2x + 3$. Find the following:

a. $(f \circ g)(-2)$ **b.** $(g \circ f)(-2)$

Solution ►

a. $(\boldsymbol{f} \circ \boldsymbol{g})(-2) = f(g(-2)) = f(\underbrace{2(-2) + 3}_{g(-2)}) = f(-1) = \underbrace{(-1)^2}_{f(-1)} = \mathbf{1}$

b. $(\boldsymbol{g} \circ \boldsymbol{f})(-2) = g(f(-2)) = g(\underbrace{(-2)^2}_{f(-2)}) = g(4) = \underbrace{2(4) + 3}_{g(4)} = \mathbf{11}$

Observation: *In the above example, $(f \circ g)(-2) \neq (g \circ f)(-2)$. This shows that **composition of functions** is **not commutative**.*

Example 2 ► **Finding the Composition of Two Functions**

Given functions f and g, find $(f \circ g)(x)$.

a. $f(x) = 2 - 3x$; $g(x) = x^2 - 2x$ **b.** $f(x) = x^2 - x + 3$; $g(x) = x + 4$

Solution ►

a. $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = f(g(x)) = f(\underbrace{x^2 - 2x}_{g(x)}) = 2 - 3(\underbrace{x^2 - 2x}_{g(x)}) = \boldsymbol{-3x^2 + 6x + 2}$

b. $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = f(g(x)) = f(\underbrace{x+4}_{g(x)}) = \underbrace{(x+4)^2 - (x+4) + 3}_{f(x+4)}$

$$= x^2 + 8x + 16 - x - 4 + 3 = \boldsymbol{x^2 + 7x + 15}$$

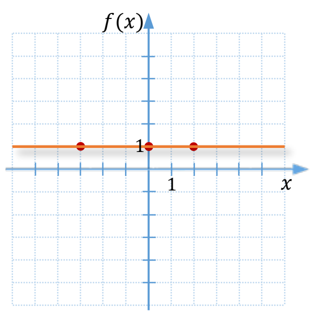
Attention! *Do not confuse the **composition** of functions, $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$, with the **multiplication** of functions, $(\boldsymbol{fg})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})$.*

Graphs of Basic Polynomial Functions

Since polynomials are functions, they can be evaluated for different x-values and graphed in a system of coordinates. How do polynomial functions look like? Below, we graph several basic polynomial functions up to the third degree, and observe their shape, domain, and range.

CONSTANT

Let us start with a **constant function**, which is defined by a zero degree polynomial, such as $f(x) = 1$. In this example, for any real x-value, the corresponding y-value is constantly equal to 1. So, the graph of this function is a **horizontal line** with the y-intercept at 1.

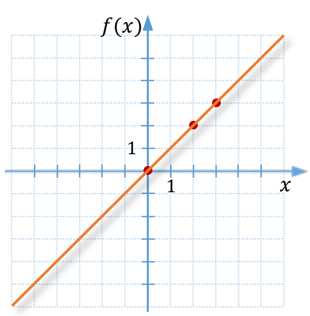


Domain: $\mathbb{R}$
Range: $\{1\}$

Generally, the graph of a **constant function**, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}$, is a horizontal line with the y-intercept at c. The domain of this function is $\mathbb{R}$ and the range is $\{c\}$.

LINEAR

The basic first degree polynomial function is the **identity function** given by the formula $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$. Since both coordinates of any point satisfying this equation are the same, the graph of the identity function is the diagonal line, as shown below.



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of any first degree polynomial function, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{mx} + \boldsymbol{b}$ with $m \neq 0$, is a slanted line. So, the domain and range of such function is $\mathbb{R}$.

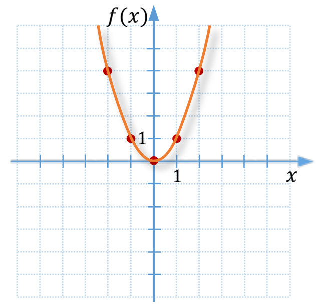
QUADRATIC

The basic second degree polynomial function is the **squaring function** given by the formula $f(x) = x^2$. The shape of the graph of this function is refered to as the **basic parabola**. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

vertex

symmetry about the y-axis



Domain: $\mathbb{R}$
Range: $[0, \infty)$

Generally, the graph of any second degree polynomial function, $f(x) = ax^2 + bx + c$ with $a \neq 0$, is a **parabola**. The domain of such function is $\mathbb{R}$ and the range depends on how the parabola is directed, with arms up or down.

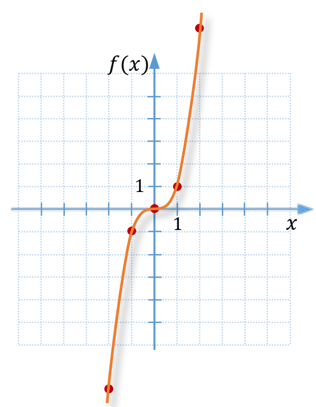
CUBIC

The basic third degree polynomial function is the **cubic function**, given by the formula $f(x) = x^3$. The graph of this function has a shape of a 'snake'. The reader is encouraged to observe the relations between the five points calculated in the table of values below.

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8

center

symmetry about the origin



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Generally, the graph of a third degree polynomial function, $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$, has a shape of a 'snake' with different size waves in the middle. The domain and range of such function is $\mathbb{R}$.

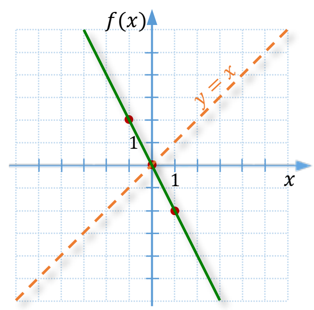
Example 3 ► Graphing Polynomial Functions

Graph each function using a table of values. Give the domain and range of each function by observing its graph. Then, on the same grid, graph the corresponding basic polynomial function. Observe and name the transformation(s) that can be applied to the basic shape in order to obtain the desired function.

a. $f(x) = -2x$ **b.** $f(x) = (x+2)^2$ **c.** $f(x) = x^3 - 2$

Solution ► **a.** The graph of $f(x) = -2x$ is a line passing through the origin and falling from left to right, as shown below in green.

x	$f(x) = -2x$
-1	2
0	0
1	-2



Domain of f: $\mathbb{R}$
Range of f: $\mathbb{R}$

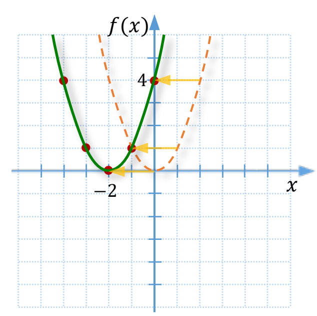
Observe that to obtain the green line, we multiply y-coordinates of the orange line by a factor of -2. Such a transformation is called a **dilation** in the **y-axis** by a factor of **-2**. This dilation can also be achieved by applying a **symmetry in the x-axis** first, and then **stretching** the resulting graph **in the y-axis** by a factor of **2**.

b. The graph of $f(x) = (x+2)^2$ is a parabola with a vertex at $(-2, 0)$, and its arms are directed upwards as shown below in green.

x	$f(x) = (x+2)^2$
-4	4
-3	1
-2	0
-1	1
0	4

vertex

symmetry about the $x = -2$



Domain: $\mathbb{R}$
Range: $[0, \infty)$

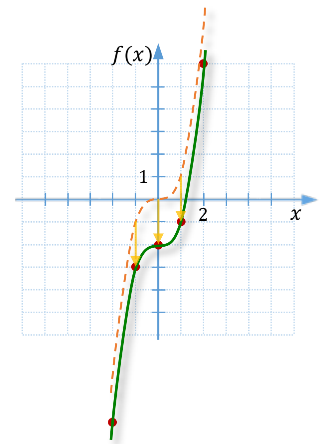
Observe that to obtain the green shape, it is enough to move the graph of the **basic parabola** by two units to the left. This transformation is called a **horizontal translation** by two units to the left. The translation to the left reflects the fact that the vertex of the parabola $f(x) = (x+2)^2$ is located at $x + 2 = 0$, which is equivalent to $x = -2$.

c. The graph of $f(x) = x^3 - 2$ has the shape of a basic cubic function with a center at $(0, -2)$.

$\boldsymbol{x}$	$\boldsymbol{f(x) = x^3 - 2}$
-2	-10
-1	-3
0	-2
1	-1
2	6

center

symmetry about $(0, -2)$



Domain: $\mathbb{R}$
Range: $\mathbb{R}$

Observe that the green graph can be obtained by shifting the graph of the **basic cubic function** by two units down. This transformation is called a **vertical translation** by two units down.

P.4 Exercises

Suppose $f(x) = x^2 + 2$, $g(x) = 5 - x$, *and* $h(x) = 2x - 3$. *Find the following values or expressions.*

1. $(f \circ g)(1)$ **2.** $(g \circ f)(1)$ **3.** $(f \circ g)(x)$ **4.** $(g \circ f)(x)$

5. $(f \circ h)(-1)$ **6.** $(h \circ f)(-1)$ **7.** $(f \circ h)(x)$ **8.** $(h \circ f)(x)$

9. $(h \circ g)(-2)$ **10.** $(g \circ h)(-2)$ **11.** $(h \circ g)(x)$ **12.** $(g \circ h)(x)$

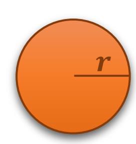
13. $(f \circ f)(2)$ **14.** $(f \circ h)\left(\frac{1}{2}\right)$ **15.** $(h \circ h)(x)$ **16.** $(g \circ g)(x)$

Solve each problem.

17. The function $f(x) = 12x$ can be used to calculate the number of inches in x feet, and the function $g(x) = 2.54x$ can be used to calculate the number of centimeters in x inches. Find the function $(g \circ f)(x)$ and suggest its application.

18. If sandwiches are sold for \$3.50 each, then $C(n) = 3.5n$ can be used to calculate the pre-tax cost in dollars of n sandwiches. If 12% of sales tax needs to be applied to the pre-tax cost x, we can use the function $T(x) = 1.12x$ to calculate the overall cost in dollars. Determine the function that calculates the overall cost of n sandwiches.

19. The circumference C of a circle with the radius r is given by the formula $C = 2\pi r$.



a. Solve the above formula for r in terms of C.
b. If A represents the area of the circle, write A as a function of the circumference C.
c. Use the function found in part (**b**) to find the area of a circle with a circumference of 6π.

20. After a collision at 10:00 a.m., a tanker began spilling oil into the ocean. The spill created a circular oil slick. It was observed that the radius r of the circular slick could be represented by the function $r(t) = 3t$, where t is the elapsed time, in hours, since the collision occurred, and r is in kilometers. Let $A(t) = \pi r^2$ represent the area of a circle with radius r. Find and interpret the composed function $(A \circ r)(t)$.

21. A clothing store announced a Christmas in July sale which offers 15% off for all items, even the ones already on sale. On the day of this sale, Julia bought a dress that was previously marked 30% down. Does this mean that Julia received an overall 45% discount on the dress? Which of the function operations can be applied to determine the final price of the dress and how? What was the final percent discount applied to the price of this dress?

***Graph** each function and state its **domain** and **range**.*

22. $f(x) = -2x + 3$

23. $f(x) = 3x - 4$

24. $f(x) = -x^2 + 4$

25. $f(x) = x^2 - 2$

26. $f(x) = \frac{1}{2}x^2$

27. $f(x) = -2x^2 + 1$

28. $f(x) = (x + 1)^2 - 2$

29. $f(x) = -x^3 + 1$

30. $f(x) = (x - 3)^3$

*Guess the **transformations** needed to apply to the graph of a basic parabola $f(x) = x^2$ to obtain the graph of the given function $g(x)$. Then **graph** both $f(x)$ and $g(x)$ on the same grid and confirm the the original guess.*

31. $g(x) = -x^2$

32. $g(x) = x^2 - 3$

33. $g(x) = x^2 + 2$

34. $g(x) = (x + 2)^2$

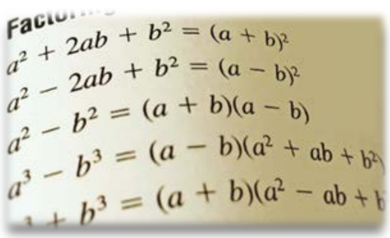
35. $g(x) = (x - 3)^2$

36. $g(x) = (x + 2)^2 - 1$

Attributions

p.**187** Roller Coaster in a Park by Priscilla Du Preez / Unsplash Licence

Factoring



Factoring is the reverse process of multiplication. Factoring polynomials in algebra has similar role as factoring numbers in arithmetic. Any number can be expressed as a product of prime numbers. For example, $6 = 2 \cdot 3$. Similarly, any polynomial can be expressed as a product of **prime** polynomials, which are polynomials that cannot be factored any further. For example, $x^2 + 5x + 6 = (x + 2)(x + 3)$. Just as factoring numbers helps in simplifying or adding fractions, factoring polynomials is very useful in simplifying or adding algebraic fractions. In addition, it helps identify zeros of polynomials, which in turn allows for solving higher degree polynomial equations.

In this chapter, we will examine the most commonly used factoring strategies with particular attention to special factoring. Then, we will apply these strategies in solving polynomial equations.

F1 Greatest Common Factor and Factoring by Grouping

Prime Factors

When working with integers, we are often interested in their factors, particularly prime factors. Likewise, we might be interested in factors of polynomials.

Definition 1.1 ► To **factor** a polynomial means to write the polynomial as a **product** of 'simpler' polynomials. For example,

$$5x + 10 = 5(x + 2), \quad \text{or} \quad x^2 - 9 = (x + 3)(x - 3).$$

In the above definition, 'simpler' means polynomials of **lower degrees** or polynomials with coefficients that **do not contain common factors** other than 1 or -1. If possible, we would like to see the polynomial factors, other than monomials, having **integral coefficients** and a **positive leading term**.

When is a polynomial factorization complete?

In the case of natural numbers, the complete factorization means a factorization into prime numbers, which are numbers divisible only by their own selves and 1. We would expect that similar situation is possible for polynomials. So, which polynomials should we consider as prime?

Observe that a polynomial such as $-4x + 12$ can be written as a product in many different ways, for instance

$$-(4x + 12), \quad 2(-2x + 6), \quad 4(-x + 3), \quad -4(x - 3), \quad -12\left(\frac{1}{3}x + 1\right), \text{ etc.}$$

Since the terms of $4x + 12$ and $-2x + 6$ still contain common factors different than 1 or -1, these polynomials are not considered to be factored completely, which means that they should not be called prime. The next two factorizations, $4(-x + 3)$ and $-4(x - 3)$ are both complete, so both polynomials $-x + 3$ and $x - 3$ should be considered as prime. But what about the last factorization, $-12\left(\frac{1}{3}x + 1\right)$? Since the remaining binomial $\frac{1}{3}x + 1$ does not have integral coefficients, such a factorization is not always desirable.

Here are some examples of **prime polynomials**:

- any monomials such as $-2x^2$, πr^2, or $\frac{1}{3}xy$;
- any linear polynomials with integral coefficients that have no common factors other than 1 or -1, such as $x-1$ or $2x+5$;
- some quadratic polynomials with integral coefficients that cannot be factored into any lower degree polynomials with integral coefficients, such as x^2+1 or x^2+x+1.

For the purposes of this course, we will assume the following definition of a prime polynomial.

Definition 1.2 ► A polynomial with integral coefficients is called **prime** if one of the following conditions is true
- it is a **monomial**, or
- the only **common factors** of its terms are **1 or $-$1** and it **cannot be factored into any lower degree polynomials** with integral coefficients.

Definition 1.3 ► A **factorization** of a polynomial with integral coefficients is **complete** if all of its **factors** are **prime**.

Here is an example of a polynomial factored completely:

$$-6x^3-10x^2+4x=-2x(3x-1)(x+2)$$

In the next few sections, we will study several factoring strategies that will be helpful in finding complete factorizations of various polynomials.

Greatest Common Factor

The first strategy of factoring is to factor out the **greatest common factor** (**GCF**).

Definition 1.4 ► The **greatest common factor** (**GCF**) of two or more terms is the largest expression that is a factor of all these terms.

In the above definition, the "largest expression" refers to the expression with the most factors, disregarding their signs.

To find the greatest common factor, we take the product of the least powers of each type of common factor out of all the terms. For example, suppose we wish to find the GCF of the terms

$$6x^2y^3,\ -18x^5y, \text{ and } 24x^4y^2.$$

First, we look for the GCF of 6, 18, and 24, which is 6. Then, we take the lowest power out of x^2, x^5, and x^4, which is x^2. Finally, we take the lowest power out of y^3, y, and y^2, which is y. Therefore,

$$\mathrm{GCF}(6x^2y^3,\ -18x^5y,\ 24x^4y^2)=6x^2y$$

This GCF can be used to factor the polynomial $6x^2y^3-18x^5y+24x^4y^2$ by first seeing it as

$$6x^2y\cdot y^2-6x^2y\cdot 3x^3+6x^2y\cdot 4x^2y,$$

and then, using the **reverse distributing property**, 'pulling' the $6x^2y$ out of the bracket to obtain

$$6x^2y(y^2 - 3x^3 + 4x^2y).$$

Note 1: Notice that since 1 and -1 are factors of any expression, the GCF is defined up to the sign. Usually, we choose the positive GCF, but sometimes it may be convenient to choose the negative GCF. For example, we can claim that

$$\mathrm{GCF}(-2x, -4y) = 2 \quad \text{or} \quad \mathrm{GCF}(-2x, -4y) = -2,$$

depending on what expression we wish to leave after factoring the GCF out:

$$-2x - 4y = \underbrace{2}_{\substack{positive\\GCF}} \underbrace{(-x - 2y)}_{\substack{negative\\leading\\term}} \text{ or } -2x - 4y = \underbrace{-2}_{\substack{negative\\GCF}} \underbrace{(x + 2y)}_{\substack{positive\\leading\\term}}$$

Note 2: If the GCF of the terms of a polynomial is equal to 1, we often say that these terms do not have any common factors. What we actually mean is that the terms do not have a common factor other than 1, as factoring 1 out does not help in breaking the original polynomial into a product of simpler polynomials. See *Definition 1.1*.

Example 1 ► **Finding the Greatest Common Factor**

Find the greatest common factor for the given expressions.

a. $6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2$ **b.** $4\pi(y-x),\ 8\pi(x-y)$

c. $ab^2,\ a^2b,\ b,\ a$ **d.** $3x^{-1}y^{-3},\ x^{-2}y^{-2}z$

Solution ►

a. Since $\mathrm{GCF}(6,3,9) = 3$, the lowest power out of x^4, x^3, and x is x, and the lowest power out of $(x+1)^3$, $(x+1)$, and $(x+1)^2$ is $(x+1)$, then

$$\mathrm{GCF}(6x^4(x+1)^3,\ 3x^3(x+1),\ 9x(x+1)^2) = \mathbf{3x(x+1)}$$

b. Since $y - x$ is opposite to $x - y$, then $y - x$ can be written as $-(x - y)$. So 4, π, and $(x - y)$ is common for both expressions. Thus,

$$\mathrm{GCF}\big(4\pi(y-x),\ 8\pi(x-y)\big) = \mathbf{4\pi(x-y)}$$

Note: The greatest common factor is unique up to its sign. Notice that in the above example, we could write $x - y$ as $-(y - x)$ and choose the GCF to be $4\pi(y - x)$.

c. The terms $ab^2,\ a^2b,\ b,$ and a have no common factor other than 1, so

$$\mathrm{GCF}(ab^2,\ a^2b,\ b,\ a) = \mathbf{1}$$

d. The lowest power out of x^{-1} and x^{-2} is x^{-2}, and the lowest power out of y^{-3} and y^{-2} is y^{-3}. Therefore,

$$\mathrm{GCF}(3x^{-1}y^{-3},\ x^{-2}y^{-2}z) = \boldsymbol{x^{-2}y^{-3}}$$

Example 2 ► **Factoring out the Greatest Common Factor**

Factor each expression by taking the greatest common factor out. Simplify the factors, if possible.

a. $54x^2y^2 + 60xy^3$

b. $ab - a^2b(a-1)$

c. $-x(x-5) + x^2(5-x) - (x-5)^2$

d. $x^{-1} + 2x^{-2} - x^{-3}$

Solution ► **a.** To find the greatest common factor of 54 and 60, we can use the method of dividing by any common factor, as presented below.

all common factors are listed in this column

$$\begin{array}{r|ll} 2 & 54, & 60 \\ 3 & 27, & 30 \\ \hline & 9, & 10 \end{array}$$

no more common factors for 9 and 10

So, $\mathrm{GCF}(54, 60) = 2 \cdot 3 = 6$.

Since $\mathrm{GCF}(54x^2y^2,\ 60xy^3) = 6xy^2$, we factor the $6xy^2$ out by dividing each term of the polynomial $54x^2y^2 + 60xy^3$ by $6xy^2$, as below.

$$\begin{aligned} &54x^2y^2 + 60xy^3 \\ &= \boldsymbol{6xy^2(9x + 10y)} \end{aligned}$$

$\frac{54x^2y^2}{6xy^2} = 9x$

$\frac{60xy^3}{6xy^2} = 10y$

Note: Since factoring is the reverse process of multiplication, it can be checked by finding the product of the factors. If the product gives us the original polynomial, the factorization is correct.

b. First, notice that the polynomial has two terms, ab and $-a^2b(a-1)$. The greatest common factor for these two terms is ab, so we have

$$ab - a^2b(a-1) = ab(\mathbf{1} - a(a-1))$$

remember to leave **1** for the first term

$$= ab(1 - a^2 + a)$$

simplify and arrange in decreasing powers

$$= ab(-a^2 + a + 1)$$

$$= \boldsymbol{-ab(a^2 - a - 1)}$$

take the "–" out

Note: Both factorizations, $ab(-a^2+a+1)$ and $-ab(a^2-a-1)$ are correct. However, we customarily leave the polynomial in the bracket with a positive leading coefficient.

c. Observe that if we write the middle term $x^2(5-x)$ as $-x^2(x-5)$ by factoring the negative out of the $(5-x)$, then $(5-x)$ is the common factor of all the terms of the equivalent polynomial

$$-x(x-5)-x^2(x-5)-(x-5)^2.$$

Then notice that if we take $-(x-5)$ as the GCF, then the leading term of the remaining polynomial will be positive. So, we factor

$$-x(x-5)+x^2(5-x)-(x-5)^2$$
$$=-x(x-5)-x^2(x-5)-(x-5)^2$$
$$=-(x-5)\big(x+x^2+(x-5)\big)$$

simplify and arrange in decreasing powers

$$=-(x-5)(\mathbf{x^2+2x-5})$$

d. The $\text{GCF}(x^{-1}, 2x^{-2}, -x^{-3})=x^{-3}$, as -3 is the lowest exponent of the common factor x. So, we factor out x^{-3} as below.

$$x^{-1}+2x^{-2}-x^{-3}$$
$$=\mathbf{x^{-3}}\,(\mathbf{x^2+2x-1})$$

the exponent 2 is found by subtracting -3 from -1

the exponent 1 is found by subtracting -3 from -2

To check if the factorization is correct, we multiply

add exponents

$$x^{-3}\,(x^2+2x-1)$$
$$=x^{-3}x^2+2x^{-3}x-1x^{-3}$$
$$=x^{-1}+2x^{-2}-x^{-3}$$

Since the product gives us the original polynomial, the factorization is correct.

Factoring by Grouping

When referring to a common factor, we have in mind a common factor other than 1.

Consider the polynomial $x^2+x+xy+y$. It consists of four terms that do not have any common factors. Yet, it can still be factored if we group the first two and the last two terms. The first group of two terms contains the common factor of x and the second group of two terms contains the common factor of y. Observe what happens when we factor each group.

$$\underbrace{x^2+x}+\underbrace{xy+y}$$
$$=x(x+1)+y(x+1)$$
$$=(x+1)(x+y)$$

now $(x+1)$ is the common factor of the entire polynomial

This method is called **factoring by grouping**, in particular, two-by-two grouping.

Warning: After factoring each group, make sure to write the "+" or "−" between the terms. Failing to write these signs leads to the false impression that the polynomial is already factored. For example, if in the second line of the above calculations we would fail to write the middle "+", the expression would look like a product $x(x+1)\ y(x+1)$, which is not the case. Also, since the expression $x(x+1)+y(x+1)$ is a sum, not a product, we should not stop at this step. We need to factor out the common bracket $(x+1)$ to leave it as a product.

A two-by-two grouping leads to a factorization only if **the binomials**, after factoring out the common factors in each group, **are the same**. Sometimes a rearrangement of terms is necessary to achieve this goal.

For example, the attempt to factor $x^3-15+5x^2-3x$ by grouping the first and the last two terms,

$$\underbrace{x^3-15}+\underbrace{5x^2-3x}$$

$$=(x^3-15)+x(5x-3)$$

does not lead us to a common binomial that could be factored out.

However, rearranging terms allows us to factor the original polynomial in the following ways:

$$x^3-15+5x^2-3x$$
$$=\underbrace{x^3+5x^2}+\underbrace{-3x-15}$$
$$=x^2(x+5)-3(x+5)$$
$$=(x+5)(x^2-3)$$

or

$$x^3-15+5x^2-3x$$
$$=\underbrace{x^3-3x}+\underbrace{5x^2-15}$$
$$=x(x^2-3)+5(x^2-3)$$
$$=(x^2-3)(x+5)$$

Factoring by grouping applies to polynomials with more than three terms. However, not all such polynomials can be factored by grouping. For example, if we attempt to factor x^3+x^2+2x-2 by grouping, we obtain

$$\underbrace{x^3+x^2}+2\underbrace{x-2}$$

$$=x^2(x+1)+2(x-1).$$

Unfortunately, the expressions $x+1$ and $x-1$ are not the same, so there is no common factor to factor out. One can also check that no other rearrangments of terms allows us for factoring out a common binomial. So, this polynomial cannot be factored by grouping.

Example 3 ► **Factoring by Grouping**

Factor each polynomial by grouping, if possible. Remember to check for the GCF first.

a. $2x^3 - 6x^2 + x - 3$ **b.** $5x - 5y - ax + ay$

c. $2x^2y - 8 - 2x^2 + 8y$ **d.** $x^2 - x + y + 1$

Solution ► **a.** Since there is no common factor for all four terms, we will attempt the two-by-two grouping method.

$$\underbrace{2x^3 - 6x^2} + \underbrace{x - 3}$$

$$= 2x^2(x-3) + 1(x-3)$$

$$= (\boldsymbol{x - 3})(\boldsymbol{2x^2 + 1})$$

write the **1** for the second term

b. As before, there is no common factor for all four terms. The two-by-two grouping method works only if the remaining binomials after factoring each group are exactly the same. We can achieve this goal by factoring $-a$, rather than a, out of the last two terms. So,

$$\underbrace{5x - 5y} \ \underbrace{-ax + ay}$$

$$= 5(x-y) - a(x-y)$$

$$= (\boldsymbol{x - 3})(\boldsymbol{2x^2 + 1})$$

reverse signs when 'pulling' a "–" out

c. Notice that 2 is the GCF of all terms, so we factor it out first.

$$2x^2y - 8 - 2x^2 + 8y$$

$$= 2(x^2y - 4 - x^2 + 4y)$$

Then, observe that grouping the first and last two terms of the remaining polynomial does not help, as the two groups do not have any common factors. However, exchanging for example the second with the fourth term will help, as shown below.

$$= 2(\underbrace{x^2y + 4y} \ \underbrace{-x^2 - 4})$$

$$= 2[y(x^2+4) - (x^2+4)]$$

$$= \boldsymbol{2(x^2 + 4)(y - 1)}$$

the square bracket is essential here because of the factor of 2

reverse signs when 'pulling' a "–" out

now, there is no need for the square bracket as multiplication is associative

d. The polynomial $x^2 - x + y + 1$ does not have any common factors for all four terms. Also, only the first two terms have a common factor. Unfortunately, when attempting to factor using the two-by-two grouping method, we obtain

$$x^2 - x + y + 1$$

$$= x(x-1) + (y+1),$$

which cannot be factored, as the expressions $x - 1$ and $y + 1$ are different.

One can also check that no other arrangement of terms allows for factoring of this polynomial by grouping. So, this polynomial cannot be factored by grouping.

Example 4 ► **Factoring in Solving Formulas**

Solve $ab = 3a + 5$ for a.

Solution ► First, we move the terms containing the variable a to one side of the equation,

$$ab = 3a + 5$$
$$ab - 3a = 5,$$

and then factor a out

$$a(b - 3) = 5.$$

So, after dividing by $b - 3$, we obtain $\boldsymbol{a = \frac{5}{b-3}}$.

F.1 Exercises

*In problems **1-2**, state whether the given sentence is **true** or **false**.*

1. The polynomial $6x + 8y$ is ***prime***.

2. The **GCF** of the terms of the polynomial $3(x - 2) + x(2 - x)$ is $(x - 2)(2 - x)$.

3. Observe the two factorizations of the polynomial $\frac{1}{2}x - \frac{3}{4}y$ performed by different students:

Student A: $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{2}(x - \frac{3}{2}y)$ *Student B:* $\frac{1}{2}x - \frac{3}{4}y = \frac{1}{4}(2x - 3y)$

Are the two factorizations correct? Which one is preferable, and why?

Find the GCF with a positive coefficient for the given expressions.

4. $8xy,\ 10xz,\ -14xy$

5. $21a^3b^6,\ -35a^7b^5,\ 28a^5b^8$

6. $4x(x - 1),\ 3x^2(x - 1)$

7. $-x(x - 3)^2,\ x^2(x - 3)(x + 2)$

8. $9(a - 5),\ 12(5 - a)$

9. $(x - 2y)(x - 1),\ (2y - x)(x + 1)$

10. $-3x^{-2}y^{-3},\ 6x^{-3}y^{-5}$

11. $x^{-2}(x + 2)^{-2},\ -x^{-4}(x + 2)^{-1}$

Factor out the greatest common factor. Leave the remaining polynomial with a positive leading coeficient. Simplify the factors, if possible.

12. $9x^2 - 81x$

13. $8k^3 + 24k$

14. $6p^3 - 3p^2 - 9p^4$

15. $6a^3 - 36a^4 + 18a^2$

16. $-10r^2s^2 + 15r^4s^2$

17. $5x^2y^3 - 10x^3y^2$

18. $a(x - 2) + b(x - 2)$

19. $a(y^2 - 3) - 2(y^2 - 3)$

20. $(x - 2)(x + 3) + (x - 2)(x + 5)$

21. $(n - 2)(n + 3) + (n - 2)(n - 3)$

22. $y(x-1)+5(1-x)$

23. $(4x-y)-4x(y-4x)$

24. $4(3-x)^2-(3-x)^3+3(3-x)$

25. $2(p-3)+4(p-3)^2-(p-3)^3$

Factor out the least power of each variable.

26. $3x^{-3}+x^{-2}$

27. $k^{-2}+2k^{-4}$

28. $x^{-4}-2x^{-3}+7x^{-2}$

29. $3p^{-5}+p^{-3}-2p^{-2}$

30. $3x^{-3}y-x^{-2}y^2$

31. $-5x^{-2}y^{-3}+2x^{-1}y^{-2}$

Factor by grouping, if possible.

32. $20+5x+12y+3xy$

33. $2a^3+a^2-14a-7$

34. $ac-ad+bc-bd$

35. $2xy-x^2y+6-3x$

36. $3x^2+4xy-6xy-8y^2$

37. $x^3-xy+y^2-x^2y$

38. $3p^2+9pq-pq-3q^2$

39. $3x^2-x^2y-yz^2+3z^2$

40. $2x^3-x^2+4x-2$

41. $x^2y^2+ab-ay^2-bx^2$

42. $xy+ab+by+ax$

43. $x^2y-xy+x+y$

44. $xy-6y+3x-18$

45. x^ny-3x^n+y-5

46. $a^nx^n+2a^n+x^n+2$

Factor completely. Remember to check for the GCF first.

47. $5x-5ax+5abc-5bc$

48. $6rs-14s+6r-14$

49. $x^4(x-1)+x^3(x-1)-x^2+x$

50. $x^3(x-2)^2+2x^2(x-2)-(x+2)(x-2)$

51. One of possible factorizations of the polynomial $4x^2y^5-8xy^3$ is $2xy^3(2xy^2-4)$. Is this a complete factorization?

Use factoring the GCF strategy to ***solve*** *each formula* ***for the indicated variable****.*

52. $A=\boldsymbol{P}+\boldsymbol{P}r$, for $\boldsymbol{P}$

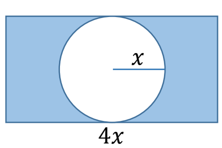
53. $M=\frac{1}{2}\boldsymbol{p}q+\frac{1}{2}\boldsymbol{p}r$, for $\boldsymbol{p}$

54. $2\boldsymbol{t}+c=k\boldsymbol{t}$, for $\boldsymbol{t}$

55. $w\boldsymbol{y}=3\boldsymbol{y}-x$, for $\boldsymbol{y}$

Write the area of each shaded region in factored form.

56.



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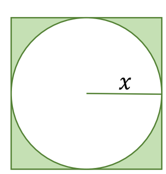

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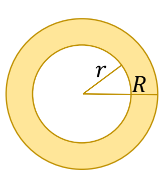
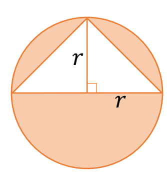
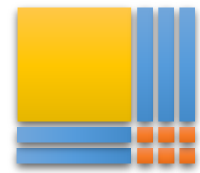


59.



F2 Factoring Trinomials



In this section, we discuss factoring trinomials. We start with factoring quadratic trinomials of the form $x^2 + bx + c$, then quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, and finally trinomials reducible to quadratic by means of substitution.

Factorization of Quadratic Trinomials $x^2 + bx + c$

GUESSING METHOD

Factorization of a quadratic trinomial $x^2 + bx + c$ is the reverse process of the FOIL method of multiplying two linear binomials. Observe that

$$(x + p)(x + q) = x^2 + qx + px + pq = x^2 + (p + q)x + pq$$

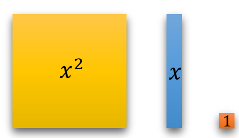
So, to reverse this multiplication, we look for two numbers p and q, such that the product pq equals to the free term c and the sum $p + q$ equals to the middle coefficient b of the trinomial.

$$x^2 + \underbrace{b}_{(p+q)} x + \underbrace{c}_{pq} = (x + p)(x + q)$$

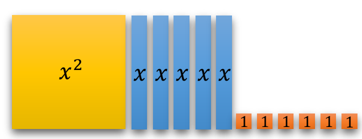
For example, to factor $x^2 + 5x + 6$, we think of two integers that multiply to 6 and add to 5. Such integers are 2 and 3, so $x^2 + 5x + 6 = (x + 2)(x + 3)$. Since multiplication is commutative, the order of these factors is not important.

VISUALIZATION OF FACTORING

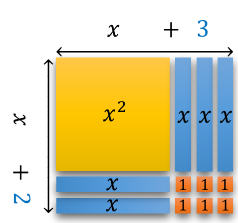
This could also be illustrated geometrically, using algebra tiles.



The area of a square with the side length x is equal to x^2. The area of a rectangle with the dimensions x by 1 is equal to x, and the area of a unit square is equal to 1. So, the trinomial $x^2 + 5x + 6$ can be represented as



To factor this trinomial, we would like to rearrange these tiles to fulfill a rectangle.



The area of such rectangle can be represented as the product of its length, $(x + 3)$, and width, $(x + 2)$ which becomes the factorization of the original trinomial.

In the trinomial examined above, the signs of the middle and the last terms are both positive. To analyse how different signs of these terms influence the signs used in the factors, observe the next three examples.

To factor $x^2 - 5x + 6$, we look for two integers that multiply to 6 and add to -5. Such integers are -2 and -3, so $x^2 - 5x + 6 = (x-2)(x-3)$.

To factor $x^2 + x - 6$, we look for two integers that multiply to -6 and add to 1. Such integers are -2 and 3, so $x^2 + x - 6 = (x-2)(x+3)$.

To factor $x^2 - x - 6$, we look for two integers that multiply to -6 and add to -1. Such integers are 2 and -3, so $x^2 - x - 6 = (x+2)(x-3)$.

Observation: The **positive constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x+p)(x+q)$ are both of the **same sign** and their **sum** is the middle coefficient b. In addition, if b is positive, both p and q are positive, and if b is negative, both p and q are negative.

The **negative constant** $\boldsymbol{c}$ in a trinomial $x^2 + bx + c$ tells us that the integers p and q in the factorization $(x+p)(x+q)$ are of **different signs** and a **difference** of their absolute values is the middle coefficient b. In addition, the integer whose absolute value is larger takes the sign of the middle coefficient b.

These observations are summarized in the following **Table of Signs**.

Assume that $|p| \geq |q|$.

sum $\boldsymbol{b}$	product $\boldsymbol{c}$	$\boldsymbol{p}$	$\boldsymbol{q}$	**comments**
+	+	+	+	b is the ***sum*** of p and q
−	+	−	−	b is the ***sum*** of p and q
+	−	+	−	b is the ***difference*** $|p| - |q|$
−	−	−	+	b is the ***difference*** $|q| - |p|$

Example 1 ► **Factoring Trinomials with the Leading Coefficient Equal to 1**

Factor each trinomial, if possible.

a. $x^2 - 10x + 24$

b. $x^2 + 9x - 36$

c. $x^2 - 39xy - 40y^2$

d. $x^2 + 7x + 9$

Solution ►

a. To factor the trinomial $x^2 - 10x + 24$, we look for two integers with a product of 24 and a sum of -10. The two integers are fairly easy to guess, -4 and -6. However, if one wishes to follow a more methodical way of finding these numbers, one can list the possible two-number factorizations of 24 and observe the sums of these numbers.

For simplicity, the table doesn't include signs of the integers. The signs are determined according to the **Table of Signs**.

product = 24 (pairs of factors of 24)	**sum = −10** (sum of factors)
1 · 24	25
2 · 12	14
3 · 8	11
4 · 6	10

Bingo!

Since the product is positive and the sum is negative, both integers must be negative. So, we take -4 and -6.

Thus, $x^2 - 10x + 24 = (\boldsymbol{x} - \mathbf{4})(\boldsymbol{x} - \mathbf{6})$. The reader is encouraged to check this factorization by multiplying the obtained binomials.

b. To factor the trinomial $x^2 + 9x - 36$, we look for two integers with a product of -36 and a sum of 9. So, let us list the possible factorizations of 36 into two numbers and observe the differences of these numbers.

product $= -36$ (pairs of factors of 36)	**sum $= 9$** (difference of factors)
$\mathbf{1 \cdot 36}$	35
$\mathbf{2 \cdot 18}$	16
$\mathbf{3 \cdot 12}$	9
$\mathbf{4 \cdot 9}$	5
$\mathbf{6 \cdot 6}$	0

This row contains the solution, so there is no need to list any of the subsequent rows.

Since the product is negative and the sum is positive, the integers are of different signs and the one with the larger absolute value assumes the sign of the sum, which is positive. So, we take 12 and -3.

Thus, $x^2 + 9x - 36 = (\boldsymbol{x} + \mathbf{12})(\boldsymbol{x} - \mathbf{3})$. Again, the reader is encouraged to check this factorization by mltiplying the obtained binomials.

c. To factor the trinomial $x^2 - 39xy - 40y^2$, we look for two binomials of the form $(x + ?\,y)(x + ?\,y)$ where the question marks are two integers with a product of -40 and a sum of 39. Since the two integers are of different signs and the absolute values of these integers differ by 39, the two integers must be -40 and 1.

Therefore, $x^2 - 39xy - 40y^2 = (\boldsymbol{x} - \mathbf{40}\boldsymbol{y})(\boldsymbol{x} + \boldsymbol{y})$.

Suggestion: Create a table of pairs of factors only if guessing the two integers with the given product and sum becomes too difficult.

d. When attempting to factor the trinomial $x^2 + 7x + 9$, we look for a pair of integers that would multiply to 9 and add to 7. There are only two possible factorizations of 9: $9 \cdot 1$ and $3 \cdot 3$. However, neither of the sums, $9 + 1$ or $3 + 3$, are equal to 7. So, there is no possible way of factoring $x^2 + 7x + 9$ into two linear binomials with integral coefficients. Therefore, if we admit only integral coefficients, this polynomial is **not factorable**.

Factorization of Quadratic Trinomials $ax^2 + bx + c$ with $a \neq 0$

Before discussing factoring quadratic trinomials with a leading coefficient different than 1, let us observe the multiplication process of two linear binomials with integral coefficients.

$$(mx + p)(nx + q) = mnx^2 + mqx + npx + pq = \underbrace{a}_{mn}\,x^2 + \underbrace{b}_{(mq+np)}\,x + \underbrace{c}_{pq}$$

GUESSING METHOD

To reverse this process, notice that this time, we are looking for four integers m, n, p, and q that satisfy the conditions

$$mn = a, \quad pq = c, \quad mq + np = b,$$

where a, b, c are the coefficients of the quadratic trinomial that needs to be factored. This produces a lot more possibilities to consider than in the guessing method used in the case of the leading coefficient equal to 1. However, if at least one of the outside coefficients, a or c, are prime, the guessing method still works reasonably well.

For example, consider $2x^2 + x - 6$. Since the coefficient $a = 2 = mn$ is a prime number, there is only one factorization of a, which is $1 \cdot 2$. So, we can assume that $m = 2$ and $n = 1$. Therefore,

$$2x^2 + x - 6 = (2x \pm |p|)(x \mp |q|)$$

Since the constant term $c = -6 = pq$ is negative, the binomial factors have different signs in the middle. Also, since pq is negative, we search for such p and q that the inside and outside products **differ** by the middle term $b = x$, up to its sign. The only factorizations of 6 are $1 \cdot 6$ and $2 \cdot 3$. So we try

$$2x^2 + x - 6 = (2x \pm 1)(x \mp 6)$$

x

$12x$ differs by $11x \to$ too much

Observe that these two trials can be disregarded at once as 2 is not a common factor of all the terms of the trinomial, while it is a common factor of the terms of one of the binomials.

$$2x^2 + x - 6 = (2x \pm 6)(x \mp 1)$$

$6x$

$2x$ differs by $4x \to$ still too much

$$2x^2 + x - 6 = (2x \pm 2)(x \mp 3)$$

$2x$

$6x$ differs by $4x \to$ still too much

$$2x^2 + x - 6 = (2x \pm 3)(x \mp 2)$$

$3x$

$4x$ differs by $x \to$ perfect!

Then, since the difference between the inner and outer products should be positive, the larger product must be positive and the smaller product must be negative. So, we distribute the signs as below.

$$2x^2 + x - 6 = (2x - 3)(x + 2)$$

$-3x$

$4x$

In the end, it is a good idea to multiply the product to check if it results in the original polynomial. We leave this task to the reader.

What if the outside coefficients of the quadratic trinomial are both composite? Checking all possible distributions of coefficients m, n, p, and q might be too cumbersome. Luckily, there is another method of factoring, called **decomposition**.

The decomposition method is based on the reverse FOIL process.
Suppose the polynomial $6x^2 + 19x + 15$ factors into $(mx + p)(nx + q)$. Observe that the FOIL multiplication of these two binomials results in the four term polynomial,

$$mnx^2 + mqx + npx + pq,$$

which after combining the two middle terms gives us the original trinomial. So, reversing these steps would lead us to the factored form of $6x^2 + 19x + 15$.

To reverse the FOIL process, we would like to:

This product is often referred to as the **master product** or the ***ac*-product**.

- Express the middle term, $19x$, as a sum of two terms, mqx and npx, such that the product of their coefficients, $mnpq$, is equal to the product of the outside coefficients $ac = 6 \cdot 15 = 90$.
- Then, factor the four-term polynomial by grouping.

Thus, we are looking for two integers with the product of 90 and the sum of 19. One can check that 9 and 10 satisfy these conditions. Therefore,

DECOMPOSITION METHOD

$$\begin{aligned} &6x^2 + 19x + 15 \\ &= 6x^2 + 9x + 10x + 15 \\ &= 3x(2x+3) + 5(2x+3) \\ &= (2x+3)(3x+5) \end{aligned}$$

Example 2 ► **Factoring Trinomials with the Leading Coefficient Different than 1**

Factor completely each trinomial.

a. $6x^3 + 14x^2 + 4x$ **b.** $-6y^2 - 10 + 19y$
c. $18a^2 - 19ab - 12b^2$ **d.** $2(x+3)^2 + 5(x+3) - 12$

Solution ► **a.** First, we factor out the GCF, which is $2x$. This gives us

$$6x^3 + 14x^2 + 4x = 2x(3x^2 + 7x + 2)$$

The outside coefficients of the remaining trinomial are prime, so we can apply the guessing method to factor it further. The first terms of the possible binomial factors must be $3x$ and x while the last terms must be 2 and 1. Since both signs in the trinomial are positive, the signs used in the binomial factors must be both positive as well. So, we are ready to give it a try:

$$\begin{matrix} 2x(3x+2)(x+1) & \text{or} & 2x(3x+1)(x+2) \\ 2x & & x \\ 3x & & 6x \end{matrix}$$

The first distribution of coefficients does not work as it would give us $2x + 3x = 5x$ for the middle term. However, the second distribution works as $x + 6x = 7x$, which matches the middle term of the trinomial. So,

$$6x^3 + 14x^2 + 4x = \mathbf{2x(3x+1)(x+2)}$$

b. Notice that the trinomial is not arranged in decreasing order of powers of y. So, first, we rearrange the last two terms to achieve the decreasing order. Also, we factor out the -1, so that the leading term of the remaining trinomial is positive.

$$-6y^2 - 10 + 19y = -6y^2 + 19y - 10 = -(6y^2 - 19y + 10)$$

Then, since the outside coefficients are composite, we will use the decomposition method of factoring. The ac-product equals to 60 and the middle coefficient equals to -19. So, we are looking for two integers that multiply to 60 and add to -19. The integers that satisfy these conditions are -15 and -4. Hence, we factor

$$-(6y^2 - 19y + 10)$$

$$= -(6y^2 - 15y - 4y + 10)$$

the **square bracket** is essential because of the negative sign outside

$$= -[3y(2y - 5) - 2(2y - 5)]$$

remember to **reverse the sign!**

$$= -(\mathbf{2y - 5})(\mathbf{3y - 2})$$

c. There is no common factor to take out of the polynomial $18a^2 - 19ab - 12b^2$. So, we will attempt to factor it into two binomials of the type $(ma \pm pb)(na \mp qb)$, using the decomposition method. The ac-product equals $-12 \cdot 18 = -2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ and the middle coefficient equals -19. To find the two integers that multiply to the ac-product and add to -19, it is convenient to group the factors of the product

$$2 \cdot 2 \cdot 2 \; 3 \cdot 3 \cdot 3$$

in such a way that the products of each group differ by 19. It turns out that grouping all the 2's and all the 3's satisfy this condition, as 8 and 27 differ by 19. Thus, the desired integers are -27 and 8, as the sum of them must be -19. So, we factor

$$18a^2 - 19ab - 12b^2$$

$$= 18a^2 - 27ab + 8ab - 12b^2$$

$$= 9a(2a - 3b) + 4b(2a - 3b)$$

$$= (\mathbf{2a - 3b})(\mathbf{9a + 4b})$$

d. To factor $2(x + 3)^2 + 5(x + 3) - 12$, first, we notice that treating the group $(x + 3)$ as another variable, say a, simplify the problem to factoring the quadratic trinomial

$$2a^2 + 5a - 12$$

This can be done by the guessing method. Since

$$2a^2 + 5a - 12 = (2a - 3)(a + 4),$$

$$-3a$$

$$8a$$

then

$$2(x + 3)^2 + 5(x + 3) - 12 = [2(x + 3) - 3][(x + 3) + 4]$$

$$= (2x + 6 - 3)(x + 3 + 4)$$

$$= (\mathbf{2x + 3})(\mathbf{x + 7})$$

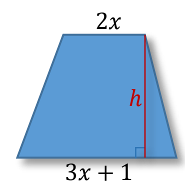
QUADRATIC IN FORM

Note 1: Polynomials that can be written in the form $\boldsymbol{a}(\quad)^2 + \boldsymbol{b}(\quad) + \boldsymbol{c}$, where $a \neq 0$ and $(\quad)$ represents any nonconstant polynomial expression, are referred to as **quadratic in form**. To factor such polynomials, it is convenient to **replace** the expression in the bracket by **a single variable**, different than the original one. This was illustrated in *Example 2d* by substituting a for $(x + 3)$. However, when using this **substitution method**, we must **remember to leave the final answer in terms of the original variable**. So, after factoring, we replace a back with $(x + 3)$, and then simplify each factor.

Note 2: Some students may feel comfortable factoring polynomials quadratic in form directly, without using substitution.

Example 3 ► **Application of Factoring in Geometry Problems**

Suppose that the area in square meters of a trapezoid is given by the polynomial $5x^2 - 9x - 2$. If the two bases are $2x$ and $(3x + 1)$ meters long, then what polynomial represents the height of the trapezoid?



Solution ► Using the formula for the area of a trapezoid, we write the equation

$$\frac{1}{2}h(a + b) = 5x^2 - 9x - 2$$

Since $a + b = 2x + (3x + 1) = 5x + 1$, then we have

$$\frac{1}{2}h(5x + 1) = 5x^2 - 9x - 2,$$

which after factoring the right-hand side gives us

$$\frac{1}{2}h(5x + 1) = (5x + 1)(x - 2).$$

To find h, it is enough to divide the above equation by the common factor $(5x + 1)$ and then multiply it by 2. So,

$$h = 2(x - 2) = \mathbf{2x - 4}.$$

F.2 Exercises

1. If $ax^2 + bx + c$ has no monomial factor, can either of the possible binomial factors have a monomial factor?
2. Is $(2x + 5)(2x - 4)$ a complete factorization of the polynomial $4x^2 + 2x - 20$?

3. When factoring the polynomial $-2x^2 - 7x + 15$, students obtained the following answers:

$$(-2x+3)(x+5),\ (2x-3)(-x-5),\ \text{or}\ -(2x-3)(x+5)$$

Which of the above factorizations are correct?

4. Is the polynomial $x^2 - x + 2$ factorable or is it prime?

Fill in the missing factor.

5. $x^2 - 4x + 3 = (\qquad)(x-1)$

6. $x^2 + 3x - 10 = (\qquad)(x-2)$

7. $x^2 - xy - 20y^2 = (x+4y)(\qquad)$

8. $x^2 + 12xy + 35y^2 = (x+5y)(\qquad)$

Factor, if possible.

9. $x^2 + 7x + 12$

10. $x^2 - 12x + 35$

11. $y^2 + 2y - 48$

12. $a^2 - a - 42$

13. $x^2 + 2x + 3$

14. $p^2 - 12p - 27$

15. $m^2 - 15m + 56$

16. $y^2 + 3y - 28$

17. $18 - 7n - n^2$

18. $20 + 8p - p^2$

19. $x^2 - 5xy + 6y^2$

20. $p^2 + 9pq + 20q^2$

Factor completely.

21. $-x^2 + 4x + 21$

22. $-y^2 + 14y + 32$

23. $n^4 - 13n^3 - 30n^2$

24. $y^3 - 15y^2 + 54y$

25. $-2x^2 + 28x - 80$

26. $-3x^2 - 33x - 72$

27. $x^4y + 7x^2y - 60y$

28. $24ab^2 + 6a^2b^2 - 3a^3b^2$

29. $40 - 35t^{15} - 5t^{30}$

30. $x^4y^2 + 11x^2y + 30$

31. $64n - 12n^5 - n^9$

32. $24 - 5x^a - x^{2a}$

33. If a polynomial $x^2 + \boxed{}x + 36$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of b?

Fill in the missing factor.

34. $2x^2 + 7x + 3 = (\qquad)(x+3)$

35. $3x^2 - 10x + 8 = (\qquad)(x-2)$

36. $4x^2 + 8x - 5 = (2x-1)(\qquad)$

37. $6x^2 - x - 15 = (2x+3)(\qquad)$

Factor completely.

38. $2x^2 - 5x - 3$

39. $6y^2 - y - 2$

40. $4m^2 + 17m + 4$

41. $6t^2 - 13t + 6$

42. $10x^2 + 23x - 5$

43. $42n^2 + 5n - 25$

44. $3p^2 - 27p + 24$

45. $-12x^2 - 2x + 30$

46. $6x^2 + 41xy - 7y^2$

47. $18x^2 + 27xy + 10y^2$

48. $8 - 13a + 6a^2$

49. $15 - 14n - 8n^2$

50. $30x^4 + 3x^3 - 9x^2$

51. $10x^3 - 6x^2 + 4x^4$

52. $2y^6 + 7xy^3 + 6x^2$

53. $9x^2y^2 - 4 + 5xy$

54. $16x^2y^3 + 3y - 16xy^2$

55. $4p^4 - 28p^2q + 49q^2$

56. $4(x-1)^2 - 12(x-1) + 9$

57. $2(a+2)^2 + 11(a+2) + 15$

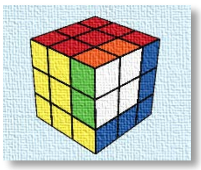
58. $4x^{2a} - 4x^a - 3$

59. If a polynomial $3x^2 + \boxed{}x - 20$ with an unknown coefficient b by the middle term can be factored into two binomials with integral coefficients, then what are the possible values of $\boxed{b}$?

60. The volume of a case of apples is $2x^3 - 3x^2 - 2x$ cubic feet, and the height of the case is $(x-2)$ feet. Find a polynomial representing the area of the bottom of the case?

61. Suppose the width of a rectangular runner carpet is $(x+5)$ feet. If the area of the carpet is $(3x^2 + 17x + 10)$ square feet, find the polynomial that represents the length of the carpet.

F3 Special Factoring and a General Strategy of Factoring



Recall that in *Section P2*, we considered formulas that provide a shortcut for finding special products, such as a product of two **conjugate** binomials,

$$(a+b)(a-b) = \boldsymbol{a^2 - b^2},$$

or the **perfect square** of a binomial,

$$(a \pm b)^2 = \boldsymbol{a^2 \pm 2ab + b^2}.$$

Since factoring reverses the multiplication process, these formulas can be used as shortcuts in factoring binomials of the form $\boldsymbol{a^2 - b^2}$ (**difference of squares)**, and trinomials of the form $\boldsymbol{a^2 \pm 2ab + b^2}$ **(perfect square)**. In this section, we will also introduce a formula for factoring binomials of the form $\boldsymbol{a^3 \pm b^3}$ (**sum or difference of cubes)**. These special product factoring techniques are very useful in simplifying expressions or solving equations, as they allow for more efficient algebraic manipulations.

At the end of this section, we give a summary of all the factoring strategies shown in this chapter.

Difference of Squares

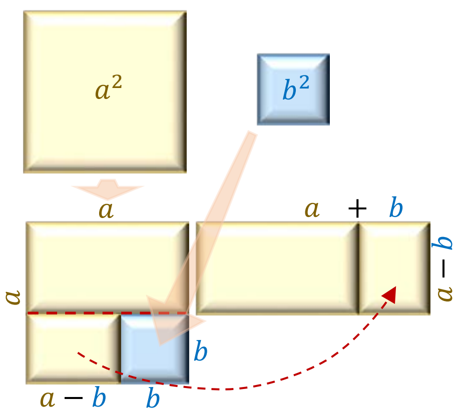


Figure 3.1

Out of the special factoring formulas, the easiest one to use is the difference of squares,

$$\boxed{\boldsymbol{a^2 - b^2 = (a+b)(a-b)}}$$

Figure 3.1 shows a geometric interpretation of this formula. The area of the yellow square, a^2, diminished by the area of the blue square, b^2, can be rearranged to a rectangle with the length of $(a+b)$ and the width of $(a-b)$.

To factor a difference of squares $\boldsymbol{a^2 - b^2}$, first, identify $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared, and then, form two factors, the sum $(\boldsymbol{a+b})$, and the difference $(\boldsymbol{a-b})$, as illustrated in the example below.

Example 1 ► **Factoring Differences of Squares**

Factor each polynomial completely.

a. $25x^2 - 1$ **b.** $3.6x^4 - 0.9y^6$

c. $x^4 - 81$ **d.** $16 - (a-2)^2$

Solution ► **a.** First, we rewrite each term of $25x^2 - 1$ as a perfect square of an expression.

$$25x^2 - 1 = (\overset{a}{\overset{\downarrow}{5x}})^2 - \overset{b}{\overset{\downarrow}{1}}{}^2$$

Then, treating $5x$ as the a and 1 as the b in the difference of squares formula $a^2 - b^2 = (a+b)(a-b)$, we factor:

$$a^2 - b^2 = (a + b)(a - b)$$

$$25x^2 - 1 = (5x)^2 - 1^2 = (\mathbf{5x + 1})(\mathbf{5x - 1})$$

b. First, we factor out 0.9 to leave the coefficients in a perfect square form. So,

$$3.6x^4 - 0.9y^6 = 0.9(4x^4 - y^6)$$

Then, after writing the terms of $4x^4 - y^6$ as perfect squares of expressions that correspond to a and b in the difference of squares formula $a^2 - b^2 = (a + b)(a - b)$, we factor

$$0.9(4x^4 - y^6) = 0.9[(\overset{a}{2x^2})^2 - (\overset{b}{y^3})^2] = \mathbf{0.9(2x^2 + y^3)(2x^2 - y^3)}$$

c. Similarly as in the previous two examples, $x^4 - 81$ can be factored by following the difference of squares pattern. So,

$$x^4 - 81 = (x^2)^2 - (9)^2 = (x^2 + 9)(x^2 - 9)$$

However, this factorization is not complete yet. Notice that $x^2 - 9$ is also a difference of squares, so the original polynomial can be factored further. Thus,

$$x^4 - 81 = (x^2 + 9)(x^2 - 9) = \mathbf{(x^2 + 9)(x + 3)(x - 3)}$$

Attention: The sum of squares, $x^2 + 9$, cannot be factored using real coefficients.

Generally, except for a common factor, a quadratic binomial of the form $\boldsymbol{a^2 + b^2}$ is **not factorable** over the real numbers.

Recall that
$\boldsymbol{a^2 + b^2 \neq (a + b)^2}$

d. Following the difference of squares formula, we have

$$\begin{aligned} 16 - (a - 2)^2 &= 4^2 - (a - 2)^2 \\ &= [4 + (a - 2)][4 - (a - 2)] \\ &= (4 + a - 2)(4 - a + 2) && \text{work out the inner brackets} \\ &= \mathbf{(2 + a)(6 - a)} && \text{combine like terms} \end{aligned}$$

Remember to use brackets after the negative sign!

Perfect Squares

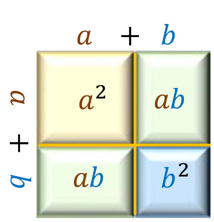


Figure 3.2

Another frequently used special factoring formula is the **perfect square** of a sum or a difference.

$$\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$$

or

$$\boldsymbol{a^2 - 2ab + b^2 = (a - b)^2}$$

Figure 3.2 shows the geometric interpretation of the perfect square of a sum. We encourage the reader to come up with a similar interpretation of the perfect square of a difference.

To factor a perfect square trinomial $\boldsymbol{a^2 \pm 2ab + b^2}$, we find $\boldsymbol{a}$ and $\boldsymbol{b}$, which are the expressions being squared. Then, depending on the middle sign, we use $\boldsymbol{a}$ and $\boldsymbol{b}$ to form the perfect square of the sum $(\boldsymbol{a + b})^{\boldsymbol{2}}$, or the perfect square of the difference $(\boldsymbol{a - b})^{\boldsymbol{2}}$.

Example 2 ► **Identifying Perfect Square Trinomials**

Decide whether the given polynomial is a perfect square.

a. $9x^2 + 6x + 4$ **b.** $9x^2 + 4y^2 - 12xy$

c. $25p^4 + 40p^2 - 16$ **d.** $49y^6 + 84xy^3 + 36x^2$

Solution ► **a.** Observe that the outside terms of the trinomial $9x^2 + 6x + 4$ are perfect squares, as $9x^2 = (3x)^2$ and $4 = 2^2$. So, the trinomial would be a perfect square if the middle terms would equal $2 \cdot 3x \cdot 2 = 12x$. Since this is not the case, our trinomial is **not a perfect square**.

Attention: Except for a common factor, trinomials of the type $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable** over the real numbers!

b. First, we arrange the trinomial in decreasing order of the powers of x. So, we obtain $9x^2 - 12xy + 4y^2$. Then, since $9x^2 = (3x)^2$, $4y^2 = (2y)^2$, and the middle term (except for the sign) equals $2 \cdot 3x \cdot 2 = 12x$, we claim that the trinomial **is a perfect square**. Since the middle term is negative, this is the perfect square of a difference. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{ccccccccccccc}
a^2 & - & 2 & a & & b & + & b^2 & = & (a & - & b)^2 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(3x)^2 & - & 2 \cdot & 3x & \cdot & 2y & + & (2y)^2 & = & (3x & - & 2y)^2
\end{array}$$

c. Even though the coefficients of the trinomial $25p^4 + 40p^2 - 16$ and the distribution of powers seem to follow the pattern of a perfect square, the last term is negative, which makes it **not a perfect square**.

d. Since $49y^6 = (7y^3)^2$, $36x^2 = (6x)^2$, and the middle term equals $2 \cdot 7y^3 \cdot 6x = 84xy^3$, we claim that the trinomial **is a perfect square**. Since the middle term is positive, this is the perfect square of a sum. So, the trinomial $9x^2 - 12xy + 4y^2$ can be seen as

$$\begin{array}{ccccccccccccc}
a^2 & + & 2 & a & & b & + & b^2 & = & (a & - & b)^2 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(7y^3)^2 & + & 2 \cdot & 7y^3 & \cdot & 6x & + & (6x)^2 & = & (7y^3 & - & 6x)^2
\end{array}$$

Example 3 ► **Factoring Perfect Square Trinomials**

Factor each polynomial completely.

a. $25x^2 + 10x + 1$ **b.** $a^2 - 12ab + 36b^2$

c. $m^2 - 8m + 16 - 49n^2$ **d.** $-4y^2 - 144y^8 + 48y^5$

Solution ► **a.** The outside terms of the trinomial $25x^2 + 10x + 1$ are perfect squares of $5a$ and 1, and the middle term equals $2 \cdot 5x \cdot 1 = 10x$, so we can follow the perfect square formula. Therefore,

$$25x^2 + 10x + 1 = (\mathbf{5x} + \mathbf{1})^2$$

b. The outside terms of the trinomial $a^2 - 12ab + 36b^2$ are perfect squares of a and $6b$, and the middle term (disregarding the sign) equals $2 \cdot a \cdot 6b = 12ab$, so we can follow the perfect square formula. Therefore,

$$a^2 - 12ab + 36b^2 = (\boldsymbol{a} - \mathbf{6}\boldsymbol{b})^2$$

c. Observe that the first three terms of the polynomial $m^2 - 8m + 16 - 49n^2$ form a perfect square of $m - 6$ and the last term is a perfect square of $7n$. So, we can write

$$m^2 - 8m + 16 - 49n^2 = (m - 6)^2 - (7n)^2$$

This is not in factored form yet!

Notice that this way we have formed a difference of squares. So we can factor it by following the difference of squares formula

$$(m - 6)^2 - (7n)^2 = (\boldsymbol{m} - \mathbf{6} - \mathbf{7}\boldsymbol{n})(\boldsymbol{m} - \mathbf{6} + \mathbf{7}\boldsymbol{n})$$

d. As in any factoring problem, first we check the polynomial $-4y^2 - 144y^8 + 48y^5$ for a common factor, which is $4y^2$. To leave the leading term of this polynomial positive, we factor out $-4y^2$. So, we obtain

$$-4y^2 - 144y^8 + 48y^5$$

$$= -4y^2\,(1 + 36y^6 - 12y^3)$$

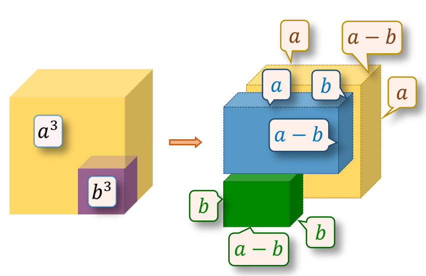
$$= -4y^2\,(36y^6 - 12y^3 + 1)$$

arrange the polynomial in decreasing powers

$$= \mathbf{-4}\boldsymbol{y}^{\mathbf{2}}\,(\mathbf{6}\boldsymbol{y}^{\mathbf{3}} - \mathbf{1})^{\mathbf{2}}$$

fold to the perfect square form

Sum or Difference of Cubes



$$a^3 - b^3 = a^2(a - b) + ab(a - b) + b^2(a - b)$$

$$= (a - b)(a^2 + ab + b^2)$$

Figure 3.3

The last special factoring formula to discuss in this section is the **sum or difference of cubes**.

$$\boldsymbol{a}^{\mathbf{3}} + \boldsymbol{b}^{\mathbf{3}} = (\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a}^{\mathbf{2}} - \boldsymbol{ab} + \boldsymbol{b}^{\mathbf{2}})$$

or

$$\boldsymbol{a}^{\mathbf{3}} - \boldsymbol{b}^{\mathbf{3}} = (\boldsymbol{a} - \boldsymbol{b})(\boldsymbol{a}^{\mathbf{2}} + \boldsymbol{ab} + \boldsymbol{b}^{\mathbf{2}})$$

The reader is encouraged to confirm these formulas by multiplying the factors in the right-hand side of each equation. In addition, we offer a geometric visualization of one of these formulas, the difference of cubes, as shown in *Figure 3.3*.

Hints for memorization of the sum or difference of cubes formulas:

- The binomial factor is a copy of the sum or difference of the terms that were originally cubed.
- The trinomial factor follows the pattern of a perfect square, except that the **middle term is single**, not doubled.
- The signs in the factored form follow the pattern ***S**ame-**O**pposite-**P**ositive* (**SOP**).

Example 4 ► **Factoring Sums or Differences of Cubes**

Factor each polynomial completely.

a. $8x^3 + 1$ **b.** $27x^7y - 125xy^4$

c. $2n^6 - 128$ **d.** $(p-2)^3 + q^3$

Solution ►

a. First, we rewrite each term of $8x^3 + 1$ as a perfect cube of an expression.

$$8x^3 + 1 = (\overset{\overset{a}{\downarrow}}{2x})^2 + \overset{\overset{b}{\downarrow}}{1}{}^2$$

Then, treating $2x$ as the a and 1 as the b in the sum of cubes formula $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, we factor:

$$\boldsymbol{a^3 + b^3 = (a + b)\left(a^2 - a\,b + b^2\right)}$$

$$8x^3 + 1 = (2x)^2 + 1^2 = (2x + 1)((2x)^2 - 2x \cdot 1 + 1^2)$$

$$= \boldsymbol{(2x+1)\left(4x^2 - 2x + 1\right)}$$

Quadratic trinomials of the form $\boldsymbol{a^2 \pm ab + b^2}$ are **not factorable**!

Notice that the trinomial $4x^2 - 2x + 1$ in not factorable anymore.

b. Since the two terms of the polynomial $27x^7y - 125xy^4$ contain the common factor xy, we factor it out and obtain

$$27x^7y - 125xy^4 = xy(27x^6 - 125y^3)$$

Observe that the remaining polynomial is a difference of cubes, $(3x^2)^3 - (5y)^3$. So, we factor,

$$27x^7y - 125xy^4 = xy[(3x^2)^3 - (5y)^3]$$

$$\boldsymbol{(a + b)\left(a^2 - a \quad b + b^2\right)}$$

$$= xy(3x^2 - 5y)[(3x^2)^2 + 3x^2 \cdot 5y + (5y)^2]$$

$$= \boldsymbol{xy\left(3x^2 - 5y\right)\left(9x^4 + 15x^2y + 25y^2\right)}$$

c. After factoring out the common factor 2, we obtain

$$2n^6 - 128 = 2(n^6 - 64)$$

Difference of squares or **difference of cubes?**

Notice that $n^6 - 64$ can be seen either as a difference of squares, $(n^3)^2 - 8^2$, or as a difference of cubes, $(n^2)^3 - 4^3$. It turns out that applying the **difference of squares** formula first **leads us to a complete factorization** while starting with the difference of cubes does not work so well here. See the two approaches below.

$$(n^3)^2 - 8^2$$
$$= (n^3 + 8)(n^3 - 8)$$
$$= (n+2)(n^2 - 2n + 4)(n-2)(n^2 + 2n + 4)$$

4 prime factors, so the factorization is complete

$$(n^2)^3 - 4^3$$
$$= (n^2 - 4)(n^4 + 4n^2 + 16)$$
$$= (n+2)(n-2)(n^4 + 4n^2 + 16)$$

There is no easy way of factoring this trinomial!

Therefore, the original polynomial should be factored as follows:

$$2n^6 - 128 = 2(n^6 - 64) = 2[(n^3)^2 - 8^2] = 2(n^3 + 8)(n^3 - 8)$$
$$= \mathbf{2(n+2)(n^2 - 2n + 4)(n-2)(n^2 + 2n + 4)}$$

d. To factor $(p-2)^3 + q^3$, we follow the sum of cubes formula $(a+b)(a^2 - ab + b^2)$ by assuming $a = p - 2$ and $b = q$. So, we have

$$(p-2)^3 + q^3 = (p - 2 + q)\,[(p-2)^2 - (p-2)q + q^2]$$
$$= (p - 2 + q)\,[p^2 - 2pq + 4 - pq + 2q + q^2]$$
$$= \mathbf{(p - 2 + q)\,[p^2 - 3pq + 4 + 2q + q^2]}$$

General Strategy of Factoring

Recall that a polynomial with integral coefficients is factored completely if all of its factors are prime over the integers.

How to Factorize Polynomials Completely?

1. Factor out all **common factors**. Leave the remaining polynomial with a positive leading term and integral coefficients, if possible.

2. Check the number of terms. If the polynomial has

- **more than three terms**, try to factor by **grouping**; a four term polynomial may require 2-2, 3-1, or 1-3 types of grouping.
- **three terms**, factor by **guessing**, **decomposition**, or follow the **perfect square** formula, if applicable.
- **two terms**, follow the **difference of squares**, or **sum or difference of cubes** formula, if applicable. Remember that sum of squares, $\boldsymbol{a^2 + b^2}$, is **not factorable** over the real numbers, except for possibly a common factor.

3. Keep in mind the **special factoring formulas**:

Difference of Squares	$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$
Perfect Square of a Sum	$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$
Perfect Square of a Difference	$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$
Sum of Cubes	$\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$
Difference of Cubes	$\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

4. **Keep factoring** each of the obtained factors until all of them are **prime** over the integers.

Example 5 ► **Multiple-step Factorization**

Factor each polynomial completely.

a. $80x^5 - 5x$ **b.** $4a^2 - 4a + 1 - b^2$

c. $(5r+8)^2 - 6(5r+8) + 9$ **d.** $(p-2q)^3 + (p+2q)^3$

Solution ►

a. First, we factor out the GCF of $80x^5$ and $-5x$, which equals to $5x$. So, we obtain

$$80x^5 - 5x = 5x(16x^4 - 1)$$

repeated difference of squares

Then, we notice that $16x^4 - 1$ can be seen as the difference of squares $(4x^2)^2 - 1^2$. So, we factor further

$$80x^5 - 5x = 5x(4x^2+1)(4x^2-1)$$

The first binomial factor, $4x^2 + 1$, cannot be factored any further using integral coefficients as it is the sum of squares, $(2x)^2 + 1^2$. However, the second binomial factor, $4x^2 - 1$, is still factorable as a difference of squares, $(2x)^2 - 1^2$. Therefore,

$$80x^5 - 5x = \boldsymbol{5x(4x^2+1)(2x+1)(2x-1)}$$

This is a complete factorization as all the factors are prime over the integers.

3-1 type of grouping

b. The polynomial $4a^2 - 4a + 1 - b^2$ consists of four terms, so we might be able to factor it by grouping. Observe that the 2-2 type of grouping has no chance to succeed, as the first two terms involve only the variable a while the second two terms involve only the variable b. This means that after factoring out the common factor in each group, the remaining binomials would not be the same. So, the 2-2 grouping would not lead us to a factorization. However, the 3-1 type of grouping should help. This is because the first three terms form the perfect square, $(2a-1)^2$, and there is a subtraction before the last term b^2, which is also a perfect square. So, in the end, we can follow the difference of squares formula to complete the factoring process.

$$\begin{aligned} \underbrace{4a^2 - 4a + 1} - \underbrace{b^2} &= (2a-1)^2 - b^2 \\ &= \boldsymbol{(2a-1-b)(2a-1+b)} \end{aligned}$$

factoring by substitution

c. To factor $(5r+8)^2 - 6(5r+8) + 9$, it is convenient to substitute a new variable, say $\boldsymbol{a}$, for the expression $5r+8$. Then,

$$\begin{aligned}(5r+8)^2 - 6(5r+8) + 9 &= \boldsymbol{a}^2 - 6\boldsymbol{a} + 9 \\ &= (\boldsymbol{a}-3)^2 \\ &= (5r+8-3)^2 \\ &= (5r+5)^2\end{aligned}$$

perfect square!

Remember to represent the new variable by a **different letter** than the original variable!

go back to the original variable

Notice that $5r+5$ can still be factored by taking the 5 out. So, for a complete factorization, we factor further

$$(5r+5)^2 = \left(5(r+1)\right)^2 = \mathbf{25}(\boldsymbol{r}+\mathbf{1})^{\mathbf{2}}$$

multiple special formulas and simplifying

d. To factor $(p-2q)^3 + (p+2q)^3$, we follow the sum of cubes formula $(a+b)(a^2 - ab + b^2)$ by assuming $a = p-2q$ and $b = p+2q$. So, we have

$$(p-2q)^3 + (p+2q)^3$$

$$= (p - \cancel{2q} + p + \cancel{2q})\,[(p-2q)^2 - (p-2q)(p+2q) + (p+2q)^2]$$

$$= 2p\,[p^2 - \cancel{4pq} + 4q^2 - (p^2 - 4q^2) + p^2 + \cancel{4pq} + 4q^2]$$

$$= 2p\,(2p^2 + 8q^2 - p^2 + 4q^2) = \mathbf{2}\boldsymbol{p}\left(\boldsymbol{p}^{\mathbf{2}} + \mathbf{12}\boldsymbol{q}^{\mathbf{2}}\right)$$

F.3 Exercises

Determine whether each polynomial in problems 7-18 is a perfect square, a difference of squares, a sum or difference of cubes, or neither.

1. $0.25x^2 - 0.16y^2$

2. $x^2 - 14x + 49$

3. $9x^4 + 4x^2 + 1$

4. $4x^2 - (x+4)^2$

5. $125x^3 - 64$

6. $y^{12} + 0.008x^3$

7. $-y^4 + 16x^4$

8. $64 + 48x^3 + 9x^6$

9. $25x^6 - 10x^3y^2 + y^4$

10. $-4x^6 - y^6$

11. $-8x^3 + 27y^6$

12. $81x^2 - 16x$

13. Generally, the sum of squares is not factorable. For example, $x^2 + 9$ cannot be factored in integral coefficients. However, some sums of squares can be factored. For example, the binomial $25x^2 + 100$ can be factored. Factor the above example and discuss what makes a sum of two squares factorable.

14. Insert the correct signs into the blanks.

a. $8 + a^3 = (2 \_\_ a)(4 \_\_ 2a \_\_ a^2)$

b. $b^3 - 1 = (b \_\_ 1)(b^2 \_\_ b \_\_ 1)$

Factor each polynomial completely, if possible.

15. $x^2 - y^2$

16. $x^2 + 2xy + y^2$

17. $x^3 - y^3$

18. $16x^2 - 100$

19. $4z^2 - 4z + 1$

20. $x^3 + 27$

21. $4z^2 + 25$

22. $y^2 + 18y + 81$

23. $125 - y^3$

24. $144x^2 - 64y^2$

25. $n^2 + 20nm + 100m^2$

26. $27a^3b^6 + 1$

27. $9a^4 - 25b^6$

28. $25 - 40x + 16x^2$

29. $p^6 - 64q^3$

30. $16x^2z^2 - 100y^2$

31. $4 + 49p^2 + 28p$

32. $x^{12} + 0.008y^3$

33. $r^4 - 9r^2$

34. $9a^2 - 12ab - 4b^2$

35. $\frac{1}{8} - a^3$

36. $0.04x^2 - 0.09y^2$

37. $x^4 + 8x^2 + 1$

38. $-\frac{1}{27} + t^3$

39. $16x^6 - 121x^2y^4$

40. $9 + 60pq + 100p^2q^2$

41. $-a^3b^3 - 125c^6$

42. $36n^{2t} - 1$

43. $9a^8 - 48a^4b + 64b^2$

44. $9x^3 + 8$

45. $(x+1)^2 - 49$

46. $\frac{1}{4}u^2 - uv + v^2$

47. $2t^4 - 128t$

48. $81 - (n+3)^2$

49. $x^{2n} + 6x^n + 9$

50. $8 - (a+2)^3$

51. $16z^4 - 1$

52. $5c^3 + 20c^2 + 20c$

53. $(x+5)^3 - x^3$

54. $a^4 - 81b^4$

55. $0.25z^2 - 0.7z + 0.49$

56. $(x-1)^3 + (x+1)^3$

57. $(x-2y)^2 - (x+y)^2$

58. $0.81p^8 + 9p^4 + 25$

59. $(x+2)^3 - (x-2)^3$

Factor each polynomial completely.

60. $3y^3 - 12x^2y$

61. $2x^2 + 50a^2 - 20ax$

62. $x^3 - xy^2 + x^2y - y^3$

63. $y^2 - 9a^2 + 12y + 36$

64. $64u^6 - 1$

65. $7m^3 + m^6 - 8$

66. $-7n^2 + 2n^3 + 4n - 14$

67. $a^8 - b^8$

68. $y^9 - y$

69. $(x^2-2)^2 - 4(x^2-2) - 21$

70. $8(p-3)^2 - 64(p-3) + 128$

71. $a^2 - b^2 - 6b - 9$

72. $25(2a-b)^2 - 9$

73. $3x^2y^2z + 25xyz^2 + 28z^3$

74. $x^{8a} - y^2$

75. $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$

76. $4x^2y^4 - 9y^4 - 4x^2z^4 + 9z^4$

77. $c^{2w+1} + 2c^{w+1} + c$

F4 Solving Polynomial Equations and Applications of Factoring

Many application problems involve solving polynomial equations. In Chapter L, we studied methods for solving linear, or first-degree, equations. Solving higher degree polynomial equations requires other methods, which often involve factoring. In this chapter, we study solving polynomial equations using the zero-product property, graphical connections between roots of an equation and zeros of the corresponding function, and some application problems involving polynomial equations or formulas that can be solved by factoring.

Zero-Product Property

Recall that to solve a linear equation, for example $2x + 1 = 0$, it is enough to isolate the variable on one side of the equation by applying reverse operations. Unfortunately, this method usually does not work when solving higher degree polynomial equations. For example, we would not be able to solve the equation $x^2 - x = 0$ through the reverse operation process, because the variable x appears in different powers.

So … how else can we solve it?

In this particular example, it is possible to guess the solutions. They are $x = 0$ and $x = 1$.

But how can we solve it algebraically?

It turns out that factoring the left-hand side of the equation $x^2 - x = 0$ helps. Indeed, $x(x - 1) = 0$ tells us that the product of x and $x - 1$ is 0. Since the product of two quantities is 0, at least one of them must be 0. So, either $x = 0$ or $x - 1 = 0$, which solves to $x = 1$.

The equation discussed above is an example of a second degree polynomial equation, more commonly known as a quadratic equation.

Definition 4.1 ► A **quadratic equation** is a second degree polynomial equation in one variable that can be written in the form,

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $a \neq 0$. This form is called **standard form**.

One of the methods of solving such equations involves factoring and the zero-product property that is stated below.

Zero-Product Theorem ► For any real numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{ab = 0} \text{ if and only if } \boldsymbol{a = 0} \text{ or } \boldsymbol{b = 0}$$

This means that any product containing a factor of 0 is equal to 0, and conversely, if a product is equal to 0, then at least one of its factors is equal to 0.

Proof ► The implication "if $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$, then $\boldsymbol{ab = 0}$" is true by the multiplicative property of zero.

To prove the implication "if $\boldsymbol{ab = 0}$, then $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$", let us assume first that $a \neq 0$. (As, if $a = 0$, then the implication is already proven.)

Since $a \neq 0$, then $\frac{1}{a}$ exists. Therefore, both sides of $ab = 0$ can be multiplied by $\frac{1}{a}$ and we obtain

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

$$b = 0,$$

which concludes the proof.

Attention: The zero-product property works only for a product equal to **0**. For example, the fact that $\boldsymbol{ab = 1}$ does not mean that either a or b equals to 1.

Example 1 ► **Using the Zero-Product Property to Solve Polynomial Equations**

Solve each equation.

a. $(x-3)(2x+5) = 0$ **b.** $2x(x-5)^2 = 0$

Solution ►

a. Since the product of $x-3$ and $2x+5$ is equal to zero, then by the zero-product property at least one of these expressions must equal to zero. So,

$$x - 3 = 0 \quad \text{or} \quad 2x + 5 = 0$$

which results in

$$x = 3 \quad \text{or} \quad 2x = -5$$

$$x = -\frac{5}{2}$$

Thus, $\left\{-\frac{5}{2}, 3\right\}$ is the solution set of the given equation.

b. Since the product $2x(x-5)^2$ is zero, then either $x = 0$ or $x - 5 = 0$, which solves to $x = 5$. Thus, the solution set is equal to $\{\mathbf{0}, \mathbf{5}\}$.

Note 1: The factor of 2 does not produce any solution, as 2 is never equal to 0.

Note 2: The perfect square $(x-5)^2$ equals to 0 if and only if the base $x - 5$ equals to 0.

Solving Polynomial Equations by Factoring

To solve polynomial equations of second or higher degree by factoring, we

- **arrange** the polynomial **in decreasing order** of powers **on one side** of the equation,
- keep the **other side** of the equation **equal to 0**,
- **factor** the polynomial **completely**,
- use the zero-product property to **form linear equations for each factor**,
- **solve** the linear equations to find the roots (solutions) to the original equation.

Example 2 ► **Solving Quadratic Equations by Factoring**

Solve each equation by factoring.

a. $x^2 + 9 = 6x$

b. $15x^2 - 12x = 0$

c. $(x + 2)(x - 1) = 4(3 - x) - 8$

d. $(x - 3)^2 = 36x^2$

Solution ►

a. To solve $x^2 + 9 = 6x$ by factoring we need one side of this equation equal to 0. So, first, we move the $6x$ term to the left side of the equation,

$$x^2 + 9 - 6x = 0,$$

and arrange the terms in decreasing order of powers of x,

$$x^2 - 6x + 9 = 0.$$

Then, by observing that the resulting trinomial forms a perfect square of $x - 3$, we factor

$$(x - 3)^2 = 0,$$

which is equivalent to

$$x - 3 = 0,$$

and finally

$$x = 3.$$

So, the solution is $x = \mathbf{3}$.

b. After factoring the left side of the equation $15x^2 - 12x = 0$,

$$3x(5x - 4) = 0,$$

we use the zero-product property. Since 3 is never zero, the solutions come from the equations

$$x = \mathbf{0} \quad \text{or} \quad 5x - 4 = 0.$$

Solving the second equation for x, we obtain

$$5x = 4,$$

and finally

$$x = \mathbf{\frac{4}{5}}.$$

So, the solution set consists of **0** and $\mathbf{\frac{4}{5}}$.

c. To solve $(x + 2)(x - 1) = 4(3 - x) - 8$ by factoring, first, we work out the brackets and arrange the polynomial in decreasing order of exponents on the left side of the equation. So, we obtain

$$x^2 + x - 2 = 12 - 4x - 8$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

Now, we can read the solutions from each bracket, that is, $x = \mathbf{-6}$ and $x = \mathbf{1}$.

Observation: In the process of solving a linear equation of the form $ax + b = 0$, first we subtract b and then we divide by a. So the solution, sometimes referred to as the root, is $x = -\frac{b}{a}$. This allows us to read the solution directly from the equation. For example, the solution to $x - 1 = 0$ is $x = 1$ and the solution to $2x - 1 = 0$ is $x = \frac{1}{2}$.

d. To solve $(x-3)^2 = 36x^2$, we bring all the terms to one side and factor the obtained difference of squares, following the formula $a^2 - b^2 = (a+b)(a-b)$. So, we have

$$(x-3)^2 - 36x^2 = 0$$

$$(x-3+6x)(x-3-6x) = 0$$

$$(7x-3)(-5x-3) = 0$$

Then, by the zero-product property,

$$7x - 3 = 0 \text{ or } -5x - 3,$$

which results in

$$x = \frac{\mathbf{3}}{\mathbf{7}} \text{ or } x = -\frac{\mathbf{3}}{\mathbf{5}}.$$

Example 3 ► **Solving Polynomial Equations by Factoring**

Solve each equation by factoring.

a. $2x^3 - 2x^2 = 12x$ **b.** $x^4 + 36 = 13x^2$

Solution ► **a.** First, we bring all the terms to one side of the equation and then factor the resulting polynomial.

$$2x^3 - 2x^2 = 12x$$

$$2x^3 - 2x^2 - 12x = 0$$

$$2x(x^2 - x - 6) = 0$$

$$2x(x-3)(x+2) = 0$$

By the zero-product property, the factors $x, (x-3)$ and $(x+2)$, give us the corresponding solutions, $0, 3$, and -2. So, the solution set of the given equation is $\{\mathbf{0}, \mathbf{3}, \mathbf{-2}\}$.

b. Similarly as in the previous examples, we solve $x^4 + 36 = 13x^2$ by factoring and using the zero-product property. Since

$$x^4 - 13x^2 + 36 = 0$$

$$(x^2-4)(x^2-9)=0$$

$$(x+2)(x-2)(x+3)(x-3)=0,$$

then, the solution set of the original equation is $\{\mathbf{-2,2,-3,3}\}$

Observation: n-th degree polynomial equations may have up to n roots (solutions).

Factoring in Applied Problems

Factoring is a useful strategy when solving applied problems. For example, factoring is often used in **solving formulas** for a variable, in **finding roots** of a polynomial function, and generally, in any problem involving **polynomial equations** that can be solved by factoring.

Example 4 ► **Solving Formulas with the Use of Factoring**

Solve each formula for the specified variable.

a. $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$, for $\boldsymbol{h}$ **b.** $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solution ► **a.** To solve $A=2\boldsymbol{h}w+2wl+2l\boldsymbol{h}$ for $\boldsymbol{h}$, we want to keep both terms containing $\boldsymbol{h}$ on the same side of the equation and bring the remaining terms to the other side. Here is an equivalent equation,

$$A-2wl=2\boldsymbol{h}w+2l\boldsymbol{h},$$

which, for convenience, could be written starting with h-terms:

$$2\boldsymbol{h}w+2l\boldsymbol{h}=A-2wl$$

Now, factoring $\boldsymbol{h}$ out causes that $\boldsymbol{h}$ appears in only one place, which is what we need to isolate it. So,

$$(2w+2l)\boldsymbol{h}=A-2wl \qquad /\div(2w+2l)$$

$$\boldsymbol{h}=\frac{\boldsymbol{A-2wl}}{\boldsymbol{2w+2l}}$$

Notice: In the above formula, there is nothing that can be simplified. Trying to reduce 2 or $2w$ or l would be an error, as there is no essential common factor that can be carried out of the numerator.

b. When solving $s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}}$ for $\boldsymbol{t}$, our goal is to, firstly, keep the variable $\boldsymbol{t}$ in the numerator and secondly, to keep it in a single place. So, we have

$$s=\frac{2\boldsymbol{t}+3}{\boldsymbol{t}} \qquad /\cdot t$$

$$s\boldsymbol{t}=2\boldsymbol{t}+3 \qquad /-2t$$

factor t

$$st - 2t = 3$$

$$t(s-2) = 3 \qquad / \div (s-2)$$

$$t = \frac{3}{s-2}.$$

Example 5 ► **Finding Roots of a Polynomial Function**

A toy-rocket is launched vertically with an initial velocity of 40 meters per second. If its height in meters after t seconds is given by the function

$$h(t) = -5t^2 + 40t,$$

in how many seconds will the rocket hit the ground?

Solution ► The rocket hits the ground when its height is 0. So, we need to find the time t for which $h(t) = 0$. Therefore, we solve the equation

$$-5t^2 + 40t = 0$$

for t. From the factored form

$$-5t(t-8) = 0$$

we conclude that the rocket is on the ground at times 0 and 8 seconds. So, the rocket hits the ground **8 seconds** after it was launched.

Example 6 ► **Solving an Application Problem with the Use of Factoring**

The height of a triangle is 1 meter less than twice the length of the base. If the area of the triangle is 14 m^2, how long are the base and the height?

Solution ► Let b and h represent the base and the height of the triangle, correspondingly. The first sentence states that h is 1 less than 2 times b. So, we record

$$h = 2b - 1.$$

Using the formula for area of a triangle, $A = \frac{1}{2}bh$, and the fact that $A = 14$, we obtain

$$14 = \frac{1}{2}b(2b-1).$$

Since this is a one-variable quadratic equation, we will attempt to solve it by factoring, after bringing all the terms to one side of the equation. So, we have

to clear the fraction, multiply each term by 2 before working out the bracket

$$0 = \frac{1}{2}b(2b-1) - 14 \qquad / \cdot 2$$

$$0 = b(2b-1) - 28$$

$$0 = 2b^2 - b - 28$$

$$0 = (2b + 7)(b - 4),$$

which by the zero-product property gives us $b = -\frac{7}{2}$ or $b = 4$. Since b represents the length of the base, it must be positive. So, the base is **4** meters long and the height is $h = 2b - 1 = 2 \cdot 4 - 1 = \mathbf{7}$ meters long.

F.4 Exercises

True or false.

1. If $xy = 0$ then $x = 0$ or $y = 0$.

2. If $ab = 1$ then $a = 1$ or $b = 1$.

3. If $x + y = 0$ then $x = 0$ or $y = 0$.

4. If $a^2 = 0$ then $a = 0$.

5. If $x^2 = 1$ then $x = 1$.

6. Which of the following equations is **not** in proper form for using the zero-product property.

a. $x(x - 1) + 3(x - 1) = 0$

b. $(x + 3)(x - 1) = 0$

c. $x(x - 1) = 3(x - 1)$

d. $(x + 3)(x - 1) = -3$

Solve each equation.

7. $3(x - 1)(x + 4) = 0$

8. $2(x + 5)(x - 7) = 0$

9. $(3x + 1)(5x + 4) = 0$

10. $(2x - 3)(4x - 1) = 0$

11. $x^2 + 9x + 18 = 0$

12. $x^2 - 18x + 80 = 0$

13. $2x^2 = 7 - 5x$

14. $3k^2 = 14k - 8$

15. $x^2 + 6x = 0$

16. $6y^2 - 3y = 0$

17. $(4 - a)^2 = 0$

18. $(2b + 5)^2 = 0$

19. $0 = 4n^2 - 20n + 25$

20. $0 = 16x^2 + 8x + 1$

21. $p^2 - 32 = -4p$

22. $19a + 36 = 6a^2$

23. $x^2 + 3 = 10x - 2x^2$

24. $3x^2 + 9x + 30 = 2x^2 - 2x$

25. $(3x + 4)(3x - 4) = -10x$

26. $(5x + 1)(x + 3) = -2(5x + 1)$

27. $4(y - 3)^2 - 36 = 0$

28. $3(a + 5)^2 - 27 = 0$

29. $(x-3)(x+5)=-7$

30. $(x+8)(x-2)=-21$

31. $(2x-1)(x-3)=x^2-x-2$

32. $4x^2+x-10=(x-2)(x+1)$

33. $4(2x+3)^2-(2x+3)-3=0$

34. $5(3x-1)^2+3=-16(3x-1)$

35. $x^3+2x^2-15x=0$

36. $6x^3-13x^2-5x=0$

37. $25x^3=64x$

38. $9x^3=49x$

39. $y^4-26y^2+25=0$

40. $n^4-50n^2+49=0$

41. $x^3-6x^2=-8x$

42. $x^3-2x^2=3x$

43. $a^3+a^2-9a-9=0$

44. $2x^3-x^2-2x+1=0$

45. $5x^3+2x^2-20x-8=0$

46. $2x^3+3x^2-18x-27=0$

47. Discuss the validity of the following solution:

$$x^3=9x$$
$$x^2=9$$
$$x=3$$

How many solutions should we expect? What is the solution set of the original equation? What went wrong in the above procedure?

48. Given that $f(x)=x^2+14x+50$, find all values of x such that $f(x)=5$.

49. Given that $g(x)=2x^2-15x$, find all values of x such that $g(x)=-7$.

50. Given that $f(x)=2x^2+3x$ and $g(x)=-6x+5$, find all values of x such that $f(x)=g(x)$.

51. Given that $g(x)=2x^2+11x-16$ and $h(x)=5+9x-x^2$, find all values of x such that $g(x)=h(x)$.

Solve each equation for the specified variable.

52. $\boldsymbol{P}rt=A-\boldsymbol{P}$, for $\boldsymbol{P}$

53. $3\boldsymbol{s}+2p=5-r\boldsymbol{s}$, for $\boldsymbol{s}$

54. $5a+b\boldsymbol{r}=\boldsymbol{r}-2c$, for $\boldsymbol{r}$

55. $E=\frac{R+\boldsymbol{r}}{\boldsymbol{r}}$, for $\boldsymbol{r}$

56. $z=\frac{x+2\boldsymbol{y}}{\boldsymbol{y}}$, for $\boldsymbol{y}$

57. $c=\frac{-2\boldsymbol{t}+4}{\boldsymbol{t}}$, for $\boldsymbol{t}$

Solve each problem.

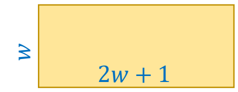
58. Bartek threw down a small rock from the top of a 120 m high observation tower. Suppose the distance travelled by the rock, in meters, is modelled by the function $d(t)=vt+4t^2$, where v is the initial velocity in m/s, and t is the time in seconds. In how many seconds will the rock hit the ground if it was thrown with the initial velocity of 4 m/s?

59. A camera is dropped from a hot-air balloon 320 meters above the ground. Suppose the height of the camera above the ground, in meters, is given by the function $h(t)=320-5t^2$, where t is the time in seconds. How long will it take for the camera to hit the ground?

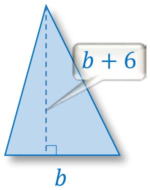
60. The sum of squares of two consecutive numbers is 85. Find the smaller number.

61. The difference between a number and its square is -156. Find the number.

62. The length of a rectangle is 1 centimeter more than twice the width. If the area of this rectangle is 105 cm^2, find its width and length.



63. A postcard is 7 cm longer than it is wide. The area of this postcard is 144 cm^2. Find its length and width.



64. A triangle with the area of 80 cm^2 is 6 cm taller than the length of its base. Find the dimensions of the triangle.

65. A triangular house is 3 m taller than it is wide. If the cross-sectional area (see the accompanying picture) of the house is 35 m^2, what are the width and the height of this house?

66. Amira designs a rectangular flower bed with a pathway of uniform width around it. She has 42 square meters of ground available for the whole project (including the path). If the flower bed is planned to be 3 meters by 4 meters, how wide would be the pathway around it?

67. Suppose a rectangular flower bed is 5 m longer than it is wide. What are the dimensions of the flower bed if its area is 84 m^2 ?

68. Suppose a picture frame measures 10 cm by 18 cm, and it frames a picture with 48 cm^2 of area. How wide is the frame?

69. When 187 cm^2 picture is framed, its outside dimensions become 15 cm by 21 cm. How wide is the frame?

70. After lengthening each side of a square by 4 cm, the area of the enlarged square turns out to be 225 cm^2. How long is the side of the original square?

71. A square piece of drywall was used to fix a hole in a wall. The sides of the piece of drywall had to be shortened by 2 inches in order to cover the required area of 49 in^2. What were the dimensions of the original piece of drywall?

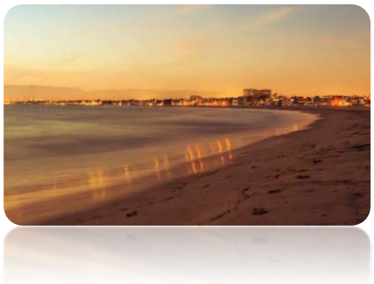
Attributions

65. The force that keeps a car from skidding on a curve is inversely proportional to the radius of the curve and jointly proportional to the weight of the car and the square of its speed. Knowing that a force of 880 N (Newtons) keeps an 800-kg car moving at 50 km/h from skidding on a curve of radius 160 m, estimate the force that would keep the same car moving at 80 km/h from skidding on a curve of radius 200 meters.

66. Suppose that the renovation time is inversely proportional to the number of workers hired for the job. Will the renovation time decrease more when hiring additional 2 workers in a 4-worker company or a 6-worker company? Justify your answer.

Attributions

Rational Expressions and Functions



In the previous two chapters we discussed algebraic expressions, equations, and functions related to polynomials. In this chapter, we will examine a broader category of algebraic expressions, *rational expressions*, also referred to as *algebraic fractions*. Similarly as in arithmetic, where a rational number is a quotient of two integers with a denominator that is different than zero, a rational expression is a quotient of two polynomials, also with a denominator that is different than zero.

We start by introducing the related topic of integral exponents, including scientific notation. Then, we discuss operations on algebraic fractions, solving rational equations, and properties and graphs of rational functions with an emphasis on such features as domain, range, and asymptotes. At the end of this chapter, we show examples of applied problems, including work problems, that require solving rational equations.

RT1 Integral Exponents and Scientific Notation

Integral Exponents

In *section P.2*, we discussed the following power rules, using whole numbers for the exponents.

product rule	$a^m \cdot a^n = a^{m+n}$	$(ab)^n = a^n b^n$
quotient rule	$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
power rule	$(a^m)^n = a^{mn}$	$a^0 = 1$ for $a \neq 0$ 0^0 is undefined

Observe that these rules gives us the following result.

$$a^{-1} = a^{n-(n+1)} \underset{\text{quotient rule}}{=} \frac{a^n}{a^{n+1}} \underset{\text{product rule}}{=} \frac{\cancel{a^n}}{\cancel{a^n} \cdot a} = \frac{1}{a}$$

Consequantly, $a^{-n} \underset{\text{power rule}}{=} (a^n)^{-1} = \frac{1}{a^n}$.

Since $\boxed{a^{-n} = \frac{1}{a^n}}$, then the expression a^n is meaningful for any integral exponent n and a nonzero real base a. So, the above rules of exponents can be extended to include integral exponents.

In practice, to work out the negative sign of an exponent, take the **reciprocal of the base**, or equivalently, "**change the level**" of the power. For example,

$$3^{-2} = \left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9} \quad \text{and} \quad \frac{2^{-3}}{3^{-1}} = \frac{3^1}{2^3} = \frac{3}{8}.$$

Attention! Exponent refers to the immediate number, letter, or expression in a bracket. For example,

$$\boldsymbol{x^{-2} = \frac{1}{x^2}}, \quad (-\boldsymbol{x})^{-2} = \frac{1}{(-x)^2} = \boldsymbol{\frac{1}{x^2}}, \quad but \quad \boldsymbol{-x^{-2} = -\frac{1}{x^2}}.$$

Example 1 ► **Evaluating Expressions with Integral Exponents**

Evaluate each expression.

a. $3^{-1} + 2^{-1}$

b. $\frac{5^{-2}}{2^{-5}}$

c. $\frac{-2^2}{2^{-7}}$

d. $\frac{-2^{-2}}{3\cdot 2^{-3}}$

Solution ►

a. $3^{-1} + 2^{-1} = \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \boldsymbol{\frac{5}{6}}$

Caution! $3^{-1} + 2^{-1} \neq (3+2)^{-1}$, because the value of $3^{-1} + 2^{-1}$ is $\frac{5}{6}$, as shown in the example, while the value of $(3+2)^{-1}$ is $\frac{1}{5}$.

b. $\frac{5^{-2}}{2^{-5}} = \frac{2^5}{5^2} = \boldsymbol{\frac{32}{25}}$

Note: To work out the negative exponent, move the power from the numerator to the denominator or vice versa.

c. $\frac{-2^2}{2^{-7}} = -2^2 \cdot 2^7 = \boldsymbol{-2^9}$

Attention! The role of a negative sign in front of a base number or in front of an exponent is different. To work out the negative in 2^{-7}, we either take the reciprocal of the base, or we change the position of the power to a different level in the fraction. So, $2^{-7} = \left(\frac{1}{2}\right)^7$ or $2^{-7} = \frac{1}{2^7}$. However, the negative sign in -2^2 just means that the number is negative. So, $-2^2 = -4$. ***Caution!*** $-2^2 \neq \frac{1}{4}$

d. $\frac{-2^{-2}}{3\cdot 2^{-3}} = \frac{-2^{\cancel{3}^{1}}}{3\cdot 2^{\cancel{2}}} = \boldsymbol{-\frac{2}{3}}$

Note: Exponential expressions can be simplified in many ways. For example, to simplify $\frac{2^{-2}}{2^{-3}}$, we can work out the negative exponents first by moving the powers to a different level, $\frac{2^3}{2^2}$, and then reduce the common factors as shown in the example; or we can employ the quotient rule of powers to obtain

$$\frac{2^{-2}}{2^{-3}} = 2^{-2-(-3)} = 2^{-2+3} = 2^1 = 2.$$

Example 2 ► **Simplifying Exponential Expressions Involving Negative Exponents**

Simplify the given expression. Leave the answer with only positive exponents.

a. $4x^{-5}$

b. $(x+y)^{-1}$

c. $x^{-1}+y^{-1}$

d. $(-2^3x^{-2})^{-2}$

e. $\frac{x^{-4}y^2}{x^2y^{-5}}$

f. $\left(\frac{-4m^5n^3}{24mn^{-6}}\right)^{-2}$

Solution ►

a. $4x^{-5}=\frac{4}{x^5}$ exponent -5 refers to x only!

b. $(x+y)^{-1}=\frac{1}{x+y}$

these expressions are NOT equivalent!

c. $x^{-1}+y^{-1}=\frac{1}{x}+\frac{1}{y}$

d. $(-2^3x^{-2})^{-2}=\left(\frac{-2^3}{x^2}\right)^{-2}=\left(\frac{x^2}{-2^3}\right)^{2}=\frac{(x^2)^2}{(-1)^2(2^3)^2}=\frac{x^4}{2^6}$

power rule – multiply exponents

work out the negative exponents inside the bracket

work out the negative exponents outside the bracket

a "−" sign can be treated as a factor of -1

e. $\frac{x^{-4}y^2}{x^2y^{-5}}=\frac{y^2y^5}{x^2x^4}=\frac{y^7}{x^6}$

product rule – add exponents

f. $\left(\frac{-\cancel{4}^{\,1}m^{\cancel{5}\,4}n^3}{\cancel{24}_{\,6}\cancel{m}n^{-6}}\right)^{-2}=\left(\frac{-m^4n^3n^6}{6}\right)^{-2}=\left(\frac{(-1)m^4n^9}{6}\right)^{-2}=\left(\frac{6}{(-1)m^4n^9}\right)^{2}=\frac{36}{m^8n^{18}}$

$(-1)^2=1$

Scientific Notation

Integral exponents allow us to record numbers with a very large or very small absolute value in a shorter, more convenient form.

For example, the average distance from the Sun to the Saturn is 1,430,000,000 km, which can be recorded as $1.43\cdot 10{,}000{,}000$ or more concisely as $1.43\cdot 10^9$.

Similarly, the mass of an electron is 0.0000000000000000000000000009 grams, which can be recorded as $9\cdot 0.0000000000000000000000000001$, or more concisely as $9\cdot 10^{-28}$.

This more concise representation of numbers is called **scientific notation** and it is frequently used in sciences and engineering.

Definition 1.1 ► A real number $\boldsymbol{x}$ is written in **scientific notation** iff $\boldsymbol{x}=\boldsymbol{a}\cdot\mathbf{10}^{\boldsymbol{n}}$,
where the coefficient $\boldsymbol{a}$ is such that $|\boldsymbol{a}|\in[\mathbf{1},\mathbf{10})$, and the exponent $\boldsymbol{n}$ is an integer.

Example 3 ► **Converting Numbers to Scientific Notation**

Convert each number to scientific notation.

a. $520{,}000$ **b.** -0.000102 **c.** $12.5 \cdot 10^3$

Solution ►

an integer has its **decimal** dot **after the last digit**

a. To represent $520{,}000$ in scientific notation, we place a decimal point after the first nonzero digit,

$$5\,.\,2\,0\,0\,0\,0$$

and then count the number of decimal places needed for the decimal point to move to its original position, which by default was after the last digit. In our example the number of places we need to move the decimal place is 5. This means that 5.2 needs to be multiplied by 10^5 in order to represent the value of 520,000. So, $\mathbf{520{,}000 = 5.2 \cdot 10^5}$.

Note: To comply with the scientific notation format, we always place the decimal point after the first nonzero digit of the given number. This will guarantee that the coefficient $\boldsymbol{a}$ satisfies the condition $\mathbf{1 \le |a| < 10}$.

b. As in the previous example, to represent -0.000102 in scientific notation, we place a decimal point after the first nonzero digit,

$$-\,0\,.\,0\,0\,0\,1\,.\,0\,2$$

and then count the number of decimal places needed for the decimal point to move to its original position. In this example, we move the decimal 4 places to the left. So the number 1.02 needs to be divided by 10^4, or equivalently, multiplied by 10^{-4} in order to represent the value of -0.000102. So, $\mathbf{-0.000102 = -1.02 \cdot 10^{-4}}$.

Observation: Notice that moving the decimal to the **right** corresponds to using a **positive** exponent, as in *Example 3a*, while moving the decimal to the **left** corresponds to using a **negative** exponent, as in *Example 3b*.

c. Notice that $12.5 \cdot 10^3$ is not in scientific notation as the coefficient 12.5 is <u>not</u> smaller than 10. To convert $12.5 \cdot 10^3$ to scientific notation, first, convert 12.5 to scientific notation and then multiply the powers of 10. So,

$$12.5 \cdot 10^3 = 1.25 \cdot 10 \cdot 10^3 = \mathbf{1.25 \cdot 10^4}$$

multiply powers by adding exponents

Example 4 ► **Converting from Scientific to Decimal Notation**

Convert each number to decimal notation.

a. $-6.57 \cdot 10^6$ **b.** $4.6 \cdot 10^{-7}$

Solution ►

a. The exponent 6 indicates that the decimal point needs to be moved 6 places to the right. So,

$$-6.57 \cdot 10^6 = -6\,.\,5\,7\,\_\,\_\,\_\,\_\,. = \mathbf{-6{,}570{,}000}$$

fill the empty places by zeros

b. The exponent -7 indicates that the decimal point needs to be moved 7 places to the left. So,

$$4.6 \cdot 10^{-7} = 0.\_\,\_\,\_\,\_\,\_\,\_\,4\,.\,6 = \mathbf{0.00000046}$$

fill the empty places by zeros

Example 5 ► **Using Scientific Notation in Computations**

Evaluate. Leave the answer in scientific notation.

a. $6.5 \cdot 10^7 \cdot 3 \cdot 10^5$ **b.** $\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}}$

Solution ►

a. Since the product of the coefficients $6.5 \cdot 3 = 19.5$ is larger than 10, we convert it to scientific notation and then multiply the remaining powers of 10. So,

$$6.5 \cdot 10^7 \cdot 3 \cdot 10^5 = 19.5 \cdot 10^7 \cdot 10^5 = 1.95 \cdot 10 \cdot 10^{12} = \mathbf{1.95 \cdot 10^{13}}$$

b. Similarly as in the previous example, since the quotient $\frac{3.6}{9} = 0.4$ is smaller than 1, we convert it to scientific notation and then work out the remaining powers of 10. So,

$$\frac{3.6 \cdot 10^3}{9 \cdot 10^{14}} = 0.4 \cdot 10^{-11} = 4 \cdot 10^{-1} \cdot 10^{-11} = \mathbf{4 \cdot 10^{-12}}$$

divide powers by subtracting exponents

Example 6 ► **Using Scientific Notation to Solve Problems**

Earth is approximately $1.5 \cdot 10^8$ kilometers from the Sun. Estimate the time in days needed for a space probe moving at an average rate of $2.4 \cdot 10^4$ km/h to reach the Sun? *Assume that the probe moves along a straight line.*

Solution ► To find time T needed for the space probe travelling at the rate $R = 2.4 \cdot 10^4$ km/h to reach the Sun that is at the distance $D = 1.5 \cdot 10^8$ km from Earth, first, we solve the motion formula $R \cdot T = D$ for T. Since $T = \frac{D}{R}$, we calculate,

$$T = \frac{1.5 \cdot 10^8}{2.4 \cdot 10^4} = 0.625 \cdot 10^4 = 6.25 \cdot 10^3$$

So, it will take $6.25 \cdot 10^3$ hours = $\frac{6250}{24}$ days $\cong \mathbf{260.4}$ **days** for the space probe to reach the Sun.

RT.1 Exercises

True or false.

1. $\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ **2.** $10^{-4} = 0.00001$ **3.** $(0.25)^{-1} = 4$

4. $-4^5 = \frac{1}{4^5}$ **5.** $(-2)^{-10} = 4^{-5}$ **6.** $2 \cdot 2 \cdot 2^{-1} = \frac{1}{8}$

7. $3x^{-2} = \frac{1}{3x^2}$ **8.** $-2^{-2} = -\frac{1}{4}$ **9.** $\frac{5^{10}}{5^{-12}} = 5^{-2}$

10. The number $0.68 \cdot 10^{-5}$ is written in scientific notation. **11.** $98.6 \cdot 10^7 = 9.86 \cdot 10^6$

12. *Match each expression in Row I with the equivalent expression(s) in Row II, if possible.*

a. 5^{-2} **b.** -5^{-2} **c.** $(-5)^{-2}$ **d.** $-(-5)^{-2}$ **e.** $-5 \cdot 5^{-2}$

A. 25 **B.** $\frac{1}{25}$ **C.** -25 **D.** $-\frac{1}{5}$ **E.** $-\frac{1}{25}$

Evaluate each expression.

13. $4^{-6} \cdot 4^3$ **14.** $-9^3 \cdot 9^{-5}$ **15.** $\frac{2^{-3}}{2^6}$ **16.** $\frac{2^{-7}}{2^{-5}}$

17. $\frac{-3^{-4}}{5^{-3}}$ **18.** $-\left(\frac{3}{2}\right)^{-2}$ **19.** $2^{-2} + 2^{-3}$ **20.** $(2^{-1} - 3^{-1})^{-1}$

Simplify each expression, if possible. Leave the answer with only ***positive exponents****. Assume that all variables represent nonzero real numbers. Keep large numerical coefficients as powers of prime numbers, if possible.*

21. $(-2x^{-3})(7x^{-8})$ **22.** $(5x^{-2}y^3)(-4x^{-7}y^{-2})$ **23.** $(9x^{-4n})(-4x^{-8n})$

24. $(-3y^{-4a})(-5y^{-3a})$ **25.** $-4x^{-3}$ **26.** $\frac{x^{-4n}}{x^{6n}}$

27. $\frac{3n^5}{nm^{-2}}$ **28.** $\frac{14a^{-4}b^{-3}}{-8a^8b^{-5}}$ **29.** $\frac{-18x^{-3}y^3}{-12x^{-5}y^5}$

30. $(2^{-1}p^{-7}q)^{-4}$

31. $(-3a^2b^{-5})^{-3}$

32. $\left(\frac{5x^{-2}}{y^3}\right)^{-3}$

33. $\left(\frac{2x^3y^{-2}}{3y^{-3}}\right)^{-3}$

34. $\left(\frac{-4x^{-3}}{5x^{-1}y^4}\right)^{-4}$

35. $\left(\frac{125x^2y^{-3}}{5x^4y^{-2}}\right)^{-5}$

36. $\left(\frac{-200x^3y^{-5}}{8x^5y^{-7}}\right)^{-4}$

37. $[(-2x^{-4}y^{-2})^{-3}]^{-2}$

38. $\frac{12a^{-2}(a^{-3})^{-2}}{6a^7}$

39. $\frac{(-2k)^2m^{-5}}{(km)^{-3}}$

40. $\left(\frac{2p}{q^2}\right)^3\left(\frac{3p^4}{q^{-4}}\right)^{-1}$

41. $\left(\frac{-3x^4y^6}{15x^{-6}y^7}\right)^{-3}$

42. $\left(\frac{-4a^3b^2}{12a^6b^{-5}}\right)^{-3}$

43. $\left(\frac{-9^{-2}x^{-4}y}{3^{-3}x^{-3}y^2}\right)^8$

44. $(4^{-x})^{2y}$

45. $(5^a)^{-a}$

46. x^ax^{-a}

47. $\frac{9n^{2-x}}{3n^{2-2x}}$

48. $\frac{12x^{a+1}}{-4x^{2-a}}$

49. $\left(x^{b-1}\right)^3\left(x^{b-4}\right)^{-2}$

50. $\frac{25x^{a+b}y^{b-a}}{-5x^{a-b}y^{b+a}}$

Convert each number to scientific notation.

51. $26{,}000{,}000{,}000$

52. -0.000132

53. 0.0000000105

54. 705.6

Convert each number to decimal notation.

55. $6.7 \cdot 10^8$

56. $5.072 \cdot 10^{-5}$

57. $2 \cdot 10^{12}$

58. $9.05 \cdot 10^{-9}$

59. One megabyte of computer memory equals 2^{20} bytes. Using decimal notation, write the number of bytes in 1 megabyte. Then, using scientific notation, approximate this number by rounding the scientific notation coefficient to two decimals places.

Evaluate. State your answer in scientific notation.

60. $(6.5 \cdot 10^3)(5.2 \cdot 10^{-8})$

61. $(2.34 \cdot 10^{-5})(5.7 \cdot 10^{-6})$

62. $(3.26 \cdot 10^{-6})(5.2 \cdot 10^{-8})$

63. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

64. $\frac{7.5 \cdot 10^9}{2.5 \cdot 10^4}$

65. $\frac{4 \cdot 10^{-7}}{8 \cdot 10^{-3}}$

66. $\frac{0.05 \cdot 16000}{0.0004}$

67. $\frac{0.003 \cdot 40{,}000}{0.00012 \cdot 600}$

Solve each problem. State your answer in ***scientific notation****.*

68. A *light-year* is an astronomical unit measuring the distance that light travels in one year. If light travels approximately $3 \cdot 10^5$ kilometers per second, how long is a light-year in kilometers?

69. In 2018, the national debt in Canada was about $6.7 \cdot 10^{11}$ dollars. If the Canadian population in 2018 was approximately $3.7 \cdot 10^7$, what was the share of this debt per person?

70. One of the brightest stars in the night sky, Vega, is about $2.365 \cdot 10^{14}$ kilometers from Earth. If one light-year is approximately $9.46 \cdot 10^{12}$ kilometers, how many light-years is it from Earth to Vega?

71. The Columbia River discharges its water to the Pacific Ocean at approximately 265,000 ft^3/sec. What is the supply of water that comes from the Columbia River in one minute? in one day? *State the answer in scientific notation.*

72. Assuming the current trends continue, the population P of Canada, in millions, can be modelled by the equation $P = 34(1.011)^x$, where x is the number of years passed after the year 2010. According to this model, what is the predicted Canadian population for the years 2025 and 2030?

73. The mass of the Moon is $7.348 \cdot 10^{22}$ kg while the mass of Earth is $5.976 \cdot 10^{24}$ kg. How many times heavier is Earth than the Moon?

74. Most calculators cannot handle operations on numbers outside of the interval $(10^{-100}, 10^{100})$. How can we compute $(5 \cdot 10^{120})^3$ without the use of a calculator?

RT2 Rational Expressions and Functions; Multiplication and Division of Rational Expressions

In arithmetic, a rational number is a quotient of two integers with denominator different than zero. In algebra, a *rational expression*, often called an *algebraic fraction*, is a quotient of two polynomials, also with denominator different than zero. In this section, we will examine rational expressions and functions, paying attention to their domains. Then, we will simplify, multiply, and divide rational expressions, employing the factoring skills developed in *Chapter P*.

Rational Expressions and Functions

Here are some examples of rational expressions:

$$-\frac{x^2}{2xy}, \qquad x^{-1}, \qquad \frac{x^2-4}{x-2}, \qquad \frac{8x^2+6x-5}{4x^2+5x}, \qquad \frac{x-3}{3-x}, \qquad x^2-25, \qquad 3x(x-1)^{-2}$$

Definition 2.1 ► A **rational expression** (**algebraic fraction**) is a quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. Since division by zero is not permitted, a rational expression is defined only for the x-values that make the denominator of the expression different than zero. The set of such x-values is referred to as the **domain** of the expression.

Note 1: Negative exponents indicate hidden fractions and therefore represent rational expressions. For instance, $x^{-1} = \frac{1}{x}$.

Note 2: A single polynomial can also be seen as a rational expression because it can be considered as a fraction with a denominator of 1.

For instance, $x^2 - 25 = \frac{x^2-25}{1}$.

Definition 2.2 ► A **rational function** is a function defined by a rational expression,

$$\boldsymbol{f(x) = \frac{P(x)}{Q(x)}}.$$

The **domain** of such function consists of all real numbers except for the x-values that make the denominator $Q(x)$ equal to 0. So, the domain $\boldsymbol{D} = \mathbb{R} \setminus \{\boldsymbol{x} | \boldsymbol{Q(x) = 0}\}$

For example, the domain of the rational function $f(x) = \frac{1}{x-3}$ is the set of all real numbers except for 3 because 3 would make the denominator equal to 0. So, we write $D = \mathbb{R} \setminus \{3\}$. Sometimes, to make it clear that we refer to function f, we might denote the domain of f by D_f, rather than just D.

Figure 1 shows a graph of the function $f(x) = \frac{1}{x-3}$. Notice that the graph does not cross the dashed vertical line whose equation is $x = 3$. This is because $f(3)$ is not defined. A closer look at the graphs of rational functions will be given in *Section RT5*.

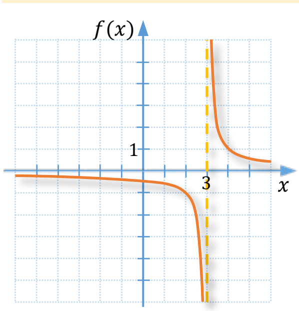


Figure 1

Example 1 ► **Evaluating Rational Expressions or Functions**

Evaluate the given expression or function for $x = -1, 0, 1$. If the value cannot be calculated, write ***undefined***.

a. $3x(x-1)^{-2}$

b. $f(x) = \frac{x}{x^2+x}$

Solution ► **a.** If $x = -1$, then $3x(x-1)^{-2} = 3(-1)(-1-1)^{-2} = -3(-2)^{-2} = \frac{-3}{(-2)^2} = -\mathbf{\frac{3}{4}}$.

If $x = 0$, then $3x(x-1)^{-2} = 3(0)(0-1)^{-2} = \mathbf{0}$.

If $x = 1$, then $3x(x-1)^{-2} = 3(1)(1-1)^{-2} = 3 \cdot 0^{-2} = \boldsymbol{undefined}$, as division by zero is not permitted.

Note: Since the expression $3x(x-1)^{-2}$ cannot be evaluated at $x = 1$, the number 1 does not belong to its domain.

b. $f(-1) = \frac{-1}{(-1)^2+(-1)} = \frac{-1}{1-1} = \boldsymbol{undefined}$.

$f(0) = \frac{0}{(0)^2+(0)} = \frac{0}{0} = \boldsymbol{undefined}$.

$f(1) = \frac{1}{(1)^2+(1)} = \mathbf{\frac{1}{2}}$.

Observation: Function $f(x) = \frac{x}{x^2+x}$ is undefined at $x = 0$ and $x = -1$. This is because the denominator $x^2 + x = x(x+1)$ becomes zero when the x-value is 0 or -1.

Example 2 ► **Finding Domains of Rational Expressions or Functions**

Find the domain of each expression or function.

a. $\frac{4}{2x+5}$

b. $\frac{x-2}{x^2-2x}$

c. $f(x) = \frac{x^2-4}{x^2+4}$

d. $g(x) = \frac{2x-1}{x^2-4x-5}$

Solution ► **a.** The domain of $\frac{4}{2x+5}$ consists of all real numbers except for those that would make the denominator $2x + 5$ equal to zero. To find these numbers, we solve the equation

$$2x + 5 = 0$$
$$2x = -5$$
$$x = -\frac{5}{2}$$

So, the domain of $\frac{4}{2x+5}$ is the set of all real numbers except for $-\frac{5}{2}$. This can be recorded in set notation as $\mathbb{R} \setminus \left\{-\frac{5}{2}\right\}$, or in set-builder notation as $\left\{\boldsymbol{x} \middle| \boldsymbol{x} \neq -\frac{5}{2}\right\}$, or in interval notation as $\left(-\infty, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$.

DOMAIN

b. To find the domain of $\frac{x-2}{x^2-2x}$, we want to exclude from the set of real numbers all the x-values that would make the denominator $x^2 - 2x$ equal to zero. After solving the equation

$$x^2 - 2x = 0$$

via factoring

$$x(x-2) = 0$$

and zero-product property

$$x = \mathbf{0} \quad \text{or} \quad x = \mathbf{2},$$

we conclude that the domain is the set of all real numbers except for 0 and 2, which can be recorded as $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$. This is because the x-values of 0 or 2 make the denominator of the expression $\frac{x-2}{x^2-2x}$ equal to zero.

c. To find the domain of the function $f(x) = \frac{x^2-4}{x^2+4}$, we first look for all the x-values that make the denominator $x^2 + 4$ equal to zero. However, $x^2 + 4$, as a sum of squares, is never equal to 0. So, the domain of function f is the set of all real numbers $\mathbb{R}$.

d. To find the domain of the function $g(x) = \frac{2x-1}{x^2-4x-5}$, we first solve the equation $x^2 - 4x - 5 = 0$ to find which x-values make the denominator equal to zero. After factoring, we obtain

$$(x-5)(x+1) = 0$$

which results in

$$x = 5 \quad \text{and} \quad x = -1$$

Thus, the domain of g equals to $\boldsymbol{D_g} = \mathbb{R} \setminus \{-\mathbf{1}, \mathbf{5}\}$.

Equivalent Expressions

Definition 2.3 ► Two expressions are **equivalent** in the **common domain** iff (if and only if) they produce the same values for every input from the domain.

Consider the expression $\frac{x-2}{x^2-2x}$ from *Example 2b*. Notice that this expression can be simplified to $\frac{\cancel{x-2}}{x(\cancel{x-2})} = \frac{1}{x}$ by reducing common factors in the numerator and the denominator. However, the domain of the simplified fraction, $\frac{1}{x}$, is the set $\mathbb{R} \setminus \{0\}$, which is different than the domain of the original fraction, $\mathbb{R} \setminus \{0,2\}$. Notice that for $x = 2$, the expression $\frac{x-2}{x^2-2x}$ is undefined while the value of the expression $\frac{1}{x}$ is $\frac{1}{2}$. So, the two expressions are not equivalent in the set of real numbers. However, if the domain of $\frac{1}{x}$ is

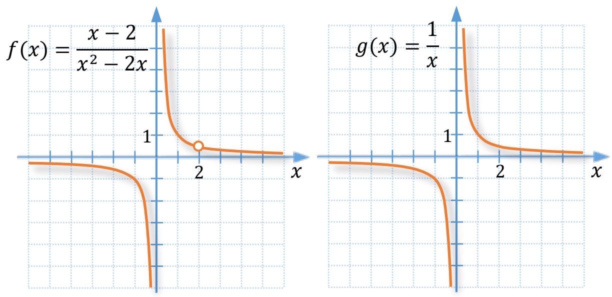


Figure 2

restricted to the set $\mathbb{R} \setminus \{0,2\}$, then the two expressions produce the same values and as such, they are equivalent. We say that the two expressions are **equivalent** in the **common domain**.

The above situation can be illustrated by graphing the related functions, $f(x) = \frac{x-2}{x^2-2x}$ and $g(x) = \frac{1}{x}$, as in *Figure 2*. The graphs of both functions are exactly the same except for the hole in the graph of f at the point $\left(2, \frac{1}{2}\right)$.

So, from now on, when writing statements like $\frac{x-2}{x^2-2x} = \frac{1}{x}$, we keep in mind that they apply only to real numbers which make both denominators different than zero. Thus, by saying in short that two **expressions are equivalent**, we really mean that they are **equivalent in the common domain**.

Note: The **domain** of $f(x) = \frac{x-2}{x^2-2x} = \frac{\cancel{x-2}}{x(\cancel{x-2})} = \frac{1}{x}$ is still $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$, even though the $(x-2)$ term was simplified.

The process of simplifying expressions involves creating equivalent expressions. In the case of rational expressions, equivalent expressions can be obtained by multiplying or dividing the numerator and denominator of the expression by the same nonzero polynomial. For example,

$$\frac{-\boldsymbol{x} - \mathbf{3}}{-\mathbf{5}\boldsymbol{x}} = \frac{(-x-3)\cdot(-1)}{(-5x)\cdot(-1)} = \frac{\boldsymbol{x} + \mathbf{3}}{\mathbf{5}\boldsymbol{x}}$$

$$\frac{\boldsymbol{x} - \mathbf{3}}{\mathbf{3} - \boldsymbol{x}} = \frac{\cancel{(x-3)}}{-1\cancel{(x-3)}} = \frac{1}{-1} = -\mathbf{1}$$

To simplify a rational expression:

- **Factor** the numerator and denominator **completely**.
- **Eliminate all common factors** by following the property of multiplicative identity. *Do not eliminate common terms - they must be factors!*

Example 3 ► **Simplifying Rational Expressions**

Simplify each expression.

a. $\frac{7a^2b^2}{21a^3b-14a^3b^2}$ **b.** $\frac{x^2-9}{x^2-6x+9}$ **c.** $\frac{20x-15x^2}{15x^3-5x^2-20x}$

Solution ► **a.** First, we factor the denominator and then reduce the common factors. So,

$$\frac{7a^2b^2}{21a^3b - 14a^3b^2} = \frac{\cancel{7}\cancel{a^2}\cancel{b^2}^{\,1}}{\cancel{7}\cancel{a^3}^{\,1}\cancel{b}(3a-2ab)} = \frac{\boldsymbol{b}}{\boldsymbol{a}(\mathbf{3} - \mathbf{2}\boldsymbol{b})}$$

b. As before, we factor and then reduce. So,

$$\frac{x^2-9}{x^2-6x+9} = \frac{\cancel{(x-3)}(x+3)}{(x-3)^{\cancel{2}\,1}} = \frac{\boldsymbol{x+3}}{\boldsymbol{x-3}}$$

Neither x nor 3 can be reduced, as they are NOT factors !

c. Factoring and reducing the numerator and denominator gives us

$$\frac{20x-15x^2}{15x^3-5x^2-20x} = \frac{\cancel{5x}(4-3x)}{\cancel{5x}(3x^2-x-4)} = \frac{4-3x}{(3x-4)(x+1)}$$

Since $\frac{4-3x}{3x-4} = \frac{-(3x-4)}{3x-4} = -1$, the above expression can be reduced further to

$$\frac{\cancel{4-3x}^{\,-1}}{\cancel{(3x-4)}(x+1)} = \frac{\boldsymbol{-1}}{\boldsymbol{x+1}}$$

Notice: An opposite expression in the numerator and denominator can be reduced to -1. For example, since $a-b$ is opposite to $b-a$, then

$$\frac{\boldsymbol{a-b}}{\boldsymbol{b-a}} = \boldsymbol{-1}, \text{ as long as } a \neq b.$$

Caution: Note that $a-b$ is NOT opposite to $a+b$!

Multiplication and Division of Rational Expressions

Recall that to multiply common fractions, we multiply their numerators and denominators, and then simplify the resulting fraction. Multiplication of algebraic fractions is performed in a similar way.

To multiply rational expressions:

- **factor** each numerator and denominator **completely**,
- **reduce all common factors** in any of the numerators and denominators,
- **multiply** the remaining expressions by writing the product of their numerators over the product of their denominators.

  For instance,

$$\frac{3x}{x^2+5x}\cdot\frac{3x+15}{6x} = \frac{3\cancel{x}}{\cancel{x}\cancel{(x+5)}}\cdot\frac{\cancel{3}\cancel{(x+5)}}{\underset{2}{\cancel{6}}x} = \frac{3}{2x}$$

Example 4 ► **Multiplying Algebraic Fractions**

Multiply and simplify. Assume nonzero denominators.

a. $\frac{2x^2y^3}{3xy^2}\cdot\frac{(2x^3y)^2}{2(xy)^3}$

b. $\frac{x^3-y^3}{x+y}\cdot\frac{3x+3y}{x^2-y^2}$

Solution ► **a.** To multiply the two algebraic fractions, we use appropriate rules of powers to simplify each fraction, and then reduce all the remaining common factors. So,

$$\frac{2x^{\cancel{2}}\cancel{y^3}}{3\cancel{x}\cancel{y^2}}\cdot\frac{(2x^3y)^2}{2(xy)^3}=\frac{2xy}{3}\cdot\frac{\cancel{4}x^{\cancel{6}}\cancel{y^2}}{\cancel{2}\cancel{x^3}y^{\cancel{3}}}=\frac{2x\cancel{y}\cdot 2x^3}{3\cdot\cancel{y}}=\frac{\mathbf{4x^4}}{\mathbf{3}}=\frac{\mathbf{4}}{\mathbf{3}}\boldsymbol{x^4}$$

equivalent answers

b. After factoring and simplifying, we have

> *Recall:* $x^3-y^3=(x-y)(x^2+xy+y^2)$
>
> $x^2-y^2=(x+y)(x-y)$

$$\frac{x^3-y^3}{x+y}\cdot\frac{3x+3y}{x^2-y^2}=\frac{\cancel{(x-y)}(x^2+xy+y^2)}{x+y}\cdot\frac{3\cancel{(x+y)}}{\cancel{(x-y)}\cancel{(x+y)}}=\frac{\mathbf{3(x^2+xy+y^2)}}{\boldsymbol{x+y}}$$

To divide rational expressions, **multiply** the first, the *dividend*, **by the reciprocal** of the second, the *divisor*.

For instance,

$$\frac{5x-10}{3x}\div\frac{3x-6}{2x^2}=\frac{5x-10}{3x}\cdot\frac{2x^2}{3x-6}=\frac{5\cancel{(x-2)}}{3\cancel{x}}\cdot\frac{2x^{\cancel{2}}}{3\cancel{(x-2)}}=\frac{10x}{9}$$

multiply by the **reciprocal**

follow multiplication rules

Example 5 ► Dividing Algebraic Fractions

Perform operations and simplify. Assume nonzero denominators.

a. $\frac{2x^2+2x}{x-1}\div(x+1)$ **b.** $\frac{x^2-25}{x^2+5x+4}\div\frac{x^2-10x+25}{2x^2+8x}\cdot\frac{x^2+x}{4x^2}$

Solution ► **a.** To divide by $(x+1)$ we multiply by the reciprocal $\frac{1}{(x+1)}$. So,

$$\frac{2x^2+2x}{x-1}\div(x+1)=\frac{2x\cancel{(x+1)}}{x-1}\cdot\frac{1}{\cancel{(x+1)}}=\frac{2x}{x-1}$$

b. The order of operations indicates to perform the division first. To do this, we convert the division into multiplication by the reciprocal of the middle expression. Therefore,

$$\frac{x^2-25}{x^2+5x+4}\div\frac{x^2-10x+25}{2x^2+8x}\cdot\frac{x^2+x}{4x^2}$$

$$=\frac{(x-5)(x+5)}{(x+4)\cancel{(x+1)}}\cdot\frac{2x^2+8x}{x^2-10x+25}\cdot\frac{\cancel{x}\cancel{(x+1)}}{4x^{\cancel{2}}}$$

$$=\frac{\cancel{(x-5)}(x+5)}{\cancel{(x+4)}}\cdot\frac{\cancel{2x}\cancel{(x+4)}}{(x-5)^{\cancel{2}}}\cdot\frac{1}{\cancel{4x}}=\frac{\boldsymbol{(x+5)}}{\mathbf{2}\boldsymbol{(x-5)}}$$

> Reduction of common factors can be done gradually, especially if there is many common factors to divide out.

RT.2 Exercises

True or false.

1. $f(x) = \frac{4}{\sqrt{x}-4}$ is a rational function.

2. The domain of $f(x) = \frac{x-2}{4}$ is the set of all real numbers.

3. $\frac{x-3}{4-x}$ is equivalent to $-\frac{x-3}{x-4}$.

4. $\frac{n^2+1}{n^2-1}$ is equivalent to $\frac{n+1}{n-1}$.

Given the rational function f, find $f(-1)$, $f(0)$, and $f(2)$.

5. $f(x) = \frac{x}{x-2}$

6. $f(x) = \frac{5x}{3x-x^2}$

7. $f(x) = \frac{x-2}{x^2+x-6}$

*For each rational function, find all numbers that are not in the domain. Then give the **domain**, using both **set notation** and **interval notation**.*

8. $f(x) = \frac{x}{x+2}$

9. $g(x) = \frac{x}{x-6}$

10. $h(x) = \frac{2x-1}{3x+7}$

11. $f(x) = \frac{3x+2}{5x-4}$

12. $g(x) = \frac{x+2}{x^2-4}$

13. $h(x) = \frac{x-2}{x^2+4}$

14. $f(x) = \frac{5}{3x-x^2}$

15. $g(x) = \frac{x^2+x-6}{x^2+12x+35}$

16. $h(x) = \frac{7}{|4x-3|}$

17. *Which rational expressions are equivalent and what is their simplest form?*

a. $\frac{2x+3}{2x-3}$ **b.** $\frac{2x-3}{3-2x}$ **c.** $\frac{2x+3}{3+2x}$ **d.** $\frac{2x+3}{-2x-3}$ **e.** $\frac{3-2x}{2x-3}$

18. *Which rational expressions can be simplified?*

a. $\frac{x^2+2}{x^2}$ **b.** $\frac{x^2+2}{2}$ **c.** $\frac{x^2-x}{x^2}$ **d.** $\frac{x^2-y^2}{y^2}$ **e.** $\frac{x}{x^2-x}$

Simplify each expression, if possible.

19. $\frac{24a^3b}{3ab^3}$

20. $\frac{-18x^2y^3}{8x^3y}$

21. $\frac{7-x}{x-7}$

22. $\frac{x+2}{x-2}$

23. $\frac{a-5}{-5+a}$

24. $\frac{(3-y)(x+1)}{(y-3)(x-1)}$

25. $\frac{12x-15}{21}$

26. $\frac{18a-2}{22}$

27. $\frac{4y-12}{4y+12}$

28. $\frac{7x+14}{7x-14}$

29. $\frac{6m+18}{7m+21}$

30. $\frac{3z^2+z}{18z+6}$

31. $\frac{m^2-25}{20-4m}$

32. $\frac{9n^2-3}{4-12n^2}$

33. $\frac{t^2-25}{t^2-10t+25}$

34. $\frac{p^2-36}{p^2+12t+36}$

35. $\frac{x^2-9x+8}{x^2+3x-4}$ **36.** $\frac{p^2+8p-9}{p^2-5p+4}$ **37.** $\frac{x^3-y^3}{x^2-y^2}$ **38.** $\frac{b^2-a^2}{a^3-b^3}$

Perform operations and simplify. Assume nonzero denominators.

39. $\frac{18a^4}{5b^2} \cdot \frac{25b^4}{9a^3}$ **40.** $\frac{28}{xy} \div \frac{63x^3}{2y^2}$ **41.** $\frac{12x}{49(xy^2)^3} \cdot \frac{(7xy)^2}{8}$

42. $\frac{x+1}{2x-3} \cdot \frac{2x-3}{2x}$ **43.** $\frac{10a}{6a-12} \cdot \frac{20a-40}{30a^3}$ **44.** $\frac{a^2-1}{4a} \cdot \frac{2}{1-a}$

45. $\frac{y^2-25}{4y} \cdot \frac{2}{5-y}$ **46.** $(8x-16) \div \frac{3x-6}{10}$ **47.** $(y^2-4) \div \frac{2-y}{8y}$

48. $\frac{3n-9}{n^2-9} \cdot (n^3+27)$ **49.** $\frac{x^2-16}{x^2} \cdot \frac{x^2-4x}{x^2-x-12}$ **50.** $\frac{y^2+10y+25}{y^2-9} \cdot \frac{y^2-3y}{y+5}$

51. $\frac{b-3}{b^2-4b+3} \div \frac{b^2-b}{b-1}$ **52.** $\frac{x^2-6x+9}{x^2+3x} \div \frac{x^2-9}{x}$ **53.** $\frac{x^2-2x}{3x^2-5x-2} \cdot \frac{9x^2-4}{9x^2-12x+4}$

54. $\frac{t^2-49}{t^2+4t-21} \cdot \frac{t^2+8t+15}{t^2-2t-35}$ **55.** $\frac{a^3-b^3}{a^2-b^2} \div \frac{2a-2b}{2a+2b}$ **56.** $\frac{64x^3+1}{4x^2-100} \cdot \frac{4x+20}{64x^2-16x+4}$

57. $\frac{x^3y-64y}{x^3y+64y} \div \frac{x^2y^2-16y^2}{x^2y^2-4xy^2+16y^2}$ **58.** $\frac{p^3-27q^3}{p^2+pq-12q^2} \cdot \frac{p^2-2pq-24q^2}{p^2-5pq-6q^2}$

59. $\frac{4x^2-9y^2}{8x^3-27y^3} \cdot \frac{4x^2+6xy+9y^2}{4x^2+12xy+9y^2}$ **60.** $\frac{2x^2+x-1}{6x^2+x-2} \div \frac{2x^2+5x+3}{6x^2+13x+6}$

61. $\frac{6x^2-13x+6}{14x^2-25x+6} \div \frac{14-21x}{49x^2+7x-6}$ **62.** $\frac{4y^2-12y+36}{27-3y^2} \div (y^3+27)$

63. $\frac{3y}{x^2} \div \frac{y^2}{x} \div \frac{y}{5x}$ **64.** $\frac{x+1}{y-2} \div \frac{2x+2}{y-2} \div \frac{x}{y}$

65. $\frac{a^2-4b^2}{a+2b} \div (a+2b) \cdot \frac{2b}{a-2b}$ **66.** $\frac{9x^2}{x^2-16y^2} \div \frac{1}{x^2+4xy} \cdot \frac{x-4y}{3x}$

67. $\frac{x^2-25}{x-4} \div \frac{x^2-2x-15}{x^2-10x+24} \cdot \frac{x+3}{x^2+10x+25}$ **68.** $\frac{y-3}{y^2-8y+16} \cdot \frac{y^2-16}{y+4} \div \frac{y^2+3y-18}{y^2+11y+30}$

Given $f(x)$ and $g(x)$, find $f(x) \cdot g(x)$ and $f(x) \div g(x)$.

69. $f(x) = \frac{x-4}{x^2+x}$ and $g(x) = \frac{2x}{x+1}$ **70.** $f(x) = \frac{x^3-3x^2}{x+5}$ and $g(x) = \frac{4x^2}{x-3}$

71. $f(x) = \frac{x^2-7x+12}{x+3}$ and $g(x) = \frac{9-x^2}{x-4}$ **72.** $f(x) = \frac{x+6}{4-x^2}$ and $g(x) = \frac{2-x}{x^2+8x+12}$

RT3 Addition and Subtraction of Rational Expressions

Many real-world applications involve adding or subtracting algebraic fractions. Like in the case of common fractions, to add or subtract algebraic fractions, we first need to change them equivalently to fractions with the same denominator. Thus, we begin by discussing the techniques of finding the least common denominator.

Least Common Denominator

The **least common denominator (LCD)** for fractions with given denominators is the same as the **least common multiple (LCM)** of these denominators. The methods of finding the LCD for fractions with numerical denominators were reviewed in *Section R3*. For example,

$$LCD(4,6,8) = 24,$$

because 24 is a multiple of 4, 6, and 8, and there is no smaller natural number that would be divisible by all three numbers, 4, 6, and 8.

Suppose the denominators of three algebraic fractions are $4(x^2 - y^2)$, $-6(x + y)^2$, and $8x$. The numerical factor of the least common multiple is 24. The variable part of the LCM is built by taking the product of all the different variable factors from each expression, with each factor raised to the **greatest** exponent that occurs in any of the expressions. In our example, since $4(x^2 - y^2) = 4(x + y)(x - y)$, then

$$LCD(\ 4(x+y)(x-y),\ \ -6(x+y)^2,\ \ 8x\ ) = \mathbf{24x(x+y)^2(x-y)}$$

Notice that we do not worry about the negative sign of the middle expression. This is because a negative sign can always be written in front of a fraction or in the numerator rather than in the denominator. For example,

$$\frac{1}{-6(x+y)^2} = -\frac{1}{6(x+y)^2} = \frac{-1}{6(x+y)^2}$$

In summary, to find the LCD for algebraic fractions, follow the steps:

- **Factor** each denominator **completely**.
- Build the LCD for the denominators by including the following as factors:
  - **LCD of all numerical coefficients**,
  - all of the **different factors** from each denominator, with each factor **raised to the greatest exponent** that occurs in any of the denominators.
    Note: Disregard any factor of -1.

Example 1 ► **Determining the LCM for the Given Expressions**

Find the LCM for the given expressions.

a. $12x^3y$ and $15xy^2(x-1)$ **b.** $x^2 - 2x - 8$ and $x^2 + 3x + 2$

c. $y^2 - x^2$, $2x^2 - 2xy$, and $x^2 + 2xy + y^2$

Solution ► **a.** Notice that both expressions, $12x^3y$ and $15xy^2(x-1)$, are already in factored form. The $LCM(12{,}15) = 60$, as

divide by 3

$$\begin{array}{c|cc} 3 & 12 & 15 \\ \hline \cdot & 4 & \cdot\, 5 \end{array} = \mathbf{60}$$

no more common factors, so we multiply the numbers in the letter L

The highest power of x is 3, the highest power of y is 2, and $(x-1)$ appears in the first power. Therefore,

$$LCM\left(12x^3y,\ \ 15xy^2(x-1)\right) = \mathbf{60x^3y^2(x-1)}$$

b. To find the LCM of x^2-2x-8 and x^2+3x+2, we factor each expression first:

$$x^2-2x-8 = (x-4)(x+2)$$
$$x^2+3x+2 = (x+1)(x+2)$$

There are three different factors in these expressions, $(x-4)$, $(x+2)$, and $(x+1)$. All of these factors appear in the first power, so

$$LCM(\,x^2-2x-8,\ \ x^2+3x+2\,) = \mathbf{(x-4)(x+2)(x+1)}$$

notice that $(x+2)$ is taken only ones!

c. As before, to find the LCM of y^2-x^2, $2x^2-2xy$, and $x^2+2xy+y^2$, we factor each expression first:

$$y^2-x^2 = (y+x)(y-x) = -(x+y)(x-y)$$
$$2x^2-2xy = 2x(x-y)$$
$$x^2+2xy+y^2 = (x+y)^2$$

as $\boldsymbol{y-x=-(x-y)}$ and $\boldsymbol{y+x=x+y}$

Since the factor of -1 can be disregarded when finding the LCM, the opposite factors can be treated as the same by factoring the -1 out of one of the expressions. So, there are four different factors to consider, 2, x, $(x+y)$, and $(x-y)$. The highest power of $(x+y)$ is 2 and the other factors appear in the first power. Therefore,

$$LCM(\,y^2-x^2,\ \ 2x^2-2xy,\ \ x^2+2xy+y^2\,) = \mathbf{2x(x-y)(x+y)^2}$$

Addition and Subtraction of Rational Expressions

Observe addition and subtraction of common fractions, as review in *Section R3*.

work out the numerator

$$\frac{1}{2}+\frac{2}{3}-\frac{5}{6} = \frac{1\cdot 3+2\cdot 2-5}{6} = \frac{3+4-5}{6} = \frac{2}{6} = \mathbf{\frac{1}{3}}$$

convert fractions to the lowest common denominator

simplify, if possible

To add or subtract algebraic fractions, follow the steps:

- **Factor** the denominators of all algebraic fractions **completely**.
- **Find the LCD** of all the denominators.
- **Convert each algebraic fraction to the lowest common denominator** found in the previous step and write the sum (or difference) as a single fraction.
- **Simplify** the numerator and the whole fraction, if possible.

Example 2 ► **Adding and Subtracting Rational Expressions**

Perform the operations and simplify if possible.

a. $\frac{a}{5}-\frac{3b}{2a}$ **b.** $\frac{x}{x-y}+\frac{y}{y-x}$

c. $\frac{3x^2+3xy}{x^2-y^2}-\frac{2-3x}{x-y}$ **d.** $\frac{y+1}{y^2-7y+6}+\frac{y+2}{y^2-5y-6}$

e. $\frac{2x}{x^2-4}+\frac{5}{2-x}-\frac{1}{2+x}$ **f.** $(2x-1)^{-2}+(2x-1)^{-1}$

Solution ►

Multiplying the numerator and denominator of a fraction by the same factor is equivalent to multiplying the whole fraction by 1, which does not change the value of the fraction.

a. Since $LCM(5,2a)=10a$, we would like to rewrite expressions, $\frac{a}{5}$ and $\frac{3b}{2a}$, so that they have a denominator of $10a$. This can be done by multiplying the numerator and denominator of each expression by the factors of $10a$ that are missing in each denominator. So, we obtain

$$\frac{a}{5}-\frac{3b}{2a}=\frac{a}{5}\cdot\frac{2a}{2a}-\frac{3b}{2a}\cdot\frac{5}{5}=\frac{\mathbf{2a^2-15b}}{\mathbf{10a}}$$

b. Notice that the two denominators, $x-y$ and $y-x$, are opposite expressions. If we write $y-x$ as $-(x-y)$, then

$$\frac{x}{x-y}+\frac{y}{y-x}=\frac{x}{x-y}+\frac{y}{-(x-y)}=\frac{x}{x-y}-\frac{y}{x-y}=\frac{x-y}{x-y}=\mathbf{1}$$

combine the signs

c. To find the LCD, we begin by factoring $x^2-y^2=(x-y)(x+y)$. Since this expression includes the second denominator as a factor, the LCD of the two fractions is $(x-y)(x+y)$. So, we calculate

$$\frac{3x^2+3xy}{x^2-y^2}-\frac{2-3x}{x-y}=\frac{(3x^2+3xy)\cdot 1+(2+3x)\cdot(x+y)}{(x-y)(x+y)}=$$

keep the bracket after a "–" sign

$$\frac{3x^2+3xy-(2x+2y+3x^2+3xy)}{(x-y)(x+y)}=\frac{3x^2+3xy-2x-2y-3x^2-3xy}{(x-y)(x+y)}=$$

$$\frac{-2x-2y}{(x-y)(x+y)}=\frac{-2\cancel{(x+y)}}{(x-y)\cancel{(x+y)}}=\frac{\mathbf{-2}}{\mathbf{(x-y)}}$$

d. To find the LCD, we first factor each denominator. Since

$$y^2 - 7y + 6 = (y-6)(y-1) \text{ and } y^2 - 5y - 6 = (y-6)(y+1),$$

then $LCD = (y-6)(y-1)(y+1)$ and we calculate

$$\frac{y+1}{y^2-7y+6} + \frac{y-1}{y^2-5y-6} = \frac{y+1}{(y-6)(y-1)} + \frac{y-1}{(y-6)(y+1)} =$$

multiply by the missing bracket

$$\frac{(y+1)\cdot(y+1) + (y-1)\cdot(y-1)}{(y-6)(y-1)(y+1)} = \frac{y^2+2y+1+(y^2-1)}{(y-6)(y-1)(y+1)} =$$

$$\frac{2y^2+2y}{(y-6)(y-1)(y+1)} = \frac{2y\cancel{(y+1)}}{(y-6)(y-1)\cancel{(y+1)}} = \frac{\mathbf{2y}}{\mathbf{(y-6)(y-1)}}$$

e. As in the previous examples, we first factor the denominators, including factoring out a negative from any opposite expression. So,

$$\frac{2x}{x^2-4} + \frac{5}{2-x} - \frac{1}{2+x} = \frac{2x}{(x-2)(x+2)} \overset{+}{-} \frac{5}{(x-2)} - \frac{1}{x+2} =$$

$LCD = (x-2)(x+2)$

$$\frac{2x - 5(x+2) - 1(x-2)}{(x-2)(x+2)} = \frac{2x - 5x - 10 - x + 2}{(x-2)(x+2)} =$$

$$\frac{-4x-8}{(x-2)(x+2)} = \frac{-4\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} = \frac{\mathbf{-4}}{\mathbf{(x-2)}}$$

e. Recall that a negative exponent really represents a hidden fraction. So, we may choose to rewrite the negative powers as fractions, and then add them using techniques as shown in previous examples.

$$3(2x-1)^{-2} + (2x-1)^{-1} = \frac{1}{(2x-1)^2} + \frac{1}{2x-1} = \frac{1 + 1\cdot(2x-1)}{(2x-1)^2} =$$

$$\frac{3+2x-1}{(2x-1)^2} = \frac{2x+2}{(2x-1)^2} = \frac{\mathbf{2(x+1)}}{\mathbf{(2x-1)^2}}$$

nothing to simplify this time

Note: Since addition (or subtraction) of rational expressions results in a rational expression, from now on the term "rational expression" will include sums of rational expressions as well.

Example 3 ► **Adding Rational Expressions in Application Problems**

Assume that a boat travels n kilometers up the river and then returns back to the starting point. If the water in the river flows with a constant current of c km/h, the total time for the round-trip can be calculated via the expression $\frac{n}{r+c} + \frac{n}{r-c}$, where r is the speed of the boat in still water in kilometers per hour. Write a single rational expression representing the total time of this trip.

Solution ► To find a single rational expression representing the total time, we perform the addition using $(r+c)(r-c)$ as the lowest common denominator. So,

$$\frac{n}{r+c}+\frac{n}{r-c}=\frac{n(r-c)+n(r+c)}{(r+c)(r-c)}=\frac{nr-nc+nr+nc}{(r+c)(r-c)}=\boldsymbol{\frac{2nr}{r^2-c^2}}$$

Example 4 ► **Adding and Subtracting Rational Functions**

Given $f(x)=\frac{1}{x^2+10x+24}$ and $g(x)=\frac{2}{x^2+4x}$, find

a. $(f+g)(x)$ **b.** $(f-g)(x)$.

Solution ► **a.** $(\boldsymbol{f}+\boldsymbol{g})(\boldsymbol{x})=f(x)+g(x)=\frac{1}{x^2+10x+24}+\frac{2}{x^2+4x}$

$$=\frac{1}{(x+6)(x+4)}+\frac{2}{x(x+4)}=\frac{1\cdot x+2(x+6)}{x(x+6)(x+4)}=\frac{x+2x+12}{x(x+6)(x+4)}$$

$$=\frac{3x+12}{x(x+6)(x+4)}=\frac{3(x+4)}{x(x+6)(x+4)}=\boldsymbol{\frac{3}{x(x+6)}}$$

b. $(\boldsymbol{f}-\boldsymbol{g})(\boldsymbol{x})=f(x)-g(x)=\frac{1}{x^2+10x+24}-\frac{2}{x^2+4x}$

$$=\frac{1}{(x+6)(x+4)}-\frac{2}{x(x+4)}=\frac{1\cdot x-2(x+6)}{x(x+6)(x+4)}=\frac{x-2x-12}{x(x+6)(x+4)}$$

$$=\boldsymbol{\frac{-x-12}{x(x+6)(x+4)}}$$

RT.3 Exercises

1. **a.** What is the LCM for 6 and 9? **b.** What is the LCD for $\frac{1}{6}$ and $\frac{1}{9}$?

2. **a.** What is the LCM for x^2-25 and $x+5$? **b.** What is the LCD for $\frac{1}{x^2-25}$ and $\frac{1}{x+5}$?

Find the LCD and then perform the indicated operations. Simplify the resulting fraction.

3. $\frac{5}{12}+\frac{13}{18}$ **4.** $\frac{11}{30}-\frac{19}{75}$ **5.** $\frac{3}{4}+\frac{7}{30}-\frac{1}{16}$ **6.** $\frac{5}{8}-\frac{7}{12}+\frac{11}{40}$

*Find the **least common multiple** (**LCM**) for each group of expressions.*

7. $24a^3b^4,\ 18a^5b^2$

8. $6x^2y^2,\ 9x^3y,\ 15y^3$

9. $x^2-4,\ x^2+2x$

10. $10x^2,\ 25(x^2-x)$

11. $(x-1)^2,\ 1-x$

12. $y^2-25,\ 5-y$

13. $x^2-y^2,\ xy+y^2$

14. $5a-15,\ a^2-6a+9$

15. $x^2+2x+1,\ x^2-4x-1$

16. $n^2-7n+10,\ n^2-8n+15$

17. $2x^2-5x-3,\ 2x^2-x-1,\ x^2-6x+9$

18. $1-2x,\ 2x+1,\ 4x^2-1$

19. $x^5-4x^4+4x^3,\ 12-3x^2,\ 2x+4$

True or false? If true, explain why. If false, correct it.

20. $\frac{1}{2x}+\frac{1}{3x}=\frac{1}{5x}$

21. $\frac{1}{x-3}+\frac{1}{3-x}=0$

22. $\frac{1}{x}+\frac{1}{y}=\frac{1}{x+y}$

23. $\frac{3}{4}+\frac{x}{5}=\frac{3+x}{20}$

Perform the indicated operations and simplify if possible.

24. $\frac{x-2y}{x+y}+\frac{3y}{x+y}$

25. $\frac{a+3}{a+1}-\frac{a-5}{a+1}$

26. $\frac{4a+3}{a-3}-1$

27. $\frac{n+1}{n-2}+2$

28. $\frac{x^2}{x-y}+\frac{y^2}{y-x}$

29. $\frac{4a-2}{a^2-49}+\frac{5+3a}{49-a^2}$

30. $\frac{2y-3}{y^2-1}-\frac{4-y}{1-y^2}$

31. $\frac{a^3}{a-b}+\frac{b^3}{b-a}$

32. $\frac{1}{x+h}-\frac{1}{x}$

33. $\frac{x-2}{x+3}+\frac{x+2}{x-4}$

34. $\frac{x-1}{3x+1}+\frac{2}{x-3}$

35. $\frac{4xy}{x^2-y^2}+\frac{x-y}{x+y}$

36. $\frac{x-1}{3x+15}-\frac{x+3}{5x+25}$

37. $\frac{y-2}{4y+8}-\frac{y+6}{5y+10}$

38. $\frac{4x}{x-1}-\frac{2}{x+1}-\frac{4}{x^2-1}$

39. $\frac{-2}{y+2}+\frac{5}{y-2}+\frac{y+3}{y^2-4}$

40. $\frac{y}{y^2-y-20}+\frac{2}{y+4}$

41. $\frac{5x}{x^2-6x+8}-\frac{3x}{x^2-x-12}$

42. $\frac{9x+2}{3x^2-2x-8}+\frac{7}{3x^2+x-4}$

43. $\frac{3y+2}{2y^2-y-10}+\frac{8}{2y^2-7y+5}$

44. $\frac{6}{y^2+6y+9}+\frac{5}{y^2-9}$

45. $\frac{3x-1}{x^2+2x-3}-\frac{x+4}{x^2-9}$

46. $\frac{1}{x+1}-\frac{x}{x-2}+\frac{x^2+2}{x^2-x-2}$

47. $\frac{2}{y+3}-\frac{y}{y-1}+\frac{y^2+2}{y^2+2y-3}$

48. $\frac{4x}{x^2-1}+\frac{3x}{1-x}-\frac{4}{x-1}$

49. $\frac{5y}{1-2y}-\frac{2y}{2y+1}+\frac{3}{4y^2-1}$

50. $\frac{x+5}{x-3}-\frac{x+2}{x+1}-\frac{6x+10}{x^2-2x-3}$

Perform the indicated operations and simplify if possible.

51. $2x^{-3}+(3x)^{-1}$

52. $(x^2-9)^{-1}+2(x-3)^{-1}$

53. $\left(\frac{x+1}{3}\right)^{-1}-\left(\frac{x-4}{2}\right)^{-1}$

54. $\left(\frac{a-3}{a^2}-\frac{a-3}{9}\right)\div\frac{a^2-9}{3a}$

55. $\frac{x^2-4x+4}{2x+1}\cdot\frac{2x^2+x}{x^3-4x}-\frac{3x-2}{x+1}$

56. $\frac{2}{x-3}-\frac{x}{x^2-x-6}\cdot\frac{x^2-2x-3}{x^2-x}$

Given $f(x)$ and $g(x)$, find $(f+g)(x)$ and $(f-g)(x)$. Leave the answer in simplified single fraction form.

57. $f(x)=\frac{x}{x+2},\ \ g(x)=\frac{4}{x-3}$

58. $f(x)=\frac{x}{x^2-4},\ \ g(x)=\frac{1}{x^2+4x+4}$

59. $f(x)=\frac{3x}{x^2+2x-3},\ \ g(x)=\frac{1}{x^2-2x+1}$

60. $f(x)=x+\frac{1}{x-1},\ \ g(x)=\frac{1}{x+1}$

Solve each problem.

61. There are two part-time waitresses at a restaurant. One waitress works every fourth day, and the other one works every sixth day. Both waitresses were hired and start working on the same day. How often do they both work on the same day?

62. A cylindrical water tank is being filled and drained at the same time. To find the rate of change of the water level one could use the expression $\frac{H}{T_{in}} - \frac{H}{T_{out}}$, where H is the height of the water in the full tank while T_{in} and T_{out} represent the time needed to fill and empty the tank, respectively. Write the rate of change of the water level as a single algebraic fraction.

63. To determine the Canadian population percent growth over the past year, one could use the expression $100\left(\frac{P_1}{P_0} - 1\right)$, where P_1 represents the current population and P_0 represents the last year's population. Write this expression as a single algebraic fraction.

64. A boat travels k kilometers against a c km/h current. Assuming the current remains constant, one could calculate the total time, in hours, needed for the entire trip via the expression $\frac{k}{s-c} + \frac{k}{s+c}$, where s represents the speed of the boat in calm water. Write this expression as a single algebraic fraction.

RT4 Complex Fractions

When working with algebraic expressions, sometimes we come across needing to simplify expressions like these:

$$\frac{\frac{x^2-9}{x+1}}{\frac{x+3}{x^2-1}},\quad \frac{1+\frac{1}{x}}{1-\frac{1}{y}},\quad \frac{\frac{1}{x+2}-\frac{1}{x+h+2}}{h},\quad \frac{1}{\frac{1}{a}-\frac{1}{b}}$$

A complex fraction is a quotient of rational expressions (including sums of rational expressions) where at least one of these expressions contains a fraction itself. In this section, we will examine two methods of simplifying such fractions.

Simplifying Complex Fractions

Definition 4.1 ► A **complex fraction** is a **quotient** of rational expressions (including their sums) that result in a fraction with more than two levels. For example, $\frac{\frac{1}{2}}{3}$ has three levels while $\frac{\frac{1}{2x}}{\frac{3}{4x}}$ has four levels. Such fractions can be **simplified to a single fraction** with only two levels. For example,

$$\frac{\frac{1}{2}}{3}=\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6},\qquad or \qquad \frac{\frac{1}{2x}}{\frac{3}{4x^2}}=\frac{1}{2\cancel{x}}\cdot\frac{4x^{\cancel{2}\,1}}{3}=\frac{2x}{3}$$

There are two common methods of simplifying complex fractions.

Method I (***multiplying by the reciprocal of the denominator***)

Replace the main division in the complex fraction with a multiplication of the numerator fraction by the reciprocal of the denominator fraction. We then simplify the resulting fraction if possible. Both examples given in *Definition 4.1* were simplified using this strategy.

Method I is the most convenient to use when both the numerator and the denominator of a complex fraction consist of single fractions. However, if either the numerator or the denominator of a complex fraction contains addition or subtraction of fractions, it is usually easier to use the method shown below.

Method II (***multiplying by LCD***)

Multiply the numerator and denominator of a complex fraction by the least common denominator of all the fractions appearing in the numerator or in the denominator of the complex fraction. Then, simplify the resulting fraction if possible. For example, to simplify $\frac{y+\frac{1}{x}}{x+\frac{1}{y}}$, multiply the numerator $y+\frac{1}{x}$ and the denominator $x+\frac{1}{y}$ by the $LCD\left(\frac{1}{x},\frac{1}{y}\right)=xy$. So,

$$\frac{\left(y+\frac{1}{x}\right)}{\left(x+\frac{1}{y}\right)}\cdot\boxed{\frac{xy}{xy}}=\frac{xy^2+y}{x^2y+x}=\frac{y\cancel{(xy+1)}}{x\cancel{(xy+1)}}=\boldsymbol{\frac{y}{x}}$$

Example 1 ► **Simplifying Complex Fractions**

Use a method of your choice to simplify each complex fraction.

a. $\dfrac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$ **b.** $\dfrac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}$

c. $\dfrac{x+\frac{1}{5}}{x-\frac{1}{3}}$ **d.** $\dfrac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}$

Solution ►

a. Since the expression $\dfrac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$ contains a single fraction in both the numerator and denominator, we will simplify it using method I, as below.

$$\frac{\frac{x^2-2x-8}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-4x-21}}=\frac{(x-4)\cancel{(x+2)}}{(x-5)\cancel{(x+3)}}\cdot\frac{(x-7)\cancel{(x+3)}}{(x+6)\cancel{(x+2)}}=\boldsymbol{\frac{(x-4)(x-7)}{(x-5)(x+6)}}$$

factor and multiply by the reciprocal

b. $\dfrac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}$ can be simplified in the following two ways:

Method I

$$\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}=\frac{a+b}{\frac{b^3+a^3}{a^3b^3}}=\frac{(a+b)a^3b^3}{a^3+b^3}$$

$$=\frac{\cancel{(a+b)}a^3b^3}{\cancel{(a+b)}(a^2-ab+b^2)}=\boldsymbol{\frac{a^3b^3}{a^2-ab+b^2}}$$

Method II

$$\frac{a+b}{\frac{1}{a^3}+\frac{1}{b^3}}\cdot\frac{a^3b^3}{a^3b^3}=\frac{(a+b)a^3b^3}{b^3+a^3}$$

$$=\frac{\cancel{(a+b)}a^3b^3}{\cancel{(a+b)}(a^2-ab+b^2)}=\boldsymbol{\frac{a^3b^3}{a^2-ab+b^2}}$$

Caution: In Method II, the factor that we multiply the complex fraction by **must be equal to 1**. This means that **the numerator and denominator of this factor must be exactly the same**.

c. To simplify $\dfrac{x+\frac{1}{5}}{x-\frac{1}{3}}$, we will use method II. Multiplying the numerator and denominator by the $LCD\left(\frac{1}{5},\frac{1}{3}\right)=15$, we obtain

$$\frac{x+\frac{1}{5}}{x-\frac{1}{3}}\cdot\frac{15}{15}=\boldsymbol{\frac{15x+3}{15x-5}}$$

d. Again, to simplify $\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}}$, we will use method II. Notice that the lowest common multiple of the denominators in blue is $(x+2)(x-2)$. So, after multiplying the numerator and denominator of the whole expression by the LCD, we obtain

$$\frac{\frac{6}{x^2-4}-\frac{5}{x+2}}{\frac{7}{x^2-4}-\frac{4}{x-2}} \cdot \frac{(x+2)(x-2)}{(x+2)(x-2)} = \frac{6-5(x-2)}{7-4(x+2)} = \frac{6-5x+10}{7-4x-8}$$

$$= \frac{-5x+16}{-4x-1} = \boldsymbol{\frac{5x-16}{4x+1}}$$

Example 2 ► **Simplifying Rational Expressions with Negative Exponents**

Simplify each expression. Leave the answer with only positive exponents.

a. $\frac{x^{-2}-y^{-1}}{y-x}$ **b.** $\frac{a^{-3}}{a^{-1}-b^{-1}}$

Solution ► **a.** If we write the expression with no negative exponents, it becomes a complex fraction, which can be simplified as in *Example 1*. So,

$$\frac{x^{-2}-y^{-1}}{y-x} = \frac{\frac{1}{x}-\frac{1}{y}}{y-x} \cdot \frac{xy}{xy} = \frac{y-x}{xy(y-x)} = \boldsymbol{\frac{1}{xy}}$$

b. As above, first, we rewrite the expression with only positive exponents and then simplify as any other complex fraction.

Remember! This factor must be = 1

$$\frac{a^{-3}}{a^{-1}-b^{-1}} = \frac{\frac{1}{a^3}}{\frac{1}{a}-\frac{1}{b}} \cdot \frac{a^3b}{a^3b} = \frac{b}{a^2b-a^3} = \boldsymbol{\frac{b}{a^2(b-a)}}$$

Example 3 ► **Simplifying the Difference Quotient for a Rational Function**

Find and simplify the expression $\frac{f(a+h)-f(a)}{h}$ for the function $f(x)=\frac{1}{x+1}$.

Solution ► Since $f(a+h)=\frac{1}{a+h+1}$ and $f(a)=\frac{1}{a+1}$, then

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h+1}-\frac{1}{a+1}}{h}$$

To simplify this expression, we can multiply the numerator and denominator by the lowest common denominator, which is $(a+h+1)(a+1)$. Thus,

This bracket is essential!

$$\frac{\frac{1}{a+h+1}-\frac{1}{a+1}}{h}\cdot\frac{(a+h+1)(a+1)}{(a+h+1)(a+1)}=\frac{a+1-(a+h+1)}{h(a+h+1)(a+1)}$$

$$=\frac{\cancel{a}+\cancel{1}-\cancel{a}-h-\cancel{1}}{h(a+h+1)(a+1)}=\frac{-\cancel{h}}{\cancel{h}(a+h+1)(a+1)}=\frac{\mathbf{-1}}{\mathbf{(a+h+1)(a+1)}}$$

keep the denominator in a factored form

RT.4 Exercises

Simplify each complex fraction.

1. $\frac{2-\frac{1}{3}}{3+\frac{7}{3}}$ **2.** $\frac{5-\frac{3}{4}}{4+\frac{1}{2}}$ **3.** $\frac{\frac{3}{8}-5}{\frac{2}{3}+6}$ **4.** $\frac{\frac{2}{3}+\frac{4}{5}}{\frac{3}{4}-\frac{1}{2}}$

Simplify each complex rational expression.

5. $\frac{\frac{x^3}{y}}{\frac{x^2}{y^3}}$ **6.** $\frac{\frac{n-5}{6n}}{\frac{n-5}{8n^2}}$ **7.** $\frac{1-\frac{1}{a}}{4+\frac{1}{a}}$ **8.** $\frac{\frac{2}{n}+3}{\frac{5}{n}-6}$

9. $\frac{\frac{9-3x}{4x+12}}{\frac{x-3}{6x-24}}$ **10.** $\frac{\frac{9}{y}}{\frac{15}{y}-6}$ **11.** $\frac{\frac{4}{x}-\frac{2}{y}}{\frac{4}{x}+\frac{2}{y}}$ **12.** $\frac{\frac{3}{a}+\frac{4}{b}}{\frac{4}{a}-\frac{3}{b}}$

13. $\frac{a-\frac{3a}{b}}{b-\frac{b}{a}}$ **14.** $\frac{\frac{1}{x}-\frac{1}{y}}{\frac{x^2-y^2}{xy}}$ **15.** $\frac{\frac{4}{y}-\frac{y}{x^2}}{\frac{1}{x}-\frac{2}{y}}$ **16.** $\frac{\frac{5}{p}-\frac{1}{q}}{\frac{1}{5q^2}-\frac{5}{p^2}}$

17. $\frac{\frac{n-12}{n}+n}{n+4}$ **18.** $\frac{2t-1}{\frac{3t-2}{t}+2t}$ **19.** $\frac{\frac{1}{a-h}-\frac{1}{a}}{h}$ **20.** $\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

21. $\frac{4+\frac{12}{2x-3}}{5+\frac{15}{2x-3}}$ **22.** $\frac{1+\frac{3}{x+2}}{1+\frac{6}{x-1}}$ **23.** $\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{b}-\frac{1}{a}}$ **24.** $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$

25. $\frac{\frac{x+3}{x}-\frac{4}{x-1}}{\frac{x}{x-1}+\frac{1}{x}}$ **26.** $\frac{\frac{3}{x^2+6x+9}+\frac{3}{x+3}}{\frac{6}{x^2-9}+\frac{6}{3-x}}$ **27.** $\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^3}+\frac{1}{b^3}}$ **28.** $\frac{\frac{4p^2-12p+9}{2p^2+7p-15}}{\frac{2p^2-15p+18}{p^2-p-30}}$

29. Are the expressions $\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$ and $\frac{x+y}{x^2+y^2}$ equivalent? Explain why or why not.

Simplify each expression. Leave the answer with only ***positive exponents****.*

30. $\frac{1}{a^{-2}-b^{-2}}$ **31.** $\frac{x^{-1}+x^{-2}}{3x^{-1}}$ **32.** $\frac{x^{-2}}{y^{-3}-x^{-3}}$ **33.** $\frac{1-(2n+1)^{-1}}{1+(2n+1)^{-1}}$

Find and simplify the ***difference quotient*** $\frac{f(a+h)-f(a)}{h}$ *for the given function.*

34. $f(x)=\frac{5}{x}$ **35.** $f(x)=\frac{2}{x^2}$ **36.** $f(x)=\frac{1}{1-x}$ **37.** $f(x)=-\frac{1}{x-2}$

Simplify each ***continued fraction****.*

38. $a-\frac{a}{1-\frac{a}{1-a}}$ **39.** $3-\frac{2}{1-\frac{2}{3-\frac{2}{x}}}$ **40.** $a+\frac{a}{2+\frac{1}{1-\frac{2}{a}}}$

RT5 Rational Equations and Graphs

In previous sections of this chapter, we worked with rational expressions. If two rational expressions are equated, a *rational equation* arises. Such equations often appear when solving application problems that involve rates of work or amounts of time considered in motion problems. In this section, we will discuss how to solve rational equations, with close attention to their domains. We will also take a look at the graphs of reciprocal functions, their properties and transformations.

Rational Equations

Definition 5.1 ► A **rational equation** is an equation involving only rational expressions and containing at least one fractional expression.

Here are some examples of rational equations:

$$\frac{x}{2}-\frac{12}{x}=-1, \qquad \frac{x^2}{x-5}=\frac{25}{x-5}, \qquad \frac{2x}{x-3}-\frac{6}{x}=\frac{18}{x^2-3x}$$

Attention! A rational equation contains an *equals* sign, while a rational expression does not. An equation can be solved for a given variable, while an expression can only be simplified or evaluated. For example, $\boldsymbol{\frac{x}{2}-\frac{12}{x}}$ is an **expression** to simplify, while $\boldsymbol{\frac{x}{2}=\frac{12}{x}}$ is an **equation** to solve.
When working with algebraic structures, it is essential to identify whether they are equations or expressions before applying appropriate strategies.

By *Definition 5.1*, rational equations contain one or more denominators. Since division by zero is not allowed, we need to pay special attention to the variable values that would make any of these denominators equal to zero. Such values would have to be excluded from the set of possible solutions. For example, neither 0 nor 3 can be solutions to the equation

$$\frac{2x}{x-3}-\frac{6}{x}=\frac{18}{x^2-3x},$$

as it is impossible to evaluate either of its sides for $x=0$ or 3. So, when solving a rational equation, it is important to find its domain first.

Definition 5.2 ► The **domain** of the variable(s) of a rational equation (in short, the **domain of a rational equation**) is the **intersection of the domains** of all rational expressions within the equation.

As stated in *Definition 2.1*, the domain of each single algebraic fraction is the set of all real numbers except for the **zeros of the denominator** (the variable values that would make the denominator equal to zero). Therefore, the **domain of a rational equation is** the set of **all real numbers except for the zeros of all the denominators** appearing in this equation.

Example 1 ► **Determining Domains of Rational Equations**

Find the domain of the variable in each of the given equations.

a. $\frac{x}{2} - \frac{12}{x} = -1$

b. $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

Solution ►

a. The equation $\frac{x}{2} - \frac{12}{x} = -1$ contains two denominators, 2 and x. 2 is never equal to zero and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{\mathbf{0}\}$.

b. The equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ contains two types of denominators, $x - 2$ and x. The $x - 2$ becomes zero when $x = 2$, and x becomes zero when $x = 0$. Thus, the domain of this equation is $\mathbb{R} \setminus \{\mathbf{0}, \mathbf{2}\}$.

c. The equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ contains three different denominators. To find the zeros of these denominators, we solve the following equations by factoring:

$y^2 - 2y - 8 = 0$	$y^2 + 6y + 8 = 0$	$y^2 - 16 = 0$
$(y-4)(y+2) = 0$	$(y+4)(y+2) = 0$	$(y-4)(y+4) = 0$
$y = 4$ or $y = -2$	$y = -4$ or $y = -2$	$y = 4$ or $y = -4$

So, -4, -2, and 4 must be excluded from the domain of this equation. Therefore, the domain $D = \mathbb{R} \setminus \{\mathbf{-4}, \mathbf{-2}, \mathbf{4}\}$.

To solve a rational equation, it is convenient to clear all the fractions first and then solve the resulting polynomial equation. This can be achieved by multiplying all the terms of the equation by the least common denominator.

Caution! Only **equations**, not expressions, can be changed equivalently by **multiplying** both of their sides by the **LCD**.
Multiplying **expressions** by any number other than 1 creates expressions that are **NOT equivalent** to the original ones. So, avoid multiplying rational expressions by the LCD.

Example 2 ► **Solving Rational Equations**

Solve each equation.

a. $\frac{x}{2} - \frac{12}{x} = -1$

b. $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$

c. $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$

d. $\frac{x-1}{x-3} = \frac{2}{x-3}$

Solution ► **a.** The domain of the equation $\frac{x}{2} - \frac{12}{x} = -1$ is the set $\mathbb{R} \setminus \{0\}$, as discussed in *Example 1a*. The $LCM(2, x) = 2x$, so we calculate

$$\frac{x}{2} - \frac{12}{x} = -1 \qquad / \cdot 2x$$

$$\cancel{2}x \cdot \frac{x}{\cancel{2}} - 2\cancel{x} \cdot \frac{12}{\cancel{x}} = -1 \cdot 2x$$

multiply each term by the **LCD**

$$x^2 - 24 = -2x$$

$$x^2 + 2x - 24 = 0$$

$$(x + 6)(x - 4) = 0$$

factor to find the **possible roots**

$$x = -6 \text{ or } x = 4$$

Since both of these numbers belong to the domain, the solution set of the original equation is $\{\mathbf{-6, 4}\}$.

b. The domain of the equation $\frac{2x}{x-2} = \frac{-3}{x} + \frac{4}{x-2}$ is the set $\mathbb{R} \setminus \{0, 2\}$, as discussed in *Example 1b*. The $LCM(x - 2, x) = x(x - 2)$, so we calculate

$$\frac{2x}{x - 2} = \frac{-3}{x} + \frac{4}{x - 2} \qquad / \cdot x(x - 2)$$

$$x\cancel{(x-2)} \cdot \frac{2x}{\cancel{x-2}} = \frac{-3}{\cancel{x}} \cdot \cancel{x}(x - 2) + \frac{4}{\cancel{x-2}} \cdot x\cancel{(x-2)}$$

$$2x^2 = -3(x - 2) + 4x$$

$$2x^2 = -3x + 6 + 4x$$

expand the bracket,
collect like terms, and
bring the terms over to one side

$$2x^2 - x + 6 = 0$$

$$(2x + 3)(x - 2) = 0$$

factor to find the **possible roots**

$$x = -\frac{3}{2} \text{ or } x = 2$$

Since 2 is excluded from the domain, there is only one solution to the original equation, $x = -\frac{3}{2}$.

c. The domain of the equation $\frac{2}{y^2-2y-8} - \frac{4}{y^2+6y+8} = \frac{2}{y^2-16}$ is the set $\mathbb{R} \setminus \{-4, -2, 4\}$, as discussed in *Example 1c*. To find the LCD, it is useful to factor the denominators first. Since
$y^2 - 2y - 8 = (y - 4)(y + 2)$,
$y^2 + 6y + 8 = (y + 4)(y + 2)$, and
$y^2 - 16 = (y - 4)(y + 4)$, then the LCD needed to clear the fractions in the original equation is $(y - 4)(y + 4)(y + 2)$. So, we calculate

$$\frac{2}{(y - 4)(y + 2)} - \frac{4}{(y + 4)(y + 2)} = \frac{2}{(y - 4)(y + 4)} \qquad / \cdot (y - 4)(y + 4)(y + 2)$$

$$\cancel{(y-4)}(y+4)\cancel{(y+2)} \cdot \frac{2}{\cancel{(y-4)}\cancel{(y+2)}} - (y-4)\cancel{(y+4)}\cancel{(y+2)} \cdot \frac{4}{\cancel{(y+4)}\cancel{(y+2)}}$$

$$= \frac{2}{\cancel{(y-4)}\cancel{(y+4)}} \cdot \cancel{(y-4)}\cancel{(y+4)}(y+2)$$

$$2(y+4) - 4(y-4) = 2(y+2)$$

$$2y + 8 - 4y + 16 = 2y + 4$$

$$20 = 4y$$

$$y = 5$$

Since 5 is in the domain, this is the true solution.

d. First, we notice that the domain of the equation $\frac{x-1}{x-3} = \frac{2}{x-3}$ is the set $\mathbb{R} \setminus \{3\}$. To solve this equation, we can multiply it by the $LCD = x - 3$, as in the previous examples, or we can apply the method of cross-multiplication, as the equation is a proportion. Here, we show both methods.

Use the method of your choice – either one is fine.

Multiplication by LCD:

$$\frac{x-1}{x-3} = \frac{2}{x-3} \qquad / \cdot (x-3)$$

this multiplication is permitted as $x - 3 \neq 0$

$$x - 1 = 2$$

$$x = 3$$

Cross-multiplication:

$$\frac{x-1}{x-3} = \frac{2}{x-3}$$

$$(x-1)(x-3) = 2(x-3) \qquad / \div (x-3)$$

this division is permitted as $x - 3 \neq 0$

$$x - 1 = 2$$

$$x = 3$$

Since 3 is excluded from the domain, there is **no solution** to the original equation.

Summary of Solving Rational Equations in One Variable

- **Determine the domain** of the variable.
- **Clear** all the **fractions by multiplying** both sides of the equation **by the LCD** of these fractions.
- **Find possible solutions** by solving the resulting equation.
- **Check** the possible solutions **against the domain**. The solution set consists of only these possible solutions that belong to the domain.

Graphs of Basic Rational Functions

So far, we discussed operations on rational expressions and solving rational equations. Now, we will look at rational functions, such as

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{-2}{x+3}, \qquad or \qquad h(x) = \frac{x-3}{x-2}.$$

Definition 5.3 ► A **rational function** is any function that can be written in the form

$$\boldsymbol{f(x) = \frac{P(x)}{Q(x)}},$$

where P and Q are polynomials and Q is not a zero polynomial.

The **domain** $\boldsymbol{D_f}$ of such function f includes all x-values for which $Q(x) \neq 0$.

Example 3 ► **Finding the Domain of a Rational Function**

Find the domain of each function.

a. $g(x) = \frac{-2}{x+3}$ **b.** $h(x) = \frac{x-3}{x-2}$

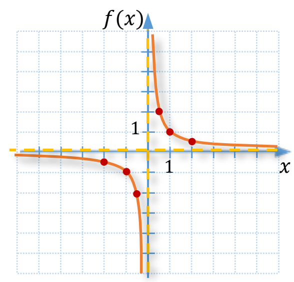
Solution ► **a.** Since $x + 3 = 0$ for $x = -3$, the domain of g is the set of all real numbers except for -3. So, the domain $\boldsymbol{D_g} = \mathbb{R} \setminus \{\boldsymbol{-3}\}$.

b. Since $x - 2 = 0$ for $x = 2$, the domain of h is the set of all real numbers except for 2. So, the domain $\boldsymbol{D_h} = \mathbb{R} \setminus \{\boldsymbol{2}\}$.

Note: The subindex f in the notation D_f indicates that the domain is of function f.

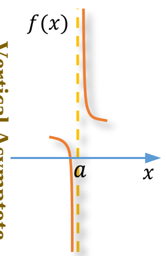
To graph a rational function, we usually start by making a table of values. Because the graphs of rational functions are typically nonlinear, it is a good idea to plot at least 3 points on each side of each x-value where the function is undefined. For example, to graph the basic rational function, $f(x) = \frac{1}{x}$, called the ***reciprocal function***, we evaluate f for a few points to the right of zero and to the left of zero. This is because f is undefined at $x = 0$, which means that the graph of f does not cross the y-axis. After plotting the obtained points, we connect them within each group, to the right of zero and to the left of zero, creating two disjoint curves. To see the shape of each curve clearly, we might need to evaluate f at some additional points.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\frac{1}{2}$	2
$\boldsymbol{1}$	1
2	$\frac{1}{2}$
$\boldsymbol{0}$	*undefined*
$-\frac{1}{2}$	-2
$\boldsymbol{-1}$	-1
$\boldsymbol{-2}$	$-\frac{1}{2}$

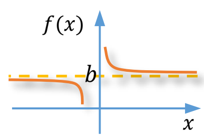


The **domain** of the reciprocal function $f(x) = \frac{1}{x}$ is $\mathbb{R} \setminus \{\boldsymbol{0}\}$, as the denominator x must be different than zero. Projecting the graph of this function onto the y-axis helps us determine the **range**, which is also $\mathbb{R} \setminus \{\boldsymbol{0}\}$.

There is another interesting feature of the graph of the reciprocal function $f(x) = \frac{1}{x}$. Observe that the graph approaches two lines, $y = 0$, the x-axis, and $x = 0$, the y-axis. These lines are called **asymptotes**. They effect the shape of the graph, but they themselves do not belong to the graph. To indicate the fact that asymptotes do not belong to the graph, we use a dashed line when graphing them.



Vertical Asymptote



Horizontal Asymptote

In general, if the y-values of a rational function approach ∞ or $-\infty$ as the x-values approach a real number a, the vertical line $x = a$ is a vertical asymptote of the graph. This can be recorded with the use of arrows, as follows:

read: approaches

$x = a$ is a **vertical asymptote** $\Leftrightarrow y \to \infty$ (or $-\infty$) when $x \to a$.

Also, if the y-values approach a real number b as x-values approach ∞ or $-\infty$, the horizontal line $y = b$ is a horizontal asymptote of the graph. Again, using arrows, we can record this statement as:

$y = a$ is a **horizontal asymptote** $\Leftrightarrow y \to b$ when $x \to \infty$ (or $-\infty$).

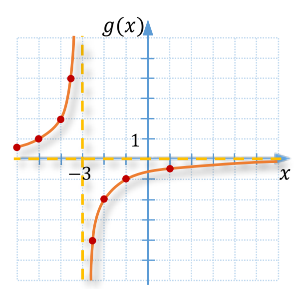
Example 4 ► **Graphing and Analysing the Graphs of Basic Rational Functions**

For each function, state its domain and the equation of the vertical asymptote, graph it, and then state its range and the equation of the horizontal asymptote.

a. $g(x) = \frac{-2}{x+3}$ **b.** $h(x) = \frac{x-3}{x-2}$

Solution ► **a.** The domain of function $g(x) = \frac{-2}{x+3}$ is $\boldsymbol{D_g} = \mathbb{R} \setminus \{-\mathbf{3}\}$, as discussed in *Example 3a*. Since -3 is excluded from the domain, we expect the vertical asymptote to be at $\boldsymbol{x} = -\mathbf{3}$.

x	$g(x)$
$-\frac{5}{2}$	-4
-2	-2
-1	-1
1	$-\frac{1}{2}$
-3	*undefined*
$-\frac{7}{2}$	4
-4	2
-5	1
-6	$\frac{2}{3}$

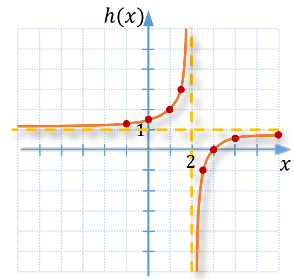


To graph function g, we evaluate it at some points to the right and to the left of -3. The reader is encouraged to check the values given in the table. Then, we draw the vertical asymptote $x = -3$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the x-axis. Indeed, the value of zero cannot be attained by the function $g(x) = \frac{-2}{x+3}$, as in order for a fraction to become zero, its numerator would have to be zero. So, the range of function g is $\mathbb{R} \setminus \{\mathbf{0}\}$ and $\boldsymbol{y} = \mathbf{0}$ is the equation of the horizontal asymptote.

b. The domain of function $h(x) = \frac{x-3}{x-2}$ is $\boldsymbol{D_h} = \mathbb{R} \setminus \{\mathbf{2}\}$, as discussed in *Example 3b*. Since 2 is excluded from the domain, we expect the vertical asymptote to be at $\boldsymbol{x} = \mathbf{2}$.

x	$h(x)$
-1	$\frac{4}{3}$
0	$\frac{3}{2}$
1	2
$\frac{3}{2}$	3
2	*undefined*
$\frac{5}{2}$	-1
3	0
4	$\frac{1}{2}$
6	$\frac{3}{4}$

As before, to graph function h, we evaluate it at some points to the right and to the left of 2. Then, we draw the vertical asymptote $x = 2$ and plot and join the obtained points on each side of this asymptote. The graph suggests that the horizontal asymptote is the line $\boldsymbol{y = 1}$. Thus, the range of function h is $\mathbb{R} \setminus \{\mathbf{1}\}$.



Notice that $\frac{x-3}{x-2} = \frac{x-2-1}{x-2} = \frac{x-2}{x-2} - \frac{1}{x-2} = 1 - \frac{1}{x-2}$. Since $\frac{1}{x-2}$ is never equal to zero than $1 - \frac{1}{x-2}$ is never equal to 1. This confirms the range and the horizontal asymptote stated above.

Example 5 ► **Connecting the Algebraic and Graphical Solutions of Rational Equations**

Given that $f(x) = \frac{x+2}{x-1}$, find all the x-values for which $f(x) = 2$. Illustrate the situation with a graph.

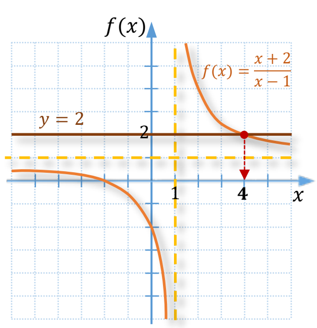
Solution ► To find all the x-values for which $f(x) = 2$, we replace $f(x)$ in the equation $f(x) = \frac{x+2}{x-1}$ with 2 and solve the resulting equation. So, we have

$$2 = \frac{x+2}{x-1} \qquad / \cdot (x-1)$$

$$2x - 2 = x + 2 \qquad / -x, \quad +2$$

$$x = 4$$

Thus, $f(x) = 2$ for $\boldsymbol{x = 4}$.



The geometrical connection can be observed by graphing the function $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$ and the line $y = 2$ on the same grid, as illustrated by the accompanying graph. The x-coordinate of the intersection of the two graphs is the solution to the equation $2 = \frac{x+2}{x-1}$. This also means that $f(4) = \frac{4+2}{4-1} = 2$. So, we can say that $f(4) = 2$.

Example 6 ► **Graphing the Reciprocal of a Linear Function**

Suppose $f(x) = 2x - 3$.

a. Determine the reciprocal function $g(x) = \frac{1}{f(x)}$ and its domain D_g.

b. Determine the equation of the vertical asymptote of the reciprocal function g.

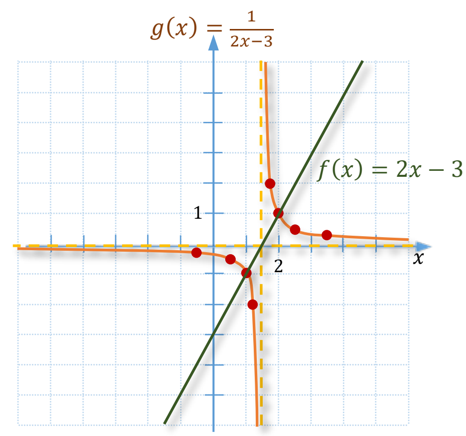
c. Graph the function f and its reciprocal function g on the same grid. Then, describe the relations between the two graphs.

Solution ► **a.** The reciprocal of $f(x) = 2x - 3$ is the function $\boldsymbol{g(x) = \frac{1}{2x-3}}$. Since $2x - 3 = 0$ for $x = \frac{3}{2}$, then the domain $\boldsymbol{D_g} = \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$.

b. A vertical asymptote of a rational function in simplified form is a vertical line passing through any of the x-values that are excluded from the domain of such a function. So, the equation of the vertical asymptote of function $g(x) = \frac{1}{2x-3}$ is $\boldsymbol{x = \frac{3}{2}}$.

c. To graph functions f and g, we can use a table of values as below.

x	$f(x)$	$g(x)$
$-\frac{1}{2}$	-4	$-\frac{1}{4}$
$\frac{1}{2}$	-2	$-\frac{1}{2}$
1	-1	-1
$\frac{5}{4}$	$\frac{1}{2}$	2
$\frac{3}{2}$	0	*undefined*
$\frac{7}{4}$	$-\frac{1}{2}$	-2
2	1	1
$\frac{5}{2}$	2	$\frac{1}{2}$
$\frac{7}{2}$	4	$\frac{1}{4}$



Notice that the vertical asymptote of the reciprocal function comes through the zero of the linear function. Also, the values of both functions are positive to the right of $\frac{3}{2}$ and negative to the left of $\frac{3}{2}$. In addition, $f(2) = g(2) = 1$ and $f(1) = g(1) = -1$. This is because the reciprocal of 1 is 1 and the reciprocal of -1 is -1. For the rest of the values, observe that the values of the linear function that are very close to zero become very large in the reciprocal function and conversely, the values of the linear function that are very far from zero become very close to zero in the reciprocal function. This suggests the horizontal asymptote at zero.

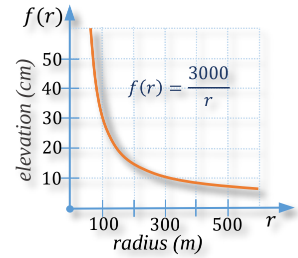
Example 7 ► Using Properties of a Rational Function in an Application Problem

Elevating the outer rail of a track allows for a safer turn of a train on a circular curve. The elevation depends on the allowable speed of the train and the radius of the curve. Suppose that a circular curve with a radius of r meters is being designed for a train travelling 100 kilometers per hour. Assume that the function $f(r) = \frac{3000}{r}$ can be used to calculate the proper elevation $y = f(r)$, in centimeters, for the outer rail.

a. Evaluate $f(300)$ and interpret the result.

b. Suppose that the outer rail for a curve is elevated 12 centimeters. Find the radius of this curve.

c. Observe the accompanying graph of the function f and discuss how the elevation of the outer rail changes as the radius r increases.



Solution

a. $f(300) = \frac{3000}{300} = 10$. Thus, the outer rail on a curve with a 300-meter radius should be elevated **10 centimeters** for a train to travel through it at 100 km/hr safely.

b. Since the elevation $y = f(r) = 12$ centimeters, to find the corresponding value of r, we need to solve the equation

$$12 = \frac{3000}{r}.$$

After multiplying this equation by r and dividing it by 12, we obtain

$$r = \frac{3000}{12} = 250$$

So, the radius of this curve should be **250 meters**.

c. As the radius increases, the outer rail needs less elevation.

RT.5 Exercises

*State the **domain** for each equation. There is no need to solve it.*

1. $\frac{x+5}{4} - \frac{x+3}{3} = \frac{x}{6}$

2. $\frac{5}{6a} - \frac{a}{4} = \frac{8}{2a}$

3. $\frac{3}{x+4} = \frac{2}{x-9}$

4. $\frac{4}{3x-5} + \frac{2}{x} = \frac{9}{4x+7}$

5. $\frac{4}{y^2-25} - \frac{1}{y+5} = \frac{2}{y-7}$

6. $\frac{x}{2x-6} - \frac{3}{x^2-6x+9} = \frac{x-2}{3x-9}$

Solve each equation.

7. $\frac{3}{8} + \frac{1}{3} = \frac{x}{12}$

8. $\frac{1}{4} - \frac{5}{6} = \frac{1}{y}$

9. $x + \frac{8}{x} = -9$

10. $\frac{4}{3a} - \frac{3}{a} = \frac{10}{3}$

11. $\frac{r}{8} + \frac{r-4}{12} = \frac{r}{24}$

12. $\frac{n-2}{2} - \frac{n}{6} = \frac{4n}{9}$

13. $\frac{5}{r+20} = \frac{3}{r}$

14. $\frac{5}{a+4} = \frac{3}{a-2}$

15. $\frac{y+2}{y} = \frac{5}{3}$

16. $\frac{x-4}{x+6} = \frac{2x+3}{2x-1}$

17. $\frac{x}{x-1} - \frac{x^2}{x-1} = 5$

18. $3 - \frac{12}{x^2} = \frac{5}{x}$

19. $\frac{1}{3} - \frac{x-1}{x} = \frac{x}{3}$

20. $\frac{1}{x} + \frac{2}{x+10} = \frac{x}{x+10}$

21. $\frac{1}{y-1} + \frac{5}{12} = \frac{-2}{3y-3}$

22. $\frac{7}{6x+3} - \frac{1}{3} = \frac{2}{2x+1}$

23. $\frac{8}{3k+9} - \frac{8}{15} = \frac{2}{5k+15}$

24. $\frac{6}{m-4} + \frac{5}{m} = \frac{2}{m^2-4m}$

25. $\frac{3}{y-2} + \frac{2y}{4-y^2} = \frac{5}{y+2}$

26. $\frac{x}{x-2} + \frac{x}{x^2-4} = \frac{x+3}{x+2}$

27. $\frac{1}{2x+10} = \frac{8}{x^2-25} - \frac{2}{x-5}$

28. $\frac{5}{y+3} = \frac{1}{4y^2-36} + \frac{2}{y-3}$

29. $\frac{6}{x^2-4x+3} - \frac{1}{x-3} = \frac{1}{4x-4}$

30. $\frac{7}{x-2} - \frac{8}{x+5} = \frac{1}{2x^2+6x-20}$

31. $\frac{5}{x-4} - \frac{3}{x-1} = \frac{x^2-1}{x^2-5x+4}$

32. $\frac{y}{y+1} + \frac{3y+5}{y^2+4y+3} = \frac{2}{y+3}$

33. $\frac{3x}{x+2} + \frac{72}{x^3+8} = \frac{24}{x^2-2x+4}$

34. $\frac{4}{x+3} + \frac{7}{x^2-3x+9} = \frac{108}{x^3+27}$

35. $\frac{x}{2x-9} - 3x = \frac{10}{9-2x}$

36. $\frac{-2}{x^2+2x-3} - \frac{5}{3-3x} = \frac{4}{3x+9}$

For the given rational function f, find all values of x for which $f(x)$ has the indicated value.

37. $f(x) = 2x - \frac{15}{x}$; $f(x) = 1$

38. $f(x) = \frac{x-5}{x+1}$; $f(x) = \frac{3}{5}$

39. $g(x) = \frac{-3x}{x+3} + x$; $g(x) = 4$

40. $g(x) = \frac{4}{x} + \frac{1}{x-2}$; $g(x) = 3$

*Graph each rational function. State its **domain**, **range** and the equations of the **vertical** and **horizontal asymptotes**.*

41. $f(x) = \frac{2}{x}$

42. $g(x) = -\frac{1}{x}$

43. $h(x) = \frac{2}{x-3}$

44. $f(x) = \frac{-1}{x+1}$

45. $g(x) = \frac{x-1}{x+2}$

46. $h(x) = \frac{x+2}{x-3}$

*For each function f, find its reciprocal function $g(x) = \frac{1}{f(x)}$ and graph both functions on the same grid. Then, state the equations of the **vertical** and **horizontal asymptotes** of function g.*

47. $f(x) = \frac{1}{2}x + 1$

48. $f(x) = -x + 2$

49. $f(x) = -2x - 3$

Solve each equation.

50. $\frac{x}{1+\frac{1}{x+1}} = x - 3$

51. $\frac{2-\frac{1}{x}}{4-\frac{1}{x^2}} = 1$

Solve each problem.

52. Suppose that the number of vehicles searching for a parking place at UFV parking lot is modelled by the function

$$f(x) = \frac{x^2}{2(1-x)},$$

where $0 \le x < 1$ is a quantity known as ***traffic intensity***.

a. For each traffic intensity, find the number of vehicles searching for a parking place. *Round your answer to the nearest one.*

i. 0.2 **ii.** 0.8 **iii.** 0.98

b. Observing answers to part (**a**), conclude how does the number of vehicles searching for a parking place changes when the traffic intensity get closer to 1.

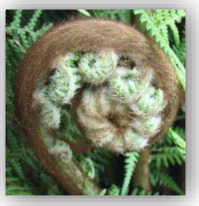
53. Suppose that the percent of deaths caused by smoking, called the ***incidence rate,*** is modelled by the rational function

$$D(x) = \frac{x-1}{x},$$

where x tells us how many times a smoker is more likely to die of lung cancer than a non-smoker.

a. Find $D(10)$ and interpret it in the context of the problem.
b. Find the x-value corresponding to the incidence rate of 0.5.
c. Under what condition would the incidence rate equal to 0?

RT6 Applications of Rational Equations



In previous sections of this chapter, we studied operations on rational expressions, simplifying complex fractions, and solving rational equations. These skills are needed when working with real-world problems that lead to a rational equation. The common types of such problems are motion or work problems. In this section, we first discuss how to solve a rational formula for a given variable, and then present several examples of application problems involving rational equations.

Formulas Containing Rational Expressions

Solving application problems often involves working with formulas. We might need to form a formula, evaluate it, or solve it for a desired variable. The basic strategies used to solve a formula for a variable were shown in *Section L2* and *F4*. Recall the guidelines that we used to isolate the desired variable:

- **Reverse operations** to clear unwanted factors or addends;
  Example: To solve $\frac{A+B}{2} = C$ for A, we multiply by 2 and then subtract B.

- **Multiply by the LCD to keep** the desired variable **in the numerator**;
  Example: To solve $\frac{A}{1+r} = P$ for r, first, we multiply by $(1 + r)$.

- **Take the reciprocal** of both sides of the equation **to keep** the desired variable **in the numerator** (this applies to proportions only);
  Example: To solve $\frac{1}{C} = \frac{A+B}{AB}$ for C, we can take the reciprocal of both sides to obtain $C = \frac{AB}{A+B}$.

- **Factor** to **keep** the desired variable **in one place**.
  Example: To solve $P + Prt = A$ for P, we first factor P out.

Below we show how to solve formulas containing rational expressions, using a combination of the above strategies.

Example 1 ► **Solving Rational Formulas for a Given Variable**

Solve each formula for the indicated variable.

a. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, for p **b.** $L = \frac{dR}{D-d}$, for D **c.** $L = \frac{dR}{D-d}$, for d

Solution ► **a.** ***Solution I:*** First, we isolate the term containing p, by 'moving' $\frac{1}{q}$ to the other side of the equation. So,

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad / -\frac{1}{q}$$

rewrite from the right to the left,

$$\frac{1}{f} - \frac{1}{q} = \frac{1}{p}$$

$$\frac{1}{p} = \frac{q-f}{fq}$$

and perform the subtraction to leave this side as a single fraction

Then, to bring p to the numerator, we can take the reciprocal of both sides of the equation, obtaining

$$\boldsymbol{p = \frac{fq}{q - f}}$$

Caution! This method can be applied only to a proportion (an equation with a **single fraction on each side**).

Solution II: The same result can be achieved by multiplying the original equation by the $LCD = fpq$, as shown below

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \qquad /\cdot fpq$$

$$pq = fq + fp \qquad /-fp$$

factor p out

$$pq - fp = fq$$

$$p(q - f) = fq \qquad /\div (q - f)$$

$$\boldsymbol{p = \frac{fq}{q - f}}$$

b. To solve $L = \frac{dR}{D-d}$ for D, we may start with multiplying the equation by the denominator to bring the variable D to the numerator. So,

This can be done in one step by interchanging L with $D - d$. The movement of the expressions resembles that of a teeter-totter.

$$L = \frac{dR}{D - d} \qquad /\cdot (D - d)$$

$$L(D - d) = d \qquad /\div L$$

$$D - d = \frac{dR}{L} \qquad /+d$$

$$\boldsymbol{D = \frac{dR}{L} + d = \frac{dR + dL}{L}}$$

Both forms are correct answers.

c. When solving $L = \frac{dR}{D-d}$ for d, we first observe that the variable d appears in both the numerator and denominator. Similarly as in the previous example, we bring the d from the denominator to the numerator by multiplying the formula by the denominator $D - d$. Thus,

$$L = \frac{dR}{D - d} \qquad /\cdot (D - d)$$

$$L(D - d) = dR.$$

Then, to keep the d in one place, we need to expand the bracket, collect terms with d, and finally factor the d out. So, we have

$$LD - Ld = dR \qquad /+Ld$$

$$LD = dR + Ld$$

$$LD = d(R + L) \qquad /\div(R + L)$$

$$\frac{LD}{R + L} = d$$

Obviously, the final formula can be written starting with d,

$$\boldsymbol{d = \frac{LD}{R + L}}.$$

Example 2 ► **Forming and Evaluating a Rational Formula**

Suppose a trip consists of two parts of the same distance d.

a. Given the speed v_1 for the first part of the trip and v_2 for the second part of the trip, find a formula for the average speed v for the whole trip. (*Make sure to leave this formula in the simplified form.*)

b. Find the average speed v for the whole trip, if the speed for the first part of the trip was 75 km/h and the speed for the second part of the trip was 105 km/h.

c. How does the v-value from (**b**) compare to the average of v_1 and v_2?

Solution ►

a. The total distance, D, for the whole trip is $d + d = 2d$. The total time, T, for the whole trip is the sum of the times for the two parts of the trip, t_1 and t_2. From the relation $rate \cdot time = distance$, we have

$$t_1 = \frac{d}{v_1} \quad \text{and} \quad t_2 = \frac{d}{v_2}.$$

Therefore,

$$t = \frac{d}{v_1} + \frac{d}{v_2},$$

which after substituting to the formula for the average speed, $V = \frac{D}{T}$, gives us

$$V = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}}.$$

Since the formula involves a complex fraction, it should be simplified. We can do this by multiplying the numerator and denominator by the $LCD = v_1 v_2$. So, we have

$$V = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}} \cdot \frac{v_1 v_2}{v_1 v_2}$$

$$V = \frac{2d v_1 v_2}{\frac{d\cancel{v_1} v_2}{\cancel{v_1}} + \frac{d v_1 \cancel{v_2}}{\cancel{v_2}}}$$

$$V=\frac{2dv_1v_2}{dv_2+dv_1}$$ factor the d

$$V=\frac{2\cancel{d}v_1v_2}{\cancel{d}(v_2+v_1)}$$

$$\boldsymbol{V=\frac{2v_1v_2}{v_2+v_1}}$$

Note 1: The average speed in this formula does not depend on the distance travelled.

Note 2: The average speed for the total trip is <u>not</u> the average (arithmetic mean) of the speeds for each part of the trip. In fact, this formula represents the **harmonic mean** of the two speeds.

b. Since $v_1=75$ km/h and $v_2=105$ km/h, using the formula developed in *Example 2a*, we calculate

$$v=\frac{2\cdot 75\cdot 105}{75+105}=\frac{15750}{180}=\mathbf{87.5\ km/h}$$

c. The average speed for the whole trip, $v=87.5$ km/h, is lower than the average of the speeds for each part of the trip, which is $\frac{75+105}{2}=90$ km/h.

Applied Problems

Many types of application problems were already introduced in *Sections L3* and *E2*. Some of these types, for example motion problems, may involve solving rational equations. Below we show examples of proportion and motion problems as well as introduce another type of problems, work problems.

Proportion Problems

When forming a proportion,

$$\boxed{\frac{category\ I\ before}{category\ II\ before}=\frac{category\ I\ after}{category\ II\ after},}$$

it is essential that the same type of data is placed in the same row or the same column.

Recall: To solve a proportion

$$\frac{a}{b}=\frac{c}{d},$$

for example, for a, it is enough to multiply the equation by b. This gives us

$$a=\frac{bc}{d}.$$

Similarly, to solve

$$\frac{a}{b} = \frac{c}{d}$$

for b, we can use the cross-multiplication method, which eventually (we encourage the reader to check this) leads us to

$$a = \frac{ad}{c}.$$

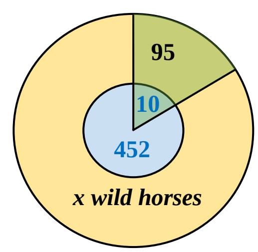
Notice that in both cases the desired variable equals the **product** of the blue variables lying **across** each other, **divided by the remaining** purple variable. This is often referred to as the 'cross multiply and divide' approach to solving a proportion.

In statistics, proportions are often used to estimate the population by analysing its sample in situations where the exact count of the population is too costly or not possible to obtain.

Example 3 ► **Estimating Numbers of Wild Animals**

To estimate the number of wild horses in a particular area in Nevada, a forest ranger catches 452 wild horses, tags them, and releases them. In a week, he catches 95 horses out of which 10 are found to be tagged. Assuming that the horses mix freely when they are released, estimate the number of wild horses in this region. *Round your answer to the nearest hundreds.*

Solution ►



Suppose there are x wild horses in region. 452 of them were tagged, so the ratio of the tagged horses to the whole population of the wild horses there is

$$\frac{452}{x}$$

The ratio of the tagged horses found in the sample of 95 horses caught in the later time is

$$\frac{10}{95}$$

So, we form the proportion:

$$\frac{452}{x} = \frac{10}{95}$$

tagged horses
all horses
population
sample

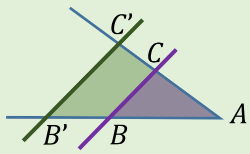
After solving for x, we have

$$x = \frac{452 \cdot 95}{10} = 4294 \approx 4300$$

So, we can say that approximately **4300** wild horses live in this region.

In geometry, proportions are the defining properties of similar figures. One frequently used theorem that involves proportions is the theorem about similar triangles, attributed to the Greek mathematician Thales.

Thales' Theorem ► Two triangles are **similar** iff the ratios of the corresponding sides are the same.



$$\triangle ABC \sim \triangle AB'C' \Leftrightarrow \frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}$$

Example 4 ► **Using Similar Triangles in an Application Problem**

A cross-section of a small storage room is in the shape of a right triangle with a height of 2 meters and a base of 1.2 meters, as shown in *Figure 6.1a*. Find the size of the largest cubic box fitting in this room when placed with its base on the floor.

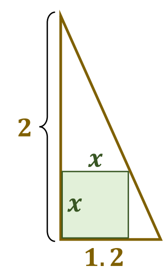


Figure 6.1a

Solution ► Suppose that the height of the box is x meters. Since the height of the storage room is 2 meters, the expression $2-x$ represents the height of the wall above the box, as shown in *Figure 6.1b*.

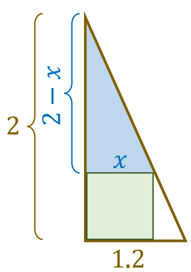


Figure 6.1b

Since the blue and brown triangles are similar, we can use the Thales' Theorem to form the proportion

$$\frac{2-x}{2} = \frac{x}{1.2}.$$

Employing cross-multiplication, we obtain

$$2.4 - 1.2x = 2x$$

$$2.4 = 3.2x$$

$$x = \frac{2.4}{3.2} = \mathbf{0.75}$$

So, the dimensions of the largest cubic box fitting in this storage room are 75 cm by 75 cm by 75 cm.

Motion Problems

Motion problems in which we compare times usually involve solving rational equations. This is because when solving the motion formula $rate\ \boldsymbol{R} \cdot time\ \boldsymbol{T} = distance\ \boldsymbol{D}$ for time, we create a fraction

$$time\ \boldsymbol{T} = \frac{distance\ \boldsymbol{D}}{rate\ \boldsymbol{R}}$$

Example 5 ► **Solving a Motion Problem Where Times are the Same**

Two bikers participate in a Cross-Mountain Crusher. One biker is 2 km/h faster than the other. The faster biker travels 35 km in the same amount of time that it takes the slower biker to cover only 30 km. Find the average speed of each biker.

Solution ► Let r represent the average speed of the slower biker. Then $r+2$ represents the average speed of the faster biker. The slower biker travels 30 km, while the faster biker travels 35 km. Now, we can complete the table

	R ·	T	= D
slower biker	r	$\frac{30}{r}$	30
faster biker	$r+2$	$\frac{35}{r+2}$	35

To complete the ***Time*** column, we divide the ***Distance*** by the ***Rate***.

Since the time of travel is the same for both bikers, we form and then solve the equation:

$$\frac{30}{r} = \frac{35}{r+2} \qquad /\div 5 \text{ and cross-multiply}$$

$$6(r+2) = 7r$$

$$6r + 12 = 7r \qquad /-6r$$

$$r = 12$$

Thus, the average speed of the slower biker is $r = \mathbf{12}$ km/h and the average speed of the faster biker is $r + 2 = \mathbf{14}$ km/h.

Example 6 ► **Solving a Motion Problem Where the Total Time is Given**

Judy and Nathan drive from Abbotsford to Kelowna, a distance of 322 km. Judy's average driving rate is 5 km/h faster than Nathan's. Judy got tired after driving the first 154 kilometers, so Nathan drove the remaining part of the trip. If the total driving time was 3 hours, what was the average rate of each driver?

Solution ► Let r represent Nathan's average rate. Then $r+5$ represents Judy's average rate. Since Judy drove 154 km, Nathan drove $322 - 154 = 168$ km. Now, we can complete the table:

	R ·	T	= D
Judy	$r+5$	$\frac{154}{r+5}$	154
Nathan	r	$\frac{168}{r}$	168
total		3	322

Note: In motion problems we may add *times* or *distances* but we usually **do not add rates!**

The equation to solve comes from the ***Time*** column.

$$\frac{154}{r+5} + \frac{168}{r} = 3 \qquad /\cdot r(r+5)$$

$$154r + 168(r+5) = 3r(r+5)$$

$$154r + 168r + 840 = 3r^2 + 15r$$

distribute; then collect like terms on one side

$$0 = 3r^2 - 307r - 840$$

factor

$$(3r+8)(r-105) = 0$$

$$r = -\frac{8}{3} \quad or \quad r = 105$$

Since a rate cannot be negative, we discard the solution $r = -\frac{8}{3}$. Therefore, Nathan's average rate was $r = \mathbf{105}$ km/h and Judy's average rate was $r + 5 = \mathbf{110}$ km/h.

Work Problems

Notice the similarity to the formula $R \cdot T = D$ used in motion problems.

When solving work problems, refer to the formula

$$\boldsymbol{Rate}\ of\ work \cdot \boldsymbol{Time} = amount\ of\ \boldsymbol{Job}\ completed$$

and organize data in a table like this:

	R ·	***T***	= ***J***
worker I			
worker II			
together			

Note: In work problems we usually *add rates* but **do not add times!**

Example 7 ► **Solving a Work Problem Involving Addition of Rates**

Adam can trim the shrubs at Centralia College in 8 hr. Bruce can do the same job in 6 hr. To the nearest minute, how long would it take them to complete the same trimming job if they work together?

Solution ► Let t be the time needed to trim the shrubs when Adam and Bruce work together. Since trimming the shrubs at Centralia College is considered to be the whole one job to complete, then the rate R in which this work is done equals

$$\boldsymbol{R} = \frac{\boldsymbol{Job}}{\boldsymbol{Time}} = \frac{\mathbf{1}}{\boldsymbol{Time}}.$$

To organize the information, we can complete the table below.

	R ·	T	= J
Adam	$\frac{1}{8}$	8	1
Bruce	$\frac{1}{6}$	6	1
together	$\frac{1}{t}$	t	1

The job column is often equal to **1**, although sometimes other values might need to be used.

To complete the ***Rate*** column, we divide the ***Job*** by the ***Time***.

Since the rate of work when both Adam and Bruce trim the shrubs is the sum of rates of individual workers, we form and solve the equation

$$\frac{1}{8}+\frac{1}{6}=\frac{1}{t} \qquad /\cdot 24t$$

$$3t+4t=24$$

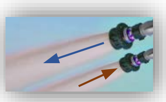
$$7t=24 \qquad /\div 7$$

$$t=\frac{24}{7}\approx 3.43$$

So, if both Adam and Bruce work together, the amount of time needed to complete the job if approximately 3.43 hours ≈ **3 hours 26 minutes**.

Note: The time needed for both workers is **shorter** than either of the individual times.

Example 8 ► **Solving a Work Problem Involving Subtraction of Rates**



The inlet pipe can fill a swimming pool in 4 hours, while the outlet pipe can empty the pool in 5 hours. If both pipes were left open, how long would it take to fill the pool?

Solution ► Suppose t is the time needed to fill the pool when both pipes are left open. If filling the pool is the whole one job to complete, then emptying the pool corresponds to -1 job. This is because when emptying the pool, we reverse the filling job.

To organize the information given in the problem, we complete the following table.

	R ·	T	= J
inlet pipe	$\frac{1}{4}$	4	1
outlet pipe	$-\frac{1}{5}$	5	-1
both pipes	$\frac{1}{t}$	t	1

The equation to solve comes from the ***Rate*** column.

$$\frac{1}{4} - \frac{1}{5} = \frac{1}{t} \qquad /\cdot 20t$$

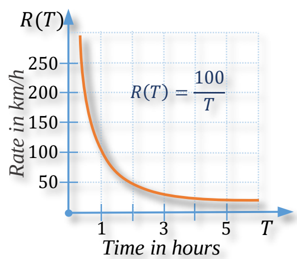
$$5t - 4t = 20$$

$$t = 20$$

So, it will take **20 hours** to fill the pool when both pipes are left open.

Inverse and Combined Variation

When two quantities vary in such a way that their **product remains constant**, we say that they are **inversely proportional**. For example, consider rate R and time T of a moving object that covers a constant distance D. In particular, if $D = 100$ km, we have



$$R = \frac{100}{T} = 100 \cdot \frac{1}{T}$$

This relation tells us that the rate is 100 times larger than the reciprocal of time. Observe though that when the time doubles, the rate is half as large. When the time triples, the rate is three times smaller, and so on. One can observe that the rate decreases proportionally to the increase of time. Such a **reciprocal relation** between the two quantities is called an **inverse variation**.

Definition 6.1 ► Two quantities, $\boldsymbol{x}$ and $\boldsymbol{y}$, are **inversely proportional** to each other (there is an **inverse variation** between them) iff there is a real constant $\boldsymbol{k} \neq \boldsymbol{0}$, such that

$$\boldsymbol{y} = \frac{\boldsymbol{k}}{\boldsymbol{x}}.$$

We say that $\boldsymbol{y}$ **varies inversely** as $\boldsymbol{x}$ with the **variation constant** $\boldsymbol{k}$.
(or equivalently: $\boldsymbol{y}$ is **inversely proportional to** $\boldsymbol{x}$ with the **proportionality constant** $\boldsymbol{k}$.)

Example 9 ► **Solving Inverse Variation Problems**

The volume V of a gas is inversely proportional to the pressure P of the gas. If a pressure of 30 kg/cm^2 corresponds to a volume of 240 cm^3, find the following:

a. The equation that relates V and P,

b. The pressure needed to produce a volume of 150 cm^3.

Solution ► **a.** To find the inverse variation equation that relates V and P, we need to find the variation constant k first. This can be done by substituting $V = 240$ and $P = 30$ into the equation $V = \frac{k}{P}$. So, we obtain

$$240 = \frac{k}{30} \qquad / \cdot 30$$

$$k = 7200.$$

Therefore, our equation is $\boldsymbol{V} = \frac{\boldsymbol{7200}}{\boldsymbol{P}}$.

b. The required pressure can be found by substituting $V = 150$ into the inverse variation equation,

$$150 = \frac{7200}{P}. \qquad / \cdot P, \quad \div 150$$

'swap' 150 and P

This gives us

$$P = \frac{7200}{150} = 48.$$

So, the pressure of the gas that assumes the volume of 150 cm^3 is **48 kg/cm^2**.

Extension: We say that $\boldsymbol{y}$ **varies inversely** as the $\boldsymbol{n}$**-th power** of $\boldsymbol{x}$ iff $\boldsymbol{y} = \frac{\boldsymbol{k}}{\boldsymbol{x^n}}$, for some nonzero constant $\boldsymbol{k}$.

Example 10 ► **Solving an Inverse Variation Problem Involving the Square of a Variable**

The intensity of light varies inversely as the square of the distance from the light source. If 4 meters from the source the intensity of light is 9 candelas, what is the intensity of this light 3 meters from the source?

Solution ► Let I represents the intensity of the light and d the distance from the source of this light. Since I varies inversely as d^2, we set the equation

$$I = \frac{k}{d^2}$$

After substituting the data given in the problem, we find the value of k:

$$9 = \frac{k}{4^2} \qquad / \cdot 16$$

$$k = 9 \cdot 16 = 144$$

So, the inverse variation equation is $I = \frac{144}{d^2}$. Hence, the light intensity at 3 meters from the source is $I = \frac{144}{3^2} = \mathbf{16}$ **candelas**.

Recall from *Section L2* that two variables, say $\boldsymbol{x}$ and $\boldsymbol{y}$, vary **directly** with a proportionality constant $\boldsymbol{k \neq 0}$ if $\boldsymbol{y = kx}$. Also, we say that one variable, say $\boldsymbol{z}$, varies **jointly** as other variables, say $\boldsymbol{x}$ and $\boldsymbol{y}$, with a proportionality constant $\boldsymbol{k \neq 0}$ if $\boldsymbol{z = kxy}$.

Definition 6.2 ► A combination of the **direct** or **joint** variation with the **inverse** variation is called a **combined variation.**

Example:
$\boldsymbol{w}$ may vary **jointly** as $\boldsymbol{x}$ and $\boldsymbol{y}$ and **inversely** as the square of $\boldsymbol{z}$. This means that there is a real constant $\boldsymbol{k \neq 0}$, such that

$$\boldsymbol{w = \frac{kxy}{z^2}}.$$

Example 11 ► **Solving Combined Variation Problems**

The resistance of a cable varies directly as its length and inversely as the square of its diameter. A 20-meter cable with a diameter of 1.2 cm has a resistance of 0.2 ohms. A 50-meter cable with a diameter of 0.6 cm is made out of the same material. What would be its resistance?

Solution ► Let R, l, and d represent respectively the resistance, length, and diameter of a cable. Since R varies directly as l and inversely as d^2, we set the combined variation equation

$$R = \frac{kl}{d^2}.$$

Substituting the data given in the problem, we have

$$0.2 = \frac{k \cdot 20}{1.2^2}, \qquad / \cdot 1.44, \quad \div 20$$

which gives us

$$k = \frac{0.2 \cdot 1.44}{20} = 0.0144$$

So, the combined variation equation is $R = \frac{0.0144l}{d^2}$. Therefore, the resistance of a 50-meter cable with the diameter of 0.6 cm is $R = \frac{0.0144 \cdot 50}{0.6^2} = \mathbf{2}$ **ohms**.

RT.6 Exercises

1. Using the formula $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, find q if $r = 6$ and $p = 10$.
2. The gravitational force between two masses is given by the formula $F = \frac{GMm}{d^2}$.
   Find M if $F = 20$, $G = 6.67 \cdot 10^{-11}$, $m = 1$, and $d = 4 \cdot 10^{-6}$. *Round your answer to one decimal place.*
3. What is the first step in solving the formula $ka + kb = a - b$ for k?
4. What is the first step in solving the formula $A = \frac{pq}{q-p}$ for p?

Solve each formula for the specified variable.

5. $m = \frac{F}{a}$ for a
6. $I = \frac{E}{R}$ for R
7. $\frac{W_1}{W_2} = \frac{d_1}{d_2}$ for d_1
8. $F = \frac{GMm}{d^2}$ for m
9. $s = \frac{(v_1+v_2)t}{2}$ for t
10. $s = \frac{(v_1+v_2)t}{2}$ for v_1

11. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for R

12. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$ for r_1

13. $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ for q

14. $\frac{t}{a} + \frac{t}{b} = 1$ for a

15. $\frac{PV}{T} = \frac{pv}{t}$ for v

16. $\frac{PV}{T} = \frac{pv}{t}$ for T

17. $A = \frac{h(a+b)}{2}$ for b

18. $a = \frac{V-v}{t}$ for V

19. $R = \frac{gs}{g+s}$ for s

20. $I = \frac{2V}{V+2r}$ for V

21. $I = \frac{nE}{E+nr}$ for n

22. $\frac{E}{e} = \frac{R+r}{r}$ for e

23. $\frac{E}{e} = \frac{R+r}{r}$ for r

24. $S = \frac{H}{m(t_1-t_2)}$ for t_1

25. $V = \frac{\pi h^2(3R-h)}{3}$ for R

26. $P = \frac{A}{1+r}$ for r

27. $\frac{V^2}{R^2} = \frac{2g}{R+h}$ for h

28. $v = \frac{d_2-d_1}{t_2-t_1}$ for t_2

Solve each problem.

29. The ratio of the weight of an object on Earth to the weight of an object on the moon is 200 to 33. What would be the weight of a 75-kg astronaut on the moon?

30. A 30-meter long ribbon is cut into two sections. How long are the two sections if the ratio of their lengths is 5 to 7?

31. Assume that burning 7700 calories causes a decrease of 1 kilogram in body mass. If walking 7 kilometers in 2 hours burns 700 calories, how many kilometers would a person need to walk at the same rate to lose 1 kg?

32. On a map of Canada, the linear distance between Vancouver and Calgary is 1.8 cm. The flight distance between the two cities is about 675 kilometers. On this same map, what would be the linear distance between Calgary and Montreal if the flight distance between the two cities is approximately 3000 kilometers?

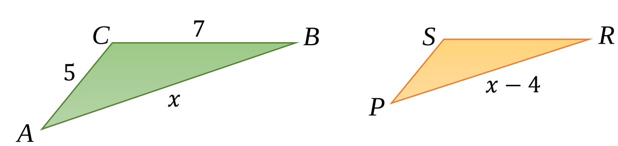
33. To estimate the population of Cape Mountain Zebra in South Africa, biologists caught, tagged, and then released 68 Cape Mountain Zebras. In a month, they caught a random sample of 84 of this type of zebras. It turned out that 5 of them were tagged. Assuming that zebras mixed freely, approximately how many Cape Mountain Zebras lived in South Africa?

34. To estimate the number of white bass fish in a particular lake, biologists caught, tagged, and then released 300 of this fish. In two weeks, they returned and collected a random sample of 196 white bass fish. This sample contained 12 previously tagged fish. Approximately how many white bass fish does the lake have?

35. Eighteen white-tailed eagles are tagged and released into the wilderness. In a few weeks, a sample of 43 white-tailed eagles was examined, and 5 of them were tagged. Estimate the white-tailed eagle population in this wilderness area.

36. A meter stick casts a 64 cm long shadow. At the same time, a 15-year old cottonwood tree casts an 18-meter long shadow. To the nearest meter, how tall is the tree?

37. The ratio of corresponding sides of similar triangles is 5 to 3. The two shorter sides of the larger triangle are 5 and 7 units long, correspondingly. Find the length of each side of the smaller triangle if its longest side is 4 units shorter than the corresponding side of the larger triangle.



38. The width of a rectangle is the same as the length of a similar rectangle. If the dimensions of the smaller rectangle are 7 cm by 12 cm, what are the dimensions of the larger rectangle?

39. Justin runs twice around a park. He averages 20 kilometers per hour during the first round and only 16 kilometers per hour during the second round. What is his average speed for the whole run? *Round your answer to one decimal place.*

40. Robert runs twice around a stadium. He averages 18 km/h during the first round. What should his average speed be during the second round to have an overall average of 20 km/h for the whole run?

41. Jim's boat moves at 20 km/h in still water. Suppose it takes the same amount of time for Jim to travel by his boat either 15 km downriver or 10 km upriver. Find the rate of the current.

42. The average speed of a plane flying west was 880 km/h. On the return trip, the same plane averaged only 620 km/h. If the total flying time in both directions was 6 hours, what was the one-way distance?

43. A plane flies 3800 kilometers with the wind, while only 3400 kilometers against the same wind. If the airplane speed in still air is 900 km/h, find the speed of the wind.

44. Walking on a moving sidewalk, Sarah could travel 40 meters forward in the same time it would take her to travel 15 meters in the opposite direction. If the rate of the moving sidewalk was 35 m/min, what was Sarah's rate of walking?

45. Arthur travelled by car from Madrid to Paris. He usually averages 100 km/h on such trips. This time, due to heavier traffic and few stops, he averaged only 85 km/h, and he reached his destination 2 hours 15 minutes later than expected. How far did Arthur travel?

46. Tony averaged 100 km/h on the first part of his trip to Lillooet, BC. The second part of his trip was 20 kilometers longer than the first, and his average speed was only 80 km/h. If the second part of the trip took him 30 minutes longer than the first part, what was the overall distance travelled by Tony?

47. Page is a college student who lives in a near-campus apartment. When she rides her bike to campus, she gets there 24 min faster than when she walks. If her average walking rate is 4 km/h and her average biking rate is 20 km/h, how far does she live from the campus?

48. Sonia can respond to all the daily e-mails in 2 hours. Betty needs 3 hours to do the same job. If they both work on responding to e-mails, what portion of this daily job can be done in 1 hour? How much more time would they need to complete the job?

49. Brenda can paint a deck in x hours, while Tony can do the same job in y hours. Write a rational expression that represents the portion of the deck that can be painted by both of them in 4 hours.

50. Aaron and Ben plan to paint a house. Aaron needs 24 hours to paint the house by himself. Ben needs 18 to do the same job. To the nearest minute, how long would it take them to paint the house if they work together?

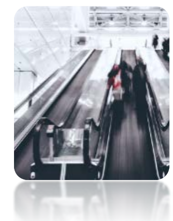
51. When working together, Adam and Brian can paint a house in 6 hours. Brian could paint this house on his own in 10 hours. How long would it take Adam to paint the house working alone?

52. An experienced floor installer can install a parquet floor twice as fast as an apprentice. Working together, it takes the two workers 2 days to install the floor in a particular house. How long would it take the apprentice to do the same job on his own?

53. A pool can be filled in 8 hr and drained in 12 hr. On one occasion, when filling the pool, the drain was accidentally left open. How long did it take to fill this pool?

54. One inlet pipe can fill a hot tub in 15 minutes. Another inlet pipe can fill the tub in 10 minutes. An outlet pipe can drain the hot tub in 18 minutes. How long would it take to fill the hot tub if all three pipes are left open?

55. Two different width escalators can empty a 1470-people auditorium in 12 min. If the wider escalator can move twice as many people as the narrower one, how many people per hour can the narrower escalator move?



56. At what times between 3:00 and 4:00 are the minute and hour hands perfectly lined up?

57. If Miranda drives to work at an average speed of 60 km/h, she is 1 min late. When she drives at an average speed of 75 km/h, she is 3 min early. How far is Miranda's workplace from her home?

58. The current in an electrical circuit at a constant potential varies inversely as the resistance of the circuit. Suppose that the current I is 9 amperes when the resistance R is 10 ohms. Find the current when the resistance is 6 ohms.

59. Assuming the same rate of work for all workers, the number of workers needed for a job varies inversely as the time required to complete the job. If it takes 3 hours for 8 workers to build a deck, how long would it take two workers to build the same deck?

60. The length of a guitar string is inversely proportional to the frequency of the string vibrations. Suppose a 60-cm long string vibrates at a frequency of 500 Hz (*1 hertz = one cycle per second*). What is the frequency of the same string when it is shortened to 50 centimeters?

61. A musical tone's pitch is inversely proportional to its wavelength. If a wavelength of 2.2 meters corresponds to a pitch of 420 vibrations per second, find the wavelength of a tone with a pitch of 660 vibrations per second.

62. The intensity, I, of a television signal is inversely proportional to the square of the distance, d, from a transmitter. If 2 km away from the transmitter the intensity is 25 W/m^2 (watts per square meter), how far from the transmitter is a TV set that receives a signal with the intensity of 2.56 W/m^2?

63. The weight W of an object is inversely proportional to the square of the distance D from the center of Earth. To the nearest kilometer, how high above the surface of Earth must a 60-kg astronaut be to weigh half as much? Assume the radius of Earth to be 6400 km.

64. The number of long-distance phone calls between two cities during a specified period in time varies jointly as the populations of the cities, P_1 and P_2, and inversely as the distance between them. Suppose 80,000 calls are made between two cities that are 400 km apart and have populations of 70,000 and 100,000. How many calls are made between Vancouver and Abbotsford that are 70 km apart and have populations of 630,000 and 140,000, respectively?

65. The force that keeps a car from skidding on a curve is inversely proportional to the radius of the curve and jointly proportional to the weight of the car and the square of its speed. Knowing that a force of 880 N (Newtons) keeps an 800-kg car moving at 50 km/h from skidding on a curve of radius 160 m, estimate the force that would keep the same car moving at 80 km/h from skidding on a curve of radius 200 meters.

66. Suppose that the renovation time is inversely proportional to the number of workers hired for the job. Will the renovation time decrease more when hiring additional 2 workers in a 4-worker company or a 6-worker company? Justify your answer.

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Radicals and Radical Functions

So far we have discussed polynomial and rational expressions and functions. In this chapter, we study algebraic expressions that contain radicals. For example, $3+\sqrt{2}$, $\sqrt[3]{x}-1$, or $\frac{1}{\sqrt{5x-1}}$. Such expressions are called ***radical expressions***. Familiarity with radical expressions is essential when solving a wide variety of problems. For instance, in algebra, some polynomial or rational equations have radical solutions that need to be simplified. In geometry, due to the frequent use of the Pythagorean equation, $a^2+b^2=c^2$, the exact distances are often radical expressions. In sciences, many formulas involve radicals.

We begin the study of radical expressions with defining radicals of various degrees and discussing their properties. Then, we show how to simplify radicals and radical expressions, and introduce operations on radical expressions. Finally, we study the methods of solving radical equations. In addition, similarly as in earlier chapters where we looked at the related polynomial and rational functions, we will also define and look at properties of radical functions.

RD1 Radical Expressions, Functions, and Graphs

Roots and Radicals

The operation of taking a **square root** of a number is the **reverse operation of squaring** a number. For example, a square root of 25 is 5 because raising 5 to the second power gives us 25.

Note: Observe that raising -5 to the second power also gives us 25. So, the square root of 25 could have two answers, 5 or -5. To avoid this duality, we choose the **nonnegative value**, called the **principal square root**, for the value of a square root of a number.

The operation of taking a square root is denoted by the symbol $\sqrt{}$. So, we have

$$\sqrt{25}=5,\quad \sqrt{0}=0,\quad \sqrt{1}=1,\quad \sqrt{9}=3, \text{ etc.}$$

What about $\sqrt{-4}=?$ Is there a number such that when it is squared, it gives us -4 ?

Since the square of any real number is nonnegative, the square root of a negative number is not a real number. So, when working in the set of real numbers, we can conclude that

$$\sqrt{positive}=positive,\quad \sqrt{0}=0,\quad \text{and}\quad \sqrt{negative}=DNE$$

does not exist

The operation of taking a **cube root** of a number is the **reverse operation of cubing** a number. For example, a cube root of 8 is 2 because raising 2 to the third power gives us 8.

This operation is denoted by the symbol $\sqrt[3]{}$. So, we have

$$\sqrt[3]{8}=2,\quad \sqrt[3]{0}=0,\quad \sqrt[3]{1}=1,\quad \sqrt[3]{27}=3, \text{ etc.}$$

Note: Observe that $\sqrt[3]{-8}$ exists and is equal to -2. This is because $(-2)^3 = -8$. Generally, a cube root can be applied to any real number and the **sign** of the resulting value **is the same** as the sign of the original number.

Thus, we have

$$\sqrt[3]{\boldsymbol{positive}} = \boldsymbol{positive}, \quad \sqrt[3]{0} = 0, \quad \text{and} \quad \sqrt[3]{\boldsymbol{negative}} = \boldsymbol{negative}$$

The square or cube roots are special cases of n-th degree radicals.

Definition 1.1 ► The ***n*-th degree radical** of a number a is a number b such that $\boldsymbol{b^n = a}$.

Notation:

degree (index, order)

radical

$$\sqrt[\boldsymbol{n}]{\boldsymbol{a}} = \boldsymbol{b} \Leftrightarrow \boldsymbol{b^n = a}$$

radical symbol

radicand

For example,
$$\sqrt[4]{16} = 2 \text{ because } 2^4 = 16,$$
$$\sqrt[5]{-32} = -2 \text{ because } (-2)^5 = -32,$$
$$\sqrt[3]{0.027} = 0.3 \text{ because } (0.3)^3 = 0.027.$$

Notice: A square root is a second degree radical, customarily denoted by $\sqrt{}$ rather than $\sqrt[2]{}$.

Example 1 ► **Evaluating Radicals**

Evaluate each radical, if possible.

a. $\sqrt{0.64}$ **b.** $\sqrt[3]{125}$ **c.** $\sqrt[4]{-16}$ **d.** $\sqrt[5]{-\frac{1}{32}}$

Solution ► **a.** Since $0.64 = (0.8)^2$, then $\sqrt{0.64} = 0.8$.

take half of the decimal places

Advice: To become fluent in evaluating square roots, it is helpful to be familiar with the following perfect square numbers:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, …, 400, …, 625, …

b. $\sqrt[3]{125} = 5$ as $5^3 = 125$

Advice: To become fluent in evaluating cube roots, it is helpful to be familiar with the following cubic numbers:
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …

c. $\sqrt[4]{-16}$ is not a real number as there is no real number which raised to the 4-th power becomes negative.

d. $\sqrt[5]{-\frac{1}{32}} = -\frac{1}{2}$ as $\left(-\frac{1}{2}\right)^5 = -\frac{1^5}{2^5} = -\frac{1}{32}$

Note: Observe that $\frac{\sqrt[5]{-1}}{\sqrt[5]{32}} = \frac{-1}{2}$, so $\sqrt[5]{-\frac{1}{32}} = \frac{\sqrt[5]{-1}}{\sqrt[5]{32}}$.

Generally, to take a radical of a quotient, $\sqrt[n]{\frac{a}{b}}$, it is the same as to take the quotient of radicals, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Example 2 ► **Evaluating Radical Expressions**

Evaluate each radical expression.

a. $-\sqrt{121}$ **b.** $-\sqrt[3]{-64}$ **c.** $\sqrt[4]{(-3)^4}$ **d.** $\sqrt[3]{(-6)^3}$

Solution ►

a. $-\sqrt{121} = -11$

b. $-\sqrt[3]{-64} = -(-4) = 4$

c. $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$

the result is positive

Note: If n is **even**, then $\sqrt[n]{\boldsymbol{a^n}} = \begin{cases} a, & if\ a \geq 0 \\ -a, & if\ a < 0 \end{cases} = |\boldsymbol{a}|$.

For example, $\sqrt{7^2} = 7$ and $\sqrt{(-7)^2} = 7$.

d. $\sqrt[3]{(-6)^3} = \sqrt[3]{-216} = -6$

the result has the same sign

Note: If n is **odd**, then $\sqrt[n]{\boldsymbol{a^n}} = \boldsymbol{a}$. For example, $\sqrt[3]{5^3} = 5$ but $\sqrt[3]{(-5)^3} = -5$.

Summary of Properties of n-th Degree Radicals

- If n is **EVEN**, then

$$\sqrt[n]{\boldsymbol{positive}} = \boldsymbol{positive}, \quad \sqrt[n]{\boldsymbol{negative}} = \boldsymbol{DNE}, \quad \text{and} \quad \sqrt[n]{\boldsymbol{a^n}} = |\boldsymbol{a}|$$

- If n is **ODD**, then

$$\sqrt[n]{\boldsymbol{positive}} = \boldsymbol{positive}, \quad \sqrt[n]{\boldsymbol{negative}} = \boldsymbol{negative}, \quad \text{and} \quad \sqrt[n]{\boldsymbol{a^n}} = \boldsymbol{a}$$

- For any natural $n \geq 0$, $\sqrt[n]{\mathbf{0}} = \mathbf{0}$ and $\sqrt[n]{\mathbf{1}} = \mathbf{1}$.

Example 3 ► **Simplifying Radical Expressions Using Absolute Value Where Appropriate**

Simplify each radical, assuming that all variables represent any real number.

a. $\sqrt{9x^2y^4}$ **b.** $\sqrt[3]{-27y^3}$ **c.** $\sqrt[4]{a^{20}}$ **d.** $-\sqrt[4]{(k-1)^4}$

Solution ► **a.** $\sqrt{9x^2y^4} = \sqrt{(3xy^2)^2} = |3xy^2| = \mathbf{3|x|y^2}$

An **even** degree radical is **nonnegative**, so we must use the **absolute value** operator.

Recall: As discussed in *Section L6*, the absolute value operator has the following properties:

$$|\boldsymbol{xy}| = |\boldsymbol{x}||\boldsymbol{y}|$$
$$\left|\frac{\boldsymbol{x}}{\boldsymbol{y}}\right| = \frac{|\boldsymbol{x}|}{|\boldsymbol{y}|}$$

Note: $|y^2| = y^2$ as y^2 is already nonnegative.

b. $\sqrt[3]{-27y^3} = \sqrt[3]{(-3y)^3} = \mathbf{-3y}$

An **odd** degree radical assumes the sign of the radicand, so we **do not apply the absolute value** operator.

c. $\sqrt[4]{a^{20}} = \sqrt[4]{(a^5)^4} = |a^5| = \mathbf{|a|^5}$

Note: To simplify an expression with an absolute value, we keep the absolute value operator as close as possible to the variable(s).

d. $-\sqrt[4]{(k-1)^4} = \mathbf{-|k-1|}$

Radical Functions

Since each nonnegative real number x has exactly one principal square root, we can define the **square root** function, $\boldsymbol{f(x)=\sqrt{x}}$. The **domain** D_f of this function is the set of nonnegative real numbers, $[\mathbf{0},\infty)$, and so is its **range** (as indicated in *Figure 1*).

To graph the square root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect square numbers. However, sometimes we want to use additional x-values that are not perfect squares. Since a square root of such a number, for example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, etc., is an irrational number, we approximate these values using a calculator.

x	y
0	0
$\frac{1}{4}$	$\frac{1}{2}$
1	1
4	2
6	$\sqrt{6}\approx 2.4$

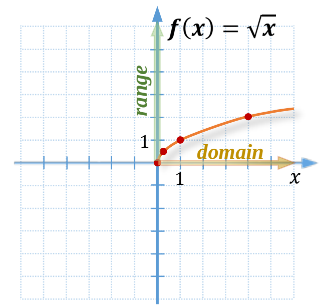


Figure 1

For example, to approximate $\sqrt{6}$, we use the sequence of keying: $\sqrt{}$ 6 *ENTER* or 6 ^ (1 / 2) *ENTER* . This is because a square root operator works the same way as the exponent of $\frac{1}{2}$.

Note: When graphing an even degree radical function, it is essential that we find its domain first. The end-point of the domain indicates the starting point of the graph, often called the vertex.

For example, since the domain of $f(x)=\sqrt{x}$ is $[0,\infty)$, the graph starts from the point $\left(0,f(0)\right)=(0,0)$, as in *Figure 1*.

Since the cube root can be evaluated for any real number, the **domain** D_f of the related **cube root** function, $\boldsymbol{f(x)=\sqrt[3]{x}}$, is the set of **all real numbers**, $\mathbb{R}$. The **range** can be observed in the graph (see *Figure 2*) or by inspecting the expression $\sqrt[3]{x}$. It is also $\mathbb{R}$.

To graph the cube root function, we create a table of values. The easiest x-values for calculation of the corresponding y-values are the perfect cube numbers. As before, sometimes we might need to estimate additional x-values. For example, to approximate $\sqrt[3]{6}$, we use the sequence of keying:

$\sqrt[3]{}$ 6 *ENTER* or

6 ^ (1 / 3) *ENTER* .

x	y
−8	−2
−6	$-\sqrt[3]{6}\approx -1.8$
−1	−1
$-\frac{1}{8}$	$-\frac{1}{2}$
0	0
$\frac{1}{8}$	$\frac{1}{2}$
1	1
6	$\sqrt[3]{6}\approx 1.8$
8	2

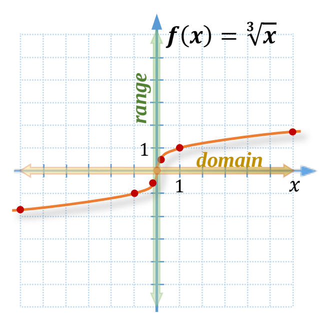


Figure 2

Example 4 ► **Finding a Calculator Approximations of Roots**

Use a calculator to approximate the given root up to three decimal places.

a. $\sqrt{3}$ **b.** $\sqrt[3]{5}$ **c.** $\sqrt[5]{100}$

Solution ►

a. $\sqrt{3} \approx 1.732$

b. $\sqrt[3]{5} \approx 1.710$

5 ^ (1 / 3) *ENTER*
or
MATH 4 5 *ENTER* on a graphing calculator

c. $\sqrt[5]{100} \approx 2.512$

100 ^ (1 / 5) *ENTER*
or
5 *MATH* 5 100 *ENTER* on a graphing calculator

Example 5 ► **Finding the Best Integer Approximation of a Square Root**

Without the use of a calculator, determine the best integer approximation of the given root.

a. $\sqrt{68}$ **b.** $\sqrt{140}$

Solution ►

a. Observe that 68 lies between the following two consecutive perfect square numbers, 64 and 81. Also, 68 lies closer to 64 than to 81. Therefore, $\sqrt{68} \approx \sqrt{64} = 8$.

b. 140 lies between the following two consecutive perfect square numbers, 121 and 144. In addition, 140 is closer to 144 than to 121. Therefore, $\sqrt{140} \approx \sqrt{144} = 12$.

Example 6 ► **Finding the Domain of a Radical Function**

Find the domain of each of the following functions.

a. $f(x) = \sqrt{2x+3}$ **b.** $g(x) = 2 - \sqrt{1-x}$

Solution ►

The **domain** of an **even degree** radical is the solution set of the inequality ***radicand* ≥ 0**

The **domain** of an **odd degree** radical is $\mathbb{R}$.

a. When finding domain D_f of function $f(x) = \sqrt{2x+3}$, we need to protect the radicand $2x+3$ from becoming negative. So, an x-value belongs to the domain D_f if it satisfies the condition

$$2x + 3 \geq 0. \qquad / -3, \quad \div 2$$

This happens for $x \geq -\frac{3}{2}$. Therefore, $\boldsymbol{D_f} = \left[-\frac{3}{2}, \infty\right)$.

b. To find the domain D_g of function $g(x) = 2 - \sqrt{1-x}$, we solve the condition

$$1 - x \geq 0 \qquad / +x$$
$$1 \geq x$$

Thus, $\boldsymbol{D_g} = (-\infty, \mathbf{1}]$.

Example 7 ► **Graphing Radical Functions**

For each function, find its domain, graph it, and find its range. Then, observe what transformation(s) of a basic root function result(s) in the obtained graph.

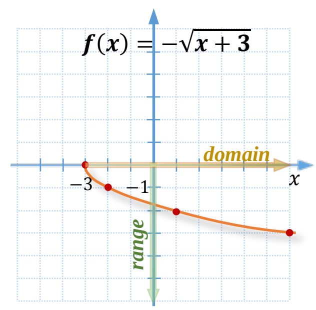
a. $f(x) = -\sqrt{x+3}$ **b.** $g(x) = \sqrt[3]{x} - 2$

Solution ►

a. The **domain** D_f is the solution set of the inequality $x + 3 \geq 0$, which is equivalent to $x \geq -3$. Hence, $\boldsymbol{D_f} = [\mathbf{-3}, \infty)$.

start from the endpoint of the domain

x	y
-3	0
-2	-1
1	-2
6	-3



The projection of the graph onto the y-axis indicates the **range** of this function, which is $(-\infty, \mathbf{0}]$.

The graph of $f(x) = -\sqrt{x+3}$ has the same shape as the graph of the basic square root function $f(x) = \sqrt{x}$, except that it is flipped over the x-axis and moved to the left by three units. These transformations are illustrated in *Figure 3*.

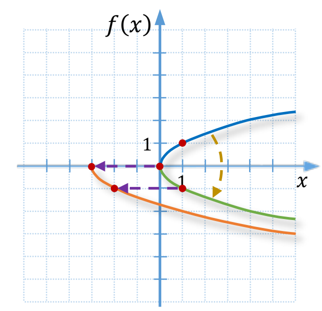
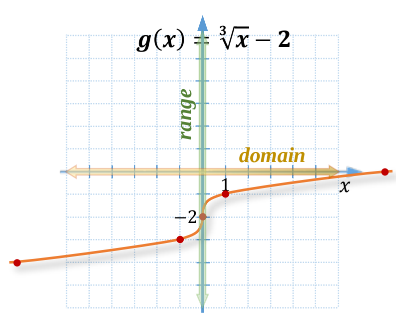


Figure 3

b. The **domain** and **range** of any odd degree radical are both the set of all real numbers. So, $\boldsymbol{D_g} = \mathbb{R}$ and $\boldsymbol{range_g} = \mathbb{R}$.

x	y
-8	-4
-1	-3
0	-2
1	-1
8	0



The graph of $g(x) = \sqrt[3]{x} - 2$ has the same shape as the graph of the basic cube root function $f(x) = \sqrt[3]{x}$, except that it is moved down by two units. This transformation is illustrated in *Figure 4*.

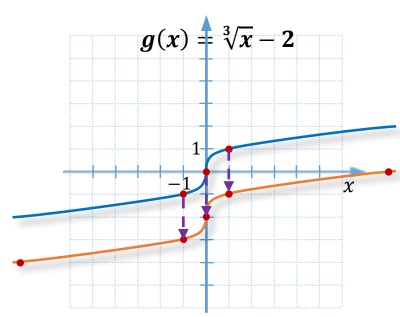


Figure 4

Radicals in Application Problems

Some application problems require evaluation of formulas that involve radicals. For example, the formula $c = \sqrt{a^2 + b^2}$ allows for finding the hypotenuse in a right angle triangle (see *Section RD3*), **Heron's** formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ allows for finding the area of any triangle given the lengths of its sides (see *Section T5*), the formula $T = 2\pi\sqrt{\frac{d^3}{Gm}}$ allows for finding the time needed for a planet to make a complete orbit around the Sun, and so on.

Example 8 ► **Using a Radical Formula in an Application Problem**

The time T, in seconds, needed for a pendulum to complete a full swing can be calculated using the formula

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where L denotes the length of the pendulum in feet, and g is the acceleration due to gravity, which is about 32 ft/sec^2. To the nearest hundredths of a second, find the time of a complete swing of an 18-inch long pendulum.

Solution ► Since $L = 18 \text{ in} = \frac{18}{12} \text{ ft} = \frac{3}{2} \text{ ft}$ and $g = 32 \text{ ft/sec}^2$, then

$$T = 2\pi\sqrt{\frac{\frac{3}{2}}{32}} = 2\pi\sqrt{\frac{3}{2 \cdot 32}} = 2\pi\sqrt{\frac{3}{64}} = 2\pi \cdot \frac{\sqrt{3}}{8} = \frac{\pi\sqrt{3}}{4} \approx 1.36$$

So, the approximate time of a complete swing of an 18-in pendulum is **1.36 seconds**.

RD.1 Exercises

Evaluate each radical, if possible.

1. $\sqrt{49}$ **2.** $-\sqrt{81}$ **3.** $\sqrt{-400}$ **4.** $\sqrt{0.09}$

5. $\sqrt{0.0016}$ **6.** $\sqrt{\frac{64}{225}}$ **7.** $\sqrt[3]{64}$ **8.** $\sqrt[3]{-125}$

9. $\sqrt[3]{0.008}$ **10.** $-\sqrt[3]{-1000}$ **11.** $\sqrt[3]{\frac{1}{0.000027}}$ **12.** $\sqrt[4]{16}$

13. $\sqrt[5]{0.00032}$ **14.** $\sqrt[7]{-1}$ **15.** $\sqrt[8]{-256}$ **16.** $-\sqrt[6]{\frac{1}{64}}$

17. Decide whether the expression $-\sqrt{-a}$ is positive, negative, 0, or not a real number, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

18. Assuming that n is odd, decide whether the expression $-\sqrt[n]{a}$ is positive, negative, or 0, given that

a. $a < 0$ **b.** $a > 0$ **c.** $a = 0$

Simplify each radical. Assume that letters can represent *<u>any real number</u>*.

19. $\sqrt{15^2}$ **20.** $\sqrt{(-15)^2}$ **21.** $\sqrt{x^2}$ **22.** $\sqrt{(-x)^2}$

23. $\sqrt{81x^2}$ **24.** $\sqrt{(-12y)^2}$ **25.** $\sqrt{(a+3)^2}$ **26.** $\sqrt{(2-x)^2y^4}$

27. $\sqrt{x^2-4x+4}$ **28.** $\sqrt{9y^2+30y+25}$ **29.** $\sqrt[3]{(-5)^3}$ **30.** $\sqrt[3]{x^3}$

31. $\sqrt[3]{-125a^3}$ **32.** $-\sqrt[3]{0.008(x-1)^3}$ **33.** $\sqrt[4]{(5x)^4}$ **34.** $\sqrt[8]{(-10)^8}$

35. $\sqrt[5]{(y-3)^5}$ **36.** $\sqrt[2017]{(a+b)^{2017}}$ **37.** $\sqrt[2018]{(2a-b)^{2018}}$ **38.** $\sqrt[6]{x^{18}}$

39. $\sqrt[4]{(a+1)^{12}}$ **40.** $\sqrt[5]{(-a)^{20}}$ **41.** $\sqrt[7]{(-k)^{35}}$ **42.** $\sqrt[4]{x^4(-y)^8}$

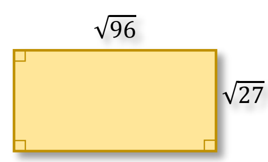
Find a decimal approximation for each radical. Round the answer to three decimal places.

43. $\sqrt{350}$ **44.** $-\sqrt{0.859}$ **45.** $\sqrt[3]{5}$ **46.** $\sqrt[5]{3}$

Without the use of a calculator, give the best integer approximation of each square root.

47. $\sqrt{67}$ **48.** $\sqrt{95}$ **49.** $\sqrt{115}$ **50.** $\sqrt{87}$

Questions in Exercises 60 and 61 refer to the accompanying rectangle. Answer these questions without the use of a calculator.



51. Give the best integer estimation of the area of the rectangle.

52. Give the best integer estimation of the perimeter of the rectangle.

Solve each problem. Do not use any calculator.

53. The length of a rectangular garden is $\sqrt{189}$ m and the width is $\sqrt{50}$ m. Give the best integral estimate for the dimentions of the garden and then estimate the perimeter of the garden to the nearest meter.

54. The sides of a triangle parking lot are $\sqrt{65}$ m, $\sqrt{78}$ m, and $\sqrt{220}$ m. Give the best integral estimate for each side and then estimate the perimeter of this parking lot to the nearest meter.

Graph *each function and give its* ***domain*** *and* ***range****. Then, discuss the* ***transformations*** *of a basic root function needed to obtain the graph of the given function.*

55. $f(x) = \sqrt{x+1}$ **56.** $g(x) = \sqrt{x} + 1$ **57.** $h(x) = -\sqrt{x}$

58. $f(x) = \sqrt{x-3}$ **59.** $g(x) = \sqrt{x} - 3$ **60.** $h(x) = 2 - \sqrt{x}$

61. $f(x) = \sqrt[3]{x-2}$ **62.** $g(x) = \sqrt[3]{x} + 2$ **63.** $h(x) = -\sqrt[3]{x} + 2$

***Graph** each function and give its **domain** and **range**.*

64. $f(x) = 2 + \sqrt{x-1}$ **65.** $g(x) = 2\sqrt{x}$ **66.** $h(x) = -\sqrt{x+3}$

67. $f(x) = \sqrt{3x+9}$ **68.** $g(x) = \sqrt{3x-6}$ **69.** $h(x) = -\sqrt{2x-4}$

70. $f(x) = \sqrt{12-3x}$ **71.** $g(x) = \sqrt{8-4x}$ **72.** $h(x) = -2\sqrt{-x}$

Graph the three given functions on the same grid and discuss the relationship between them.

73. $f(x) = 2x+1$; $g(x) = \sqrt{2x+1}$; $h(x) = \sqrt[3]{2x+1}$

74. $f(x) = -x+2$; $g(x) = \sqrt{-x+2}$; $h(x) = \sqrt[3]{-x+2}$

75. $f(x) = \frac{1}{2}x+1$; $g(x) = \sqrt{\frac{1}{2}x+1}$; $h(x) = \sqrt[3]{\frac{1}{2}x+1}$

Solve each problem.

76. The distance D, in kilometers, from the point of sight to the horizon is given by the formula $D = 4\sqrt{H}$, where H denotes the height of the point of sight above the sea level, in meters. To the nearest tenth of a kilometer, how far away is the horizon for a 180 cm tall man standing on a 40-m high cliff?

77. Let T represents the threshold body weight, in kilograms, above which the risk of death of a person increases significantly. Suppose the formula $h = 40\sqrt[3]{T}$ can be used to calculate the height h, in centimeters, of a middle age man with the threshold body weight T. To the nearest centimeter, find the height corresponding to a threshold weight of a 100 kg man at his forties.

78. The orbital period (time needed for a planet to make a complete rotation around the Sun) is given by the

formula $T = 2\pi\sqrt{\frac{r^3}{GM}}$, where r is the average distance of the planet from the Sun, G is the universal gravitational constant, and M is the mass of the Sun. To the nearest day, find the orbital period of Mercury, knowing that its average distance from the Sun is $5.791 \cdot 10^7$ km, the mass of the Sun is $1.989 \cdot 10^{30}$ kg, and $G = 6.67408 \cdot 10^{-11}$ $m^3/(kg \cdot s^2)$. (Attention: *Watch the units!*)

79. Suppose that the time t, in seconds, needed for an object to fall a certain distance can be found by using the formula $t = \sqrt{\frac{2d}{g}}$, where d is the distance in meters, and g is the acceleration due to gravity. An astronaut standing on a platform above the moon's surface drops an object, which hits the ground 2 seconds after it was dropped. Assume that the acceleration due to gravity on the moon is 1.625 m/s^2. How high above the surface was the object at the time it was dropped?

*Half of the perimeter (**semiperimeter**) of a triangle with sides a, b, and c is* $\boldsymbol{s = \frac{1}{2}(a + b + c)}$*. The area of such a triangle is given by the **Heron's Formula**:* $\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}$.
*In problems **89-90**, find the area of a triangular piece of land with the given sides.*

80. $a = 3\text{ m}, b = 4\text{ m}, c = 5\text{ m}$

81. $a = 80\text{ m}, b = 80\text{ m}, c = 140\text{ m}$

RD2 Rational Exponents

In *Sections P2* and *RT1*, we reviewed the properties of powers with natural and integral exponents. All of these properties hold for real exponents as well. In this section, we give meaning to expressions with rational exponents, such as $a^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, or $(2x)^{0.54}$, and use the rational exponent notation as an alternative way to write and simplify radical expressions.

Rational Exponents

Observe that $\sqrt{9} = 3 = 3^{2\cdot\frac{1}{2}} = 9^{\frac{1}{2}}$. Similarly, $\sqrt[3]{8} = 2 = 2^{3\cdot\frac{1}{3}} = 8^{\frac{1}{3}}$. This suggests the following generalization:

For any real number a and a natural number $n > 1$, we have

$$\sqrt[n]{\boldsymbol{a}} = \boldsymbol{a}^{\frac{1}{n}}.$$

Notice: The **denominator** of the rational exponent is the **index** of the radical.

Caution! If $a < 0$ and n is an even natural number, then $a^{\frac{1}{n}}$ is <u>not</u> a real number.

Example 1 ► **Converting Radical Notation to Rational Exponent Notation**

Convert each radical to a power with a rational exponent and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\sqrt[6]{16}$ **b.** $\sqrt[3]{27x^3}$ **c.** $\sqrt{\frac{4}{b^6}}$

Solution ► **a.** $\sqrt[6]{16} = 16^{\frac{1}{6}} = (2^4)^{\frac{1}{6}} = 2^{\frac{4}{6}} = \boldsymbol{2^{\frac{2}{3}}}$

Observation: Expressing numbers as **powers of prime numbers** often allows for further simplification.

b. $\sqrt[3]{27x^3} = (27x^3)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot (x^3)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} \cdot x = \boldsymbol{3x}$

distribution of exponents

change into a power of a prime number

Note: The above example can also be done as follows:

$$\sqrt[3]{27x^3} = \sqrt[3]{3^3x^3} = (3^3x^3)^{\frac{1}{3}} = \boldsymbol{3x}$$

c. $\sqrt{\frac{9}{b^6}} = \left(\frac{9}{b^6}\right)^{\frac{1}{2}} = \frac{(3^2)^{\frac{1}{2}}}{(b^6)^{\frac{1}{2}}} = \boldsymbol{\frac{3}{b^3}}$, as $b > 0$.

Observation: $\sqrt{a^4} = a^{\frac{4}{2}} = a^2$.

Generally, for any real number $a \neq 0$, natural number $n > 1$, and integral number m, we have

$$\sqrt[n]{\boldsymbol{a^m}} = (a^m)^{\frac{1}{n}} = \boldsymbol{a^{\frac{m}{n}}}$$

Rational exponents are introduced in such a way that they automatically agree with the rules of exponents, as listed in *Section RT1*.
Furthermore, the rules of exponents hold not only for rational but also for **real exponents**.

Observe that following the rules of exponents and the commutativity of multiplication, we have

$$\sqrt[n]{\boldsymbol{a^m}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{\boldsymbol{a}}\right)^m,$$

provided that $\sqrt[n]{a}$ exists.

Example 2 ► **Converting Rational Exponent Notation to the Radical Notation**

Convert each power with a rational exponent to a radical and simplify, if possible.

a. $5^{\frac{3}{4}}$ **b.** $(-27)^{\frac{1}{3}}$ **c.** $3x^{-\frac{2}{5}}$

Solution ►

a. $5^{\frac{3}{4}} = \sqrt[4]{5^3} = \sqrt[4]{\mathbf{125}}$

b. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = \mathbf{-3}$

Notice that $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$, so $(-27)^{\frac{1}{3}} = -27^{\frac{1}{3}}$.

However, $(-9)^{\frac{1}{2}} \neq -9^{\frac{1}{2}}$, as $(-9)^{\frac{1}{2}}$ is not a real number while $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

c. $3x^{-\frac{2}{5}} = \frac{3}{x^{\frac{2}{5}}} = \frac{\mathbf{3}}{\sqrt[5]{\boldsymbol{x^2}}}$

Caution: A **negative exponent** indicates a **reciprocal** not a negative number!

Also, the exponent refers to x only, so 3 remains in the numerator.

Observation: If $a^{\frac{m}{n}}$ is a real number, then

$$\boldsymbol{a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}},$$

provided that $a \neq 0$.

Caution! Make sure to distinguish between a negative exponent and a negative result. Negative exponent leads to a reciprocal of the base. The result can be either positive or negative, depending on the sign of the base. For example,

$$8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}, \text{ but } (-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{-2} = -\frac{1}{2} \text{ and } -8^{-\frac{1}{3}} = -\frac{1}{8^{\frac{1}{3}}} = -\frac{1}{2}.$$

Example 3 ► **Applying Rules of Exponents When Working with Rational Exponents**

Simplify each expression. Write your answer with only positive exponents. Assume that all variables represent positive real numbers.

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}}$ **b.** $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}}$ **c.** $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$

Solution ►

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}} = 2a^{\frac{3}{4}+\left(-\frac{2}{3}\right)} = 2a^{\frac{9}{12}-\frac{8}{12}} = \boldsymbol{2a^{\frac{1}{12}}}$

b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}} = 4^{\frac{1}{3}-\frac{5}{3}} = 4^{-\frac{4}{3}} = \boldsymbol{\frac{1}{4^{\frac{4}{3}}}}$

c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}} = x^{\frac{\cancel{3}}{\cancel{8}_2}\cdot\frac{\cancel{4}}{\cancel{3}}} \cdot y^{\frac{5}{2}\cdot\frac{4}{3}} = \boldsymbol{x^{\frac{1}{2}}y^{\frac{10}{3}}}$

Example 4 ► **Evaluating Powers with Rational Exponents**

Evaluate each power.

a. $64^{-\frac{1}{3}}$ **b.** $\left(-\frac{8}{125}\right)^{\frac{2}{3}}$

Solution ►

a. $64^{-\frac{1}{3}} = (2^6)^{-\frac{1}{3}} = 2^{-2} = \frac{1}{2^2} = \boldsymbol{\frac{1}{4}}$

It is helpful to change the base into a **power of prime number**, if possible.

b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}} = \left(\left(-\frac{2}{5}\right)^{\cancel{3}}\right)^{\frac{2}{\cancel{3}}} = \left(-\frac{2}{5}\right)^2 = \boldsymbol{\frac{4}{25}}$

Observe that if m in $\sqrt[n]{a^m}$ is a multiple of n, that is if $m = kn$ for some integer k, then

$$\sqrt[n]{a^{kn}} = a^{\frac{k\cancel{n}}{\cancel{n}}} = a^k$$

Example 5 ► **Simplifying Radical Expressions by Converting to Rational Exponents**

Simplify. Assume that all variables represent positive real numbers. Leave your answer in simplified single radical form.

a. $\sqrt[5]{3^{20}}$ **b.** $\sqrt{x} \cdot \sqrt[4]{x^3}$ **c.** $\sqrt[3]{2\sqrt{2}}$

Solution ► **a.** $\sqrt[5]{3^{20}} = (3^{20})^{\frac{1}{5}} = 3^4 = \mathbf{81}$

divide at the exponential level

b. $\sqrt{x} \cdot \sqrt[4]{x^3} = x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} = x^{\frac{1\cdot 2}{2\cdot 2}+\frac{3}{4}} = x^{\frac{5}{4}} = x \cdot x^{\frac{1}{4}} = \boldsymbol{x\sqrt[4]{x}}$

add exponents as $\frac{5}{4} = 1 + \frac{1}{4}$

c. $\sqrt[3]{2\sqrt{2}} = \left(2 \cdot 2^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{1+\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{2}} = \mathbf{\sqrt{2}}$

This bracket is essential!

Another solution:

$\sqrt[3]{2\sqrt{2}} = 2^{\frac{1}{3}} \cdot \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}} = 2^{\frac{1}{2}} = \mathbf{\sqrt{2}}$

RD.2 Exercises

Match each expression from Column I with the equivalent expression from Column II.

1.

Column I	Column II
a. $9^{\frac{1}{2}}$	**A.** $\frac{1}{3}$
b. $9^{-\frac{1}{2}}$	**B.** 3
c. $-9^{\frac{3}{2}}$	**C.** -27
d. $-9^{-\frac{1}{2}}$	**D.** not a real number
e. $(-9)^{\frac{1}{2}}$	**E.** $\frac{1}{27}$
f. $9^{-\frac{3}{2}}$	**F.** $-\frac{1}{3}$

2.

Column I	Column II
a. $(-32)^{\frac{2}{5}}$	**A.** 2
b. $-27^{\frac{2}{3}}$	**B.** $\frac{1}{4}$
c. $32^{\frac{1}{5}}$	**C.** -8
d. $32^{-\frac{2}{5}}$	**D.** -9
e. $-4^{\frac{3}{2}}$	**E.** not a real number
f. $(-4)^{\frac{3}{2}}$	**F.** 4

Write the base as a ***power of a prime number*** *to evaluate each expression, if possible.*

3. $32^{\frac{1}{5}}$ **4.** $27^{\frac{4}{3}}$ **5.** $-49^{\frac{3}{2}}$ **6.** $16^{\frac{3}{4}}$

7. $-100^{-\frac{1}{2}}$ **8.** $125^{-\frac{1}{3}}$ **9.** $\left(\frac{64}{81}\right)^{\frac{3}{4}}$ **10.** $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

11. $(-36)^{\frac{1}{2}}$ **12.** $(-64)^{\frac{1}{3}}$ **13.** $\left(-\frac{1}{8}\right)^{-\frac{1}{3}}$ **14.** $(-625)^{-\frac{1}{4}}$

*Rewrite **with** rational exponents and simplify, if possible. Assume that all variables represent positive real numbers.*

15. $\sqrt{5}$ **16.** $\sqrt[3]{6}$ **17.** $\sqrt{x^6}$ **18.** $\sqrt[5]{y^2}$

19. $\sqrt[3]{64x^6}$ **20.** $\sqrt[3]{16x^2y^3}$ **21.** $\sqrt{\frac{25}{x^5}}$ **22.** $\sqrt[4]{\frac{16}{a^6}}$

*Rewrite **without** rational exponents, and simplify, if possible. Assume that all variables represent positive real numbers.*

23. $4^{\frac{5}{2}}$ **24.** $8^{\frac{3}{4}}$ **25.** $x^{\frac{3}{5}}$ **26.** $a^{\frac{7}{3}}$

27. $(-3)^{\frac{2}{3}}$ **28.** $(-2)^{\frac{3}{5}}$ **29.** $2x^{-\frac{1}{2}}$ **30.** $x^{\frac{1}{3}}y^{-\frac{1}{2}}$

*Use the **laws of exponents** to simplify. Write the answers with positive exponents. Assume that all variables represent positive real numbers.*

31. $3^{\frac{3}{4}} \cdot 3^{\frac{1}{8}}$ **32.** $x^{\frac{2}{3}} \cdot x^{-\frac{1}{4}}$ **33.** $\frac{2^{\frac{5}{8}}}{2^{-\frac{1}{8}}}$ **34.** $\frac{a^{\frac{1}{3}}}{a^{\frac{2}{3}}}$

35. $\left(5^{\frac{15}{8}}\right)^{\frac{2}{3}}$ **36.** $\left(y^{\frac{2}{3}}\right)^{-\frac{3}{7}}$ **37.** $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$ **38.** $\left(a^{-\frac{2}{3}} \cdot b^{\frac{5}{8}}\right)^{-4}$

39. $\left(\frac{y^{-\frac{3}{2}}}{x^{-\frac{5}{3}}}\right)^{\frac{1}{3}}$ **40.** $\left(\frac{a^{-\frac{2}{3}}}{b^{-\frac{5}{6}}}\right)^{\frac{3}{4}}$ **41.** $x^{\frac{2}{3}} \cdot 5x^{-\frac{2}{5}}$ **42.** $x^{\frac{2}{5}} \cdot \left(4x^{-\frac{4}{5}}\right)^{-\frac{1}{4}}$

*Use rational exponents to **simplify**. Write the answer **in radical notation** if appropriate. Assume that all variables represent positive real numbers.*

43. $\sqrt[6]{x^2}$ **44.** $\left(\sqrt[3]{ab}\right)^{15}$ **45.** $\sqrt[6]{y^{-18}}$ **46.** $\sqrt{x^4y^{-6}}$

47. $\sqrt[6]{81}$ **48.** $\sqrt[14]{128}$ **49.** $\sqrt[3]{8y^6}$ **50.** $\sqrt[4]{81p^6}$

51. $\sqrt[3]{(4x^3y)^2}$ **52.** $\sqrt[5]{64(x+1)^{10}}$ **53.** $\sqrt[4]{16x^4y^2}$ **54.** $\sqrt[5]{32a^{10}d^{15}}$

*Use rational exponents to rewrite in a **single radical expression** in a simplified form. Assume that all variables represent positive real numbers.*

55. $\sqrt[3]{5} \cdot \sqrt{5}$ **56.** $\sqrt[3]{2} \cdot \sqrt[4]{3}$ **57.** $\sqrt{a} \cdot \sqrt[3]{3a}$ **58.** $\sqrt[3]{x} \cdot \sqrt[5]{2x}$

59. $\sqrt[6]{x^5} \cdot \sqrt[3]{x^2}$ **60.** $\sqrt[3]{xz} \cdot \sqrt{z}$ **61.** $\frac{\sqrt{x^5}}{\sqrt{x^8}}$ **62.** $\frac{\sqrt[3]{a^5}}{\sqrt{a^3}}$

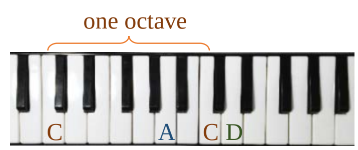
63. $\frac{\sqrt[3]{8x}}{\sqrt[4]{x^3}}$ **64.** $\sqrt[3]{\sqrt{a}}$ **65.** $\sqrt[4]{\sqrt[3]{xy}}$ **66.** $\sqrt{\sqrt[3]{(3x)^2}}$

67. $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$ **68.** $\sqrt[3]{3\sqrt{3}}$ **69.** $\sqrt[4]{x\sqrt{x}}$ **70.** $\sqrt[3]{2\sqrt{x}}$

71. Consider two expressions: $\sqrt[n]{x^n + y^n}$ and $x + y$. Observe that for $x = 1$ and $y = 0$ both expressions are equal: $\sqrt[n]{x^n + y^n} = \sqrt[n]{1^n + 0^n} = 1 = 1 + 9 = x + y$. Does this mean that $\sqrt[n]{x^n + y^n} = x + y$? Justify your answer.

Solve each problem.

72. When counting both the black and white keys on a piano, an octave contains 12 keys. The frequencies of consecutive keys increase by a factor of $2^{\frac{1}{12}}$. For example, the frequency of the tone D that is two keys above middle C is



$$2^{\frac{1}{12}} \cdot 2^{\frac{1}{12}} = \left(2^{\frac{1}{12}}\right)^2 = 2^{\frac{1}{6}} \approx 1.12$$

times the frequency of the middle C.

a. If tone G, which is five keys below the middle C, has a frequency of about 196 cycles per second, estimate the frequency of the middle C to the nearest tenths of a cycle.

b. Find the relation between frequencies of two tones that are one octave apart.

73. An animal's heart rate is related to the animal's weight. Suppose that the average heart rate R, in beats per minute, for an animal that weighs k kilograms can be estimated by using the function $R(w) = 600w^{-\frac{1}{2}}$. What is the expected average heart rate of a horse that weighs 400 kilograms?

74. Suppose that the duration of a storm T, in hours, can be determined by using the function $T(D) = 0.03D^{\frac{3}{2}}$, where D denotes the diameter of a storm in kilometers. To the nearest minute, what is the duration of a storm with a diameter of 20 kilometers?

RD3 Simplifying Radical Expressions and the Distance Formula

In the previous section, we simplified some radical expressions by replacing radical signs with rational exponents, applying the rules of exponents, and then converting the resulting expressions back into radical notation. In this section, we broaden the above method of simplifying radicals by examining products and quotients of radicals with the same indexes, as well as explore the possibilities of decreasing the index of a radical.

In the second part of this section, we will apply the skills of simplifying radicals in problems involving the Pythagorean Theorem. In particular, we will develop the distance formula and apply it to calculate distances between two given points in a plane.

Multiplication, Division, and Simplification of Radicals

Suppose we wish to multiply radicals with the same indexes. This can be done by converting each radical to a rational exponent and then using properties of exponents as follows:

PRODUCT RULE

$$\sqrt[n]{\boldsymbol{a}} \cdot \sqrt[n]{\boldsymbol{b}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{\boldsymbol{ab}}$$

This shows that the **product of same index radicals** is the **radical of the product** of their radicands.

Similarly, the **quotient of same index radicals** is the **radical of the quotient** of their radicands, as we have

QUOTIENT RULE

$$\frac{\sqrt[n]{\boldsymbol{a}}}{\sqrt[n]{\boldsymbol{b}}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{\boldsymbol{a}}{\boldsymbol{b}}}$$

So, $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = \mathbf{4}$. Similarly, $\frac{\sqrt[3]{16}}{\sqrt[3]{2}} = \sqrt[3]{\frac{16}{2}} = \sqrt[3]{8} = \mathbf{2}$.

Attention! There is no such rule for addition or subtraction of terms. For instance,

$$\sqrt{\boldsymbol{a} + \boldsymbol{b}} \neq \sqrt{\boldsymbol{a}} \pm \sqrt{\boldsymbol{b}},$$

and generally

$$\sqrt[n]{\boldsymbol{a} \pm \boldsymbol{b}} \neq \sqrt[n]{\boldsymbol{a}} \pm \sqrt[n]{\boldsymbol{b}}.$$

Here is a counterexample: $\sqrt[n]{2} = \sqrt[n]{1+1} \neq \sqrt[n]{1} + \sqrt[n]{1} = 1 + 1 = 2$

Example 1 ► **Multiplying and Dividing Radicals of the Same Indexes**

Perform the indicated operations and simplify, if possible. Assume that all variables are positive.

a. $\sqrt{10} \cdot \sqrt{15}$

b. $\sqrt{2x^3}\sqrt{6xy}$

c. $\frac{\sqrt{10x}}{\sqrt{5}}$

d. $\frac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}}$

Solution ►

a. $\sqrt{10}\cdot\sqrt{15} = \sqrt{10\cdot 15} = \sqrt{2\cdot 5\cdot 5\cdot 3} = \sqrt{5\cdot 5\cdot 2\cdot 3} = \sqrt{25}\cdot\sqrt{6} = \mathbf{5\sqrt{6}}$

product rule; prime factorization; commutativity of multiplication; product rule

b. $\sqrt{2x^3}\,\sqrt{6xy} = \sqrt{2\cdot 2\cdot 3x^4y} = \sqrt{4x^4}\cdot\sqrt{3y} = \mathbf{2x^2\sqrt{3y}}$

use commutativity of multiplication to isolate perfect square factors

Here the multiplication sign is assumed, even if it is not indicated.

c. $\dfrac{\sqrt{10x}}{\sqrt{5}} = \sqrt{\dfrac{\overset{2}{\cancel{10}}x}{\cancel{5}}} = \mathbf{\sqrt{2x}}$

quotient rule

d. $\dfrac{\sqrt[4]{32x^3}}{\sqrt[4]{2x}} = \sqrt[4]{\dfrac{\overset{16}{\cancel{32}}\cancel{x^3}^{2}}{\cancel{2x}}} = \sqrt[4]{16x^2} = \sqrt[4]{16}\cdot\sqrt[4]{x^2} = \mathbf{2\sqrt{x}}$

Recall that $\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$.

Caution! Remember to indicate the index of the radical for indexes higher than two.

The product and quotient rules are essential when simplifying radicals.

To **simplify a radical** means to:

1. Make sure that all **power factors of the radicand have exponents smaller than the index of the radical**.

   For example, $\sqrt[3]{2^4x^8y} = \sqrt[3]{2^3x^6}\cdot\sqrt[3]{2x^2y} = 2x\sqrt[3]{2x^2y}$.

2. Leave the radicand with **no fractions**.

   For example, $\sqrt{\dfrac{2x}{25}} = \dfrac{\sqrt{2x}}{\sqrt{25}} = \dfrac{\sqrt{2x}}{5}$.

3. **Rationalize** any **denominator**. (Make sure that denominators are **free from radicals**.)

   For example, $\sqrt{\dfrac{4}{x}} = \dfrac{\sqrt{4}}{\sqrt{x}} = \dfrac{2\cdot\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}} = \dfrac{2\sqrt{x}}{x}$, providing that $x > 0$.

4. **Reduce the power** of the radicand **with the index** of the radical, if possible.

   For example, $\sqrt[4]{x^2} = x^{\frac{2}{4}} = x^{\frac{1}{2}} = \sqrt{x}$.

Example 2 ► **Simplifying Radicals**

Simplify each radical. Assume that all variables are positive.

a. $\sqrt[5]{96x^7y^{15}}$ **b.** $\sqrt[4]{\dfrac{a^{12}}{16b^4}}$ **c.** $\sqrt{\dfrac{25x^2}{8x^3}}$ **d.** $\sqrt[6]{27a^{15}}$

Solution ► **a.** $\sqrt[5]{96x^7y^{15}} = \sqrt[5]{2^5 \cdot 3x^7y^{15}} = \boldsymbol{2xy^3\sqrt[5]{3x^2}}$

$\sqrt[5]{y^{15}} = y^3$

$\sqrt[5]{x^7} = x\sqrt[5]{x^2}$

Generally, to simplify $\sqrt[d]{x^a}$, we perform the division

$$a \div d = quotient\ \boldsymbol{q} + remainder\ \boldsymbol{r},$$

and then pull the $\boldsymbol{q}$-th power of x out of the radical, leaving the $\boldsymbol{r}$-th power of x under the radical. So, we obtain

$$\boldsymbol{\sqrt[d]{x^a} = x^q\sqrt[d]{x^r}}$$

b. $\sqrt[4]{\dfrac{a^{12}}{16b^4}} = \dfrac{\sqrt[4]{a^{12}}}{\sqrt[4]{2^4b^4}} = \boldsymbol{\dfrac{a^3}{2b}}$

c. $\sqrt{\dfrac{25\cancel{x^2}}{8\cancel{x^3}^{1}}} = \sqrt{\dfrac{25}{2^3x}} = \dfrac{\sqrt{25}}{\sqrt{2^3x}} = \dfrac{5}{2\sqrt{2x}} \cdot \dfrac{\sqrt{2x}}{\sqrt{2x}} = \dfrac{5\sqrt{2x}}{2 \cdot 2x} = \boldsymbol{\dfrac{5\sqrt{2x}}{4x}}$

d. $\sqrt[6]{27a^{15}} = \sqrt[6]{3^3a^{15}} = a^2\sqrt[6]{3^3a^3} = a^2 \cdot \sqrt[\cancel{6}^{2}]{(3a)^{\cancel{3}}} = \boldsymbol{a^2\sqrt{3a}}$

Example 3 ► **Simplifying Expressions Involving Multiplication, Division, or Composition of Radicals with Different Indexes**

Simplify each expression. Leave your answer in simplified single radical form. Assume that all variables are positive.

a. $\sqrt{xy^5} \cdot \sqrt[3]{x^4y}$ **b.** $\dfrac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}}$ **c.** $\sqrt[3]{x^2\sqrt{2x}}$

Solution ► **a.** $\sqrt{xy^5} \cdot \sqrt[3]{x^2y} = x^{\frac{1}{2}}y^{\frac{5}{2}} \cdot x^{\frac{2}{3}}y^{\frac{1}{3}} = x^{\frac{1\cdot3}{2\cdot3}+\frac{2\cdot2}{3\cdot2}}y^{\frac{5\cdot3}{2\cdot3}+\frac{1\cdot2}{3\cdot2}} = x^{\frac{7}{6}}y^{\frac{17}{6}} = (x^7y^{17})^{\frac{1}{6}}$

$$= \sqrt[6]{x^7y^{17}} = \boldsymbol{xy^2\sqrt[6]{xy^5}}$$

If radicals are of different indexes, convert them to exponential form.

b. $\dfrac{\sqrt[4]{a^2b^3}}{\sqrt[3]{ab}} = \dfrac{a^{\frac{\cancel{2}}{\cancel{4}_2}}b^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}} = a^{\frac{1\cdot3}{2\cdot3}-\frac{1\cdot2}{3\cdot2}}b^{\frac{3\cdot3}{4\cdot3}-\frac{1\cdot4}{3\cdot4}} = a^{\frac{1\cdot2}{6\cdot2}}b^{\frac{5}{12}} = (a^2b^5)^{\frac{1}{12}} = \boldsymbol{\sqrt[12]{a^2b^5}}$

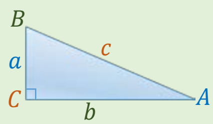
Bring the exponents to the LCD in order to leave the answer as a single radical.

c. $\sqrt[3]{x^2\sqrt{2x}} = x^{\frac{2}{3}} \cdot \left((2x)^{\frac{1}{2}}\right)^{\frac{1}{3}} = x^{\frac{2}{3}} \cdot 2^{\frac{1}{6}} \cdot x^{\frac{1}{6}} = x^{\frac{2\cdot2}{3\cdot2}+\frac{1}{6}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1}{6}}x^{\frac{5}{6}} = (2x^5)^{\frac{1}{6}} = \boldsymbol{\sqrt[6]{2x^5}}$

Pythagorean Theorem and Distance Formula

One of the most famous theorems in mathematics is the Pythagorean Theorem.

Pythagorean Theorem ►



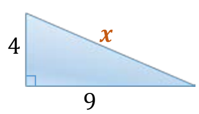
Suppose angle C in a triangle ABC is a $90°$ angle.
Then the **sum of the squares** of the lengths of the two **legs**, $\boldsymbol{a}$ and $\boldsymbol{b}$, equals to the **square** of the length of the **hypotenuse** $\boldsymbol{c}$:

$$\boldsymbol{a^2 + b^2 = c^2}$$

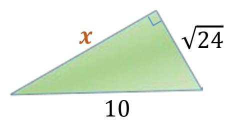
Example 4 ► **Using The Pythagorean Equation**

For the first two triangles, find the exact length x of the unknown side. For triangle (**c**), express length x in terms of the unknown n.

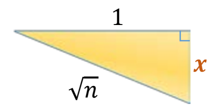
a.



b.



c.



Solution ►

Caution: Generally, $\sqrt{x^2} = |x|$ However, the length of a side of a triangle is positive. So, we can write $\sqrt{x^2} = x$

a. The length of the hypotenuse of the given right triangle is equal to x. So, the Pythagorean equation takes the form

$$x^2 = 4^2 + 9^2.$$

To solve it for x, we take a square root of each side of the equation. This gives us

$$\sqrt{x^2} = \sqrt{4^2 + 9^2}$$
$$x = \sqrt{16 + 81}$$
$$x = \boldsymbol{\sqrt{97}}$$

b. Since 10 is the length of the hypotenuse, we form the Pythagorean equation

$$10^2 = x^2 + \sqrt{24}^2.$$

To solve it for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we have

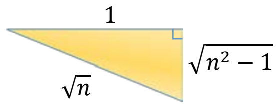
$$10^2 - \sqrt{24}^2 = x^2$$
$$100 - 24 = x^2$$
$$x^2 = 76$$
$$x = \sqrt{76} = \sqrt{4 \cdot 19} = \boldsymbol{2\sqrt{19}}$$

Customary, we simplify each root, if possible.

c. The length of the hypotenuse is $\sqrt{n}$, so we form the Pythagorean equation as below.

$$\left(\sqrt{n}\right)^2 = 1^2 + x^2$$

To solve this equation for x, we isolate the x^2 term and then apply the square root operator to both sides of the equation. So, we obtain



$$n^2 = 1 + x^2$$
$$n^2 - 1 = x^2$$
$$x = \sqrt{\boldsymbol{n^2 - 1}}$$

Note: Since the hypotenuse of length $\sqrt{n}$ must be longer than the leg of length 1, then $n > 1$. This means that $n^2 - 1 > 0$, and therefore $\sqrt{n^2-1}$ is a positive real number.

The Pythagorean Theorem allows us to find the distance between any two given points in a plane.
Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in a coordinate plane. Then $|x_2 - x_1|$ represents the horizontal distance between A and B and $|y_2 - y_1|$ represents the vertical distance between A and B, as shown in *Figure 1*. Notice that by applying the absolute value operator to each difference of the coordinates we guarantee that the resulting horizontal and vertical distance is indeed a nonnegative number.

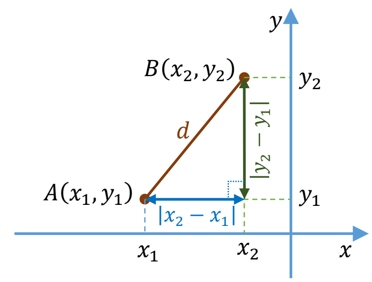


Figure 1

Applying the Pythagorean Theorem to the right triangle shown in *Figure 1*, we form the equation

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2,$$

where d is the distance between A and B.

Notice that $|x_2 - x_1|^2 = (x_2 - x_1)^2$ as a perfect square automatically makes the expression nonnegative. Similarly, $|y_2 - y_1|^2 = (y_2 - y_1)^2$. So, the Pythagorean equation takes the form

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

After solving this equation for d, we obtain the **distance formula**:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Note: Observe that due to squaring the difference of the corresponding coordinates, **the distance between two points is the same regardless of which point is chosen as first**, (x_1, y_1), and second, (x_2, y_2).

Example 5 ► **Finding the Distance Between Two Points**

Find the exact distance between the points $(-2,4)$ and $(5,3)$.

Solution ► Let $(-2,4) = (x_1, y_1)$ and $(5,3) = (x_2, y_2)$. To find the distance d between the two points, we follow the distance formula:

$$d=\sqrt{(5-(-2))^2+(3-4)^2}=\sqrt{7^2+(-1)^2}=\sqrt{49+1}=\sqrt{50}=\mathbf{5\sqrt{2}}$$

So, the points $(-2,4)$ and $(5,3)$ are $5\sqrt{2}$ units apart.

RD.3 Exercises

Multiply and simplify, if possible. Assume that all variables are positive.

1. $\sqrt{5}\cdot\sqrt{5}$
2. $\sqrt{18}\cdot\sqrt{2}$
3. $\sqrt{6}\cdot\sqrt{3}$
4. $\sqrt{15}\cdot\sqrt{6}$
5. $\sqrt{45}\cdot\sqrt{60}$
6. $\sqrt{24}\cdot\sqrt{75}$
7. $\sqrt{3x^3}\cdot\sqrt{6x^5}$
8. $\sqrt{5y^7}\cdot\sqrt{15a^3}$
9. $\sqrt{12x^3y}\,\sqrt{8x^4y^2}$
10. $\sqrt{30a^3b^4}\,\sqrt{18a^2b^5}$
11. $\sqrt[3]{4x^2}\,\sqrt[3]{2x^4}$
12. $\sqrt[4]{20a^3}\,\sqrt[4]{4a^5}$

Divide and simplify, if possible. Assume that all variables are positive.

13. $\frac{\sqrt{90}}{\sqrt{5}}$
14. $\frac{\sqrt{48}}{\sqrt{6}}$
15. $\frac{\sqrt{42a}}{\sqrt{7a}}$
16. $\frac{\sqrt{30x^3}}{\sqrt{10x}}$
17. $\frac{\sqrt{52ab^3}}{\sqrt{13a}}$
18. $\frac{\sqrt{56xy^3}}{\sqrt{8x}}$
19. $\frac{\sqrt{128x^2y}}{2\sqrt{2}}$
20. $\frac{\sqrt{48a^3b}}{2\sqrt{3}}$
21. $\frac{\sqrt[4]{80}}{\sqrt[4]{5}}$
22. $\frac{\sqrt[3]{108}}{\sqrt[3]{4}}$
23. $\frac{\sqrt[3]{96a^5b^2}}{\sqrt[3]{12a^2b}}$
24. $\frac{\sqrt[4]{48x^9y^{13}}}{\sqrt[4]{3xy^5}}$

Simplify each expression. Assume that all variables are positive.

25. $\sqrt{144x^4y^9}$
26. $-\sqrt{81m^8n^5}$
27. $\sqrt[3]{-125a^6b^9c^{12}}$
28. $\sqrt{50x^3y^4}$
29. $\sqrt[4]{\frac{1}{16}m^8n^{20}}$
30. $-\sqrt[3]{-\frac{1}{27}x^2y^7}$
31. $\sqrt{7a^7b^6}$
32. $\sqrt{75p^3q^4}$
33. $\sqrt[5]{64x^{12}y^{15}}$
34. $\sqrt[5]{p^{14}q^7r^{23}}$
35. $-\sqrt[4]{162a^{15}b^{10}}$
36. $-\sqrt[4]{32x^5y^{10}}$
37. $\sqrt{\frac{16}{49}}$
38. $\sqrt[3]{\frac{27}{125}}$
39. $\sqrt{\frac{121}{y^2}}$
40. $\sqrt{\frac{64}{x^4}}$
41. $\sqrt[3]{\frac{81a^5}{64}}$
42. $\sqrt{\frac{36x^5}{y^6}}$
43. $\sqrt[4]{\frac{16x^{12}}{y^4z^{16}}}$
44. $\sqrt[5]{\frac{32y^8}{x^{10}}}$
45. $\sqrt[4]{36}$
46. $\sqrt[6]{27}$
47. $-\sqrt[10]{x^{25}}$
48. $\sqrt[12]{x^{44}}$
49. $-\sqrt{\frac{1}{x^3y}}$
50. $\sqrt[3]{\frac{64x^{15}}{y^4z^5}}$
51. $\sqrt[6]{\frac{x^{13}}{y^6z^{12}}}$
52. $\sqrt[6]{\frac{p^9q^{24}}{r^{18}}}$

53. To simplify the radical $\sqrt{x^3+x^2}$, a student wrote $\sqrt{x^3+x^2} = x\sqrt{x} + x = x(\sqrt{x}+1)$. Is this correct? Justify your answer.

Perform operations. Leave the answer in simplified ***<u>single radical</u>*** *form. Assume that all variables are positive.*

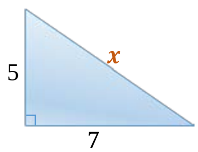
54. $\sqrt{3}\cdot\sqrt[3]{4}$ **55.** $\sqrt{x}\cdot\sqrt[5]{x}$ **56.** $\sqrt[3]{x^2}\cdot\sqrt[4]{x}$ **57.** $\sqrt[3]{4}\cdot\sqrt[5]{8}$

58. $\dfrac{\sqrt[3]{a^2}}{\sqrt{a}}$ **59.** $\dfrac{\sqrt{x}}{\sqrt[4]{x}}$ **60.** $\dfrac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$ **61.** $\dfrac{\sqrt[5]{16a^2}}{\sqrt[3]{2a^2}}$

62. $\sqrt[3]{2\sqrt{x}}$ **63.** $\sqrt{x\sqrt[3]{2x^2}}$ **64.** $\sqrt[4]{3\sqrt[3]{9}}$ **65.** $\sqrt[3]{x^2\sqrt[4]{x^3}}$

For each right triangle, find length x. Simplify the answer if possible. In problems ***73*** *and* ***74,*** *expect the length x to be an expression in terms of n.*

66.



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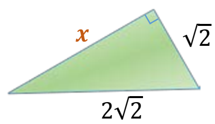

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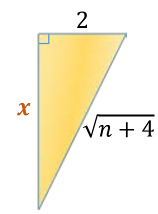

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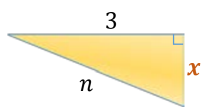

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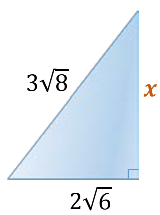

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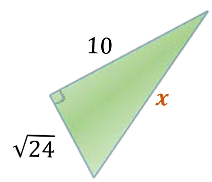


Find the exact distance between each pair of points.

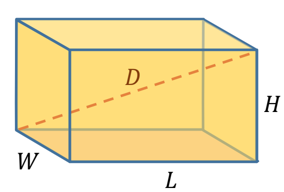
72. $(8,13)$ and $(2,5)$ **73.** $(-8,3)$ and $(-4,1)$ **74.** $(-6,5)$ and $(3,-4)$

75. $\left(\frac{5}{7},\frac{1}{14}\right)$ and $\left(\frac{1}{7},\frac{11}{14}\right)$ **76.** $(0,\sqrt{6})$ and $(\sqrt{7},0)$ **77.** $(\sqrt{2},\sqrt{6})$ and $(2\sqrt{2},-4\sqrt{6})$

78. $(-\sqrt{5},6\sqrt{3})$ and $(\sqrt{5},\sqrt{3})$ **79.** $(0,0)$ and (p,q) **80.** $(x+h,y+h)$ and (x,y) (*assume that* $h>0$)

Solve each problem.

81. To find the diagonal of a box, we can use the formula $D=\sqrt{W^2+L^2+H^2}$, where W, L, and H are, respectively, the width, length, and height of the box. Find the diagonal D of a storage container that is 6.1 meters long, 2.4 meters wide, and 2.6 meters high. *Round your answer to the nearest centimeter.*

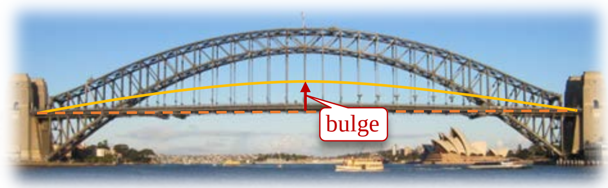


82. The screen of a 32-inch television is 27.9-inch wide. To the nearest tenth of an inch, what is the measure of its height? (Note: *TVs are measured diagonally, so a 32-inch television means that its screen measures diagonally 32 inches.*)

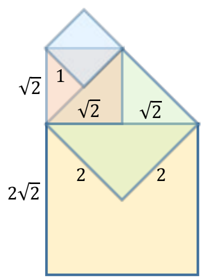
83. Suppose $A=(0,-3)$ and P is a point on the x-axis of a Cartesian coordinate system. Find all possible coordinates of P if $AP=5$.

84. Suppose $B = (1, 0)$ and P is a point on the y-axis of a Cartesian coordinate system. Find all possible coordinates of P if $BP = 2$.

85. Due to high temperatures, a 3-km bridge may expand up to 0.6 meters in length. If the maximum bulge occurs at the middle of the bridge, find the height of such a bulge. *The answer may be surprising. To avoid such situations, engineers design bridges with expansion spaces.*



RD4 Operations on Radical Expressions; Rationalization of Denominators



Unlike operations on fractions or decimals, sums and differences of many radicals cannot be simplified. For instance, we cannot combine $\sqrt{2}$ and $\sqrt{3}$, nor simplify expressions such as $\sqrt[3]{2}-1$. These types of radical expressions can only be approximated with the aid of a calculator.
However, some radical expressions can be combined (added or subtracted) and simplified. For example, the sum of $2\sqrt{2}$ and $\sqrt{2}$ is $3\sqrt{2}$, similarly as $2x + x = 3x$.
In this section, first, we discuss the addition and subtraction of radical expressions. Then, we show how to work with radical expressions involving a combination of the four basic operations. Finally, we examine how to rationalize denominators of radical expressions.

Addition and Subtraction of Radical Expressions

Recall that to perform addition or subtraction of two variable terms we need these terms to be **like**. This is because the addition and subtraction of terms are performed by factoring out the variable "like" part of the terms as a common factor. For example,

$$x^2 + 3x^2 = (1+3)x^2 = 4x^2$$

The same strategy works for addition and subtraction of the same types of radicals or **radical terms** (terms containing radicals).

Definition 4.1 ► Radical terms containing radicals with the <u>same index</u> and the <u>same radicands</u> are referred to as **like radicals** or **like radical terms**.
For example,

$\sqrt{5x}$ and $2\sqrt{5x}$ are **like** (the indexes and the radicands are the same)

while

$5\sqrt{2}$ and $2\sqrt{5}$ are **<u>not</u> like** (the radicands are different)

and

$\sqrt{x}$ and $\sqrt[3]{x}$ are **<u>not</u> like radicals** (the indexes are different).

To **add** or **subtract like radical expressions** we **factor out the common radical** and any other common factor, if applicable. For example,

$$4\sqrt{2} + 3\sqrt{2} = (4+3)\sqrt{2} = 7\sqrt{2},$$

and

$$4xy\sqrt{2} - 3x\sqrt{2} = (4y+3)x\sqrt{2}.$$

Caution! Unlike radical expressions cannot be combined. For example, we are unable to perform the addition $\sqrt{6}+\sqrt{3}$. Such a sum can only be approximated using a calculator.

Notice that unlike radicals may become like if we simplify them first. For example, $\sqrt{200}$ and $\sqrt{50}$ are not like, but $\sqrt{200} = 10\sqrt{2}$ and $\sqrt{50} = 5\sqrt{2}$. Since $10\sqrt{2}$ and $5\sqrt{2}$ are like radical terms, they can be combined. So, we can perform, for example, the addition:

$$\sqrt{200} + \sqrt{50} = 10\sqrt{2} + 5\sqrt{2} = 15\sqrt{2}$$

Example 1 ► **Adding and Subtracting Radical Expressions**

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

a. $5\sqrt{3} - 8\sqrt{3}$

b. $3\sqrt[5]{2} - 7x\sqrt[5]{2} + 6\sqrt[5]{2}$

c. $7\sqrt{45} + \sqrt{80} - \sqrt{12}$

d. $3\sqrt[3]{y^5} - 7\sqrt[3]{y^2} + \sqrt[3]{8y^8}$

e. $\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}}$

f. $\sqrt{25x - 25} - \sqrt{9x - 9}$

Solution ►

a. To subtract like radicals, we combine their coefficients via factoring.

$$5\sqrt{3} - 8\sqrt{3} = (5 - 8)\sqrt{3} = \mathbf{-3\sqrt{3}}$$

The brackets are essential here.

b. $3\sqrt[5]{2} - 7x\sqrt[5]{2} + 6\sqrt[5]{2} = (3 - 7x + 6)\sqrt[5]{2} = (9 - 7x)\sqrt[5]{2}$

Note: Even if not all coefficients are like, factoring the common radical is a useful strategy that allows us to combine like radical expressions.

c. The expression $7\sqrt{45} + \sqrt{80} - \sqrt{12}$ consists of unlike radical terms, so they cannot be combined in this form. However, if we simplify the radicals, some of them may become like and then become possible to combine.

$$7\sqrt{45} + \sqrt{80} - \sqrt{12} = 7\sqrt{9 \cdot 5} + \sqrt{16 \cdot 5} - \sqrt{4 \cdot 3} = 7 \cdot 3\sqrt{5} + 4\sqrt{5} - 2\sqrt{3}$$

$$= 21\sqrt{5} + 4\sqrt{5} - 2\sqrt{3} = \mathbf{25\sqrt{5} - 2\sqrt{3}}$$

d. As in the previous example, we simplify each radical expression before attempting to combine them.

$$3\sqrt[3]{y^5} - 5y\sqrt[3]{y^2} + \sqrt[5]{32y^7} = 3y\sqrt[3]{y^2} - 5y\sqrt[3]{y^2} + 2y\sqrt[5]{y^2}$$

$$= (3y - 5y)\sqrt[3]{y^2} + 2y\sqrt[5]{y^2} = \mathbf{-2y\sqrt[3]{y^2} + 2y\sqrt[5]{y^2}}$$

Remember to write the <u>index</u> with each radical.

Note: The last two radical expressions cannot be combined because of different indexes.

e. To perform the addition $\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}}$, we may simplify each radical expression first. Then, we add the expressions by bringing them to the least common denominator and finally, factor the common radical, as shown below.

$$\sqrt{\frac{x}{16}} + 2\sqrt{\frac{x^3}{9}} = \frac{\sqrt{x}}{\sqrt{16}} + 2\frac{\sqrt{x^3}}{\sqrt{9}} = \frac{\sqrt{x}}{4} + 2\frac{\sqrt{x^3}}{3} = \frac{3\sqrt{x} + 2 \cdot 4 \cdot x\sqrt{x}}{12} = \left(\frac{\mathbf{3+8x}}{\mathbf{12}}\right)\sqrt{\boldsymbol{x}}$$

f. In an attempt to simplify radicals in the expression $\sqrt{25x^2-25}-\sqrt{9x^2-9}$, we factor each radicand first. So, we obtain

$$\sqrt{25x^2-25}-\sqrt{9x^2-9}=\sqrt{25(x^2-1)}-\sqrt{9(x^2-1)}=5\sqrt{x^2-1}-3\sqrt{x^2-1}$$

$$=\mathbf{2\sqrt{x^2-1}}$$

Caution! **The root of a sum <u>does not</u> equal the sum of the roots.** For example,

$$\sqrt{5}=\sqrt{\mathbf{1+4}}\neq\sqrt{\mathbf{1}}+\sqrt{\mathbf{4}}=1+2=3$$

So, radicals such as $\sqrt{25x^2-25}$ or $\sqrt{9x^2-9}$ can be simplified only via factoring a perfect square out of their radicals while $\sqrt{x^2-1}$ cannot be simplified any further.

Multiplication of Radical Expressions with More than One Term

Similarly as in the case of multiplication of polynomials, multiplication of radical expressions where at least one factor consists of more than one term is performed by applying the distributive property.

Example 2 ► **Multiplying Radical Expressions with More than One Term**

Multiply and then simplify each product. Assume that all variables represent positive real numbers.

a. $5\sqrt{2}\left(3\sqrt{2x}-\sqrt{6}\right)$

b. $\sqrt[3]{x}\left(\sqrt[3]{3x^2}-\sqrt[3]{81x^2}\right)$

c. $\left(2\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-3\sqrt{2}\right)$

d. $\left(x\sqrt{x}-\sqrt{y}\right)\left(x\sqrt{x}+\sqrt{y}\right)$

e. $\left(3\sqrt{2}+2\sqrt[3]{x}\right)\left(3\sqrt{2}-2\sqrt[3]{x}\right)$

f. $\left(\sqrt{5y}+y\sqrt{y}\right)^2$

Solution ► **a.**

$$5\sqrt{2}\left(3\sqrt{2x}-\sqrt{6}\right)=15\sqrt{4x}-5\sqrt{2\cdot2\cdot3}=15\cdot2\sqrt{x}-5\cdot2\sqrt{3}=\mathbf{30\sqrt{x}-10\sqrt{3}}$$

$5\sqrt{2}\cdot3\sqrt{2x}=5\cdot3\sqrt{2\cdot2x}$

These are unlike terms. So, they cannot be combined.

b.

simplification

$$\sqrt[3]{x}\left(\sqrt[3]{3x^2}-\sqrt[3]{81x^2}\right)=\sqrt[3]{3x^2\cdot x}-\sqrt[3]{81x^2\cdot x}=x\sqrt[3]{3}-3x\sqrt[3]{3}=\mathbf{-2x\sqrt[3]{3}}$$

distribution

combining like terms

c. To multiply two binomial expressions involving radicals we may use the **FOIL** method. Recall that the acronym **FOIL** refers to multiplying the **F**irst, **O**uter, **I**nner, and **L**ast terms of the binomials.

$$\left(2\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-3\sqrt{2}\right)=\overset{\textbf{F}}{2\cdot 3}-\overset{\textbf{O}}{6\sqrt{3\cdot 2}}+\overset{\textbf{I}}{\sqrt{2\cdot 3}}-\overset{\textbf{L}}{3\cdot 2}=\cancel{6}-6\sqrt{6}+\sqrt{6}-\cancel{6}$$

$$=\boldsymbol{-5\sqrt{6}}$$

d. To multiply two conjugate binomial expressions we follow the difference of squares formula, $(a-b)(a+b)=a^2-b^2$. So, we obtain

$$\left(\sqrt{x}\right)^2=x$$

$$\left(x\sqrt{x}-\sqrt{y}\right)\left(x\sqrt{x}+\sqrt{y}\right)=\left(x\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2=x^2\cdot x-y=\boldsymbol{x^3-y}$$

square each factor

e. Similarly as in the previous example, we follow the difference of squares formula.

$$\left(3\sqrt{2}+2\sqrt[3]{x}\right)\left(3\sqrt{2}-2\sqrt[3]{x}\right)=\left(3\sqrt{2}\right)^2-\left(2\sqrt[3]{x}\right)^2=9\cdot 2-4\sqrt[3]{x^2}=\boldsymbol{18-4\sqrt[3]{x^2}}$$

f. To multiply two identical binomial expressions we follow the perfect square formula, $(a+b)(a+b)=a^2+2ab+b^2$. So, we obtain

$$\left(\sqrt{5y}+y\sqrt{y}\right)^2=\left(\sqrt{5y}\right)^2+2\left(\sqrt{5y}\right)\left(y\sqrt{y}\right)+\left(y\sqrt{y}\right)^2=5y+2y\sqrt{5y^2}+y^2y$$

$$=\boldsymbol{5y+2\sqrt{5}y^2+y^3}$$

Rationalization of Denominators

As mentioned in *Section RD3*, a process of simplifying radicals involves rationalization of any emerging denominators. Similarly, a radical expression is not in its simplest form unless all its denominators are rational. This agreement originated before the days of calculators when computation was a tedious process performed by hand. Nevertheless, even in present time, the agreement of keeping denominators rational does not lose its validity, as we often work with variable radical expressions. For example, the expressions $\frac{2}{\sqrt{2}}$ and $\sqrt{2}$ are equivalent, as

$$\frac{2}{\sqrt{2}}=\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{2\sqrt{2}}{2}=\sqrt{2}$$

Similarly, $\frac{x}{\sqrt{x}}$ is equivalent to $\sqrt{x}$, as

$$\frac{x}{\sqrt{x}}=\frac{x}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}=\frac{x\sqrt{x}}{x}=\sqrt{x}$$

While one can argue that evaluating $\frac{2}{\sqrt{2}}$ is as easy as evaluating $\sqrt{2}$ when using a calculator, the expression $\sqrt{x}$ is definitely easier to use than $\frac{x}{\sqrt{x}}$ in any further algebraic manipulations.

Definition 4.2 ► The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalization** of the denominator.

Rationalization of denominators is carried out by multiplying the given fraction by a factor of 1, as shown in the next two examples.

Example 3 ► **Rationalizing Monomial Denominators**

Simplify, if possible. Leave the answer with a rational denominator. Assume that all variables represent positive real numbers.

a. $\frac{-1}{3\sqrt{5}}$ **b.** $\frac{5}{\sqrt[3]{32x}}$ **c.** $\sqrt[4]{\frac{81x^5}{y}}$

Solution ►

a. Notice that $\sqrt{5}$ can be converted to a rational number by multiplying it by another $\sqrt{5}$. Since the denominator of a fraction cannot be changed without changing the numerator in the same way, we multiply both, the numerator and denominator of $\frac{-1}{3\sqrt{5}}$ by $\sqrt{5}$. So, we obtain

$$\frac{-1}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{-\sqrt{5}}{3 \cdot 5} = -\frac{\sqrt{\mathbf{5}}}{\mathbf{15}}$$

b. First, we may want to simplify the radical in the denominator. So, we have

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{\sqrt[3]{8 \cdot 4x}} = \frac{5}{2\sqrt[3]{4x}}$$

Then, notice that since $\sqrt[3]{4x} = \sqrt[3]{2^2x}$, it is enough to multiply it by $\sqrt[3]{2x^2}$ to nihilate the radical. This is because $\sqrt[3]{2^2x} \cdot \sqrt[3]{2x^2} = \sqrt[3]{2^3x^3} = 2x$. So, we proceed

$$\frac{5}{\sqrt[3]{32x}} = \frac{5}{2\sqrt[3]{4x}} \cdot \frac{\sqrt[3]{2x^2}}{\sqrt[3]{2x^2}} = \frac{5\sqrt[3]{2x^2}}{2 \cdot 2x} = \frac{\mathbf{5}\sqrt[3]{\mathbf{2x^2}}}{\mathbf{4x}}$$

Caution: A common mistake in the rationalization of $\sqrt[3]{4x}$ is the attempt to multiply it by a copy of $\sqrt[3]{4x}$. However, $\sqrt[3]{4x} \cdot \sqrt[3]{4x} = \sqrt[3]{16x^2} = 2\sqrt[3]{3x^2}$ is still not rational. This is because we work with a cubic root, not a square root. So, to rationalize $\sqrt[3]{4x}$ we must look for 'filling' the radicand to a perfect cube. This is achieved by multiplying $4x$ by $2x^2$ to get $8x^3$.

c. To simplify $\sqrt[4]{\frac{81x^5}{y}}$, first, we apply the quotient rule for radicals, then simplify the radical in the numerator, and finally, rationalize the denominator. So, we have

$$\sqrt[4]{\frac{81x^5}{y}} = \frac{\sqrt[4]{81x^5}}{\sqrt[4]{y}} = \frac{3x\sqrt[4]{x}}{\sqrt[4]{y}} \cdot \frac{\sqrt[4]{y^3}}{\sqrt[4]{y^3}} = \frac{\mathbf{3x}\sqrt[4]{\mathbf{xy^3}}}{\mathbf{y}}$$

To rationalize a binomial containing square roots, such as $2-\sqrt{x}$ or $\sqrt{2}-\sqrt{3}$, we need to find a way to square each term separately. This can be achieved through multiplying by a conjugate binomial, in order to benefit from the difference of squares formula. In particular, we can rationalize denominators in expressions below as follows:

$$\frac{1}{2-\sqrt{x}}=\frac{1}{(2-\sqrt{x})}\cdot\frac{(2+\sqrt{x})}{(2+\sqrt{x})}=\frac{\mathbf{2}+\sqrt{\boldsymbol{x}}}{\mathbf{4}-\boldsymbol{x}}$$

Apply the difference of squares formula: $(\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{a}^2-\boldsymbol{b}^2$

or

$$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})}\cdot\frac{(\sqrt{2}-\sqrt{3})}{(\sqrt{2}-\sqrt{3})}=\frac{2-\sqrt{6}}{2-3}=\frac{2-\sqrt{6}}{-1}=\sqrt{\mathbf{6}}-\mathbf{2}$$

Example 4 ► Rationalizing Binomial Denominators

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\frac{1-\sqrt{3}}{1+\sqrt{3}}$ **b.** $\frac{\sqrt{xy}}{2\sqrt{x}-\sqrt{y}}$

Solution ►

factor

a. $\frac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\frac{(1-\sqrt{3})}{(1-\sqrt{3})}=\frac{1-2\sqrt{3}+3}{1-3}=\frac{4-2\sqrt{3}}{-2}=\frac{\cancel{-2}(-2+\sqrt{3})}{\cancel{-2}}=\sqrt{\mathbf{3}}-\mathbf{2}$

b. $\frac{\sqrt{xy}}{2\sqrt{x}-\sqrt{y}}\cdot\frac{(2\sqrt{x}+\sqrt{y})}{(2\sqrt{x}+\sqrt{y})}=\frac{2x\sqrt{y}+y\sqrt{x}}{4x-y}$

Some of the challenges in algebraic manipulations involve simplifying quotients with radical expressions, such as $\frac{4-2\sqrt{3}}{-2}$, which appeared in the solution to *Example 4a*. The key concept that allows us to simplify such expressions is **factoring**, as only common factors can be reduced.

Example 5 ► Writing Quotients with Radicals in Lowest Terms

Write each quotient in lowest terms.

a. $\frac{15-6\sqrt{5}}{6}$ **b.** $\frac{3x+\sqrt{8x^2}}{9x}$

Solution ► **a.** To reduce this quotient to the lowest terms we may factor the numerator first,

$$\frac{15-6\sqrt{5}}{6}=\frac{\cancel{3}(5-2\sqrt{5})}{\cancel{6}_2}=\mathbf{\frac{5-2\sqrt{5}}{2}},$$

or alternatively, rewrite the quotient into two fractions and then simplify,

$$\frac{15-6\sqrt{5}}{6}=\frac{15}{6}-\frac{6\sqrt{5}}{6}=\mathbf{\frac{5}{2}-\sqrt{5}}.$$

Caution: Here are the common errors to avoid:

$\cancel{\frac{15-\cancel{6}\sqrt{5}}{\cancel{6}}=15-\sqrt{5}}$ - only common factors can be reduced!

$\cancel{\frac{15-6\sqrt{5}}{6}=\frac{\overset{3}{\cancel{9}}\sqrt{5}}{\cancel{6}}}=\frac{3\sqrt{5}}{2}$ - subtraction is performed after multiplication!

b. To reduce this quotient to the lowest terms, we simplify the radical and factor the numerator first. So,

$$\frac{3x+\sqrt{8x^2}}{6x}=\frac{3x+2x\sqrt{2}}{6x}=\frac{\cancel{x}(3+2\sqrt{2})}{6\cancel{x}}=\mathbf{\frac{3+2\sqrt{2}}{6}}$$

This expression cannot be simplified any further.

RD.4 Exercises

1. A student claims that $24-4\sqrt{x}=20\sqrt{x}$ because for $x=1$ both sides of the equation equal to 20. Is this a valid justification? Explain.

2. Generally, $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$. For example, if $a=b=1$, we have $\sqrt{1+1}=\sqrt{2}\neq 2=1+1=\sqrt{1}+\sqrt{1}$. Can you think of a situation when $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$?

Perform operations and simplify, if possible. Assume that all variables represent positive real numbers.

3. $2\sqrt{3}+5\sqrt{3}$

4. $6\sqrt[3]{x}-4\sqrt[3]{x}$

5. $9y\sqrt{3x}+4y\sqrt{3x}$

6. $12a\sqrt{5b}-4a\sqrt{5b}$

7. $5\sqrt{32}-3\sqrt{8}+2\sqrt{3}$

8. $-2\sqrt{48}+4\sqrt{75}-\sqrt{5}$

9. $\sqrt[3]{16}+3\sqrt[3]{54}$

10. $\sqrt[4]{32}-3\sqrt[4]{2}$

11. $\sqrt{5a}+2\sqrt{45a^3}$

12. $\sqrt[3]{24x}-\sqrt[3]{3x^4}$

13. $4\sqrt{x^3}-2\sqrt{9x}$

14. $7\sqrt{27x^3}+\sqrt{3x}$

15. $6\sqrt{18x}-\sqrt{32x}+2\sqrt{50x}$

16. $2\sqrt{128a}-\sqrt{98a}+2\sqrt{72a}$

17. $\sqrt[3]{6x^4}+\sqrt[3]{48x}-\sqrt[3]{6x}$

18. $9\sqrt{27y^2}-14\sqrt{108y^2}+2\sqrt{48y^2}$

19. $3\sqrt{98n^2}-5\sqrt{32n^2}-3\sqrt{18n^2}$

20. $-4y\sqrt{xy^3}+7x\sqrt{x^3y}$

21. $6a\sqrt{ab^5}-9b\sqrt{a^3b}$

22. $\sqrt[3]{-125p^9}+p\sqrt[3]{-8p^6}$

23. $3\sqrt[4]{x^5y}+2x\sqrt[4]{xy}$

24. $\sqrt{125a^5}-2\sqrt[3]{125a^4}$

25. $x\sqrt[3]{16x}+\sqrt{2}-\sqrt[3]{2x^4}$

26. $\sqrt{9a-9}+\sqrt{a-1}$

27. $\sqrt{4x+12}-\sqrt{x+3}$

28. $\sqrt{x^3-x^2}-\sqrt{4x-4}$

29. $\sqrt{25x-25}-\sqrt{x^3-x^2}$

30. $\frac{4\sqrt{3}}{3}-\frac{2\sqrt{3}}{9}$

31. $\frac{\sqrt{27}}{2}-\frac{3\sqrt{3}}{4}$

32. $\sqrt{\frac{49}{x^4}}+\sqrt{\frac{81}{x^8}}$

33. $2a\sqrt[4]{\frac{a}{16}}-5a\sqrt[4]{\frac{a}{81}}$

34. $-4\sqrt[3]{\frac{4}{y^9}}+3\sqrt[3]{\frac{9}{y^{12}}}$

35. A student simplifies the below expression as follows:

$$\begin{aligned}\sqrt{8}+\sqrt[3]{16} &\stackrel{?}{=} \sqrt{4\cdot 2}+\sqrt[3]{8\cdot 2}\\ &\stackrel{?}{=} \sqrt{4}\cdot\sqrt{2}+\sqrt[3]{8}\cdot\sqrt[3]{2}\\ &\stackrel{?}{=} 2\sqrt{2}+2\sqrt[3]{2}\\ &\stackrel{?}{=} 4\sqrt{4}\\ &\stackrel{?}{=} 8\end{aligned}$$

Check each equation for correctness and disscuss any errors that you can find. What would you do differently and why?

36. Match each expression from ***Column I*** with the equivalent expression in ***Column II***. Assume that A and B represent positive real numbers.

Column I	***Column II***
A. $(A+\sqrt{B})(A-\sqrt{B})$	**a.** $A-B$
B. $(\sqrt{A}+B)(\sqrt{A}-B)$	**b.** $A+2B\sqrt{A}+B^2$
C. $(\sqrt{A}+\sqrt{B})(\sqrt{A}-\sqrt{B})$	**c.** $A-B^2$
D. $(\sqrt{A}+\sqrt{B})^2$	**d.** $A-2\sqrt{AB}+B$
E. $(\sqrt{A}-\sqrt{B})^2$	**e.** A^2-B
F. $(\sqrt{A}+B)^2$	**f.** $A+2\sqrt{AB}+B$

Multiply, and then simplify each product. Assume that all variables represent positive real numbers.

37. $\sqrt{5}(3-2\sqrt{5})$

38. $\sqrt{3}(3\sqrt{3}-\sqrt{2})$

39. $\sqrt{2}(5\sqrt{2}-\sqrt{10})$

40. $\sqrt{3}(-4\sqrt{3}+\sqrt{6})$

41. $\sqrt[3]{2}(\sqrt[3]{4}-2\sqrt[3]{32})$

42. $\sqrt[3]{3}(\sqrt[3]{9}+2\sqrt[3]{21})$

43. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$

44. $(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})$

45. $(2\sqrt{3}+5)(2\sqrt{3}-5)$

46. $(6+3\sqrt{2})(6-3\sqrt{2})$

47. $(5-\sqrt{5})^2$

48. $(\sqrt{2}+3)^2$

49. $(\sqrt{a}+5\sqrt{b})(\sqrt{a}-5\sqrt{b})$

50. $(2\sqrt{x}-3\sqrt{y})(2\sqrt{x}+3\sqrt{y})$

51. $(\sqrt{3}+\sqrt{6})^2$

52. $(\sqrt{5}-\sqrt{10})^2$

53. $(2\sqrt{5}+3\sqrt{2})^2$

54. $(2\sqrt{3}-5\sqrt{2})^2$

55. $(4\sqrt{3}-5)(\sqrt{3}-2)$

56. $(4\sqrt{5}+3\sqrt{3})(3\sqrt{5}-2\sqrt{3})$

57. $(\sqrt[3]{2y}-5)(\sqrt[3]{2y}+1)$

58. $(\sqrt{x+5}-3)(\sqrt{x+5}+3)$

59. $(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})$

60. $(\sqrt{x+2}+\sqrt{x-2})^2$

Given $f(x)$ and $g(x)$, find $(f+g)(x)$ and $(fg)(x)$.

61. $f(x)=5x\sqrt{20x}$ and $g(x)=3\sqrt{5x^3}$

62. $f(x)=2x\sqrt[4]{64x}$ and $g(x)=-3\sqrt[4]{4x^5}$

Rationalize each denominator and simplify, if possible. Assume that all variables represent positive real numbers.

63. $\frac{\sqrt{5}}{2\sqrt{2}}$

64. $\frac{3}{5\sqrt{3}}$

65. $\frac{12}{\sqrt{6}}$

66. $-\frac{15}{\sqrt{24}}$

67. $-\frac{10}{\sqrt{20}}$

68. $\sqrt{\frac{3x}{20}}$

69. $\sqrt{\frac{5y}{32}}$

70. $\frac{\sqrt[3]{7a}}{\sqrt[3]{3b}}$

71. $\frac{\sqrt[3]{2y^4}}{\sqrt[3]{6x^4}}$

72. $\frac{\sqrt[3]{3n^4}}{\sqrt[3]{5m^2}}$

73. $\frac{pq}{\sqrt[4]{p^3q}}$

74. $\frac{2x}{\sqrt[5]{18x^8}}$

75. $\frac{17}{6+\sqrt{2}}$

76. $\frac{4}{3-\sqrt{5}}$

77. $\frac{2\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

78. $\frac{6\sqrt{3}}{3\sqrt{2}-\sqrt{3}}$

79. $\frac{3}{3\sqrt{5}+2\sqrt{3}}$

80. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+5\sqrt{2}}$

81. $\frac{m-4}{\sqrt{m}+2}$

82. $\frac{4}{\sqrt{x}-2\sqrt{y}}$

83. $\frac{\sqrt{3}+2\sqrt{x}}{\sqrt{3}-2\sqrt{x}}$

84. $\frac{\sqrt{x}-2}{3\sqrt{x}+\sqrt{y}}$

85. $\frac{2\sqrt{a}}{\sqrt{a}-\sqrt{b}}$

86. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Write each quotient in lowest terms. Assume that all variables represent positive real numbers.

87. $\frac{10-20\sqrt{5}}{10}$

88. $\frac{12+6\sqrt{3}}{6}$

89. $\frac{12-9\sqrt{72}}{18}$

90. $\frac{2x+\sqrt{8x^2}}{2x}$

91. $\frac{6p-\sqrt{24p^3}}{3p}$

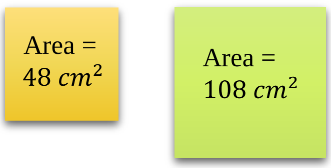
92. $\frac{9x+\sqrt{18}}{15}$

93. When solving one of the trigonometry problems, a student come up with the answer $\frac{\sqrt{3}-1}{1+\sqrt{3}}$. The textbook answer to this problem was $2-\sqrt{3}$. Was the student's answer equivalent to the textbook answer?

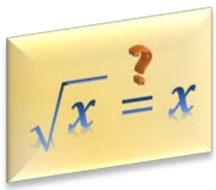
Solve each problem.

94. The base of the second tallest of the Pyramids of Giza is a square with an area of 46,225 m^2. What is its perimeter?

95. The areas of two types of square wall tiles sold at the local Home Depot store are 48 cm^2 and 108 cm^2, respectively. What is the difference in the length of sides of the two tiles? *Give the exact answer in a simplified radical form and its approximation to the nearest tenth.*



RD5 Radical Equations



In this section, we discuss techniques for solving radical equations. These are equations containing at least one radical expression with a variable, such as $\sqrt{3x-2} = x$, or a variable expression raised to a fractional exponent, such as $(2x)^{\frac{1}{3}} + 1 = 5$.
At the end of this section, we revisit working with formulas involving radicals as well as application problems that can be solved with the use of radical equations.

Radical Equations

Definition 5.1 ► A **radical equation** is an equation in which a variable appears in one or more radicands. This includes radicands 'hidden' under fractional exponents.
For example, since $(x-1)^{\frac{1}{2}} = \sqrt{x-1}$, then the base $x-1$ is, in fact, the 'hidden' radicand.

Some examples of radical equations are

$$x = \sqrt{2x}, \quad \sqrt{x} + \sqrt{x-2} = 5, \quad (x-4)^{\frac{3}{2}} = 8, \quad \sqrt[3]{3+x} = 5$$

Note that $x = \sqrt{2}$ is not a radical equation since there is no variable under the radical sign.

The process of solving radical equations involves clearing radicals by raising both sides of an equation to an appropriate power. This method is based on the following property of equality.

Power Rule: For any **odd** natural number $\boldsymbol{n}$, the equation $\boldsymbol{a} = \boldsymbol{b}$ is equivalent to the equation $\boldsymbol{a}^{\boldsymbol{n}} = \boldsymbol{b}^{\boldsymbol{n}}$.

For any **even** natural number $\boldsymbol{n}$, if an equation $\boldsymbol{a} = \boldsymbol{b}$ is true, then $\boldsymbol{a}^{\boldsymbol{n}} = \boldsymbol{b}^{\boldsymbol{n}}$ is true.

When rephrased, the power rule for odd powers states that the solution sets to both equations, $a = b$ and $a^n = b^n$, are exactly the same.

However, the power rule for even powers states that the solutions to the original equation $a = b$ are among the solutions to the 'power' equation $a^n = b^n$.

Unfortunately, the reverse implication does not hold for even numbers n. We cannot conclude that $a = b$ from the fact that $a^n = b^n$ is true. For instance, $3^2 = (-3)^2$ is true but $3 \neq -3$. This means that not all solutions of the equation $a^n = b^n$ are in fact true solutions to the original equation $a = b$. Solutions that do not satisfy the original equation are called **extraneous solutions** or **extraneous roots**. Such solutions must be rejected.

For example, to solve $\sqrt{2-x} = x$, we may square both sides of the equation to obtain the quadratic equation

$$2 - x = x^2.$$

Then, we solve it via factoring and the zero-product property:

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

So, the possible solutions are $x = -2$ and $x = 1$.

Notice that $x = 1$ satisfies the original equation, as $\sqrt{2-1} = 1$ is true. However, $x = -2$ does not satisfy the original equation as its left side equals to $\sqrt{2-(-2)} = \sqrt{4} = 2$, while the right side equals to -2. Thus, $x = -2$ is the extraneous root and as such, it does not belong to the solution set of the original equation. So, the solution set of the original equation is $\{1\}$.

Caution: When the power rule for **even powers** is used to solve an equation, **every solution** of the 'power' equation **must be checked in the original equation**.

Example 1 ► **Solving Equations with One Radical**

Solve each equation.

a. $\sqrt{3x+4} = 4$

b. $\sqrt{2x-5} + 4 = 0$

c. $2\sqrt{x+1} = x - 7$

d. $\sqrt[3]{x-8} + 2 = 0$

Solution ►

a. Since the radical in $\sqrt{3x+4} = 4$ is isolated on one side of the equation, squaring both sides of the equation allows for clearing (reversing) the square root. Then, by solving the resulting polynomial equation, one can find the possible solution(s) to the original equation.

$$\left(\sqrt{a}\right)^2 = \left(a^{\frac{1}{2}}\right)^2 = a$$

$$\left(\sqrt{3x+4}\right)^2 = (4)^2$$

$$3x + 4 = 16$$

$$3x = 12$$

$$x = 4$$

To check if 4 is a true solution, it is enough to check whether or not $x = 4$ satisfies the original equation.

$$\sqrt{3 \cdot 4 + 4} \overset{?}{=} 4$$

$$\sqrt{16} \overset{?}{=} 4$$

$$4 = 4 \quad \checkmark$$

true

Since $x = 4$ satisfies the original equation, the solution set is $\{\mathbf{4}\}$.

b. To solve $\sqrt{2x-5} + 4 = 0$, it is useful to isolate the radical on one side of the equation. So, consider the equation

$$\sqrt{2x-5} = -4$$

Notice that the left side of the above equation is nonnegative for any x-value while the right side is constantly negative. Thus, such an equation cannot be satisfied by any x-value. Therefore, this equation has **no solution**.

c. Squaring both sides of the equation gives us

$$\left(2\sqrt{x+1}\right)^2 = (x-7)^2$$

$$4(x+1) = x^2 - 14x + 49$$

the bracket is essential here

apply the **perfect square** formula $(a-b)^2 = \boldsymbol{a^2 - 2ab + b^2}$

$$4x + 4 = x^2 - 14x + 49$$

$$x^2 - 18x + 45 = 0$$

$$(x-3)(x-15) = 0$$

So, the possible solutions are $x = 3$ or $x = 15$. We check each of them by substituting them into the original equation.

If $x = 3$, then

$$2\sqrt{3+1} \stackrel{?}{=} 3 - 7$$

$$2\sqrt{4} \stackrel{?}{=} -4$$

$$4 \neq -4$$ false

So $x = 3$ is the **extraneous** root.

If $x = 15$, then

$$2\sqrt{15+1} \stackrel{?}{=} 15 - 7$$

$$2\sqrt{16} \stackrel{?}{=} 8$$

$$8 = 8$$ true

Since only 15 satisfies the original equation, the solution set is $\{15\}$.

d. To solve $\sqrt[3]{x-8} + 2 = 0$, we first isolate the radical by subtracting 2 from both sides of the equation.

$$\sqrt[3]{x-8} = -2$$

Then, to clear the cube root, we raise both sides of the equation to the third power.

$$\left(\sqrt[3]{x-8}\right)^3 = (-2)^3$$

So, we obtain

$$x - 8 = -8$$

$$x = 0$$

Since we applied the power rule for odd powers, the obtained solution is the true solution. So the solution set is $\{0\}$.

Observation: When using the power rule for odd powers checking the obtained solutions against the original equation is not necessary. This is because there is no risk of obtaining extraneous roots when applying the power rule for odd powers.

To solve radical equations with more than one radical term, we might need to apply the power rule repeatedly until all radicals are cleared. In an efficient solution, each application of the power rule should cause clearing of at least one radical term. For that reason, it is a good idea to isolate a single radical term on one side of the equation before each application of the power rule. For example, to solve the equation

$$\sqrt{x-3}+\sqrt{x+5}=4,$$

we isolate one of the radicals before squaring both sides of the equation. So, we have

$$\left(\sqrt{x-3}\right)^2=\left(4-\sqrt{x+5}\right)^2$$

$$\cancel{x}-3=\underbrace{16}_{a^2}-\underbrace{8\sqrt{x+5}}_{2ab}+\underbrace{\cancel{x}+5}_{b^2}$$

Remember that the **perfect square** formula consists of **three terms**.

Then, we isolate the remaining radical term and simplify, if possible. This gives us

$$8\sqrt{x+5}=24 \qquad / \div 8$$

$$\sqrt{x+5}=3 \qquad / \ (\,)^2$$

Squaring both sides of the last equation gives us

$$x+5=9 \qquad / -5$$

$$x=4$$

The reader is encouraged to check that $x=\mathbf{4}$ is the true solution to the original equation.

A general strategy for solving radical equations, including those with two radical terms, is as follows.

Summary of Solving a Radical Equation

- **Isolate one of the radical terms**. Make sure that one radical term is alone on one side of the equation.
- **Apply an appropriate power rule.** Raise each side of the equation to a power that is the same as the index of the isolated radical.
- **Solve the resulting equation.** If it still contains a radical, repeat the first two steps.
- **Check** all proposed solutions in the original equation.
- **State the solution set** to the original equation.

Example 2 ► **Solving Equations Containing Two Radical Terms**

Solve each equation.

a. $\sqrt{3x+1}-\sqrt{x+4}=1$ **b.** $\sqrt[3]{4x-5}=2\sqrt[3]{x+1}$

Solution ► **a.** We start solving the equation $\sqrt{3x+1}-\sqrt{x+4}=1$ by isolating one radical on one side of the equation. This can be done by adding $\sqrt{x+4}$ to both sides of the equation. So, we have

$$\sqrt{3x+1}=1+\sqrt{x+4}$$

which after squaring give us

$$\left(\sqrt{3x+1}\right)^2 = \left(1+\sqrt{x+4}\right)^2$$

$$3x+1 = 1+2\sqrt{x+4}+x+4$$

$$2x-4 = 2\sqrt{x+4} \qquad / \div 2$$

$$x-2 = \sqrt{x+4}$$

To clear the remaining radical, we square both sides of the above equation again.

$$(x-2)^2 = \left(\sqrt{x+4}\right)^2$$

$$x^2-4x+4 = x+4$$

$$x^2-5x = 0$$

The resulting polynomial equation can be solved by factoring and applying the zero-product property. Thus,

$$x(x-5)=0.$$

So, the possible roots are $x=0$ or $x=5$.

We check each of them by substituting to the original equation.

If $x=0$, then

$$\sqrt{3\cdot 0+1}-\sqrt{0+4} \stackrel{?}{=} 1$$

$$\sqrt{1}-\sqrt{4} \stackrel{?}{=} 1$$

$$1-2 \stackrel{?}{=} 1$$

$$-1 \neq 1$$

false

If $x=5$, then

$$\sqrt{3\cdot 5+1}-\sqrt{5+4} \stackrel{?}{=} 1$$

$$\sqrt{16}-\sqrt{9} \stackrel{?}{=} 1$$

$$4-3 \stackrel{?}{=} 1$$

$$1=1$$

true

Since $x=0$ is the **extraneous** root, it does not belong to the solution set.

Only 5 satisfies the original equation. So, the solution set is $\{\mathbf{5}\}$.

b. To solve the equation $\sqrt[3]{4x-5} = 2\sqrt[3]{x+1}$, we would like to clear the cubic roots. This can be done by cubing both of its sides, as shown below.

$$\left(\sqrt[3]{4x-5}\right)^3 = \left(2\sqrt[3]{x+1}\right)^3$$

$$4x-5 = 2^3(x+1)$$

the bracket is essential here

$$4x-5 = 8x+8 \qquad / -4x, -8$$

$$-13 = 4x \qquad / \div(-13)$$

$$x = -\frac{\mathbf{4}}{\mathbf{13}}$$

Since we applied the power rule for cubes, the obtained root is the true solution of the original equation.

Formulas Containing Radicals

Many formulas involve radicals. For example, the period T, in seconds, of a pendulum of length L, in feet, is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Sometimes, we might need to solve a radical formula for a specified variable. In addition to all the strategies for solving formulas for a variable, discussed in *Sections L2*, *F4*, and *RT6*, we may need to apply the power rule to clear the radical(s) in the formula.

Example 3 ► **Solving Radical Formulas for a Specified Variable**

Solve each formula for the indicated variable.

a. $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for $\boldsymbol{a}$ **b.** $r = \sqrt[3]{\frac{A}{P}} - 1$ for $\boldsymbol{P}$

Solution ►

a. Since $\boldsymbol{a}$ appears in the radicand, to solve $N = \frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for $\boldsymbol{a}$, we may want to clear the radical by squaring both sides of the equation. So, we have

$$N^2 = \left(\frac{1}{2\pi}\sqrt{\frac{a}{r}}\right)^2$$

$$N^2 = \frac{1}{(2\pi)^2} \cdot \frac{a}{r} \qquad / \cdot 4\pi^2 r$$

$$\boldsymbol{4\pi^2 N^2 r = a}$$

Note: We could also first multiply by 2π and then square both sides of the equation.

b. First, observe the position of $\boldsymbol{P}$ in the equation $r = \sqrt[3]{\frac{A}{P}} - 1$. It appears in the denominator of the radical. Therefore, to solve for $\boldsymbol{P}$, we may plan to isolate the cube root first, cube both sides of the equation to clear the radical, and finally bring $\boldsymbol{P}$ to the numerator. So, we have

$$r = \sqrt[3]{\frac{A}{P}} - 1 \qquad / +1$$

$$(r+1)^3 = \left(\sqrt[3]{\frac{A}{P}}\right)^3$$

$$(r+1)^3 = \frac{A}{P} \qquad / \cdot P, \ \div (r+1)^3$$

$$\boldsymbol{P = \frac{A}{(r+1)^3}}$$

Radicals in Applications

Many application problems in sciences, engineering, or finances translate into radical equations.

Example 4 ► **Finding the Velocity of a Skydiver**

After d meters of a free fall from an airplane, a skydiver's velocity v, in kilometers per hour, can be estimated according to the formula $v = 15.9\sqrt{d}$. Approximately how far, in meters, does a skydiver need to fall to attain the velocity of 100 km/h?

Solution ► We may substitute $v = 100$ into the equation $v = 15.9\sqrt{d}$ and solve it for d, as below.

$$100 = 15.9\sqrt{d} \qquad / \div 15.9$$

$$6.3 \approx \sqrt{d} \qquad / \textit{ square both sides}$$

$$\boldsymbol{40 \approx d}$$

Thus, a skydiver falls at 100 kph approximately after 40 meters of free falling.

RD.5 Exercises

True or false.

1. $\sqrt{2x} = x^2 - \sqrt{5}$ is a radical equation.
2. When raising each side of a radical equation to a power, the resulting equation is equivalent to the original equation.
3. $\sqrt{3x+9} = x$ cannot have negative solutions.
4. -9 is a solution to the equation $\sqrt{x} = -3$.

Solve each equation.

5. $\sqrt{7x-3}=6$

6. $\sqrt{5y+2}=7$

7. $\sqrt{6x}+1=3$

8. $\sqrt{2k}-4=6$

9. $\sqrt{x+2}=-6$

10. $\sqrt{y-3}=-2$

11. $\sqrt[3]{x}=-3$

12. $\sqrt[3]{a}=-1$

13. $\sqrt[4]{y-3}=2$

14. $\sqrt[4]{n+1}=3$

15. $5=\frac{1}{\sqrt{a}}$

16. $\frac{1}{\sqrt{y}}=3$

17. $\sqrt{3r+1}-4=0$

18. $\sqrt{5x-4}-9=0$

19. $4-\sqrt{y-2}=0$

20. $9-\sqrt{4a+1}=0$

21. $x-7=\sqrt{x-5}$

22. $x+2=\sqrt{2x+7}$

23. $2\sqrt{x+1}-1=x$

24. $3\sqrt{x-1}-1=x$

25. $y-4=\sqrt{4-y}$

26. $x+3=\sqrt{9-x}$

27. $x=\sqrt{x^2+4x-20}$

28. $x=\sqrt{x^2+3x+9}$

29. Discuss the validity of the following solution:

$$\sqrt{2x+1}=4-x$$

$$2x+1=16+x^2$$

$$x^2-2x+15=0$$

$$(x-5)(x+3)=0$$

so $x=5$ or $x=-3$

30. Discuss the validity of the following solution:

$$\sqrt{3x+1}-\sqrt{x+4}=1$$

$$(3x+1)-(x+4)=1$$

$$2x-3=1$$

$$2x=4$$

$$x=2$$

Solve each equation.

31. $\sqrt{5x+1}=\sqrt{2x+7}$

32. $\sqrt{5y-3}=\sqrt{2y+3}$

33. $\sqrt[3]{p+5}=\sqrt[3]{2p-4}$

34. $\sqrt[3]{x^2+5x+1}=\sqrt[3]{x^2+4x}$

35. $2\sqrt{x-3}=\sqrt{7x+15}$

36. $\sqrt{6x-11}=3\sqrt{x-7}$

37. $3\sqrt{2t+3}-\sqrt{t+10}=0$

38. $2\sqrt{y-1}-\sqrt{3y-1}=0$

39. $\sqrt{x-9}+\sqrt{x}=1$

40. $\sqrt{y-5}+\sqrt{y}=5$

41. $\sqrt{3n}+\sqrt{n-2}=4$

42. $\sqrt{x+5}-2=\sqrt{x-1}$

43. $\sqrt{14-n}=\sqrt{n+3}+3$

44. $\sqrt{p+15}-\sqrt{2p+7}=1$

45. $\sqrt{4a+1}-\sqrt{a-2}=3$

46. $4-\sqrt{a+6}=\sqrt{a-2}$

47. $\sqrt{x-5}+1=-\sqrt{x+3}$

48. $\sqrt{3x-5}+\sqrt{2x+3}+1=0$

49. $\sqrt{2m-3}+2-\sqrt{m+7}=0$

50. $\sqrt{x+2}+\sqrt{3x+4}=2$

51. $\sqrt{6x+7}-\sqrt{3x+3}=1$

52. $\sqrt{4x+7}-4=\sqrt{4x-1}$

53. $\sqrt{5y+4}-3=\sqrt{2y-2}$

54. $\sqrt{2\sqrt{x+11}}=\sqrt{4x+2}$

55. $\sqrt{1+\sqrt{24+10x}}=\sqrt{3x+5}$

56. $(2x-9)^{\frac{1}{2}}=2+(x-8)^{\frac{1}{2}}$

57. $(3k+7)^{\frac{1}{2}}=1+(k+2)^{\frac{1}{2}}$

58. $(x+1)^{\frac{1}{2}}-(x-6)^{\frac{1}{2}}=1$

59. $\sqrt{(x^2-9)^{\frac{1}{2}}}=2$

60. $\sqrt{\sqrt{x}+4}=\sqrt{x}-2$

61. $\sqrt{a^2+30a}=a+\sqrt{5a}$

62. Discuss how to evaluate the expression $\sqrt{5+3\sqrt{3}}-\sqrt{5-3\sqrt{3}}$ without the use of a calculator.

Solve each formula for the indicated variable.

63. $Z=\sqrt{\frac{L}{C}}$ for L

64. $V=\sqrt{\frac{2K}{m}}$ for K

65. $V=\sqrt{\frac{2K}{m}}$ for m

66. $r=\sqrt{\frac{Mm}{F}}$ for M

67. $r=\sqrt{\frac{Mm}{F}}$ for F

68. $Z=\sqrt{L^2+R^2}$ for R

69. $F=\frac{1}{2\pi\sqrt{LC}}$ for C

70. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for a

71. $N=\frac{1}{2\pi}\sqrt{\frac{a}{r}}$ for r

Solve each problem.

72. One of Einstein's special relativity principles states that time passes faster for bodies that travel with greater speed. The ratio of the time that passes for a body that moves with a speed v to the elapsed time that passes on Earth is called the ***aging rate*** and can be calculated by using the formula $\boldsymbol{r}=\frac{\sqrt{c^2-v^2}}{\sqrt{c^2}}$, where c is the speed of light, and v is the speed of the travelling body. For example, the aging rate of 0.5 means that one year for the person travelling at the speed v corresponds to two years spent on Earth.

   a. Find the aging rate for a person travelling at 80% of the speed of light.

   b. Find the elapsed time on Earth for 20 days of travelling time at 60% of the speed of light.

73. Assume that the formula $\boldsymbol{BSA} = \sqrt{\frac{\mathbf{11}\boldsymbol{wh}}{\mathbf{18000}}}$ can be used to calculate the ***Body Surface Area***, in square meters, of a person with the weight w, in kilograms, and the height h, in centimeters. Greg weighs 78 kg and has a BSA of 3 m^2. To the nearest centimeter, how tall is he?

74. The distance d, in kilometers, to the horizon for an object h kilometers above the Earth's surface can be approximated by using the equation $\boldsymbol{d} = \sqrt{\mathbf{12800}\boldsymbol{h} + \boldsymbol{h}^{\mathbf{2}}}$. Estimate the distance between a satellite that is 1000 km above the Earth's surface and the horizon.

75. The formula $\boldsymbol{S} = \frac{\mathbf{24}}{\mathbf{5}}\sqrt{\mathbf{10}\boldsymbol{fL}}$, where f is the drag factor of the road surface, and L is the length of a skid mark, in meters, allows for calculating the speed S, in kilometers per hour, of a car before it started skidding to a stop. To the nearest meter, calculate the length of the skid marks left by a stopping car on a road surface with a drag factor of 0.5, if the car was travelling at 50 km/h at the time of applying the brakes.

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Quadratic Equations and Functions

In this chapter, we discuss various ways of solving quadratic equations, $ax^2 + bx + c = 0$, including equations quadratic in form, such as $x^{-2} + x^{-1} - 20 = 0$, and solving formulas for a variable that appears in the first and second power, such as k in $k^2 - 3k = 2N$. Frequently used strategies of solving quadratic equations include the **completing the square** procedure and its generalization in the form of the **quadratic formula**. Completing the square allows for rewriting quadratic functions in vertex form, $f(x) = a(x - h)^2 + k$, which is very useful for graphing as it provides information about the location, shape, and direction of the parabola.

In the second part of this chapter, we examine properties and graphs of quadratic functions, including basic transformations of these graphs.

Finally, these properties are used in solving application problems, particularly problems involving **optimization**.

In the last section of this chapter, we study how to solve polynomial and rational inequalities using **sign analysis**.

Q1 Methods of Solving Quadratic Equations

As defined in *Section F4*, a quadratic equation is a second-degree polynomial equation in one variable that can be written in standard form as

$$\boldsymbol{ax^2 + bx + c = 0},$$

where a, b, and c are real numbers and $\boldsymbol{a \neq 0}$. Such equations can be solved in many different ways, as presented below.

Solving by Graphing

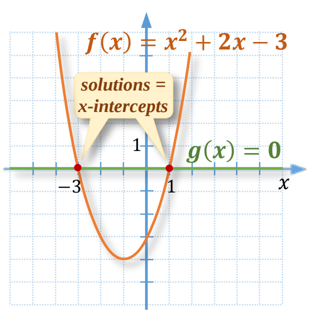


Figure 1.1

To solve a quadratic equation, for example $x^2 + 2x - 3 = 0$, we can consider its left side as a function $f(x) = x^2 + 2x - 3$ and the right side as a function $g(x) = 0$. To satisfy the original equation, both function values must be equal. After graphing both functions on the same grid, one can observe that this happens at points of intersection of the two graphs.

So the **solutions** to the original equation are the x-coordinates of the intersection points of the two graphs. In our example, these are the **x-intercepts** or the **roots** of the function $f(x) = x^2 + 2x - 3$, as indicated in *Figure 1.1*.

Thus, the solutions to $x^2 + 2x - 3 = 0$ are $\boldsymbol{x = -3}$ and $\boldsymbol{x = 1}$.

Note: Notice that the graphing method, although visually appealing, is not always reliable. For example, the solutions to the equation $49x^2 - 4 = 0$ are $x = \frac{2}{7}$ and $x = -\frac{2}{7}$. Such numbers would be very hard to read from the graph.

Thus, the graphing method is advisable to use when searching for integral solutions or estimations of solutions.

To find exact solutions, we can use one of the algebraic methods presented below.

Solving by Factoring

Many quadratic equations can be solved by factoring and employing the zero-product property, as in *Section F4*.

For example, the equation $x^2 + 2x - 3 = 0$ can be solved as follows:

$$(x+3)(x-1) = 0$$

so, by zero-product property,

$$x + 3 = 0 \text{ or } x - 1 = 0,$$

which gives us the solutions

$$\boldsymbol{x = -3} \text{ or } \boldsymbol{x = 1}.$$

Solving by Using the Square Root Property

Quadratic equations of the form $\boldsymbol{ax^2 + c = 0}$ can be solved by applying the **square root property**.

Square Root Property:

For any positive real number a, if $\boldsymbol{x^2 = a}$, then $\boldsymbol{x = \pm\sqrt{a}}$.

This is because $\sqrt{x^2} = |x|$. So, after applying the square root operator to both sides of the equation $x^2 = a$, we have

$$\sqrt{x^2} = \sqrt{a}$$
$$|x| = \sqrt{a}$$
$$x = \pm\sqrt{a}$$

The $\pm\sqrt{a}$ is a shorter recording of two solutions: $\sqrt{a}$ and $-\sqrt{a}$.

For example, the equation $49x^2 - 4 = 0$ can be solved as follows:

$$49x^2 - 4 = 0 \qquad / +4$$

$$49x^2 = 4 \qquad / \div 49$$

$$x^2 = \frac{4}{49}$$

apply square root to both sides of the equation

$$\sqrt{x^2} = \sqrt{\frac{4}{49}}$$

Here we use the **square root property**. Remember the $\pm$ sign!

$$x = \pm\sqrt{\frac{4}{49}}$$

$$\boldsymbol{x = \pm\frac{2}{7}}$$

Note: Using the square root property is a common solving strategy for quadratic equations where **one side is a perfect square** of an unknown quantity and the **other side is a constant** number.

Example 1 ► **Solve by the Square Root Property**

Solve each equation using the square root property.

a. $(x-3)^2 = 49$ **b.** $2(3x-6)^2 - 54 = 0$

Solution ► **a.** Applying the square root property, we have

$$\sqrt{(x-3)^2} = \sqrt{49}$$

$$x - 3 = \pm 7 \qquad / +3$$

$$x = 3 \pm 7$$

so

$$\boldsymbol{x = 10} \text{ or } \boldsymbol{x = -4}$$

b. To solve $2(3x-6)^2 - 54 = 0$, we isolate the perfect square first and then apply the square root property. So,

$$2(3x-6)^2 - 54 = 0 \qquad / +54, \div 2$$

$$(3x-6)^2 = \frac{54}{2}$$

$$\sqrt{(3x-6)^2} = \sqrt{27}$$

$$3x - 6 = \pm 3\sqrt{3} \qquad / +6$$

$$3x = 6 \pm 3\sqrt{3} \qquad / \div 3$$

$$x = \frac{6 \pm 3\sqrt{3}}{3}$$

$$x = \frac{\cancel{3}(2 \pm \sqrt{3})}{\cancel{3}}$$

$$x = 2 \pm \sqrt{3}$$

Thus, the solution set is $\{\mathbf{2 - \sqrt{3}},\ \mathbf{2 + \sqrt{3}}\}$.

Caution: To simplify expressions such as $\frac{6+3\sqrt{3}}{3}$, we **factor the numerator** first. The common errors to avoid are

incorrect order of operations $\frac{6+3\sqrt{3}}{3} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$

or

incorrect canceling $\frac{6+\cancel{3}\sqrt{3}}{\cancel{3}} = 6 + \sqrt{3}$

or

incorrect canceling $\frac{\overset{2}{\cancel{6}}+3\sqrt{3}}{\cancel{3}} = 2 + 3\sqrt{3}$

Solving by Completing the Square

So far, we have seen how to solve quadratic equations, $ax^2 + bx + c = 0$, if the expression $ax^2 + bx + c$ is factorable or if the coefficient b is equal to zero. To solve other quadratic equations, we may try to rewrite the variable terms in the form of a perfect square, so that the resulting equation can already be solved by the square root property.

For example, to solve $x^2 + 6x - 3 = 0$, we observe that the variable terms $x^2 + 6x$ could be written in **perfect square** form if we add 9, as illustrated in *Figure 1.2*. This is because

$$x^2 + 6x + 9 = (x + 3)^2$$

observe that 3 comes from taking half of 6

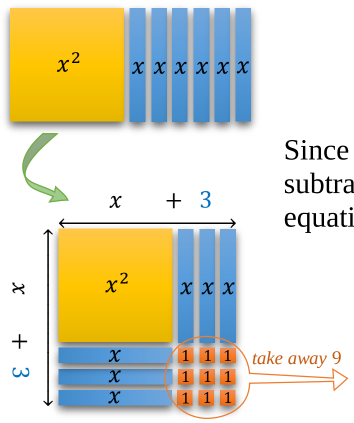


Figure 1.2

Since the original equation can only be changed to an equivalent form, if we add 9, we must subtract 9 as well. *(Alternatively, we could add 9 to both sides of the equation.)* So, the equation can be transformed as follows:

Completing the Square Procedure

$$x^2 + 6x - 3 = 0 \quad / +12$$

$$\underbrace{x^2 + 6x + 9}_{perfect\ square} - 9 - 3 = 0$$

$$(x + 3)^2 = 12$$

square root property

$$\sqrt{(x + 3)^2} = \sqrt{12}$$

$$x + 3 = \pm 2\sqrt{3}$$

$$x = \mathbf{-3 \pm 2\sqrt{3}}$$

Generally, to **complete the square** for the first two terms of the equation

$$\boldsymbol{x^2 + bx + c = 0},$$

we take half of the x-coefficient, which is $\frac{b}{2}$, and square it. Then, we add and subtract that number, $\left(\frac{b}{2}\right)^2$. *(Alternatively, we could add* $\left(\frac{b}{2}\right)^2$ *to both sides of the equation.)*
This way, we produce an equivalent equation

$$\boldsymbol{x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0},$$

and consequently,

$$\boldsymbol{\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0}.$$

We can write this equation directly, by following the rule:
Write the sum of x and **half of the middle coefficient**, **square the binomial**, and **subtract the perfect square of the constant** appearing in the bracket.

To **complete the square** for the first two terms of a quadratic equation with a leading coefficient of $a \neq 1$,

$$\boldsymbol{ax^2 + bx + c = 0},$$

we

- divide the equation by $\boldsymbol{a}$ *(alternatively, we could factor a out of the first two terms)* so that the leading coefficient is 1, and then
- complete the square as in the previous case, where $\boldsymbol{a = 1}$.

So, after division by $\boldsymbol{a}$, we obtain

$$\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}.$$

Since half of $\boldsymbol{\frac{b}{a}}$ is $\boldsymbol{\frac{b}{2a}}$, then we complete the square as follows:

$$\boldsymbol{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0}.$$

Remember to **subtract the perfect square of the constant** appearing in the bracket!

Example 2 ► **Solve by Completing the Square**

Solve each equation using the completing the square method.

a. $x^2 + 5x - 1 = 0$ **b.** $3x^2 - 12x - 5 = 0$

Solution ► **a.** First, we complete the square for $x^2 + 5x$ by adding and subtracting $\left(\frac{5}{2}\right)^2$ and then we apply the square root property. So, we have

$$\underbrace{x^2 + 5x + \left(\frac{5}{2}\right)^2}_{perfect\ square} - \left(\frac{5}{2}\right)^2 - 1 = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} - 1 \cdot \frac{4}{4} = 0$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{29}{4} = 0 \qquad / +\frac{29}{4}$$

apply square root to both sides of the equation

$$\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$$

remember to use the $\pm$ sign!

$$x + \frac{5}{2} = \pm\sqrt{\frac{29}{4}}$$

$$x+\frac{5}{2}=\pm\frac{\sqrt{29}}{2} \quad / -\frac{5}{2}$$

$$x=\frac{-5\pm\sqrt{29}}{2}$$

Thus, the solution set is $\left\{\frac{-5-\sqrt{29}}{2}, \frac{-5+\sqrt{29}}{2}\right\}$.

Note: Unless specified otherwise, we are expected to state the **exact solutions** rather than their calculator approximations. Sometimes, however, especially when solving application problems, we may need to use a calculator to approximate the solutions. The reader is encouraged to check that the two decimal **approximations** of the above solutions are

$$\frac{-5-\sqrt{29}}{2}\approx\mathbf{-5.19} \quad \text{and} \quad \frac{-5+\sqrt{29}}{2}\approx\mathbf{0.19}$$

b. In order to apply the strategy as in the previous example, we divide the equation by the leading coefficient, 3. So, we obtain

$$3x^2-12x-5=0 \quad / \div 3$$

$$x^2-4x-\frac{5}{3}=0$$

Then, to complete the square for x^2-4x, we may add and subtract 4. This allows us to rewrite the equation equivalently, with the variable part in perfect square form.

$$(x-2)^2-4-\frac{5}{3}=0$$

$$(x-2)^2=4\cdot\frac{3}{3}+\frac{5}{3}$$

$$(x-2)^2=\frac{17}{3}$$

$$x-2=\pm\sqrt{\frac{17}{3}}$$

$$x=\mathbf{2}\pm\frac{\sqrt{17}}{\sqrt{3}}$$

Note: The final answer could be written as a single fraction as shown below:

$$x=\frac{2\sqrt{3}\pm\sqrt{17}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{6\pm\sqrt{51}}{3}$$

Solving with Quadratic Formula

Applying the completing the square procedure to the quadratic equation

$$ax^2+bx+c=0,$$

with real coefficients $a\neq 0$, b, and c, allows us to develop a general formula for finding the solution(s) to any such equation.

Quadratic Formula ► The solution(s) to the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are real coefficients, are given by the formula

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Here $x_{1,2}$ denotes the two solutions, $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Proof: ► First, since $a \neq 0$, we can divide the equation $ax^2 + bx + c = 0$ by a. So, the equation to solve is

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then, we complete the square for $x^2 + \frac{b}{a}x$ by adding and subtracting the perfect square of half of the middle coefficient, $\left(\frac{b}{2a}\right)^2$. So, we obtain

$$\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{perfect\ square} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \qquad / +\left(\frac{b}{2a}\right)^2, -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad / -\frac{b}{2a}$$

and finaly,

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which concludes the proof.

Example 3 ► **Solving Quadratic Equations with the Use of the Quadratic Formula**

Using the Quadratic Formula, solve each equation, if possible. Then visualize the solutions graphically.

a. $2x^2 + 3x - 20 = 0$ **b.** $3x^2 - 4 = 2x$ **c.** $x^2 - \sqrt{2}x + 3 = 0$

Solution ►

a. To apply the quadratic formula, first, we identify the values of a, b, and c. Since the equation is in standard form, $a = 2$, $b = 3$, and $c = -20$. The solutions are equal to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2(-20)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{9 + 160}}{4}$$

$$= \frac{-3 \pm 13}{4} = \begin{cases} \frac{-3 + 13}{4} = \frac{10}{4} = \mathbf{\frac{5}{2}} \\ \frac{-3 - 13}{4} = \frac{-16}{4} = \mathbf{-4} \end{cases}$$

Thus, the solution set is $\left\{\mathbf{-4}, \mathbf{\frac{5}{2}}\right\}$.

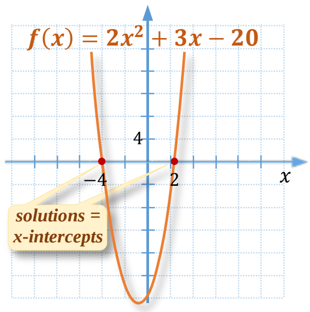


Figure 1.3

These solutions can be seen as x-intercepts of the function $f(x) = 2x^2 + 3x - 20$, as shown in *Figure 1.3*.

b. Before we identify the values of a, b, and c, we need to write the given equation $3x^2 - 4 = 2x$ in standard form. After subtracting $4x$ from both sides of the given equation, we obtain

$$3x^2 - 2x - 4 = 0$$

Since $a = 3$, $b = -2$, and $c = -5$, we evaluate the quadratic formula,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

$$= \frac{2 \pm \sqrt{4 \cdot 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{\cancel{2}(1 \pm \sqrt{13})}{\cancel{6}} = \mathbf{\frac{1 \pm \sqrt{13}}{3}}$$

So, the solution set is $\left\{\mathbf{\frac{1-\sqrt{13}}{3}}, \mathbf{\frac{1+\sqrt{13}}{3}}\right\}$.

simplify by factoring

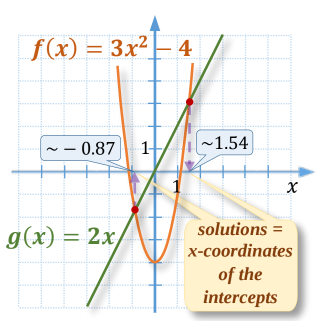


Figure 1.4

We may visualize solutions to the original equation, $3x^2 - 4 = 2x$, by graphing functions $f(x) = 3x^2 - 4$ and $g(x) = 2x$. The x-coordinates of the intersection points are the solutions to the equation $f(x) = g(x)$, and consequently to the original equation. As indicated in *Figure 1.4*, the approximations of these solutions are $\frac{1-\sqrt{13}}{3} \approx -0.87$ and $\frac{1+\sqrt{13}}{3} \approx 1.54$.

c. Substituting $a = 1$, $b = -\sqrt{2}$, and $c = 3$ into the Quadratic Formula, we obtain

$$x_{1,2} = \frac{\sqrt{2} \pm \sqrt{\left(-\sqrt{2}\right)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{\sqrt{2} \pm \sqrt{4 - 12}}{2} = \frac{\sqrt{2} \pm \sqrt{-8}}{2}$$

not a real number!

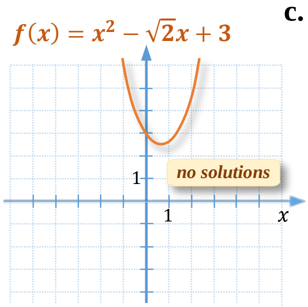


Figure 1.5

Since a square root of a negative number is not a real value, we have **no real solutions**. Thus the solution set to this equation is $\emptyset$. In a graphical representation, this means that the graph of the function $f(x) = x^2 - \sqrt{2}x + 3$ does not cross the x-axis. See *Figure 1.5*.

Observation: Notice that we could find the solution set in *Example 3c* just by evaluating the radicand $b^2 - 4ac$. Since this radicand was negative, we concluded that there is no solution to the given equation as a root of a negative number is not a real number. There was no need to evaluate the whole expression of Quadratic Formula.

So, the radicand in the Quadratic Formula carries important information about the number and nature of roots. Because of it, this radicand earned a special name, the discriminant.

Definition 1.1 ► The radicand $\boldsymbol{b^2 - 4ac}$ in the Quadratic Formula is called the **discriminant** and it is denoted by Δ.

Notice that in terms of Δ, the Quadratic Formula takes the form

$$\boxed{\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}}}$$

Observing the behaviour of the expression $\sqrt{\Delta}$ allows us to classify the number and type of solutions (roots) of a quadratic equation with rational coefficients.

Characteristics of Roots (Solutions) Depending on the Discriminant

Suppose $\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c} = 0$ has **rational** coefficients $\boldsymbol{a} \neq 0$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\Delta = \boldsymbol{b^2 - 4ac}$.

- If $\Delta < \mathbf{0}$, then the equation has **no real solutions**, as $\sqrt{negative}$ is not a real number.
- If $\Delta = \mathbf{0}$, then the equation has **one rational solution**, $\boldsymbol{\frac{-b}{2a}}$.
- If $\Delta > \mathbf{0}$, then the equation has **two solutions**, $\boldsymbol{\frac{-b-\sqrt{\Delta}}{2a}}$ and $\boldsymbol{\frac{-b+\sqrt{\Delta}}{2a}}$.
  These solutions are
  - **irrational**, if Δ is **not** a **perfect square number**
  - **rational**, if Δ is a **perfect square number** (as $\sqrt{perfect\ square} = integer$)

  In addition, if $\Delta \geq \mathbf{0}$ is a **perfect square number**, then the equation could be solved by **factoring**.

Example 4 ► **Determining the Number and Type of Solutions of a Quadratic Equation**

Using the discriminant, determine the number and type of solutions of each equation without solving the equation. If the equation can be solved by factoring, show the factored form of the trinomial.

a. $2x^2 + 7x - 15 = 0$

b. $4x^2 - 12x + 9 = 0$

c. $3x^2 - x + 1 = 0$

d. $2x^2 - 7x + 2 = 0$

Solution ► **a.** $\Delta = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$

Since 169 is a perfect square number, the equation has **two rational solutions** and it can be solved by factoring. Indeed, $2x^2 + 7x - 15 = (2x - 3)(x + 5)$.

b. $\Delta = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$

$\Delta = 0$ indicates that the equation has **one rational solution** and it can be solved by factoring. Indeed, the expression $4x^2 - 12x + 9$ is a perfect square, $(2x - 3)^2$.

c. $\Delta = (-1)^2 - 4 \cdot 3 \cdot 1 = 1 - 12 = -11$

Since $\Delta < 0$, the equation has **no real solutions** and therefore it can not be solved by factoring.

d. $\Delta = (-7)^2 - 4 \cdot 2 \cdot 2 = 49 - 16 = 33$

Since $\Delta > 0$ but it is not a perfect square number, the equation has **two real solutions** but it cannot be solved by factoring.

Example 5 ► **Solving Equations Equivalent to Quadratic**

Solve each equation.

a. $2 + \frac{7}{x} = \frac{5}{x^2}$ **b.** $2x^2 = (x + 2)(x - 1) + 1$

Solution ► **a.** This is a rational equation, with the set of $\mathbb{R} \setminus \{0\}$ as its domain. To solve it, we multiply the equation by the $LCD = x^2$. This brings us to a quadratic equation

$$2x^2 + 7x = 5$$

or equivalently

$$2x^2 + 7x - 5 = 0,$$

which can be solved by following the Quadratic Formula for $a = 2$, $b = 7$, and $c = -5$. So, we have

$$x_{1,2} = \frac{-7 \pm \sqrt{(-7)^2 - 4 \cdot 2(-5)}}{2 \cdot 2} = \frac{-7 \pm \sqrt{49 + 40}}{4} = \frac{\mathbf{-7 \pm \sqrt{89}}}{\mathbf{4}}$$

Since both solutions are in the domain, the solution set is $\left\{\frac{-7-\sqrt{89}}{4}, \frac{-7+\sqrt{89}}{4}\right\}$

b. To solve $1 - 2x^2 = (x + 2)(x - 1)$, we simplify the equation first and rewrite it in standard form. So, we have

$$1 - 2x^2 = x^2 + x - 2 \qquad / -x, +2$$

$$-3x^2 - x + 3 = 0 \qquad / \cdot (-1)$$

$$3x^2 + x - 3 = 0$$

Since the left side of this equation is not factorable, we may use the Quadratic Formula. So, the solutions are

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3(-3)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1+36}}{6} = \boldsymbol{\frac{1 \pm \sqrt{37}}{6}}.$$

Q.1 Exercises

True or False.

1. A quadratic equation is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are any real numbers.

2. If the graph of $f(x) = ax^2 + bx + c$ intersects the x-axis twice, the equation $ax^2 + bx + c = 0$ has two solutions.

3. If the equation $ax^2 + bx + c = 0$ has no solution, the graph of $f(x) = ax^2 + bx + c$ does not intersect the x-axis.

4. The Quadratic Formula cannot be used to solve the equation $x^2 - 5 = 0$ because the equation does not contain a linear term.

5. The solution set for the equation $x^2 = 16$ is $\{4\}$.

6. To complete the square for $x^2 + bx$, we add $\left(\frac{b}{2}\right)^2$.

7. If the discriminant is positive, the equation can be solved by factoring.

For each function f,
a) *graph $f(x)$ using a table of values;*
b) *find the x-intercepts of the graph;*
c) *solve the equation $f(x) = 0$ by factoring and compare these solutions to the x-intercepts of the graph.*

8. $f(x) = -x^2 - 3x + 2$

9. $f(x) = x^2 + 2x - 3$

10. $f(x) = 3x + x(x-2)$

11. $f(x) = 2x - x(x-3)$

12. $f(x) = 4x^2 - 4x - 3$

13. $f(x) = -\frac{1}{2}(2x^2 + 5x - 12)$

Solve each equation using the ***square root property****.*

14. $x^2 = 49$

15. $x^2 = 32$

16. $a^2 - 50 = 0$

17. $n^2 - 24 = 0$

18. $3x^2 - 72 = 0$

19. $5y^2 - 200 = 0$

20. $(x-4)^2 = 64$

21. $(x+3)^2 = 16$

22. $(3n-1)^2 = 7$

23. $(5t+2)^2 = 12$ **24.** $x^2 - 10x + 25 = 45$ **25.** $y^2 + 8y + 16 = 44$

26. $4a^2 + 12a + 9 = 32$ **27.** $25(y-10)^2 = 36$ **28.** $16(x+4)^2 = 81$

29. $(4x+3)^2 = -25$ **30.** $(3n-2)(3n+2) = -5$ **31.** $2x - 1 = \frac{18}{2x-1}$

Solve each equation using the ***completing the square*** *procedure.*

32. $x^2 + 12x = 0$ **33.** $y^2 - 3y = 0$ **34.** $x^2 - 8x + 2 = 0$

35. $n^2 + 7n = 3n - 4$ **36.** $p^2 - 4p = 4p - 16$ **37.** $y^2 + 7y - 1 = 0$

38. $2x^2 - 8x = -4$ **39.** $3a^2 + 6a = -9$ **40.** $3y^2 - 9y + 15 = 0$

41. $5x^2 - 60x + 80 = 0$ **42.** $2t^2 + 6t - 10 = 0$ **43.** $3x^2 + 2x - 2 = 0$

44. $2x^2 - 16x + 25 = 0$ **45.** $9x^2 - 24x = -13$ **46.** $25n^2 - 20n = 1$

47. $x^2 - \frac{4}{3}x = -\frac{1}{9}$ **48.** $x^2 + \frac{5}{2}x = -1$ **49.** $x^2 - \frac{2}{5}x - 3 = 0$

In problems ***50-51****, find all values of x such that $f(x) = g(x)$ for the given functions f and g.*

50. $f(x) = x^2 - 9$ and $g(x) = 4x - 6$ **51.** $f(x) = 2x^2 - 5x$ and $g(x) = -x + 14$

52. Explain the errors in the following solutions of the equation $5x^2 - 8x + 2 = 0$:

a. $x = \frac{8 \pm \sqrt{-8^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64-40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \frac{\mathbf{1}}{\mathbf{2}} \pm \mathbf{2}\sqrt{\mathbf{6}}$

b. $x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 5 \cdot 2}}{2 \cdot 8} = \frac{8 \pm \sqrt{64-40}}{16} = \frac{8 \pm \sqrt{24}}{16} = \frac{8 \pm 2\sqrt{6}}{16} = \begin{cases} \frac{10\sqrt{6}}{16} = \frac{\mathbf{5}\sqrt{\mathbf{6}}}{\mathbf{8}} \\ \frac{6\sqrt{6}}{16} = \frac{\mathbf{3}\sqrt{\mathbf{6}}}{\mathbf{8}} \end{cases}$

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Solve each equation with the aid of the ***Quadratic Formula****, if possible. Illustrate your solutions graphically, using a table of values or a graphing utility.*

53. $x^2 + 3x + 2 = 0$ **54.** $y^2 - 2 = y$ **55.** $x^2 + x = -3$

56. $2y^2 + 3y = -2$ **57.** $x^2 - 8x + 16 = 0$ **58.** $4n^2 + 1 = 4n$

Solve each equation with the aid of the ***Quadratic Formula****. Give the* ***exact*** *and* ***approximate*** *solutions up to two decimal places.*

59. $a^2 - 4 = 2a$ **60.** $2 - 2x = 3x^2$ **61.** $0.2x^2 + x + 0.7 = 0$

62. $2t^2 - 4t + 2 = 3$ **63.** $y^2 + \frac{y}{3} = \frac{1}{6}$ **64.** $\frac{x^2}{4} - \frac{x}{2} = 1$

65. $5x^2 = 17x - 2$ **66.** $15y = 2y^2 + 16$ **67.** $6x^2 - 8x = 2x - 3$

Use the discriminant to determine the ***number and type of solutions*** *for each equation. Also, without solving, decide whether the equation can be solved by* ***factoring*** *or whether the quadratic formula should be used.*

68. $3x^2 - 5x - 2 = 0$

69. $4x^2 = 4x + 3$

70. $x^2 + 3 = -2\sqrt{3}x$

71. $4y^2 - 28y + 49 = 0$

72. $3y^2 - 10y + 15 = 0$

73. $9x^2 + 6x = -1$

In problems 74-76, find all values of constant k, so that each equation will have ***exactly one*** *rational solution.*

74. $x^2 + ky + 49 = 0$

75. $9y^2 - 30y + k = 0$

76. $kx^2 + 8x + 1 = 0$

77. Suppose that one solution of a quadratic equation with integral coefficients is irrational. Assuming that the equation has two solutions, can the other solution be a rational number? Justify your answer.

Solve each equation using any algebraic method. State the solutions in their exact form.

78. $-2x(x + 2) = -3$

79. $(x + 2)(x - 4) = 1$

80. $(x + 2)(x + 6) = 8$

81. $(2x - 3)^2 = 8(x + 1)$

82. $(3x + 1)^2 = 2(1 - 3x)$

83. $2x^2 - (x + 2)(x - 3) = 12$

84. $(x - 2)^2 + (x + 1)^2 = 0$

85. $1 + \frac{2}{x} + \frac{5}{x^2} = 0$

86. $x = \frac{2(x+3)}{x+5}$

87. $2 + \frac{1}{x} = \frac{3}{x^2}$

88. $\frac{3}{x} + \frac{x}{3} = \frac{5}{2}$

89. $\frac{1}{x} + \frac{1}{x+4} = \frac{1}{7}$

Q2 Applications of Quadratic Equations

Some polynomial, rational or even radical equations are **quadratic in form.** As such, they can be solved using techniques described in the previous section. For instance, the rational equation $\frac{1}{x^2}+\frac{1}{x}-6=0$ is quadratic in form because if we replace $\frac{1}{x}$ with a single variable, say a, then the equation becomes quadratic, $a^2+a-6=0$. In this section, we explore applications of quadratic equations in solving equations quadratic in form as well as solving formulas containing variables in the second power.

We also revisit application problems that involve solving quadratic equations. Some of the application problems that are typically solved with the use of quadratic or polynomial equations were discussed in *Sections F4* and *RT6*. However, in the previous sections, the equations used to solve such problems were all possible to solve by factoring. In this section, we include problems that require the use of methods other than factoring.

Equations Quadratic in Form

Definition 2.1 ► A nonquadratic equation is referred to as **quadratic in form** or **reducible to quadratic** if it can be written in the form

$$a\boldsymbol{u}^2+b\boldsymbol{u}+c=0,$$

where $a \neq 0$ and $\boldsymbol{u}$ represents any *algebraic expression*.

Equations quadratic in form are usually easier to solve by using strategies for solving the related quadratic equation $a\boldsymbol{u}^2+b\boldsymbol{u}+c=0$ for the expression $\boldsymbol{u}$, and then solve for the original variable, as shown in the example below.

Example 1 ► **Solving Equations Quadratic in Form**

Solve each equation.

a. $(x^2-1)^2-(x^2-1)=2$

b. $x-3\sqrt{x}=10$

c. $\frac{1}{(a+2)^2}+\frac{1}{a+2}-6=0$

Solution ► **a.** First, observe that the expression x^2-1 appears in the given equation in the first and second power. So, it may be useful to replace x^2-1 with a new variable, for example u. After this substitution, the equation becomes quadratic,

This can be any letter, as long as it is different than the original variable.

$$u^2-u=2, \qquad / -2$$

and can be solved via factoring

$$u^2-u-2=0$$

$$(u-2)(u+1)=0$$

$$u=2 \text{ or } u=-1$$

Since we need to solve the original equation for x, not for u, we replace u back with x^2-1. This gives us

$$x^2 - 1 = 2 \quad \text{or} \quad x^2 - 1 = -1$$

$$x^2 = 3 \quad \text{or} \quad x^2 = 0$$

$$x = \pm\sqrt{3} \quad \text{or} \quad x = 0$$

Thus, the solution set is $\{-\sqrt{\mathbf{3}}, \mathbf{0}, \sqrt{\mathbf{3}}\}$.

b. If we replace $\sqrt{x}$ with, for example, a, then $x = a^2$, and the equation becomes

$$a^2 - 3a = 10, \qquad / \; -10$$

which can be solved by factoring

$$a^2 - 3a - 10 = 0$$

$$(a + 2)(a - 5) = 0$$

$$a = -2 \quad \text{or} \quad a = 5$$

After replacing a back with $\sqrt{x}$, we have

$$\sqrt{x} = -2 \quad \text{or} \quad \sqrt{x} = 5.$$

The first equation, $\sqrt{x} = -2$, does not give us any solution as the square root cannot be negative. After squaring both sides of the second equation, we obtain $x = 25$. So, the solution set is $\{\mathbf{25}\}$.

c. The equation $\frac{1}{(a+2)^2} + \frac{1}{a+2} - 6 = 0$ can be solved as any other rational equation, by clearing the denominators via multiplying by the $LCD = (a + 2)^2$. However, it can also be seen as a quadratic equation as soon as we replace $\frac{1}{a+2}$ with, for example, x. By doing so, we obtain

$$x^2 + x - 6 = 0,$$

which after factoring

$$(x + 3)(x - 2) = 0,$$

gives us

$$x = -3 \quad \text{or} \quad x = 2$$

Remember to use a different letter than the variable in the original equation.

Again, since we need to solve the original equation for a, we replace x back with $\frac{1}{a+2}$. This gives us

$$\frac{1}{a+2} = -3 \quad \text{or} \quad \frac{1}{a+2} = 2 \qquad / \; \textit{take the reciprocal of both sides}$$

$$a + 2 = \frac{1}{-3} \quad \text{or} \quad a + 2 = \frac{1}{2} \qquad / \; -2$$

$$a = -\frac{7}{3} \quad \text{or} \quad a = -\frac{3}{2}$$

Since both values are in the domain of the original equation, which is $\mathbb{R} \setminus \{0\}$, then the solution set is $\left\{-\frac{7}{3}, -\frac{3}{2}\right\}$.

Solving Formulas

When solving formulas for a variable that appears in the second power, we use the same strategies as in solving quadratic equations. For example, we may use the square root property or the quadratic formula.

Example 2 ► **Solving Formulas for a Variable that Appears in the Second Power**

Solve each formula for the given variable.

a. $E = mc^2$, for c **b.** $N = \frac{k^2 - 3k}{2}$, for k

Solution ► **a.** To solve for c, first, we reverse the multiplication by m via the division by m. Then, we reverse the operation of squaring by taking the square root of both sides of the equation.

$$E = mc^2 \qquad / \div m$$

$$\frac{E}{m} = c^2$$

Then, we reverse the operation of squaring by taking the square root of both sides of the equation. So, we have

$$\sqrt{\frac{E}{m}} = \sqrt{c^2},$$

Remember that $\sqrt{c^2} = |c|$, so we use the $\pm$ sign in place of $|\ |$.

and therefore

$$\boldsymbol{c = \pm\sqrt{\frac{E}{m}}}$$

b. Observe that the variable k appears in the formula $N = \frac{k^2-3k}{2}$ in two places. Once in the first and once in the second power of k. This means that we can treat this formula as a quadratic equation with respect to k and solve it with the aid of the quadratic formula. So, we have

$$N = \frac{k^2 - 3k}{2} \qquad / \cdot 2$$

$$2N = k^2 - 3k \qquad / -2N$$

$$\boldsymbol{k}^2 - 3\boldsymbol{k} - 2N = 0$$

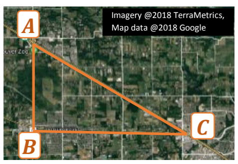
Substituting $a = 1$, $b = -3$, and $c = -2N$ to the quadratic formula, we obtain

$$\boldsymbol{k_{1,2}} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-2N)}}{2} = \boldsymbol{\frac{3 \pm \sqrt{9 + 8N}}{2}}$$

Application Problems

Many application problems require solving quadratic equations. Sometimes this can be achieved via factoring, but often it is helpful to use the quadratic formula.

Example 3 ► **Solving a Distance Problem with the Aid of the Quadratic Formula**



Three towns A, B, and C are positioned as shown in the accompanying figure. The roads at A form a right angle. The towns A and C are connected by a straight road as well. The distance from A to B is 7 kilometers less than the distance from B to C. The distance from A to C is 20 km. Approximate the remaining distances between the towns up to the tenth of a kilometer.

Solution ► Since the roads between towns form a right triangle, we can employ the Pythagorean equation

$$AC^2 = AB^2 + BC^2$$

Suppose that $BC = x$. Then $AB = x - 7$, and we have

$$20^2 = (x-7)^2 + x^2$$

$$400 = x^2 - 14x + 49 + x^2$$

$$2x^2 - 14x - 351 = 0$$

Applying the quadratic formula, we obtain

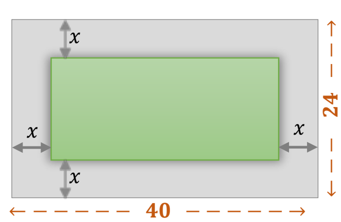
$$x_{1,2} = \frac{14 \pm \sqrt{14^2 + 4 \cdot 2 \cdot 351}}{4} = \frac{14 \pm \sqrt{196 + 2808}}{4} = \frac{14 \pm \sqrt{3004}}{4} \approx 17.2 \text{ } or \text{ } \cancel{-10.2}$$

Since x represents a distance, it must be positive. So, the only solution is $x \approx 17.2$ km. Thus, the distance $\boldsymbol{BC} \approx \mathbf{17.2}$ **km** and hence $\boldsymbol{AB} \approx 17.2 - 7 = \mathbf{10.2}$ **km**.

Example 4 ► **Solving a Geometry Problem with the Aid of the Quadratic Formula**

A city designated 24 m by 40 m rectangular area for a playground with a sidewalk of uniform width around it. The playground itself is using $\frac{2}{3}$ of the original rectangular area. To the nearest centimeter, what is the width of the sidewalk?

Solution ► To visualize the situation, we may draw a diagram as below.



Suppose x represents the width of the sidewalk. Then, the area of the playground (the green area) can be expressed as $(40-2x)(24-2x)$. Since the green area is $\frac{2}{3}$ of the original rectangular area, we can form the equation

$$(40-2x)(24-2x) = \frac{2}{3}(40\cdot 24)$$

To solve it, we may want to lower the coefficients by dividing both sides of the equation by 4 first. This gives us

$$\frac{\cancel{2}(20-x)\cdot\cancel{2}(12-x)}{\cancel{4}} = \frac{2}{\cancel{3}}\cdot\frac{(\overset{10}{\cancel{40}}\cdot\overset{8}{\cancel{24}})}{\cancel{4}}$$

$$(20-x)(12-x) = 160$$

$$240-32x+x^2 = 160 \qquad / -160$$

$$x^2-32x+80 = 0,$$

which can be solved using the Quadratic Formula:

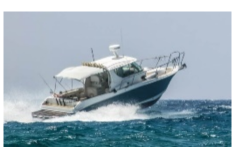
$$x_{1,2} = \frac{32\pm\sqrt{(-32)^2-4\cdot 80}}{2} = \frac{32\pm\sqrt{704}}{2} \approx \frac{32\pm 8\sqrt{11}}{2}$$

$$= 16\pm 4\sqrt{11} \approx \begin{cases}\cancel{29.27}\\ 2.73\end{cases}$$

The width x must be smaller than 12, so this value is too large to be considered.

Thus, the sidewalk is approximately $\mathbf{2.73}$ meters wide.

Example 5 ► **Solving a Motion Problem That Requires the Use of the Quadratic Formula**



The Columbia River flows at a rate of 2 mph for the length of a popular boating route. In order for a boat to travel 3 miles upriver and return in a total of 4 hours, how fast must the boat be able to travel in still water?

Solution ► Suppose the rate of the boat moving in still water is r. Then, $r-2$ represents the rate of the boat moving upriver and $r+2$ represents the rate of the boat moving downriver. We can arrange these data in the table below.

	R ·	T	= D
upriver	$r-2$	$\frac{3}{r-2}$	3
downriver	$r+2$	$\frac{3}{r+2}$	3
total		4	

We fill the time-column by following the formula $T=\frac{D}{R}$.

By adding the times needed for traveling upriver and downriver, we form the rational equation

$$\frac{3}{r-2}+\frac{3}{r+2} = 4, \qquad / \cdot(r^2-4)$$

which after multiplying by the $LCD = r^2-4$ becomes a quadratic equation.

$$3(r+2)+3(r-2) = 4(r^2-4)$$

$$3r + 6 + 3r - 6 = 4r^2 - 16 \qquad / \cdot -6r$$

$$0 = 4r^2 - 6r - 16 \qquad / \div 2$$

$$0 = 2r^2 - 3r - 8$$

Using the Quadratic Formula, we have

$$r_{1,2} = \frac{3 \pm \sqrt{(-3)^2 + 4 \cdot 2 \cdot 8}}{2 \cdot 2} = \frac{3 \pm \sqrt{9 + 64}}{4} = \frac{3 \pm \sqrt{73}}{4} \approx \begin{cases} 2.9 \\ \cancel{-1.4} \end{cases}$$

Since the rate cannot be negative, the boat moves in still water at approximately $\mathbf{2.9}$ **mph**.

Example 6 ► **Solving a Work Problem That Requires the Use of the Quadratic Formula**

Krista and Joanna work in the same office. Krista can file the daily office documents in 1 hour less time than Joanna can. Working together, they can do the job in 1 hr 45 min. To the nearest minute, how long would it take each person working alone to file these documents?

Solution ► Suppose the time needed for Joanna to complete the job is t, in hours. Then, $t - 1$ represents the time needed for Krista to complete the same job. Since we keep time in hours, we need to convert 1 hr 45 min into $1\frac{3}{4}$ hr = $\frac{7}{4}$ hr. Now, we can arrange the given data in a table, as below.

	R ·	T	= Job
Joanna	$\frac{1}{t}$	t	1
Krista	$\frac{1}{t-1}$	$t - 1$	1
together	$\frac{4}{7}$	$\frac{7}{4}$	1

We fill the rate-column by following the formula $R = \frac{Job}{T}$.

By adding the rates of work for each person, we form the rational equation

$$\frac{1}{t} + \frac{1}{t-1} = \frac{4}{7}, \qquad / \cdot 7t(t-1)$$

which after multiplying by the $LCD = 7t(t-1)$ becomes a quadratic equation.

$$7(t - 1) + 7t = 4(t^2 - t)$$

$$7t - 7 + 7t = 4t^2 - 4t \qquad / -6t, +3$$

$$0 = 4t^2 - 18t + 7$$

Using the Quadratic Formula, we have

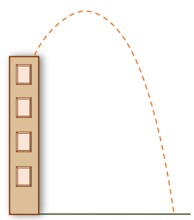
$$t_{1,2} = \frac{18 \pm \sqrt{(-18)^2 - 4 \cdot 4 \cdot 7}}{2 \cdot 4} = \frac{18 \pm \sqrt{212}}{8} = \frac{18 \pm 2\sqrt{53}}{8} = \frac{9 \pm \sqrt{53}}{4} \approx \begin{cases} 4.07 \\ \cancel{0.43} \end{cases}$$

Since the time needed for Joanna cannot be shorter than 1 hr, we reject the 0.43 possibility. So, working alone, **Joanna** requires approximately 4.07 hours ≈ **4 hours 4 minutes**, while **Krista** can do the same job in approximately 3.07 hours ≈ **3 hours 4 minutes**.

Example 7 ► **Solving a Projectile Problem Using a Quadratic Function**

A ball is projected upward from the top of a 96-ft building at 32 ft/sec. Its height above the ground, h, in feet, can be modelled by the function $h(t) = -16t^2 + 32t + 96$, where t is the time in seconds after the ball was projected. To the nearest tenth of a second, when does the ball hit the ground?

Solution ►



The ball hits the ground when its height h above the ground is equal to zero. So, we look for the solutions to the equation

$$h(t) = 0$$

which is equivalent to

$$-16t^2 + 32t + 96 = 0$$

Before applying the Quadratic Formula, we may want to lower the coefficients by dividing both sides of the equation by -16. So, we have

$$t^2 - 2t - 6 = 0$$

and

$$t_{1,2} = \frac{2 \pm \sqrt{(-2)^2 + 4 \cdot 6}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7} \approx \begin{cases} 3.6 \\ \cancel{-1.6} \end{cases}$$

Thus, the ball hits the ground in about **3.6 seconds**.

Q.2 Exercises

1. Discuss the validity of the following solution to the equation $\left(\frac{1}{x-2}\right)^2 - \frac{1}{x-2} - 2 = 0$:

Since this equation is quadratic in form, we solve the related equation $a^2 - a - 2 = 0$ by factoring

$$(a-2)(a+1) = 0.$$

The possible solutions are $a = 2$ and $a = -1$. Since 2 is not in the domain of the original equation, the solution set as $\{-1\}$.

Solve each equation by treating it as a quadratic in form.

2. $x^4 - 6x^2 + 9 = 0$

3. $x^8 - 29x^4 + 100 = 0$

4. $x - 10\sqrt{x} + 9 = 0$

5. $2x - 9\sqrt{x} + 4 = 0$

6. $y^{-2} - 5y^{-1} - 36 = 0$

7. $2a^{-2} + a^{-1} - 1 = 0$

8. $\left(1 + \sqrt{t}\right)^2 + \left(1 + \sqrt{t}\right) - 6 = 0$

9. $\left(2 + \sqrt{x}\right)^2 - 3\left(2 + \sqrt{x}\right) - 10 = 0$

10. $(x^2+5x)^2+2(x^2+5x)-24=0$

11. $(t^2-2t)^2-4(t^2-2t)+3=0$

12. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}-5=0$

13. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}-8=0$

14. $1-\frac{1}{2p+1}-\frac{1}{(2p+1)^2}=0$

15. $\frac{2}{(u+2)^2}+\frac{1}{u+2}=3$

16. $\left(\frac{x+3}{x-3}\right)^2-\left(\frac{x+3}{x-3}\right)=6$

17. $\left(\frac{y^2-1}{y}\right)^2-4\left(\frac{y^2-1}{y}\right)-12=0$

In problems 23-40, solve each formula for the indicated variable.

18. $F=\frac{mv^2}{r}$, for v

19. $V=\pi r^2 h$, for r

20. $A=4\pi r^2$, for r

21. $V=\frac{1}{3}s^2 h$, for s

22. $F=\frac{Gm_1m_2}{r^2}$, for r

23. $N=\frac{kq_1q_2}{s^2}$, for s

24. $a^2+b^2=c^2$, for b

25. $I=\frac{703W}{H^2}$, for H

26. $A=\pi r^2+\pi rs$, for r

27. $A=2\pi r^2+2\pi rh$, for r

28. $s=v_0t+\frac{gt^2}{2}$, for t

29. $t=\frac{a}{\sqrt{a^2+b^2}}$, for a

30. $P=\frac{A}{(1+r)^2}$, for r

31. $P=EI-RI^2$, for I

32. $s(6s-t)=t^2$, for s

33. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for v, assuming that $c>0$ and $m>0$

34. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, for c, assuming that $v>0$ and $m>0$

35. $p=\frac{E^2R}{(r+R)^2}$, for E, assuming that $(r+R)>0$

36. The "**golden**" proportions have been considered visually pleasing for the past 2900 years. A rectangle with the width w and length l has "golden" proportions if

$$\frac{\boldsymbol{w}}{\boldsymbol{l}}=\frac{\boldsymbol{l}}{\boldsymbol{w}+\boldsymbol{l}}.$$

Solve this formula for $\boldsymbol{l}$. Then, find the value of the **golden ratio** $\frac{\boldsymbol{l}}{\boldsymbol{w}}$ up to three decimal places.

Answer each question.

37. A boat moves r km/h in still water. If the rate of the current is c km/h,

a. give an expression for the rate of the boat moving upstream;

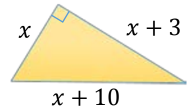
b. give an expression for the rate of the boat moving downstream.

38. **a.** Vivian marks a class test in n hours. Give an expression representing Vivian's rate of marking, in the number of marked tests per hour.

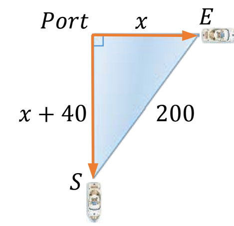
b. How many tests will she have marked in h hours?

Solve each problem.

39. Find the exact length of each side of the triangle.

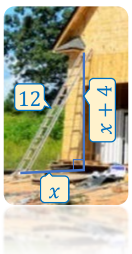


40. Two cruise ships leave a port at the same time, but they move at different rates. The faster ship is heading south, and the slower one is heading east. After a few hours, they are 200 km apart. If the faster ship went 40 km farther than the slower one, how far did each ship travelled?



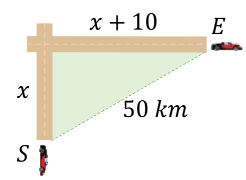
41. The length of a rectangular area carpet is 2 ft more than twice the width. Diagonally, the carpet measures 13 ft. Find the dimensions of the carpet.

42. The legs of a right triangle with 26 cm long hypotenuse differ by 14 cm. Find the lengths of the legs.



43. A 12-ft ladder is tilting against a house. The top of the ladder is 4 ft further from the ground than the bottom of the ladder is from the house. To the nearest inch, how high does the ladder reach?

44. Two cars leave an intersection, one heading south and the other heading east. In one hour the cars are 50 kilometers apart. If the faster car went 10 kilometers farther than the slower one, how far did each car travelled?

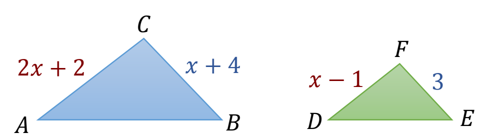


45. The length and width of a computer screen differ by 4 inches. Find the dimensions of the screen, knowing that its area is 117 square inches.

46. The length of an American flag is 1 inch shorter than twice the width. If the area of this flag is 190 square inches, find the dimensions of the flag.

47. The length of a Canadian flag is twice the width. If the area of this flag is 100 square meters, find the exact dimensions of the flag.

48. ***Thales Theorem*** states that corresponding sides of similar triangles are proportional. The accompanying diagram shows two similar triangles, ΔABC and ΔDEF. Given the information in the diagram, find the length AC.

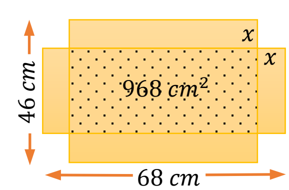


Solve each problem.

49. Sonia bought an area carpet for her 12 ft by 18 ft room. The carpet covers 135 ft^2, and when centered in the room, it leaves a strip of the bare floor of uniform width around the edges of the room. How wide is this strip?

50. Park management plans to create a rectangular 14 m by 20 m flower garden with a sidewalk of uniform width around the perimeter of the garden. There are enough funds to install 152 m^2 of a brick sidewalk. Find the width of the sidewalk.

51. Squares of equal area are cut from each corner of a 46 cm by 68 cm rectangular cardboard. Obtained this way flaps are folded up to create an open box with the area of the base equal to 968 cm^2. What is the height of the box?



52. The outside measurements of a picture frame are 22 cm and 28 cm. If the area of the exposed picture is 301 cm^2, find the width of the frame.

53. The length of a rectangle is one centimeter shorter than twice the width. The rectangle shares its longer side with a square of 169 cm^2 area. What are the dimensions of the rectangle?

54. A rectangular piece of cardboard is 15 centimeters longer than it is wide. 100 cm^2 squares are removed from each corner of the cardboard. Folding up the established flaps creates an open box of 13.5-litre volume. Find the dimensions of the original piece of cardboard. (*Hint:* 1 litre = 1000 cm^2)

55. Karin travelled 420 km by her motorcycle to visit a friend. When planning the return trip by the same road, she calculated that her driving time could be 1 hour shorter if she increases her average speed by 10 km/h. On average, how fast was she driving to her friend?

56. An average, an Airbus A380 flies 80 km/h faster than a Boeing 787 Dreamliner. Suppose an Airbus A380 flew 2600 km in half an hour shorter time than it took a Boeing 787 Dreamliner to fly 2880 mi. Determine the speed of each plane.

57. Two small planes, a Skyhawk and a Mooney Bravo, took off from the same place and at the same time. The Skyhawk flew 500 km. The Mooney Bravo flew 1050 km in one hour longer time and at a 100 km/h faster speed. If the planes fly faster than 200 km/h, find the average rate of each plane.

58. Gina drives 550 km to a conference. Due to heavier traffic, she returns at 10 km/h slower rate. If the round trip took her 10.5 hours, what was Gina’s average rate of driving to the workshop?

59. A barge travels 25 km upriver and then returns in a total of 5 hours. If the current in the river is 3 km/hr, approximately how fast would this barge move in still water?

60. A canoeist travels 3 kilometers down a river with a 3 km/h current. For the return trip upriver, the canoeist chose to use a longer branch of the river with a 2 km/hr current. If the return trip is 4 km long and the time needed for travelling both ways is 3 hours, approximate the speed of the canoe in still water.

61. Two planes take off from the same airport and at the same time. The first plane flies with an average speed r km/h and is heading North. The second plane flies faster by 40 km/h and is heading East. In thirty minutes the planes are 580 kilometers apart from each other. Determine the average speed of each plane.

62. Jack flew 650 km to visit his relatives in Alaska. On the way to Alaska, his plane encounter a 40 km/h headwind. On the returning trip, the plane flew with a 20 km/h tailwind. If the total flying time (both ways) was 5 hours 45 minutes, what was the average speed of the plane in still air?

63. Two janitors, an experienced and a newly hired one, need 4 hours to clean a school building. The newly hired worker would need 1.5 hour longer time than the experienced one to clean the school on its own. To the nearest minute, how much time is required for the experienced janitor to clean the school working alone?

64. Two workers can weed out a vegetable garden in 2 hr. On its own, one worker can do the same job in half an hour shorter time than the other. To the nearest minute, how long would it take the faster worker to weed out the garden by himself?

65. Helen and Monica are planting flowers in their garden. On her own, Helen would need an hour longer than Monica to plant all the flowers. Together, they can finish the job in 8 hr. To the nearest minute, how long would it take each person to plant all the flowers if working alone?

66. To prepare the required number of pizza crusts for a day, the owner of Ricardo's Pizza needs 40 minutes shorter time than his worker Sergio. Together, they can make these pizza crusts in 2 hours. To the nearest minute, how long would it take each of them to do this job alone?

67. A fish tank can be filled with water with the use of one of two pipes of different diameters. If only the larger-diameter pipe is used, the tank can be filled in an hour shorter time than if only the smaller-diameter pipe is used. If both pipes are open, the tank can be filled in 1 hr 12 min. How much time is needed for each pipe to fill the tank if working alone?

68. Two roofers, Garry and Larry, can install new asphalt roof shingles in 6 hours 40 min. On his own, Garry can do this job in 3 hours shorter time than Larry can. How much time each or the roofers need to install these shingles alone?

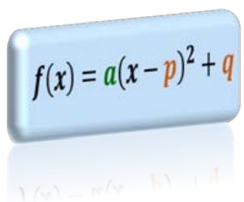
69. A ball is thrown down with the initial velocity of 6 m/sec from a balcony that is 100 m above the ground. Suppose that function $h(t) = -4.9t^2 - 6t + 100$ can be used to determine the height $h(t)$ of the ball t seconds after it was thrown down. Approximately in how many seconds the ball will be 5 meters above the ground?

70. A bakery's weekly profit, P (in dollars), for selling n poppyseed strudels can be modelled by the function $P(n) = -0.05x^2 + 7x - 200$. What is the minimum number of poppyseed strudels that must be sold to make a profit of \$200?

71. If P dollars is invested in an account that pays the annual interest rate r (in decimal form), then the amount A of money in the account after 2 years can be determined by the formula $A = P(1 + r)^2$. Suppose \$3000 invested in this account for 2 years grew to \$3257.29. What was the interest rate?

72. To determine the distance, d, of an object to the horizon we can use the equation $d = \sqrt{12800h + h^2}$, where h represents the distance of an object to the Earth's surface, and both, d and h, are in kilometers. To the nearest meter, how far above the Earth's surface is a plane if its distance to the horizon is 400 kilometers?

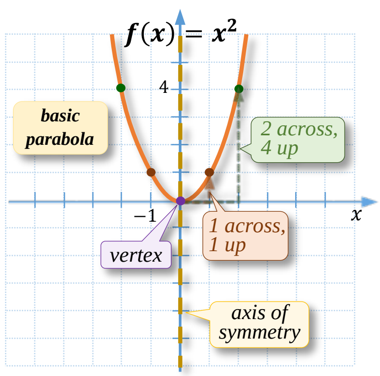
Q3 Properties and Graphs of Quadratic Functions



In this section, we explore an alternative way of graphing quadratic functions. It turns out that if a quadratic function is given in vertex form, $\boldsymbol{f(x) = a(x-p)^2 + q}$, its graph can be obtained by transforming the shape of the basic **parabola**, $f(x) = x^2$, by applying a **vertical dilation** by the factor of a, as well as a **horizontal translation** by p units and **vertical translation** by q units. This approach makes the graphing process easier than when using a table of values.

In addition, the vertex form allows us to identify the main characteristics of the corresponding graph such as **shape**, **opening**, **vertex**, and **axis of symmetry**. Then, the additional properties of a quadratic function, such as **domain** and **range**, or where the function increases or decreases can be determined by observing the obtained graph.

Properties and Graph of the Basic Parabola $f(x) = x^2$



Recall the shape of the **basic parabola**, $f(x) = x^2$, as discussed in *Section P4*.

x	x^2
-2	4
-1	1
0	0
1	1
2	4

vertex

symmetry about the y-axis

Figure 3.1

Observe the relations between the points listed in the table above. If we start with plotting the **vertex** $(\mathbf{0}, \mathbf{0})$, then the next pair of points, $(\mathbf{1}, \mathbf{1})$ and $(-\mathbf{1}, \mathbf{1})$, is plotted **1 unit across** from the vertex (both ways) and **1 unit up**. The following pair, $(\mathbf{2}, \mathbf{4})$ and $(-\mathbf{2}, \mathbf{4})$, is plotted **2 units across** from the vertex and **4 units up**. The graph of the parabola is obtained by connecting these 5 main points by a curve, as illustrated in *Figure 3.1*.

The graph of this parabola is symmetric in the y-axis, so the equation of the **axis of symmetry** is $\boldsymbol{x = 0}$.

The **domain** of the basic parabola is the set of all real numbers, $\mathbb{R}$, as $f(x) = x^2$ is a polynomial, and polynomials can be evaluated for any real x-value.

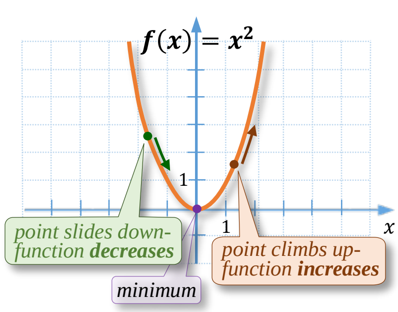


Figure 3.2

The **arms** of the parabola are directed **upwards**, which means that the vertex is the lowest point of the graph. Hence, the **range** of the basic parabola function, $f(x) = x^2$, is the interval $[\mathbf{0}, \infty)$, and the **minimum value** of the function is **0**.

Suppose a point 'lives' on the graph and travels from left to right. Observe that in the case of the basic parabola, if x-coordinates of the 'travelling' point are smaller than 0, the point slides down along the graph. Similarly, if x-coordinates are larger than 0, the point climbs up the graph. (See *Figure 3.2*) To describe this property in mathematical language, we say that the function $f(x) = x^2$ **decreases** in the interval $(-\infty, 0]$ and **increases** in the interval $[0, \infty)$.

Properties and Graphs of a Dilated Parabola $f(x) = ax^2$

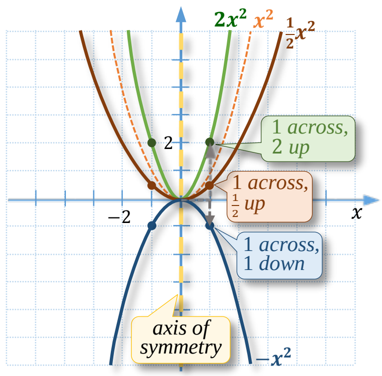


Figure 3.3

Figure 3.3 shows graphs of several functions of the form $f(x) = ax^2$. Observe how the shapes of these parabolas change for various values of a in comparison to the shape of the basic parabola $y = x^2$.

The common point for all of these parabolas is the vertex (0,0). Additional points, essential for graphing such parabolas, are shown in the table below.

x	ax^2
-2	$4a$
-1	a
0	0
1	a
2	$4a$

vertex

1 unit apart from zero, a units up

2 units apart from zero, 4a units up

For example, to graph $f(x) = 3x^2$, it is convenient to plot the **vertex** first, which is at the point $(\mathbf{0}, \mathbf{0})$. Then, we may move the pen **1** unit **across** from the vertex (either way) and **3** units **up** to plot the points $(-1,3)$ and $(1,3)$. If the grid allows, we might want to plot the next two points, $(-2,12)$ and $(2,12)$, by moving the pen **2** units **across** from the vertex and $4 \cdot 3 = \mathbf{12}$ units **up**, as in *Figure 3.4*.

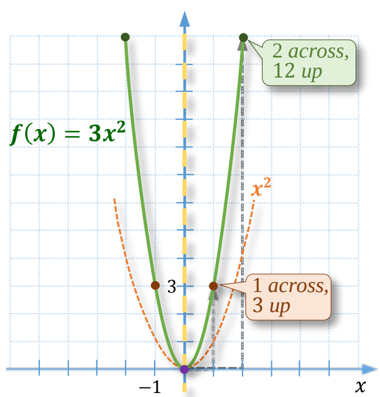


Figure 3.4

Notice that the obtained shape (in green) is **narrower** than the shape of the basic parabola (in orange). However, similarly as in the case of the basic parabola, the shape of the dilated function is still **symmetrical about the y-axis**, $\boldsymbol{x = 0}$.

Now, suppose we want to graph the function $f(x) = -\frac{1}{2}x^2$. As before, we may start by plotting the vertex at (0,0). Then, we move the pen **1** unit **across** from the vertex (either way) and **half** a unit **down** to plot the points $\left(-1, -\frac{1}{2}\right)$ and $\left(1, -\frac{1}{2}\right)$, as in *Figure 3.5*. The next pair of points can be plotted by moving the pen **2** units **across** from the vertex and **2** units **down**, as the ordered pairs $(-2,-2)$ and $(2,-2)$ satisfy the equation $f(x) = -\frac{1}{2}x^2$.

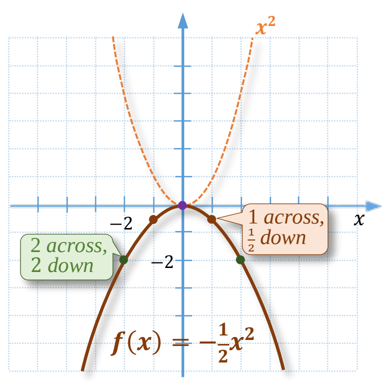


Figure 3.5

Notice that this time the obtained shape (in green) is **wider** than the shape of the basic parabola (in orange). Also, as a result of the **negative $\boldsymbol{a}$-value**, the parabola opens **down**, and the **range** of this function is $(-\infty, \mathbf{0}]$.

Generally, the **shape** of a quadratic function of the form $\boldsymbol{f(x) = ax^2}$ is
- **narrower** than the shape of the basic parabola, if $|\boldsymbol{a}| > \mathbf{1}$;
- **wider** than the shape of the basic parabola, if $\mathbf{0} < |\boldsymbol{a}| < \mathbf{1}$; and
- **the same** as the shape of the **basic parabola**, $y = x^2$, if $|\boldsymbol{a}| = \mathbf{1}$.

The parabola opens **up**, for $\boldsymbol{a > 0}$, and **down**, for $\boldsymbol{a < 0}$.
Thus the **vertex** becomes the **lowest point** of the graph, if $\boldsymbol{a > 0}$, and the **highest point** of the graph, if $\boldsymbol{a < 0}$.

The **range** of $\boldsymbol{f(x) = ax^2}$ is $[\mathbf{0}, \infty)$, if $\boldsymbol{a > 0}$, and $(-\infty, \mathbf{0}]$, if $\boldsymbol{a < 0}$.

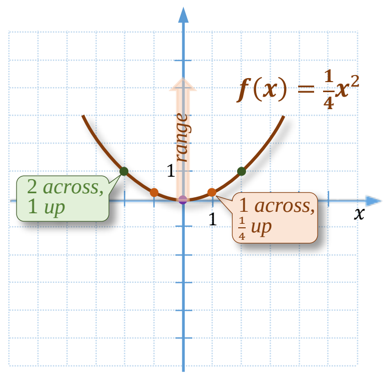
The **axis of symmetry** of the dilated parabola $\boldsymbol{f(x) = ax^2}$ remains the same as that of the basic parabola, which is $\boldsymbol{x = 0}$.

Example 1 ► **Graphing a Dilated Parabola and Describing Its Shape, Opening, and Range**

For each quadratic function, describe its shape and opening. Then graph it and determine its range.

a. $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -2x^2$

Solution ►



a. Since the leading coefficient of the function $f(x) = \frac{1}{4}x^2$ is positive, the parabola **opens up**. Also, since $0 < \frac{1}{4} < 1$, we expect the shape of the parabola to be **wider** than that of the basic parabola.

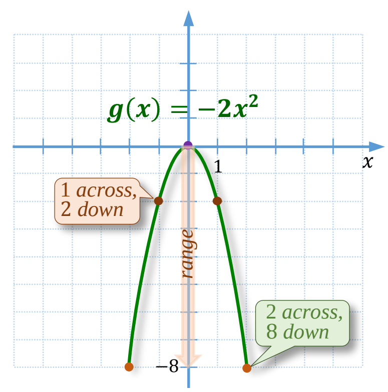
To graph $f(x) = \frac{1}{4}x^2$, first we plot the vertex at $(0,0)$ and then points $\left(\pm 1, \frac{1}{4}\right)$ and $\left(\pm 2, \frac{1}{4} \cdot 4\right) = (\pm 2, 1)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $[0, \infty)$.

b. Since the leading coefficient of the function $g(x) = -2x^2$ is negative, the parabola **opens down**. Also, since $|-2| > 1$, we expect the shape of the parabola to be **narrower** than that of the basic parabola.

To graph $g(x) = -2x^2$, first we plot the vertex at $(0,0)$ and then points $(\pm 1, -2)$ and $(\pm 2, -2 \cdot 4) = (\pm 2, -8)$. After connecting these points with a curve, we obtain the graph of the parabola.

By projecting the graph onto the y-axis, we observe that the range of the function is $(-\infty, 0]$.



Properties and Graphs of the Basic Parabola with Shifts

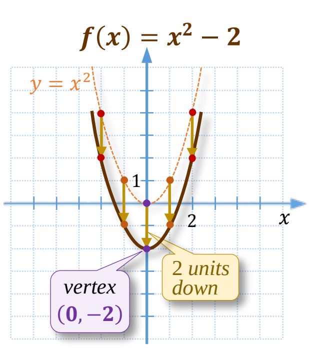


Figure 3.6

Suppose we would like to graph the function $f(x) = x^2 - 2$. We could do this via a table of values, but there is an easier way if we already know the shape of the basic parabola $y = x^2$.

Observe that for every x-value, the value of $x^2 - 2$ is obtained by subtracting 2 from the value of x^2. So, to graph $f(x) = x^2 - 2$, it is enough to **move each point** (x, x^2) of the basic parabola by **two units down**, as indicated in *Figure 3.6*.

The shift of y-values by 2 units down causes the **range** of the new function, $f(x) = x^2 - 2$, to become $[-2, \infty)$. Observe that this vertical shift also changes the minimum value of this function, from 0 to -2.

The **axis of symmetry** remains unchanged, and it is $x = 0$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = x^2 + q}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{q}$ **steps up**, if $\boldsymbol{q > 0}$;
- **shifting** the graph of the basic parabola $|\boldsymbol{q}|$ **steps down**, if $\boldsymbol{q < 0}$.

The **vertex** of such parabola is at $(\boldsymbol{0, q})$. The **range** of it is $[\boldsymbol{q}, \infty)$.
The **minimum** (lowest) **value** of the function is $\boldsymbol{q}$.
The **axis of symmetry** is $\boldsymbol{x = 0}$.

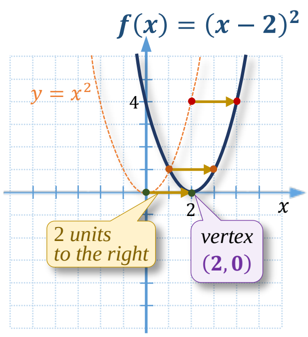


Figure 3.7

Now, suppose we wish to graph the function $f(x) = (x-2)^2$. We can graph it by joining the points calculated in the table below.

x	$(x-2)^2$
0	4
1	1
2	0
3	1
4	4

Observe that the parabola $f(x) = (x-2)^2$ assumes its lowest value at the vertex. The lowest value of the perfect square $(x-2)^2$ is zero, and it is attained at the x-value of 2. Thus, the vertex of this parabola is $(2,0)$.

Notice that the **vertex** $(\boldsymbol{2, 0})$ of $f(x) = (x-2)^2$ is positioned 2 units to the right from the vertex $(0,0)$ of the basic parabola.

This suggests that the graph of the function $f(x) = (x-2)^2$ can be obtained without the aid of a table of values. It is enough to shift the graph of the basic parabola **2 units** to the **right**, as shown in *Figure 3.7.*

Observe that the horizontal shift does not influence the **range** of the new parabola $f(x) = (x-2)^2$. It is still $[\boldsymbol{0}, \infty)$, the same as for the basic parabola. However, the **axis of symmetry** has changed to $\boldsymbol{x = 2}$.

Generally, the graph of a quadratic function of the form $\boldsymbol{f(x) = (x-p)^2}$ can be obtained by

- **shifting** the graph of the basic parabola $\boldsymbol{p}$ **steps** to the **right**, if $\boldsymbol{p > 0}$;
- **shifting** the graph of the basic parabola $|\boldsymbol{p}|$ **steps** to the **left**, if $\boldsymbol{p < 0}$.

The **vertex** of such a parabola is at $(\boldsymbol{p, 0})$. The **range** of it is $[\boldsymbol{0}, \infty)$.

The **minimum value** of the function is $\boldsymbol{0}$.

The **axis of symmetry** is $\boldsymbol{x = p}$.

Example 2 ► **Graphing Parabolas and Observing Transformations of the Basic Parabola**

Graph each parabola by plotting its vertex and following the appropriate opening and shape. Then describe transformations of the basic parabola that would lead to the obtained graph. Finally, state the range and the equation of the axis of symmetry.

a. $f(x) = (x+3)^2$

b. $g(x) = -x^2 + 1$

Solution

a. The perfect square $(x+3)^2$ attains its lowest value at $x=-3$. So, the **vertex** of the parabola $f(x)=(x+3)^2$ is $(\mathbf{-3},\mathbf{0})$. Since the leading coefficient is 1, the parabola takes the shape of $y=x^2$, and its **arms open up**.

The graph of the function f can be obtained by **shifting** the graph of the basic parabola **3 units to the left**, as shown in *Figure 3.8.*

The **range** of function f is $[\mathbf{0},\infty)$, and the equation of the **axis of symmetry** is $\boldsymbol{x=-3}$.

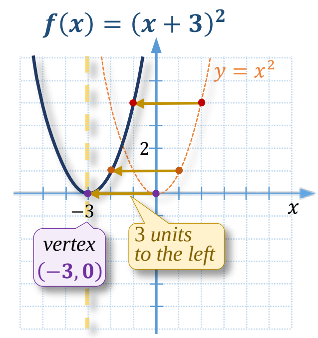


Figure 3.8

b. The expression $-x^2+1$ attains its highest value at $x=0$. So, the **vertex** of the parabola $g(x)=-x^2+1$ is $(\mathbf{0},\mathbf{1})$. Since the leading coefficient is -1, the parabola takes the shape of $y=x^2$, but its **arms open down**.

The graph of the function g can be obtained by:

- first, **flipping the graph** of the basic parabola **over the *x*-axis**, and then
- **shifting** the graph of $y=-x^2$ **1 unit up**, as shown in *Figure 3.9.*

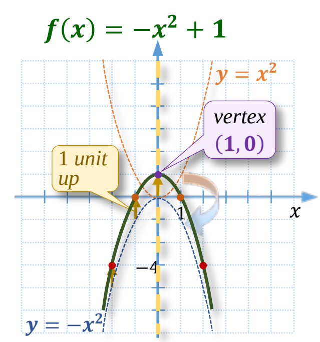


Figure 3.9

The **range** of the function g is $(-\infty,\mathbf{1}]$, and the equation of the **axis of symmetry** is $\boldsymbol{x=0}$.

Note: The order of transformations in the above example is essential. The reader is encouraged to check that **shifting** the graph of $y=x^2$ by 1 unit up first and then **flipping** it over the x-axis results in a different graph than the one in *Figure 3.9.*

Properties and Graphs of Quadratic Functions Given in the Vertex Form $f(x)=a(x-p)^2+q$

So far, we have discussed properties and graphs of quadratic functions that can be obtained from the graph of the basic parabola by applying mainly a single transformation. These transformations were: dilations (including flips over the x-axis), and horizontal and vertical shifts. Sometimes, however, we need to apply more than one transformation. We have already encountered such a situation in *Example 2b*, where a flip and a horizontal shift was applied. Now, we will look at properties and graphs of any function of the form $\boldsymbol{f(x)=a(x-p)^2+q}$, referred to as the **vertex form** of a quadratic function.

Suppose we wish to graph $f(x) = 2(x+1)^2 - 3$. This can be accomplished by connecting the points calculated in a table of values, such as the one below, or by observing the coordinates of the vertex and following the shape of the graph of $y = 2x^2$. Notice that the vertex of our parabola is at $(-1, -3)$. This information can be taken directly from the equation $f(x) = 2(x+1)^2 - 3 = 2(x - (-1))^2 - 3$,

opposite to the number in the bracket *the same last number*

without the aid of a table of values.

x	$2(x+1)^2 - 3$
-3	5
-2	-1
-1	-3
0	-1
1	5

1 unit apart from zero, 2 units up

vertex

The rest of the points follow the pattern of the shape for the $y = 2x^2$ parabola: 1 across, 2 up; 2 across, $4 \cdot 2 = 8$ up. So, we connect the points as in *Figure 3.10*.

Notice that the graph of function f could also be obtained as a result of translating the graph of $y = 2x^2$ by 1 unit left and 3 units down, as indicated in *Figure 3.10* by the blue vectors.

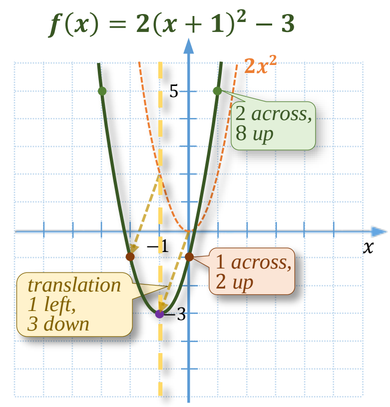


Figure 3.10

Here are the main properties of the graph of function f:
- It has a **shape** of $y = \mathbf{2x^2}$;
- It is a parabola that **opens up**;
- It has a **vertex** at $(\mathbf{-1}, \mathbf{-3})$;
- It is **symmetrical** about the line $\boldsymbol{x = -1}$;
- Its **minimum value** is $\mathbf{-3}$, and this minimum is attained at $x = -1$;
- Its domain is the set of all real numbers, and its **range** is the interval $[\mathbf{-3}, \infty)$;
- It **decreases** for $x \in (-\infty, \mathbf{-1}]$ and **increases** for $x \in [\mathbf{-1}, \infty)$.

The above discussion of properties and graphs of a quadratic function given in vertex form leads us to the following general observations:

Characteristics of Quadratic Functions Given in Vertex Form $f(x) = a(x-p)^2 + q$

1. The graph of a quadratic function given in **vertex form**

$$\boldsymbol{f(x) = a(x-p)^2 + q}, \text{ where } \boldsymbol{a \neq 0},$$

is a **parabola** with **vertex** $(\boldsymbol{p}, \boldsymbol{q})$ and **axis of symmetry** $\boldsymbol{x = p}$.

2. The graph **opens up** if $\boldsymbol{a}$ is **positive** and **down** if $\boldsymbol{a}$ is **negative**.

3. If $\boldsymbol{a > 0}$, $\boldsymbol{q}$ is the **minimum value**. If $\boldsymbol{a < 0}$, $\boldsymbol{q}$ is the **maximum value**.

3. The graph is **narrower** than that of $y = x^2$ if $|\boldsymbol{a}| > \mathbf{1}$.
The graph is **wider** than that of $y = x^2$ if $\mathbf{0} < |\boldsymbol{a}| < \mathbf{1}$.

4. The **domain** of function f is the set of real numbers, $\mathbb{R}$.
The **range** of function f is $[\boldsymbol{q}, \infty)$ if $\boldsymbol{a}$ is **positive** and $(-\infty, \boldsymbol{q}]$ if $\boldsymbol{a}$ is **negative**.

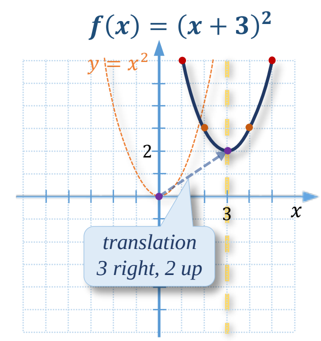
Example 3 ► **Identifying Properties and Graphing Quadratic Functions Given in Vertex Form $\boldsymbol{f(x) = a(x-p)^2 + q}$**

For each function, identify its **vertex**, **opening**, **axis of symmetry**, and **shape**. Then graph the function and state its **domain** and **range**. Finally, describe **transformations** of the basic parabola that would lead to the obtained graph.

a. $f(x) = (x-3)^2 + 2$ **b.** $g(x) = -\frac{1}{2}(x+1)^2 + 3$

Solution ►

a. The vertex of the parabola $f(x) = (x-3)^2 + 2$ is $(\boldsymbol{3}, \boldsymbol{2})$; the graph **opens up**, and the equation of the axis of symmetry is $\boldsymbol{x = 3}$. To graph this function, we can plot the vertex first and then follow the shape of the basic parabola $y = \boldsymbol{x^2}$.



The domain of function f is $\mathbb{R}$, and the range is $[\boldsymbol{2}, \infty)$.

The graph of f can be obtained by shifting the graph of the basic parabola **3 units to the right** and **2 units up**.

b. The vertex of the parabola $g(x) = -\frac{1}{2}(x+1)^2 + 3$ is $(-\boldsymbol{1}, \boldsymbol{3})$; the graph **opens down**, and the equation of the axis of symmetry is $\boldsymbol{x = -1}$. To graph this function, we can plot the vertex first and then follow the shape of the parabola $y = -\frac{1}{2}\boldsymbol{x^2}$. This means that starting from the vertex, we move the pen one unit across (both ways) and drop half a unit to plot the next two points, $\left(0, \frac{5}{2}\right)$ and symmetrically $\left(-2, \frac{5}{2}\right)$. To plot the following two points, again, we start from the vertex and move our pen two units across and 2 units down (as $-\frac{1}{2} \cdot 4 = -2$). So, the next two points are $(1,1)$ and symmetrically $(-4,1)$, as indicated in *Figure 3.11*.

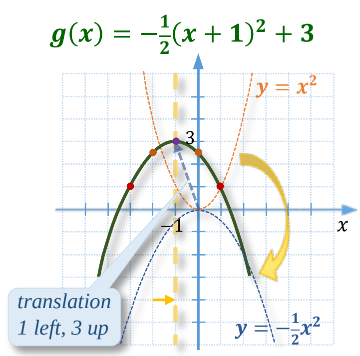


Figure 3.11

The domain of function g is $\mathbb{R}$, and the range is $(-\infty, \boldsymbol{3}]$.

The graph of g can be obtained from the graph of the basic parabola in two steps:

1. **Dilate** the basic parabola by multiplying its y-values by the factor of $-\frac{1}{2}$.
2. Shift the graph of the dilated parabola $y = -\frac{1}{2}\boldsymbol{x^2}$, **1 unit to the left** and **3 units up**, as indicated in *Figure 3.11*.

Aside from the main properties such as vertex, opening and shape, we are often interested in x- and y-intercepts of the given parabola. The next example illustrates how to find these intercepts from the vertex form of a parabola.

Example 4 ► **Finding the Intercepts from the Vertex Form $\boldsymbol{f(x) = a(x-p)^2 + q}$**

Find the x- and y-intercepts of each parabola.

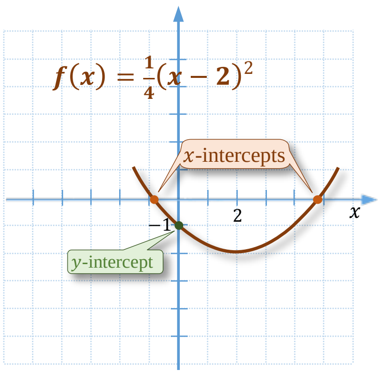
a. $f(x) = \frac{1}{4}(x-2)^2 - 2$ **b.** $g(x) = -2(x+1)^2 - 3$

Solution ► **a.** To find the y-intercept, we evaluate the function at zero. Since

$$f(0) = \frac{1}{4}(-2)^2 - 2 = 1 - 2 = -1,$$

then the y-intercept is $(\mathbf{0}, -\mathbf{1})$.

To find x-intercepts, we set $f(x) = 0$. So, we need to solve the equation



$$\frac{1}{4}(x-2)^2 - 2 = 0 \qquad / +1$$

$$\frac{1}{4}(x-2)^2 = 2 \qquad / \cdot 4$$

$$(x-2)^2 = 8 \qquad / \sqrt{\ }$$

$$\sqrt{(x-2)^2} = \sqrt{8}$$

$$|x-2| = 2\sqrt{2}$$

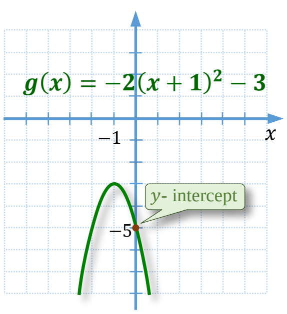
$$x - 2 = \pm 2\sqrt{2} \qquad / +2$$

$$x = 2 \pm 2 = \begin{cases} 2 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{cases}$$

Hence, the two x-intercepts are: $\left(\mathbf{2} - \mathbf{2}\sqrt{\mathbf{2}}, \mathbf{0}\right)$ and $\left(\mathbf{2} + \mathbf{2}\sqrt{\mathbf{2}}, \mathbf{0}\right)$.

b. Since $g(0) = -2(1)^2 - 3 = -5$, then the y-intercept is $(\mathbf{0}, -\mathbf{5})$.

To find x-intercepts, we etempt to solve the equation



$$-2(x+1)^2 - 3 = 0$$

$$-2(x+1)^2 = 3$$

$$\underbrace{(x+1)^2}_{\text{nonnegative}} = \underbrace{-\frac{3}{2}}_{\text{negative}}$$

nonnegative *cannot be equal* *negative*

However, since the last equation doesn't have any solution, we conclude that function $g(x)$ has no x-intercepts.

Q.3 Exercises

1. Match each quadratic function **a.-d.** with its graph **I-IV**.

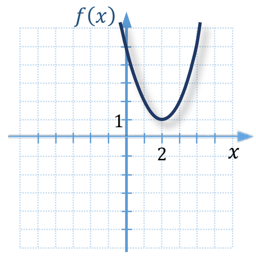
a. $f(x) = (x-2)^2 - 1$

b. $f(x) = (x-2)^2 + 1$

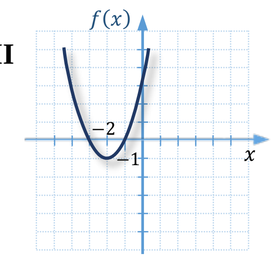
c. $f(x) = (x+2)^2 + 1$

d. $f(x) = (x+2)^2 - 1$

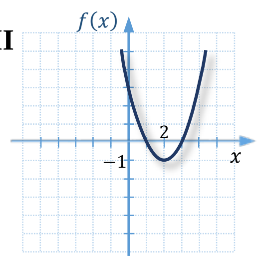
I



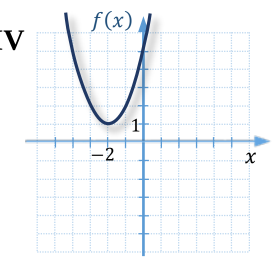
II



III



IV



2. Match each quadratic function **a.-d.** with its graph **I-IV**.

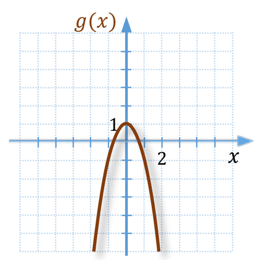
a. $g(x) = -(x-2)^2 + 1$

b. $g(x) = x^2 - 1$

c. $g(x) = -2x^2 + 1$

d. $g(x) = 2(x+2)^2 - 1$

I



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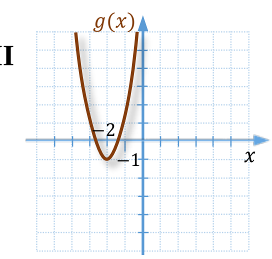
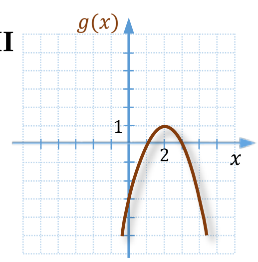


III



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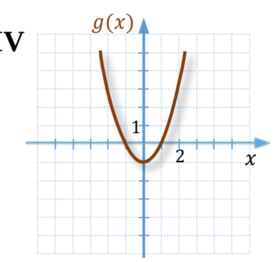


3. Match each quadratic function with the characteristics of its parabolic graph.

a. $f(x) = 5(x-3)^2 + 2$ **I** vertex $(3,2)$, opens down

b. $f(x) = -4(x+2)^2 - 3$ **II** vertex $(3,2)$, opens up

c. $f(x) = -\frac{1}{2}(x-3)^2 + 2$ **III** vertex $(-2,-3)$, opens down

d. $f(x) = \frac{1}{4}(x+2)^2 - 3$ **IV** vertex $(-2,-3)$, opens up

For each quadratic function, describe the ***shape*** *(as* ***wider***, ***narrower***, *or the* ***same*** *as the shape of $y = x^2$) and* ***opening*** *(****up*** *or* ***down****) of its graph. Then* ***graph it*** *and determine its* ***range****.*

4. $f(x) = 3x^2$ **5.** $f(x) = -\frac{1}{2}x^2$ **6.** $f(x) = -\frac{3}{2}x^2$

7. $f(x) = \frac{5}{2}x^2$ **8.** $f(x) = -x^2$ **9.** $f(x) = \frac{1}{3}x^2$

Graph *each parabola by plotting its vertex, and following its shape and opening. Then,* ***describe transformations*** *of the basic parabola that would lead to the obtained graph. Finally, state the* ***domain*** *and* ***range****, and the equation of the* ***axis of symmetry****.*

10. $f(x) = (x-3)^2$ **11.** $f(x) = -x^2 + 2$ **12.** $f(x) = x^2 - 5$

13. $f(x) = -(x+2)^2$ **14.** $f(x) = -2x^2 - 1$ **15.** $f(x) = \frac{1}{2}(x+2)^2$

For each parabola, state its ***vertex***, ***shape***, ***opening***, *and* ***x- and y-intercepts****. Then,* ***graph*** *the function and describe* ***transformations*** *of the basic parabola that would lead to the obtained graph.*

16. $f(x) = 3x^2 - 1$ **17.** $f(x) = -\frac{3}{4}x^2 + 3$

18. $f(x) = -\frac{1}{2}(x+4)^2 + 2$ **19.** $f(x) = \frac{5}{2}(x-2)^2 - 4$

20. $f(x) = 2(x-3)^2 + \frac{3}{2}$ **21.** $f(x) = -3(x+1)^2 + 5$

22. $f(x) = -\frac{2}{3}(x+2)^2 + 4$ **23.** $f(x) = \frac{4}{3}(x-3)^2 - 2$

24. Four students, **A**, **B**, **C**, and **D**, tried to graph the function $f(x) = -2(x+1)^2 - 3$ by transforming the graph of the basic parabola, $y = x^2$. Here are the transformations that each student applied

Student ***A****:*
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of -2

Student ***B****:*
- dilation of y-values by the factor of -2
- shift 1 unit left
- shift 3 units down

Student ***C****:*
- flip over the x-axis
- shift 1 unit left and 3 units down
- dilation of y-values by the factor of 2

Student ***D****:*
- shift 1 unit left
- dilation of y-values by the factor of 2
- shift 3 units down
- flip over the x-axis

With the assumption that all transformations were properly applied, discuss whose graph was correct and what went wrong with the rest of the graphs. Is there any other sequence of transformations that would result in a correct graph?

For each parabola, state the coordinates of its ***vertex*** *and then* ***graph*** *it. Finally, state the* ***extreme value*** *(****maximum*** *or* ***minimum****, whichever applies) and the* ***range*** *of the function.*

25. $f(x) = 3(x-1)^2$ **26.** $f(x) = -\frac{5}{2}(x+3)^2$

27. $f(x) = (x+2)^2 - 3$

28. $f(x) = -3(x+4)^2 + 5$

29. $f(x) = -2(x-5)^2 - 2$

30. $f(x) = 2(x-4)^2 + 1$

31. $f(x) = \frac{1}{2}(x+1)^2 + \frac{3}{2}$

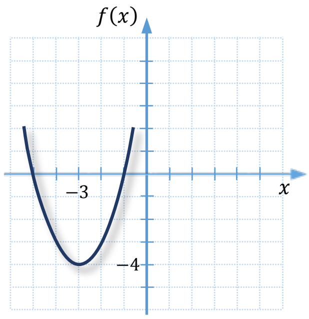
32. $f(x) = -\frac{1}{2}(x-1)^2 - 3$

33. $f(x) = -\frac{1}{4}(x-3)^2 + 4$

34. $f(x) = \frac{3}{4}\left(x+\frac{5}{2}\right)^2 - \frac{3}{2}$

Given the graph of a parabola, state the most probable ***equation*** *of the corresponding function. Hint: Use the vertex form of a quadratic function.*

35.



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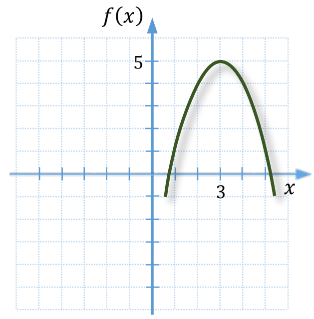

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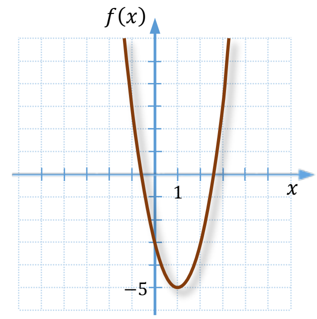

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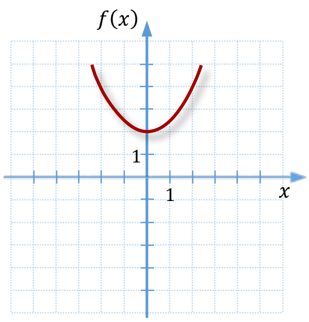

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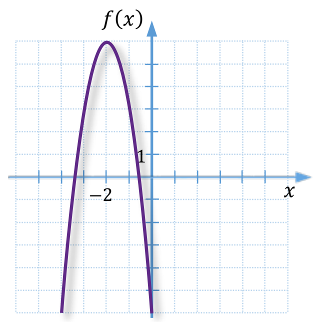

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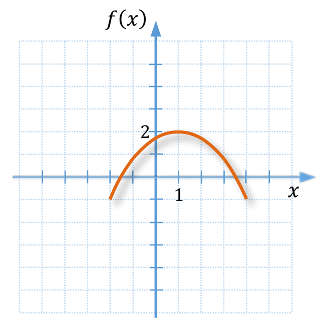


Q4 Properties of Quadratic Function and Optimization Problems

In the previous section, we examined how to graph and read the characteristics of the graph of a quadratic function given in vertex form, $f(x) = a(x-p)^2 + q$. In this section, we discuss the ways of **graphing** and reading the **characteristics** of the graph of a quadratic function given in **standard form**, $\boldsymbol{f(x) = ax^2 + bx + c}$. One of these ways is to convert standard form of the function to vertex form by **completing the square** so that the information from the vertex form may be used for graphing. The other handy way of graphing and reading properties of a quadratic function is to **factor** the defining trinomial and use the **symmetry** of a parabolic function.

At the end of this section, we apply properties of quadratic functions to solve certain **optimization problems**. To solve these problems, we look for the **maximum** or **minimum** of a particular quadratic function satisfying specified conditions called **constraints**. Optimization problems often appear in geometry, calculus, business, computer science, etc.

Graphing Quadratic Functions Given in the Standard Form $\boldsymbol{f(x) = ax^2 + bx + c}$

To graph a quadratic function given in standard form, $f(x) = ax^2 + bx + c$, we can use one of the following methods:

1. constructing a **table of values** (this would always work, but it could be cumbersome);
2. converting to **vertex form** by using the technique of completing the square (see *Example 1-3*);
3. **factoring** and employing the properties of a parabolic function. (this is a handy method if the function can be easily factored – see *Example 3 and 4*)

The table of values approach can be used for any function, and it was already discussed on various occasions throughout this textbook.

Converting to **vertex form** involves completing the square. For example, to convert the function $f(x) = 2x^2 + x - 5$ to its vertex form, we might want to start by dividing both sides of the equation by the leading coefficient 2, and then complete the square for the polynomial on the right side of the equation, as below.

$$\frac{f(x)}{2} = x^2 + \frac{1}{2}x - \frac{5}{2}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{5 \cdot 8}{2 \cdot 8}$$

$$\frac{f(x)}{2} = \left(x + \frac{1}{4}\right)^2 - \frac{41}{16}$$

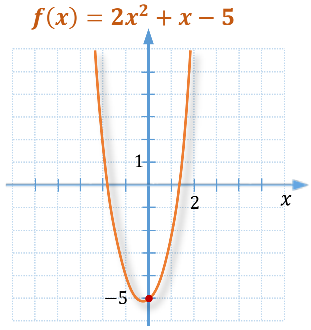


Figure 4.1

Finally, the vertex form is obtained by multiplying both sides of the equation back by 2. So, we have

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{41}{8}$$

This form lets us identify the vertex, $\left(-\frac{1}{4}, -\frac{41}{8}\right)$, and the shape, $y = 2x^2$, of the parabola, which is essential for graphing it. To create an approximate graph of function f, we may want to round the vertex to approximately $(-0.25, -5.1)$ and evaluate $f(0) = 2 \cdot 0^2 + 0 - 5 = -5$. So, the graph is as in *Figure 4.1*.

Example 1 ► **Converting the Standard Form of a Quadratic Function to the Vertex Form**

Rewrite each function in its vertex form. Then, identify the vertex.

a. $f(x) = -3x^2 + 2x$ **b.** $g(x) = \frac{1}{2}x^2 + x + 3$

Solution ► **a.** To convert f to its vertex form, we follow the completing the square procedure. After dividing the equation by the leading coefficient,

$$f(x) = -3x^2 + 2x, \qquad / \div (-3)$$

we have

$$\frac{f(x)}{-3} = x^2 - \frac{2}{3}x$$

Then, we complete the square for the right side of the equation,

$$\frac{f(x)}{-3} = \left(x - \frac{1}{3}\right)^2 - \frac{1}{9}, \qquad / \cdot (-3)$$

and finally, multiply back by the leading coefficient,

$$f(x) = -3\left(x - \frac{1}{3}\right)^2 + \frac{1}{3}.$$

Therefore, the vertex of this parabola is at the point $\left(\boldsymbol{\frac{1}{3}}, \boldsymbol{\frac{1}{3}}\right)$.

b. As in the previous example, to convert g to its vertex form, we first wish to get rid of the leading coefficient. This can be achieved by multiplying both sides of the equation $g(x) = \frac{1}{2}x^2 - x + 3$ by 2. So, we obtain

$$2g(x) = x^2 + 2x + 6$$

$$2g(x) = (x + 1)^2 - 1 + 6$$

$$2g(x) = (x + 1)^2 + 5, \qquad / \div 2$$

which can be solved back for g,

$$g(x) = \frac{1}{2}(x + 1)^2 + \frac{5}{2}.$$

Therefore, the vertex of this parabola is at the point $\left(\boldsymbol{-1}, \boldsymbol{\frac{5}{2}}\right)$.

Completing the square allows us to derive a formula for the vertex of the graph of any quadratic function given in its standard form, $f(x) = ax^2 + bx + c$, where $a \neq 0$. Applying the same procedure as in *Example 1*, we calculate

$$f(x) = ax^2 + bx + c \qquad / \div a$$

$$\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

Recall: This is the **discriminant** **Δ!**

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \qquad / \cdot a$$

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \frac{-(b^2 - 4ac)}{4a}$$

Thus, the coordinates of the vertex $(\boldsymbol{p}, \boldsymbol{q})$ are $\boldsymbol{p} = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ and $\boldsymbol{q} = \frac{-(\boldsymbol{b}^2 - \boldsymbol{4ac})}{\boldsymbol{4a}} = \frac{-\Delta}{\boldsymbol{4a}}$.

Observation: Notice that the expression for q can also be found by evaluating f at $x = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$.

So, the vertex of the parabola can also be expressed as $\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$.

VERTEX FORMULA

Summarizing, the **vertex** of a parabola defined by $f(x) = \boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c}$, where $\boldsymbol{a} \neq 0$, can be calculated by following one of the formulas:

$$\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-(\boldsymbol{b}^2 - \boldsymbol{4ac})}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \frac{-\Delta}{\boldsymbol{4a}}\right) = \left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}, \boldsymbol{f}\left(-\frac{\boldsymbol{b}}{\boldsymbol{2a}}\right)\right)$$

Example 2 ► **Using the Vertex Formula to Find the Vertex of a Parabola**

Use the vertex formula to find the vertex of the graph of $f(x) = -x^2 - x + 1$.

Solution ► The first coordinate of the vertex is equal to $-\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{\boldsymbol{1}}{\boldsymbol{2}}$.

The second coordinate can be calculated by following the formula

$\frac{-\Delta}{4a} = \frac{-((-1)^2 - 4\cdot(-1)\cdot 1)}{4\cdot(-1)} = \frac{\boldsymbol{5}}{\boldsymbol{4}}$,

or by evaluating $f\left(-\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 1 = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{\boldsymbol{5}}{\boldsymbol{4}}$.

So, the vertex is $\left(-\frac{\boldsymbol{1}}{\boldsymbol{2}}, \frac{\boldsymbol{5}}{\boldsymbol{4}}\right)$.

Example 3 ► **Graphing a Quadratic Function Given in the Standard Form**

Graph each function.

a. $g(x) = \frac{1}{2}x^2 + x + 3$ **b.** $f(x) = -x^2 - x + 1$

Solution ► **a.** The shape of the graph of function g is the same as this of $y = \frac{1}{2}x^2$. Since the leading coefficient is positive, the arms of the parabola **open up**.

The **vertex**, $\left(-\mathbf{1}, \frac{\mathbf{5}}{\mathbf{2}}\right)$, was found in *Example 1b* as a result of completing the square. Since the vertex is in quadrant II and the graph opens up, we do not expect any x-intercepts. However, without much effort, we can find the y-intercept by evaluating $g(0) = \mathbf{3}$. Furthermore, since $(0, 3)$ belongs to the graph, then by symmetry, $(-2, 3)$ must also belong to the graph. So, we graph function g is as in *Figure 4.2*.

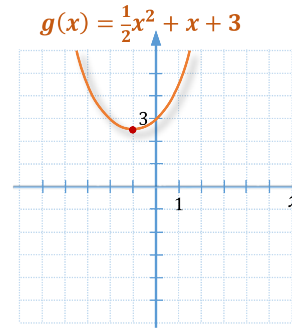


Figure 4.2

b. The graph of function f has the shape of the basic parabola. Since the leading coefficient is negative, the arms of the parabola **open down**.

When plotting points with fractional coordinates, round the values to one place value.

The **vertex**, $\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{4}}\right)$, was found in *Example 2* by using the vertex formula. Since the vertex is in quadrant II and the graph opens down, we expect two x-intercepts. Their values can be found via the quadratic formula applied to the equation $-x^2 - x + 1 = 0$. So, the x-intercepts are $x_{1,2} = \frac{1 \pm \sqrt{5}}{-2} \approx \mathbf{-1.6}$ *or* $\mathbf{0.6}$. In addition, the y-intercept of the graph is $f(0) = \mathbf{1}$.

Using all this information, we graph function f, as in *Figure 4.3*.

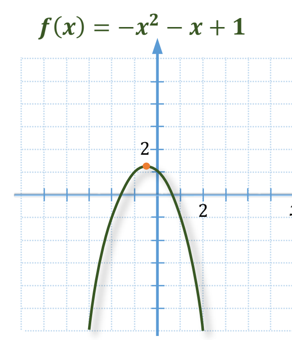


Figure 4.3

Graphing Quadratic Functions Given in the Factored Form $f(x) = a(x - r_1)(x - r_2)$

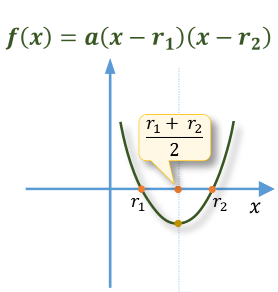


Figure 4.4

What if a quadratic function is given in factored form? Do we have to change it to vertex or standard form in order to find the vertex and graph it?

The factored form, $f(x) = a(x - r_1)(x - r_2)$, allows us to find the roots (or x-intercepts) of such a function. These are r_1 and r_2. A parabola is symmetrical about the axis of symmetry, which is the vertical line passing through its vertex. So, the first coordinate of the vertex is the same as the first coordinate of the midpoint of the line segment connecting the roots, r_1 with r_2, as indicated in *Figure 4.4*. Thus, the first coordinate of the vertex is the average of the two roots, $\frac{r_1 + r_2}{2}$. Then, the second coordinate of the vertex can be found by evaluating $f\left(\frac{r_1 + r_2}{2}\right)$.

Example 4 ► Graphing a Quadratic Function Given in a Factored Form

Graph function $g(x) = -(x-2)(x+1)$.

Solution ►

First, observe that the graph of function g has the same shape as the graph of the basic parabola, $f(x) = x^2$. Since the leading coefficient is negative, the arms of the parabola **open down**. Also, the graph intersects the x-axis at 2 and -1. So, the first coordinate of the vertex is the average of 2 and -1, which is $\frac{1}{2}$. The second coordinate is

$$g\left(\frac{1}{2}\right) = -\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} + 1\right) = -\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}$$

Therefore, function g can be graphed by connecting the vertex, $\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{9}}{\mathbf{4}}\right)$, and the x-intercepts, $(-1,0)$ and $(-1,0)$, with a parabolic curve, as in *Figure 4.5*. For a more precise graph, we may additionally plot the y-intercept, $g(0) = 2$, and the symmetrical point $g(1) = 2$.

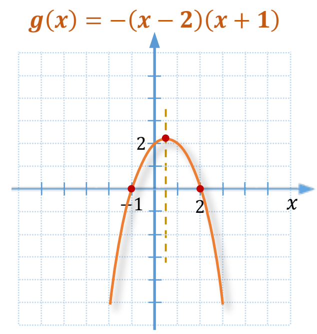


Figure 4.5

Example 5 ► Using Complete Factorization to Graph a Quadratic Function

Graph function $f(x) = 4x^2 - 2x - 6$.

Solution ►

Since the discriminant $\Delta = (-2)^2 - 4 \cdot 4 \cdot (-6) = 4 + 96 = 100$ is a perfect square number, the defining trinomial is factorable. So, to graph function f, we may want to factor it first. Notice that the GCF of all the terms is 2. So, $f(x) = 2(2x^2 - x - 3)$. Then, using factoring techniques discussed in *Section F2*, we obtain $f(x) = 2(2x-3)(x+1)$. This form allows us to identify the roots (or zeros) of function f, which are $\frac{3}{2}$ and -1. So, the first coordinate of the vertex is the average of $\frac{3}{2} = 1.5$ and -1, which is $\frac{1.5+(-1)}{2} = \frac{0.5}{2} = 0.25$. The second coordinate can be calculated by evaluating

$$f(0.25) = 2(2 \cdot 0.25 - 3)(0.25 + 1) = 2(0.5 - 3)(1.25) = 2(-2.5)(1.25) = -6.25$$

So, we can graph function f by connecting its vertex, $(\mathbf{0.25}, \mathbf{-6.25})$, and its x-intercepts, $(-1,0)$ and $(1.5,0)$, with a parabolic curve, as in *Figure 4.6*. For a more precise graph, we may additionally plot the y-intercept, $f(0) = -6$, and by symmetry, $f(0.5) = -6$.

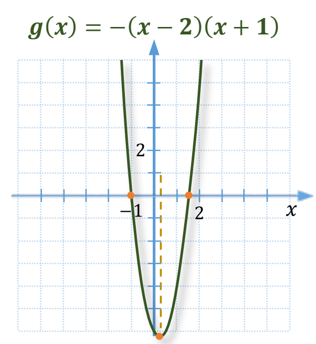


Figure 4.6

Observation: Since x-intercepts of a parabola are the solutions (zeros) of its equation, the equation of a parabola with x-intercepts at $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ can be written as

$$\boldsymbol{y = a(x - r_1)(x - r_2)},$$

for some real coefficient $\boldsymbol{a} \neq 0$.

Example 6 ► **Finding an Equation of a Quadratic Function Given Its Solutions**

a. Find an equation of a quadratic function whose graph passes the x-axis at -1 and 3.

b. Find an equation of a quadratic function whose graph passes the x-axis at -1 and 3 and the y-axis at -4.

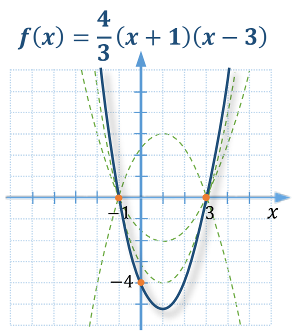
c. Write a quadratic equation with integral coefficients knowing that the solutions of this equation are $\frac{1}{2}$ and $-\frac{2}{3}$.

Solution ►

a. x-intercepts of a function are the zeros of this function. So, -1 and 3 are the zeros of the quadratic function. This means that the defining formula for such function should include factors $\big(x-(-1)\big)$ and $(x-3)$. So, it could be

$$f(x) = (x+1)(x-3).$$

Notice that this is indeed a quadratic function with x-intercepts at -1 and 3. Hence, it satisfies the conditions of the problem.



b. Using the solution to *Example 6a*, notice that any function of the form

$$f(x) = a(x+1)(x-3),$$

where a is a nonzero real number, is a quadratic function with x-intercepts at -1 and 3. To guarantee that the graph of our function passes through the point $(0,-4)$, we need to find the particular value of the coefficient a. This can be done by substituting $x = 0$ and $f(x) = -4$ into the function's equation and solving it for a. Thus,

$$-4 = a(0+1)(0-3)$$

$$-4 = -3a$$

$$a = \frac{4}{3},$$

and the desired function is $f(x) = \frac{4}{3}(x+1)(x-3)$.

c. First, observe that $\frac{1}{2}$ is a solution to the linear equation $2x-1=0$. Similarly, $-\frac{2}{3}$ is a solution to the equation $3x+2=0$. Multiplying these two equations side by side, we obtain a quadratic equation

$$(2x-1)(3x+2) = 0$$

that satisfies the conditions of the problem.

Note: Here, we could create the desired equation by writing

$$\left(x-\frac{1}{2}\right)\left(x-\left(-\frac{2}{3}\right)\right) = 0 \qquad / \cdot 6$$

and then multiplying it by the $LCD = \mathbf{6} = \mathbf{2}\cdot\mathbf{3}$

$$\mathbf{2}\left(x-\frac{1}{2}\right)\mathbf{3}\left(x+\frac{2}{3}\right) = 0$$

$$(2x-1)(3x+2) = 0$$

Optimization Problems

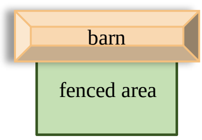
In many applied problems we are interested in **maximizing** or **minimizing** some quantity under specific conditions, called **constraints**. For example, we might be interested in finding the greatest area that can be fenced in by a given length of fence, or minimizing the cost of producing a container of a given shape and volume. These types of problems are called **optimization problems**.
Since the vertex of the graph of a quadratic function is either the highest or the lowest point of the parabola, it can be used in solving optimization problems that can be modeled by a quadratic function.

The vertex of a parabola provides the following information.

- The y-value of the vertex gives the maximum or minimum value of y.
- The x-value tells where the maximum or minimum occurs.

Example 7 ►

Maximizing Area of a Rectangular Region



John has 60 meters of fencing to enclose a rectangular field by his barn. Assuming that the barn forms one side of the rectangle, find the maximum area he can enclose and the dimensions of the enclosed field that yields this area.

Solution ►

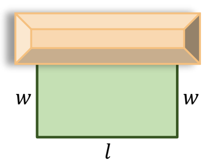


Figure 4.7

Let l and w represent the length and width of the enclosed area correspondingly, as indicated in *Figure 4.7*. The 60 meters of fencing is used to cover the distance of twice along the width and once along the length. So, we can form the constraint equation

$$2W + l = 60 \qquad (1)$$

To analyse the area of the field,

$$A = lw, \qquad (2)$$

we would like to express it as a function of one variable, for example w. To do this, we can solve the constraint equation (1) for l and substitute the obtained expression into the equation of area, (2). Since $l = 60 - 2w$, then

$$A = lw = (60 - 2w)w$$

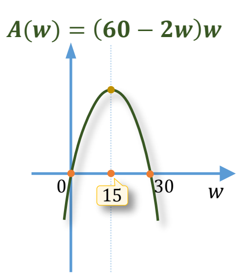


Figure 4.8

Observe that the graph of the function $A(w) = (60 - 2w)w$ is a parabola that opens down and intersects the x-axis at 0 and 30. This is because the leading coefficient of $(60 - 2w)w$ is negative and the roots to the equation $(60 - 2w)w = 0$ are 0 and 30. These roots are symmetrical in the axis of symmetry, which also passes through the vertex of the parabola, as illustrated in *Figure 4.8*. So, the first coordinate of the vertex is the average of the two roots, which is $\frac{0+30}{2} = 15$. Thus, the width that would maximize the enclosed area is $w_{max} = \mathbf{15}$ **meters**. Consequently, the length that would maximize the enclosed area is $l_{max} = 60 - 2w_{max} = 60 - 2 \cdot 15 = \mathbf{30}$ **meters**. The maximum area represented by the second coordinate of the vertex can be obtained by evaluating the function of area at the width of 15 meters.

$$A(15) = (60 - 2 \cdot 15)15 = 30 \cdot 15 = 450 \text{ m}^2$$

So, the maximum area that can be enclosed by 60 meters of fencing is **450 square meters**, and the dimensions of this rectangular area are **15 by 30 meters**.

Example 8 ► **Minimizing Average Unit Cost**

A company producing skateboards has determined that when x hundred skateboards are produced, the average cost of producing one skateboard can be modelled by the function

$$C(x) = 0.15x^2 - 0.75x + 1.5,$$

where $C(x)$ is in hundreds of dollars. How many skateboards should be produced to minimize the average cost of producing one skateboard? What would this cost be?

Solution ► Since $C(x)$ is a quadratic function, to find its minimum, it is enough to find the vertex of the parabola $C(x) = 0.15x^2 - 0.75x + 1.5$. This can be done either by completing the square method or by using the formula for the vertex, $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$. We will do the latter. So, the vertex is

$$\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = \left(\frac{0.75}{0.3}, \frac{-(0.75^2 - 4 \cdot 0.15 \cdot 1.5)}{0.6}\right) = \left(2.5, \frac{-(0.5625 - 1.35)}{0.6}\right)$$
$$= \left(2.5, \frac{0.3375}{0.6}\right) = (\mathbf{2.5}, \mathbf{0.5625}).$$

This means that the lowest average unit cost can be achieved when 250 skateboards are produced, and then the average cost of a skateboard would be \$56.25.

Q.4 Exercises

Convert each quadratic function to its ***vertex form****. Then, state the coordinates of the* ***vertex****.*

1. $f(x) = x^2 + 6x + 10$ **2.** $f(x) = x^2 - 4x - 5$ **3.** $f(x) = x^2 + x - 3$

4. $f(x) = x^2 - x + 7$ **5.** $f(x) = -x^2 + 7x + 3$ **6.** $f(x) = 2x^2 - 4x + 1$

7. $f(x) = -3x^2 + 6x + 12$ **8.** $f(x) = -2x^2 - 8x + 10$ **9.** $f(x) = \frac{1}{2}x^2 + 3x - 1$

Use the vertex formula, $\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$*, to find the coordinates of the* ***vertex*** *of each parabola.*

10. $f(x) = x^2 + 6x + 3$ **11.** $f(x) = -x^2 + 3x - 5$ **12.** $f(x) = \frac{1}{2}x^2 - 4x - 7$

13. $f(x) = -3x^2 + 6x + 5$ **14.** $f(x) = 5x^2 - 7x$ **15.** $f(x) = 3x^2 + 6x - 20$

*For each parabola, state its **vertex**, **opening** and **shape**. Then **graph** it and state the **domain** and **range**.*

16. $f(x) = x^2 - 5x$

17. $f(x) = x^2 + 3x$

18. $f(x) = x^2 - 2x - 5$

19. $f(x) = -x^2 + 6x - 3$

20. $f(x) = -x^2 - 3x + 2$

21. $f(x) = 2x^2 + 12x + 18$

22. $f(x) = -2x^2 + 3x - 1$

23. $f(x) = -2x^2 + 4x + 1$

24. $f(x) = 3x^2 + 4x + 2$

*For each quadratic function, state its **zeros** (roots), coordinates of the **vertex**, **opening** and **shape**. Then **graph** it and identify its **extreme** (minimum or maximum) **value** as well as where it occurs.*

25. $f(x) = (x - 2)(x + 2)$

26. $f(x) = -(x + 3)(x - 1)$

27. $f(x) = x^2 - 4x$

28. $f(x) = x^2 + 5x$

29. $f(x) = x^2 - 8x + 16$

30. $f(x) = -x^2 - 4x - 4$

31. $f(x) = -3(x^2 - 1)$

32. $f(x) = \frac{1}{2}(x + 3)(x - 4)$

33. $f(x) = -\frac{3}{2}(x - 1)(x - 5)$

Find an equation of a quadratic function satisfying the given conditions.

34. passes the x-axis at -2 and 5

35. has x-intercepts at 0 and $\frac{2}{5}$

36. passes the x-axis at -3 and -1 and y-axis at 2

37. $f(1) = 0$, $f(4) = 0$, $f(0) = 3$

Write a quadratic equation with the indicated solutions using only integral coefficients.

38. -5 and 6

39. 0 and $\frac{1}{3}$

40. $-\frac{2}{5}$ and $\frac{3}{4}$

41. 2

42. Suppose the x-intercepts of the graph of a parabola are $(x_1, 0)$ and $(x_2, 0)$. What is the equation of the axis of symmetry of this graph?

43. How can we determine the number of x-intercepts of the graph of a quadratic function without graphing the function?

True or false? Explain.

44. The domain and range of a quadratic function are both the set of real numbers.

45. The graph of every quadratic function has exactly one y-intercept.

46. The graph of $y = -2(x - 1)^2 - 5$ has no x-intercepts.

47. The maximum value of y in the function $y = -4(x - 1)^2 + 9$ is 9.

48. The value of the function $f(x) = x^2 - 2x + 1$ is at its minimum when $x = 0$.

49. The graph of $f(x) = 9x^2 + 12x + 4$ has one x-intercept and one y-intercept.

50. If a parabola opens down, it has two x-intercepts.

Solve each problem.

51. A ball is projected from the ground straight up with an initial velocity of 24.5 m/sec. The function $h(t) = -4.9t^2 + 24.5t$ allows for calculating the height $h(t)$, in meters, of the ball above the ground after t seconds. What is the maximum height reached by the ball? In how many seconds should we expect the ball to come back to the ground?

52. A firecracker is fired straight up and explodes at its maximum height above the ground. The function $h(t) = -4.9t^2 + 98t$ allows for calculating the height $h(t)$, in meters, of the firecracker above the ground t seconds after it was fired. In how many seconds after firing should we expect the firecracker to explode and at what height?

53. Antonio prepares and sells his favourite desserts at a market stand. Suppose his daily cost, C, in dollars, to sell n desserts can be modelled by the function $C(n) = 0.5n^2 - 30n + 350$. How many of these desserts should he sell to minimize the cost and what is the minimum cost?

54. Chris has a hot-dog stand. His daily cost, C, in dollars, to sell n hot-dogs can be modelled by the function $C(n) = 0.1n^2 - 15n + 700$. How many hotdogs should he sell to minimize the cost and what is the minimum cost?

55. Find two positive numbers with a sum of 32 that would produce the maximum product.

56. Find two numbers with a difference of 32 that would produce the minimum product.

57. Luke uses 16 meters of fencing to enclose a rectangular area for his baby goats. The enclosure shares one side with a large barn, so only 3 sides need to be fenced. If Luke wishes to enclose the greatest area, what should the dimensions of the enclosure be?

58. Ryan uses 60 meters of fencing to enclose a rectangular area for his livestock. He plans to subdivide the area by placing additional fence down the middle of the rectangle to separate different types of livestock. What dimensions of the overall rectangle will maximize the total area of the enclosure?

59. Julia works as a tour guide. She charges \$58 for an individual tour. When more people come for a tour, she charges \$2 less per person for each additional person, up to 25 people.

a. Express the price per person P as a function of the number of people n, for $n \in \{1,2, \ldots ,25\}$.

b. Express her revenue, R, as a function of the number of people on tour.

c. How many people on tour would maximize Julia's revenue?

d. What is the highest revenue she can achieve?

60. One-day adult passes for The Mission Folk Festival cost \$50. At this price, the organizers of the festival expect about 1300 people to purchase the pass. Suppose that the organizers observe that every time they increase the cost per pass by 5\$, the number of passes sold decrease by about 100.

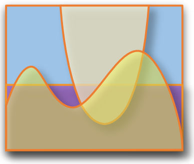
a. Express the number of passes sold, N, as a function of the cost, c, of a one-day pass.

b. Express the revenue, R, as a function of the cost, c, of a one-day pass.

c. How much should a one-day pass costs to maximize the revenue?

d. What is the maximum revenue?

Q5 Polynomial and Rational Inequalities



In *Sections L4* and *L5*, we discussed solving linear inequalities in one variable as well as solving systems of such inequalities. In this section, we examine polynomial and rational inequalities in one variable. Such inequalities can be solved using either graphical or analytic methods. Below, we discuss both types of methods with a particular interest in the analytic one.

Solving Quadratic Inequalities by Graphing

Definition 5.1 ► A **quadratic inequality** is any inequality that can be written in one of the forms

$$\boldsymbol{ax^2+bx+c>(\geq)\,0}, \quad \text{or} \quad \boldsymbol{ax^2+bx+c<(\leq)\,0}, \quad \text{or} \quad \boldsymbol{ax^2+bx+c\neq 0}$$

where a, b, and c are real numbers, with $a \neq 0$.

To solve a quadratic inequality, it is useful to solve the related quadratic equation first. For example, to solve $x^2+2x-3>(\geq)\,0$, or $x^2+2x-3<(\leq)\,0$, we may consider solving the related equation:

$$x^2+2x-3=0$$

$$(x+3)(x-1)=0$$

$$x=-3 \text{ or } x=1.$$

This helps us to sketch an approximate graph of the related function $f(x)=x^2+x-2$, as in *Figure 1.1*. The graph of function f is a parabola that crosses the x-axis at $x=-3$ and $x=1$, and is directed upwards. Observe that the graph extends below the x-axis for x-values from the interval $(-3,1)$ and above the x-axis for x-values from the set $(-\infty,-3)\cup(1,\infty)$. This allows us to read solution sets of several inequalities, as listed below.

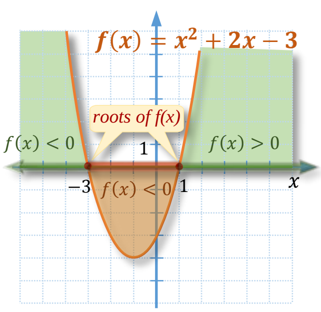


Figure 1.1

Inequality	Solution Set
$\boldsymbol{x^2+2x-3>0}$	$(-\infty,-3)\cup(1,\infty)$
$\boldsymbol{x^2+2x-3\geq 0}$	$(-\infty,-3]\cup[1,\infty)$
$\boldsymbol{x^2+2x-3<0}$	$(-3,1)$
$\boldsymbol{x^2+2x-3\leq 0}$	$[-3,1]$
$\boldsymbol{x^2+2x-3\neq 0}$	$\mathbb{R}\setminus\{-3,1\}$

Note: If the inequalities contain equations $(\geq,\leq)$, the x-values of the intercepts are included in the solution sets. Otherwise, the x-values of the intercepts are excluded from the solution sets.

Solving Polynomial Inequalities

Definition 5.2 ► A **polynomial inequality** is any inequality that can be written in one of the forms

$$\boldsymbol{P(x)>(\geq)\,0}, \quad \text{or} \quad \boldsymbol{P(x)<(\leq)\,0}, \quad \text{or} \quad \boldsymbol{P(x)\neq 0}$$

where $P(x)$ is a polynomial with real coefficients.

Note: Linear or quadratic inequalities are special cases of polynomial inequalities.

Polynomial inequalities can be solved graphically or analytically, without the use of a graph. The analytic method involves determining the sign of the polynomial by analysing signs of the polynomial factors for various x-values, as in the following example.

Example 1 ► **Solving Polynomial Inequalities Using Sign Analysis**

Solve each inequality using sign analysis.

a. $(x+1)(x-2)x > 0$ **b.** $2x^4 + 8 \le 10x^2$

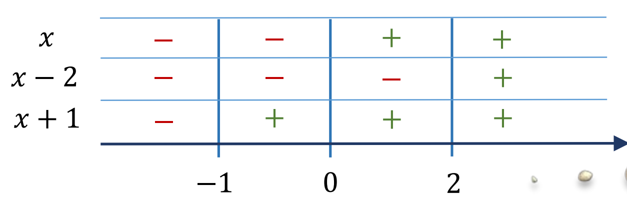
Solution ► **a.** The solution set of $(x+1)(x-2)x > 0$ consists of all x-values that make the product $(x+1)(x-2)x$ positive. To analyse how the sign of this product depends on the x-values, we can visualise the sign behaviour of each factor with respect to the x-value, by recording applicable signs in particular sections of a number line.

For example, the expression $x+1$ changes its sign at $x=-1$.
If $x > -1$, the expression $x+1$ is positive, so we mark "+" in the interval $(-1, \infty)$.
If $x < -1$, the expression $x+1$ is negative, so we mark "−" in the interval $(-\infty, -1)$.

So, the sign behaviour of the expression $x+1$ can be recorded on a number line, as below.

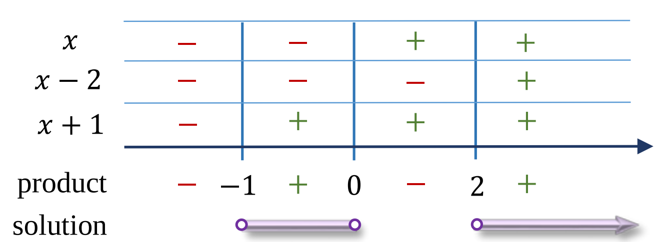
Here, the **zeros** of polynomials are referred to as **critical numbers**. This is because the polynomials may change their signs at these numbers.

A similar analysis can be conducted for the remaining factors, $x-2$ and x. These expressions change their signs at $x=2$ and $x=0$, correspondingly. The sign behaviour of all the factors can be visualised by reserving one line of signs per each factor, as shown below.



Remember to write the critical numbers in increasing order!

The sign of the product $(x+1)(x-2)x$ is obtained by multiplying signs in each column. The result is marked beneath the number line, as below.



The signs in the "product" row give us the grounds to state the solution set for the original inequality, $(x+1)(x-2)x > 0$. Before we write the final answer though, it is helpful to visualize the solution set by graphing it in the "solution" row. To satisfy the original inequality, we need the product to be positive, so we look for the intervals within which the product is positive. Since the inequality does not include an equation, the intervals are open. Therefore, the solution set is $(\mathbf{-1}, \mathbf{0}) \cup (\mathbf{2}, \infty)$.

b. To solve $2x^4 + 8 \leq 10x^2$ by sign analysis, we need to keep one side of the inequality equal to zero and factor the other side so that we may identify the critical numbers. To do this, we may change the inequality as below.

$$2x^4 + 8 \leq 10x^2 \qquad / -10x^2$$

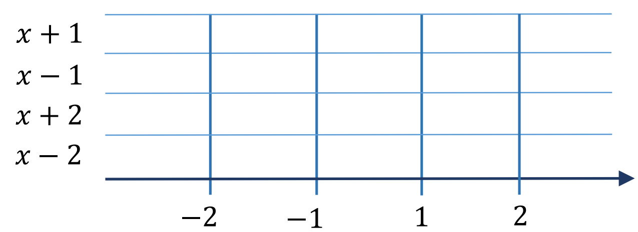
$$2x^4 - 10x^2 + 8 \leq 0 \qquad / \div 2$$

$$x^4 - 5x^2 + 4 \leq 0$$

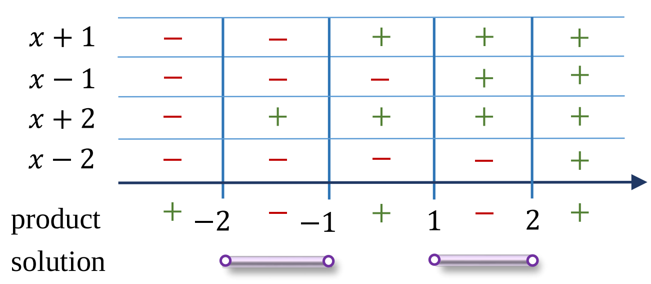
$$(x^2 - 4)(x^2 - 1) \leq 0$$

$$(x-2)(x+2)(x-1)(x+1) \leq 0$$

The critical numbers (the x-values that make the factors equal to zero) are 2, -2, 1, and -1. To create a table of signs, we arrange these numbers on a number line in increasing order and list all the factors in the left column.



Then, we can fill in the table with signs that each factor assumes for the x-values from the corresponding section of the number line.



The signs in each part of the number line can be determined either by:

- **analysing** the behaviour of each factor with respect to its critical number (For example, $x - 2 > 0$ for $x > 2$. So, the row of signs assumed by $x - 2$ consists of negative signs until the critical number 2 and positive signs after this number.)

or

- **testing** an x-value from each section of the number line (For example, since the expression $x - 2$ does not change its sign inside the interval $(-1,1)$, to determine its sign, it is enough to evaluate it for an easy to calculate **test number** from this interval. For instance, when $x = 0 \in (-1,1)$, the value of $x - 2$ is negative. This means that all the values of $x - 2$ are negative between -1 and 1.)

Finally, underneath each column, we record the sign of the product and graph the solution set to the inequality $(x-2)(x+2)(x-1)(x+1) \leq 0$

$x+1$	$-$		$-$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$+$		$+$
$x+2$	$-$		$+$		$+$		$+$		$+$
$x-2$	$-$		$-$		$-$		$-$		$+$
product	$+$	-2	$-$	-1	$+$	1	$-$	2	$+$
solution		$\bullet$	———	$\bullet$		$\bullet$	———	$\bullet$	

Note: Since the inequality contains an equation, the endpoints of the resulting intervals belong to the solution set as well. Hence, they are marked by **filled-in circles** and notated with **square brackets** in interval notation.

So, the solution set is $[\mathbf{-2}, \mathbf{-1}] \cup [\mathbf{1}, \mathbf{2}]$.

Solving Rational Inequalities

Definition 5.3 ► A **rational inequality** is any inequality that can be written in one of the forms

$$\frac{P(x)}{Q(x)} > (\geq)\ \mathbf{0}, \quad \text{or} \quad \frac{P(x)}{Q(x)} < (\leq)\ \mathbf{0}, \quad \text{or} \quad \frac{P(x)}{Q(x)} \neq \mathbf{0}$$

where $P(x)$ and $Q(x)$ are polynomials with real coefficients.

Rational inequalities can be solved similarly as polynomial inequalities. To solve a rational inequality using the sign analysis method, we need to make sure that one side of the inequality is **zero** and the other side is expressed as a **single algebraic fraction** with a completely factored numerator and denominator.

Example 2 ► **Solving Rational Inequalities Using Sign Analysis**

Solve each inequality using sign analysis.

a. $\frac{(x-2)x}{x+1} \geq 0$ **b.** $\frac{4-x}{x+2} \geq x$

Solution ► **a.** The right side of the inequality $\frac{(x-2)x}{x+1} > 0$ is zero, and the left side is a single fraction with both numerator and denominator in factored form. So, to solve this inequality, it is enough to analyse signs of the expression $\frac{(x-2)x}{x+1}$ at different intervals of the domain.

These intervals are determined by the **critical numbers** (the zeros of the numerator and denominator), which are -1, 0, and 2.

x	$-$		$-$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$x+1$	$-$		$+$		$+$		$+$
product	$-$	-1	$+$	0	$-$	2	$+$
solution							

As indicated in the above table of signs, the solution set to the inequality $\frac{(x-2)x}{x+1} \geq 0$ contains numbers between -1 and 0 and numbers higher than 2. In addition, since the inequality includes an equation, $x = 0$ and $x = 2$ are also solutions. However, $x = -1$ is not a solution because -1 does not belong to the domain of the expression $\frac{(x-2)x}{x+1}$ since it would make the denominator 0. So, the solution set is $(-\mathbf{1}, \mathbf{0}] \cup [\mathbf{2}, \infty)$.

Attention: **Solutions** to a rational inequality **must belong to the domain** of the inequality. This means that any number that makes the denominator 0 must be excluded from the solution set.

b. To solve $\frac{4-x}{x+2} \geq x$ by the sign analysis method, first, we would like to keep the right side equal to zero. So, we rearrange the inequality as below.

When working with inequalities, **avoid multiplying by the denominator** as it can be positive or negative for different x-values!

$$\frac{4-x}{x+2} \geq x \qquad / -x$$

$$\frac{4-x}{x+2} - x \geq 0$$

$$\frac{4-x-x(x+2)}{x+2} \geq 0$$

$$\frac{4-x-x^2-2x}{x+2} \geq 0$$

$$\frac{-x^2-3x+4}{x+2} \geq 0$$

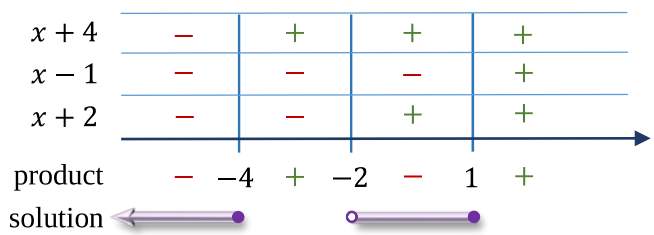
$$\frac{-(x^2+3x-4)}{x+2} \geq 0$$

$$\frac{-(x+4)(x-1)}{x+2} \geq 0 \qquad / \cdot(-1)$$

$$\frac{(x+4)(x-1)}{x+2} \leq 0$$

When multiplying by a **negative** number, remember to **reverse** the inequality sign!

Now, we can analyse the signs of the expression $\frac{(x+4)(x-1)}{x+2}$, using the table of signs with the critical numbers -4, -2, and 1.



So, the solution set for the inequality $\frac{(x+4)(x-1)}{x+2} \le 0$, which is equivalent to $\frac{4-x}{x+2} \ge x$, contains numbers lower than -4 and numbers between -2 and 1. Since the inequality includes an equation, $x = -4$ and $x = 1$ are also solutions. However, $x = -2$ is not in the domain of $\frac{(x+4)(x-1)}{x+2}$, and therefore it is not a solution.

Thus, the solution set to the original inequality is $(-\infty, -\mathbf{4}] \cup (-\mathbf{2}, \mathbf{1}]$.

Summary of Solving Polynomial or Rational Inequalities

1. **Write** the inequality so that **one of its sides is zero** and the other side is expressed as the **product or quotient of prime polynomials**.
2. **Determine the critical numbers**, which are the roots of all the prime polynomials appearing in the inequality.
3. **Divide the number line into intervals** formed by the set of critical numbers.
4. **Create a table of signs** for all prime factors in all intervals formed by the set of critical numbers. This can be done by analysing the sign of each factor, or by testing a number from each interval.
5. **Determine the sign of the overall expression** in each of the intervals.
6. **Graph the intervals of numbers that satisfy the inequality**. Make sure to **exclude endpoints that are not in the domain** of the inequality.
7. **State the solution set** to the original inequality **in interval notation**.

Solving Special Cases of Polynomial or Rational Inequalities

Some inequalities can be solved without the use of a graph or a table of signs.

Example 3 ► **Solving Special Cases of Inequalities**

Solve each inequality.

a. $(3x+2)^2 > -1$ **b.** $\frac{(x-4)^2}{x^2} \le 0$

Solution ► **a.** First, notice that the left side of the inequality $(3x+2)^2 > -1$ is a perfect square and as such, it assumes a nonnegative value for any input x. Since a nonnegative quantity is always bigger than -1, the inequality is satisfied by any real number x. So, the solution set is $\mathbb{R}$.

Note: The solution set of an inequality that is **always true** is the set of all real numbers, $\mathbb{R}$. For example, inequalities that take one of the following forms

$$\boldsymbol{nonnegative} > \boldsymbol{negative}$$

$$\boldsymbol{positive} \geq \boldsymbol{negative}$$

$$\boldsymbol{positive} > \boldsymbol{nonpositive}$$

$$\boldsymbol{positive} > \mathbf{0}$$

$$\boldsymbol{negative} < \mathbf{0}$$

are **always true**. So their solution set is $\mathbb{R}$.

b. Since the left side of the inequality $\frac{(x-4)^2}{x^2} \leq 0$ is a perfect square, it is bigger or equal to zero for all x-values. So, we have

$$0 \leq \frac{(x-4)^2}{x^2} \leq 0,$$

which can be true only if $\frac{(x-4)^2}{x^2} = 0$. Since a fraction equals to zero only when its numerator equals to zero, the solution to the last equation is $x = 4$. Thus, the solution set for the original inequality is $\{\mathbf{4}\}$.

Observation: Notice that the inequality $\frac{(x-4)^2}{x^2} < 0$ has no solution as a perfect square is never negative.

Note: The solution set of an inequality that is **never true** is the empty set, $\emptyset$. For example, inequalities that take one of the following forms

$$\boldsymbol{positive}\ (\boldsymbol{or\ nonnegative}) \leq \boldsymbol{negative}$$

$$\boldsymbol{nonnegative} < \boldsymbol{nonpositive}$$

$$\boldsymbol{positive} \leq \boldsymbol{nonpositive}$$

$$\boldsymbol{positive} \leq \mathbf{0} \ \text{ or } \ \boldsymbol{negative} \geq \mathbf{0}$$

$$\boldsymbol{nonnegative} < \mathbf{0} \ \text{ or } \ \boldsymbol{nonpositive} > \mathbf{0}$$

are **never true**. So, their solution sets are $\emptyset$.

Polynomial and Rational Inequalities in Application Problems

Some application problems involve solving polynomial or rational inequalities.

Example 4 ► **Finding the Range of Values Satisfying the Given Condition**

The manager of a shoe store observed that the weekly revenue, R, for selling rain boots at p dollars per pair can be modelled by the function $R(p) = 170p - 2p^2$. For what range of prices p will the weekly revenue be at least \$3000?

Solution ► Since the revenue must be at least \$3000, we can set up the inequality

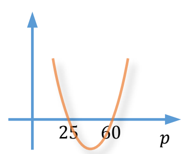
$$R(p) = 170p - 2p^2 \geq 3000,$$

and solve it for p. So, we have

$$-2p^2 + 170p - 3000 \geq 0 \qquad / \cdot (-2)$$

$$p^2 - 85p + 1500 \leq 0$$

$$(p - 25)(p - 60) \leq 0$$



Since the left-hand side expression represents a directed upwards parabola with roots at $p = 25$ and $p = 60$, its graph looks like in the accompanying figure. The graph extends below the p-axis for p-values between 25 and 60. So, to generate weekly revenue of at least \$3000, the price p of a pair of rain boots must take a value within the interval $[\mathbf{25}\$, \mathbf{60}\$]$.

Q.5 Exercises

True or False.

1. To determine if the value of an expression is greater than or less than 0 in a given interval, a test number can be used.

2. If the solution to the inequality $P(x) \geq 0$, where $P(x)$ is a polynomial with real coefficients, is $[2, 5)$, then the solution to the inequality $P(x) < 0$ is $(-\infty, 2) \cup [5, \infty)$.

3. The inequalities $(x - 1)(x + 3) \leq 0$ and $\frac{(x-1)}{(x+3)} \leq 0$ have the same solutions.

4. The solution set of the inequality $(x - 1)^2 > 0$ is the set of all real numbers.

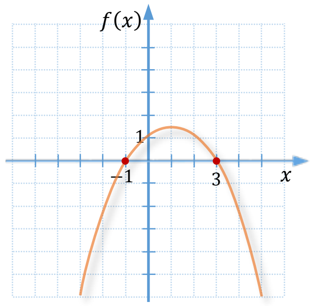
5. The inequality $x^2 + 1 \leq 0$ has no solution.

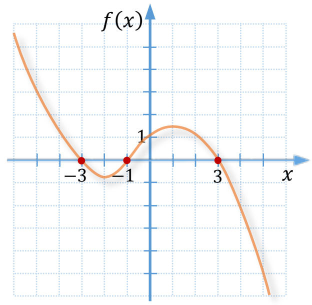
6. The solution set of the inequality $\frac{(x-1)^2}{(x+1)^2} \geq 0$ is the set of all real numbers.

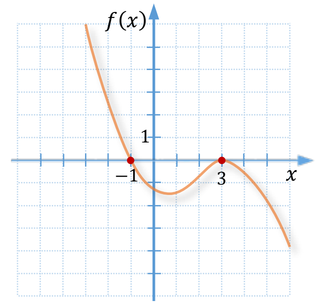
Given the graph of function f, state the solution set for each inequality

a. $f(x) \geq 0$

b. $f(x) < 0$

7. 

8. 

9. 

Solve *each inequality by sketching an approximate graph for the related equation.*

10. $(x+4)(x-2) > 0$

11. $(x+1)(x-2) < 0$

12. $x^2 - 4x + 3 \geq 0$

13. $\frac{1}{2}(x^2 - 3x - 10) \leq 0$

14. $4 - 9x^2 > 0$

15. $-x^2 - 2x < 0$

Solve *each inequality using* ***sign analysis****.*

16. $(x-3)(x+2) > 0$

17. $(x+4)(x-5) < 0$

18. $x^2 + 2x - 7 \leq 8$

19. $x^2 - x - 2 \geq 10$

20. $3x^2 + 10x > 8$

21. $2x^2 + 5x < -2$

22. $x^2 + 9 > -6x$

23. $x^2 + 4 \leq 4x$

24. $6 + x - x^2 \leq 0$

25. $20 - x - x^2 < 0$

26. $(x-1)(x+2)(x-3) \geq 0$

27. $(x+3)(x-2)(x-5) \leq 0$

28. $x(x+3)(2x-1) > 0$

29. $x^2(x-2)(2x-1) \geq 0$

30. $x^4 - 13x^2 + 36 \leq 0$

Solve each inequality using sign analysis.

31. $\frac{x}{x+1} > 0$

32. $\frac{x+1}{x-2} < 0$

33. $\frac{2x-1}{x+3} \leq 0$

34. $\frac{2x-3}{x+1} \geq 0$

35. $\frac{3}{y+5} > 1$

36. $\frac{5}{t-1} \leq 2$

37. $\frac{x-1}{x+2} \leq 3$

38. $\frac{a+4}{a+3} \geq 2$

39. $\frac{2t-3}{t+3} < 4$

40. $\frac{3y+9}{2y-3} < 3$

41. $\frac{1-2x}{2x+5} \leq 2$

42. $\frac{2x+3}{1-x} \leq 1$

43. $\frac{4x}{2x-1} \leq x$

44. $\frac{-x}{x+2} > 2x$

45. $\frac{2x-3}{(x+1)^2} \leq 0$

46. $\frac{2x-3}{(x-2)^2} \geq 0$

47. $\frac{x^2+1}{5-x^2} > 0$

48. $x < \frac{3x-8}{5-x}$

49. $\frac{1}{x+2} \geq \frac{1}{x-3}$

50. $\frac{2}{x+3} \leq \frac{1}{x-1}$

51. $\frac{(x-3)(x+1)}{4-x} \geq 0$

52. $\frac{(x+2)(x-1)}{(x+4)^2} \geq 0$

53. $\frac{x^2-2x-8}{x^2+10x+25} > 0$

54. $\frac{x^2-4x}{x^2-x-6} \leq 0$

Solve each inequality.

55. $(4-3x)^2 \geq -2$

56. $(5+2x)^2 < -1$

57. $\frac{(1-2x)^2}{2x^4} \leq 0$

58. $\frac{(1-2x)^2}{(x+2)^2} > -3$

59. $\frac{-2x^2}{(x+2)^2} \geq 0$

60. $\frac{-x^2}{(x-3)^2} < 0$

Solve each problem.

61. Sonia tossed a dice upwards with an initial velocity of 4 m/sec. Suppose the height, h, in meters, of the dice above the ground t seconds after it was tossed is modelled by the function $h(t) = -4.9t^2 + 12t + 1$. If the dice landed on a 0.7 meters high table, estimate the interval of time during which the dice was above the table? *Round the answer to two decimal places.*

62. A company producing furniture observes that the weekly cost, C, for producing n accent glass tables can be modelled by the function $C(x) = 2n^2 - 60n + 800$. How many of these tables should the company produce to decrease the weekly cost below \$400?

63. Anna wishes to create a rectangular flower bed and install a rubber edge around its perimeter. She bought 36 meters of edging. If she intends to use all the edging, how long could be the flower bed enclosure so that its area is at least 72 m^2?

64. Suppose that when a company producing furniture sells n chairs, the average cost per chair, C, in dollars, is modelled by the function $C(n) = \frac{450+2x}{x}$. For what number of chairs n will the average cost per chair be less than \$12?

65. A software developing company has a revenue of \$22 million this year. Suppose R, in millions of dollars, is the company's last year revenue. The company's percent revenue growth, P, in percent, is given by the function $P(R) = \frac{2200-100R}{R}$. For what revenues, R, would the company's revenue grow by more than 10%?

Attributions

Trigonometry

Trigonometry is the branch of mathematics that studies the relations between the sides and angles of triangles. The word "**trigonometry**" comes from the Greek **trigōnon** (triangle) and **metron** (measure.) It was first studied by the Babylonians, Greeks, and Egyptians, and used in surveying, navigation, and astronomy. Trigonometry is a powerful tool that allows us to find the measures of angles and sides of triangles, without physically measuring them, and areas of plots of land. We begin our study of trigonometry by studying angles and their degree measures.

T1 Angles and Degree Measure

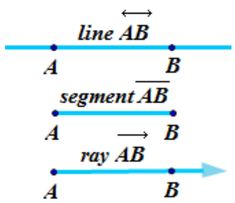


Figure 1a

Two distinct points $\boldsymbol{A}$ and $\boldsymbol{B}$ determine a line denoted $\overleftrightarrow{\boldsymbol{AB}}$. The portion of the line between $\boldsymbol{A}$ and $\boldsymbol{B}$, including the points $\boldsymbol{A}$ and $\boldsymbol{B}$, is called a **line segment** (or simply, a **segment**) $\overline{\boldsymbol{AB}}$. The portion of the line $\overleftrightarrow{\boldsymbol{AB}}$ that starts at $\boldsymbol{A}$ and continues past $\boldsymbol{B}$ is called the **ray** $\overrightarrow{\boldsymbol{AB}}$ (see *Figure 1a*.) Point $\boldsymbol{A}$ is the **endpoint** of this ray.

Two rays $\overrightarrow{\boldsymbol{AB}}$ and $\overrightarrow{\boldsymbol{AC}}$ sharing the same endpoint $\boldsymbol{A}$, cut the plane into two separate regions. The union of the two rays and one of those regions is called an **angle**, the common endpoint $\boldsymbol{A}$ is called a **vertex**, and the two **rays** are called **sides** or **arms** of this angle. Customarily, we draw a small arc connecting the two rays to indicate which of the two regions we have in mind.

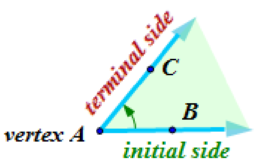
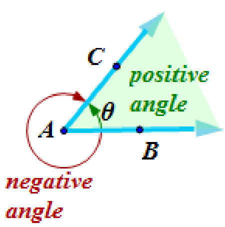


Figure 1b

In trigonometry, an **angle** is often identified with its **measure**, which is the **amount of rotation** that a ray in its initial position (called the **initial side**) needs to turn about the vertex to come to its final position (called the **terminal side**), as in *Figure 1b*. If the rotation from the initial side to the terminal side is *counterclockwise*, the angle is considered to be *positive*. If the rotation is *clockwise*, the angle is *negative* (see *Figure 1c*).



An angle is named either after its vertex, its rays, or the amount of rotation between the two rays. For example, an angle can be denoted $\angle\boldsymbol{A}$, $\angle\boldsymbol{BAC}$, or $\angle\boldsymbol{\theta}$, where the sign $\angle$ (or $\measuredangle$) simply means *an angle*. Notice that in the case of naming an angle with the use of more than one letter, like $\angle\boldsymbol{BAC}$, the middle letter ($\boldsymbol{A}$) is associated with the vertex and the angle is oriented from the ray containing the first point ($\boldsymbol{B}$) to the ray containing the third point ($\boldsymbol{C}$). Customarily, angles (often identified with their measures) are denoted by Greek letters such as $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\theta}$, etc.

Figure 1c

An angle formed by rotating a ray counterclockwise (in short, **ccw**) exactly one **complete revolution** around its vertex is defined to have a measure of 360 degrees, which is abbreviated as **360°**.

Definition 1.1 ► One **degree (1°)** is the measure of an angle that is $\frac{1}{360}$ part of a complete revolution.

One **minute** $(\mathbf{1'})$, is the measure of an angle that is $\frac{1}{60}$ part of a degree.

One **second** $(\mathbf{1''})$ is the measure of an angle that is $\frac{1}{60}$ part of a minute.

Therefore $\mathbf{1° = 60'}$ and $\mathbf{1' = 60''}$.

A fractional part of a degree can be expressed in decimals (e.g. 29.68°) or in minutes and seconds (e.g. 29°40'48"). We say that the first angle is given in **decimal form,** while the second angle is given in **DMS** (**D**egree, **M**inute, **S**econd) **form**.

Example 1 ► **Converting Between Decimal and DMS Form**

Convert as indicated.

a. $29.68°$ to DMS form

b. $46°18'21''$ to decimal degree form

Solution ► **a.** $29.68°$ can be converted to DMS form, using any calculator with **DMS** or **° ′ ″** key. To do it by hand, separate the fractional part of a degree and use the conversion factor $1° = 60'$.

$$\mathbf{29.68°} = 29° + 0.68°$$
$$= 29° + 0.68 \cdot 60' = 29° + 40.8'$$

Similarly, to convert the fractional part of a minute to seconds, separate it and use the conversion factor $1' = 60''$. So we have

$$29.68° = 29° + 40' + 0.8 \cdot 60'' = \mathbf{29°40'48''}$$

b. Similarly, $46°18'21''$ can be converted to the decimal form, using the **DMS** or **° ′ ″** key. To do it by hand, we use the conversions $1' = \left(\frac{1}{60}\right)^{°}$ and $1'' = \left(\frac{1}{3600}\right)^{°}$.

$$\mathbf{46°18'21''} = \left[46 + 18 \cdot \frac{1}{60} + 21 \cdot \frac{1}{3600}\right]^{°} \cong \mathbf{46.3058°}$$

Example 2 ► **Adding and Subtracting Angles in DMS Form**

Perform the indicated operations.

a. $36°58'21'' + 5°06'45''$ **b.** $36°17' - 15°46'15''$

Solution ► **a.** First, we add degrees, minutes, and seconds separately. Then, we convert each $60''$ into $1'$ and each $60'$ into $1°$. Finally, we add the degrees, minutes, and seconds again.

$$36°58'21'' + 5°06'45'' = 41° + 64' + 66''$$
$$= 41° + 1°04' + 1'06'' = \mathbf{42°05'06''}$$

b. We can subtract within each denomination, degrees, minutes, and seconds, even if the answer is negative. Then, if we need more minutes or seconds to perform the remaining subtraction, we convert $1°$ into $60'$ or $1'$ into $60''$ to finish the calculation.

$$36°17' - 15°46'15'' = 21° - 29' - 15''$$
$$= 20° + 60' - 29' - 15'' = 20° + 31' - 15''$$
$$= 20° + 30' + 60'' - 15'' = \mathbf{20°30'45''}$$

Angles in Standard Position

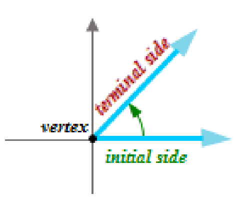


Figure 2

In trigonometry, we often work with angles in **standard position**, which means angles located in a rectangular system of coordinates with the vertex at the origin and the initial

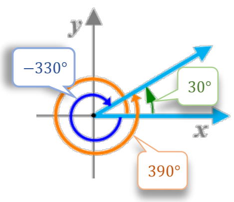


Figure 3

side on the positive x-axis, as in *Figure 2*. With the notion of angle as an amount of rotation of a ray to move from the initial side to the terminal side of an angle, the standard position allows us to represent infinitely many angles with the same terminal side. Those are the angles produced by rotating a ray from the initial side by full revolutions beyond the terminal side, either in a positive or negative direction. Such angles share the same initial and terminal sides and are referred to as **coterminal** angles.

For example, angles $-330°, 30°, 390°, 750°$, and so on, are coterminal.

Definition 1.2 ► Angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are **coterminal**, if and only if there is an integer $\boldsymbol{k}$, such that

$$\boldsymbol{\alpha = \beta + k \cdot 360°}$$

Example 3 ► **Finding Coterminal Angles**

Find one positive and one negative angle that is closest to $0°$ and coterminal with

a. $80°$

b. $-530°$

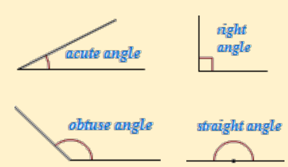
Solution ►

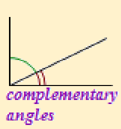
a. To find the closest to $0°$ positive angle coterminal with $80°$ we add one complete revolution, so we have $80° + 360° = \mathbf{440°}$.
Similarly, to find the closest to $0°$ negative angle coterminal with $80°$ we subtract one complete revolution, so we have $80° - 360° = \mathbf{-280°}$.

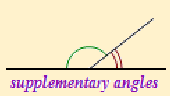
b. This time, to find the closest to $0°$ positive angle coterminal with $-530°$ we need to add two complete revolutions: $-530° + 2 \cdot 360° = \mathbf{190°}$.
To find the closest to $0°$ negative angle coterminal with $-530°$, it is enough to add one revolution: $-530° + 360° = \mathbf{-170°}$.

Definition 1.3 ► Let $\boldsymbol{\alpha}$ be the measure of an angle. Such an angle is called

acute, if $\boldsymbol{\alpha} \in (\mathbf{0°}, \mathbf{90°})$;
right, if $\boldsymbol{\alpha} = \mathbf{90°}$; (right angle is marked by the symbol ∟)
obtuse, if $\boldsymbol{\alpha} \in (\mathbf{90°}, \mathbf{180°})$; and
straight, if $\boldsymbol{\alpha} = \mathbf{180°}$.







Angles that sum to $\mathbf{90°}$ are called **complementary**.
Angles that sum to $\mathbf{180°}$ are called **supplementary**.

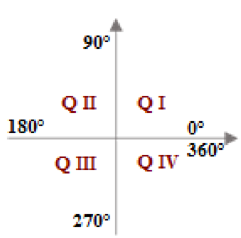


Figure 4

The two axes divide the plane into 4 regions, called **quadrants**. They are numbered counterclockwise, starting with the top right one, as in *Figure 4*.
An angle in standard position is said to lie in the quadrant in which its terminal side lies.
For example, an **acute** angle is in *quadrant I* and an **obtuse** angle is in *quadrant II*.
Angles in standard position with their terminal sides along the x-axis or y-axis, such as $\mathbf{0°}, \mathbf{90°}, \mathbf{180°}, \mathbf{270°}$, and so on, are called **quadrantal angles**.

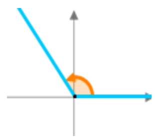
Example 4 ► **Classifying Angles by Quadrants**

Draw each angle in standard position. Determine the quadrant in which each angle lies or classify the angle as quadrantal.

a. $125°$ **b.** $-50°$ **c.** $270°$ **d.** $210°$

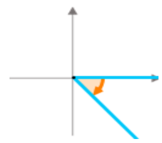
Solution ►

a.



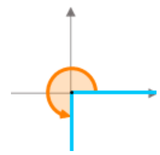
$125°$ is in **QII**

b.



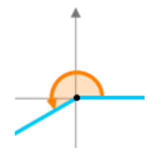
$-50°$ is in **QIV**

c.



quadrantal angle

d.



$210°$ is in **QIII**

Example 5 ► **Finding Complementary and Supplementary Angles**

Find the complement and the supplement of $57°$.

Solution ► Since complementary angles add to $90°$, the complement of $57°$ is $90° - 57° = 33°$.
Since supplementary angles add to $180°$, the supplement of $57°$ is $180° - 57° = 123°$.

T.1 Exercises

Convert each angle measure to ***decimal degrees****. Round the answer to the nearest thousandth of a degree.*

1. $20°04'30''$ **2.** $71°45'$ **3.** $274°18'15''$

4. $34°41'07''$ **5.** $15°10'05''$ **6.** $64°51'35''$

Convert each angle measure to ***degrees, minutes, and seconds****. Round the answer to the nearest second.*

7. $18.0125°$ **8.** $89.905°$ **9.** $65.0015°$

10. $184.3608°$ **11.** $175.3994°$ **12.** $102.3771°$

Perform each calculation.

13. $62°18' + 21°41'$ **14.** $71°58' + 47°29'$ **15.** $65°15' - 31°25'$

16. $90° - 51°28'$ **17.** $15°57'45'' + 12°05'18''$ **18.** $90° - 36°18'47''$

Give the complement and the supplement of each angle.

19. $30°$ **20.** $60°$ **21.** $45°$ **22.** $86.5°$ **23.** $15°30'$

24. Give an expression representing the complement of a $\boldsymbol{\theta}°$ angle.

25. Give an expression representing the supplement of a $\boldsymbol{\theta}°$ angle.

Sketch each angle in standard position. Draw an arrow representing the correct amount of rotation. Give the quadrant of each angle or identify it as a quadrantal angle.

26. $75°$ **27.** $135°$ **28.** $-60°$ **29.** $270°$ **30.** $390°$

31. $315°$ **32.** $510°$ **33.** $-120°$ **34.** $240°$ **35.** $-180°$

Find the angle of least positive measure coterminal with each angle.

36. $-30°$ **37.** $375°$ **38.** $-203°$ **39.** $855°$ **40.** $1020°$

Give an expression that generates all angles coterminal with the given angle. Use $\boldsymbol{k}$ *to represent any integer.*

41. $30°$ **42.** $45°$ **43.** $0°$ **44.** $90°$ **45.** $\alpha°$

Find the degree measure of the smaller angle formed by the hands of a clock at the following times.

46.

47. $3:15$

48. $1:45$

T2 Trigonometric Ratios of an Acute Angle and of Any Angle

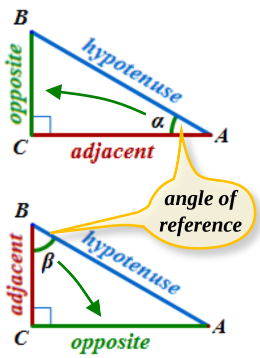


Figure 2.1

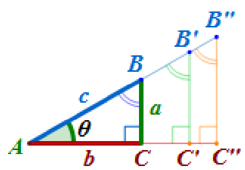


Figure 2.2

Generally, trigonometry studies ratios between sides in right angle triangles. When working with right triangles, it is convenient to refer to the side **opposite** to an angle, the side **adjacent** to (next to) an angle, and the **hypotenuse**, which is the longest side, opposite to the right angle. Notice that the opposite and adjacent sides depend on the **angle of reference** (one of the two acute angles.) However, the hypotenuse stays the same, regardless of the choice of the angle or reference. See *Figure 2.1*.

Notice that any two right triangles with the same acute angle $\boldsymbol{\theta}$ are **similar**. See *Figure 2.2*. **Similar** means that their corresponding angles are **congruent** and their corresponding sides are **proportional**. For instance, assuming notation as on *Figure 2.2*, we have

$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'},$$

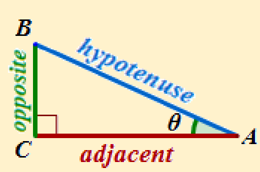
or equivalently

$$\frac{BC}{AB} = \frac{B'C'}{AB'}, \qquad \frac{AC}{AB} = \frac{AC'}{AB'}, \qquad \frac{BC}{AC} = \frac{B'C'}{AC'}.$$

Therefore, the ratios of any two sides of a right triangle does not depend on the size of the triangle but only on the size of the angle of reference. See the following demonstration. This means that we can study those **ratios** of sides as **functions** of an acute angle.

Trigonometric Functions of Acute Angles

Definition 2.1 ► Given a **right angle triangle** with an **acute** angle $\boldsymbol{\theta}$, the three **primary trigonometric ratios** of the angle $\boldsymbol{\theta}$, called ***sine***, ***cosine***, and ***tangent*** (abbreviation: *sin*, *cos*, *tan*) are defined as follows:

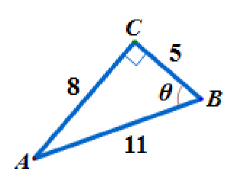


$$\sin\boldsymbol{\theta} = \frac{\boldsymbol{Opposite}}{\boldsymbol{Hypotenuse}}, \qquad \cos\boldsymbol{\theta} = \frac{\boldsymbol{Adjacent}}{\boldsymbol{Hypotenuse}}, \qquad \tan\boldsymbol{\theta} = \frac{\boldsymbol{Opposite}}{\boldsymbol{Adjacent}}$$

For easier memorization, we can use the acronym **SOH – CAH – TOA** (read: *so - ka - toe - ah*), formed from the first letter of the function and the corresponding ratio.

Example 1 ► **Identifying Sides of a Right Triangle to Form Trigonometric Ratios**

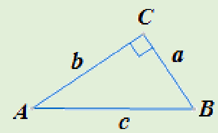
Identify the hypotenuse, opposite, and adjacent side of angle θ and state values of the three trigonometric ratios.



Solution ► Side AB is the hypotenuse, as it lies across the right angle.
Side BC is the adjacent, as it is part of the angle θ, other than hypotenuse.
Side AC is the opposite, as it lies across angle θ.
Therefore, $\sin\theta = \frac{opp.}{hyp.} = \frac{\mathbf{8}}{\mathbf{11}}$, $\cos\theta = \frac{adj.}{hyp.} = \frac{\mathbf{5}}{\mathbf{11}}$, and $\tan\theta = \frac{opp.}{adj.} = \frac{\mathbf{8}}{\mathbf{5}}$.

The three **primary trigonometric ratios** together with the **Pythagorean Theorem** allow us to **solve** any right angle triangle. That means that given the measurements of two sides, or one side and one angle, with a little help of algebra, we can find the measurements of all remaining sides and angles of any right triangle. See *Section T.4*.

Pythagorean Theorem ► A **triangle** $\boldsymbol{ABC}$ **is right** with $\angle C = 90°$ if and only if $\boldsymbol{a^2 + b^2 = c^2}$.

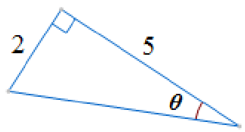


Convention: *The **side opposite** the given **vertex** (or angle) is named after the vertex, except that by a small rather than a capital letter. For example, the side opposite vertex **A** is called **a**.*

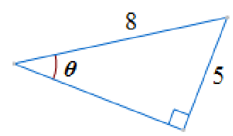
Example 2 ► **Finding Values of Trigonometric Ratios With the Aid of Pythagorean Theorem**

Given the triangle, find the exact values of the sine, cosine, and tangent ratios for angle θ.

a.



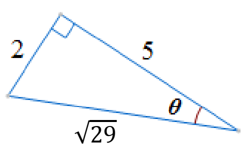
b.



Solution ► **a.** Let $\boldsymbol{h}$ denote the hypotenuse. By **Pythagorean Theorem**, we have

$$h^2 = 2^2 + 5^2$$
$$h = \sqrt{4 + 25} = \sqrt{29}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{\mathbf{2\sqrt{29}}}{\mathbf{29}}$$

$$\cos\theta = \frac{5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{\mathbf{5\sqrt{29}}}{\mathbf{29}}$$

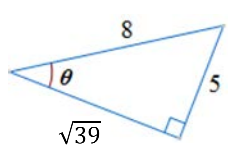
Note:
It is customary to rationalize the denominator.

$$\tan\theta = \frac{\mathbf{2}}{\mathbf{5}}$$

b. Let $\boldsymbol{a}$ denote the adjacent side. By the **Pythagorean Theorem**, we have

$$a^2 + 5^2 = 8^2$$
$$a = \sqrt{8^2 - 5^2} = \sqrt{64 - 25} = \sqrt{39}$$

Now, we are ready to state the exact values of the three trigonometric ratios:



$$\sin\theta = \frac{\mathbf{5}}{\mathbf{8}}$$

$$\cos\theta = \frac{\mathbf{\sqrt{39}}}{\mathbf{8}}$$

$$\tan\theta = \frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}} = \frac{\mathbf{5\sqrt{39}}}{\mathbf{39}}$$

Trigonometric Functions of Any Angle

Notice that any angle of a right triangle, other than the right angle, is acute. Thus, the "SOH – CAH – TOA" definition of the trigonometric ratios refers to acute angles only. However, we can extend this definition to include all angles. This can be done by observing our right triangle within the Cartesian Coordinate System.

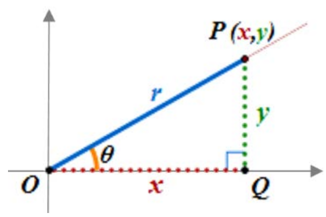


Figure 2.3

Let triangle OPQ with $\angle Q = 90°$ be placed in the coordinate system so that O coincides with the origin, Q lies on the positive part of the x-axis, and P lies in the first quadrant. See *Figure 2.3*. Let (x, y) be the coordinates of the point P, and let θ be the measurement of $\angle QOP$. This way, angle θ is in standard position and the triangle OPQ is obtained by **projecting** point P perpendicularly onto the x-axis. Thus in this setting, the position of point P actually determines both the angle θ and the ΔOPQ. Observe that the coordinates of point P ($\boldsymbol{x}$ and $\boldsymbol{y}$) really represent the length of the **adjacent** and the **opposite** side, correspondingly. Since the length of the **hypotenuse** represents the distance of the point P from the origin, it is often denoted by $\boldsymbol{r}$ (from *radius*.)

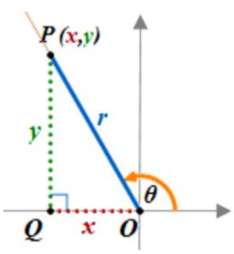


Figure 2.4

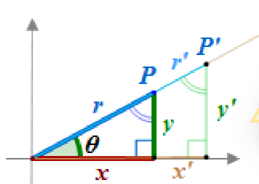
By rotating the radius $\boldsymbol{r}$ and projecting the point P perpendicularly onto x-axis (follow the green dotted line from P to Q in *Figure 2.4*), we can obtain a right triangle corresponding to any angle θ, not only an acute angle. Since the coordinates of a point in a plane can be negative, to establish a correcpondence between the coordinates x and y of the point P, and the distances OQ and QP, it is convenient to think of **directed distances** rather than just distances. Distance becomes directed if we assign a sign to it. So, lets assign a positive sign to horizontal or vertical distances that follow the directions of the corresponding number lines, and a negative sign otherwise. For example, the directed distance $OQ = x$ in *Figure 2.3* is positive because the direction from O to Q follows the order on the x-axis while the directed distance $OQ = x$ in *Figure 2.4* is negative because the direction from O to Q is against the order on the x-axis. Likewise, the directed distance $QP = y$ is positive for angles in the first and second quadrant (as in *Figure 2.3* and *2.4*), and it is negative for angles in the third and fourth quadrant (convince yourself by drawing a diagram).

Definition 2.2 ► Let $P(\boldsymbol{x}, \boldsymbol{y})$ be any point, different than the origin, on the terminal side of an angle $\boldsymbol{\theta}$ in standard position. Also, let $\boldsymbol{r} = \sqrt{\boldsymbol{x^2} + \boldsymbol{y^2}}$ be the distance of the point P from the origin. We define

$$\sin\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{r}}, \qquad \cos\boldsymbol{\theta} = \frac{\boldsymbol{x}}{\boldsymbol{r}}, \qquad \tan\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}} \ \ (\text{for } x \neq 0)$$

Observations:

- For acute angles, *Definition 2.2* agrees with the "**SOH – CAH – TOA**" *Definition 2.1*.



$$\frac{y}{r} = \frac{y'}{r'}$$
$$\frac{x}{r} = \frac{x'}{r'}$$
$$\frac{y}{x} = \frac{y'}{x'}$$

- Proportionality of similar triangles guarantees that each point of the same terminal ray defines the same trigonometric ratio. This means that the above definition assigns a unique value to each trigonometric ratio for any given angle regardless of the point chosen on the terminal side of this angle. Thus, the above trigonometric ratios are in fact **functions of any real angle** and these functions are properly defined in terms of x, y, and r.

- Since $r > 0$, the first two trigonometric functions, **sine** $\left(\frac{y}{r}\right)$ and **cosine** $\left(\frac{x}{r}\right)$, are defined for any real angle θ.

- The third trigonometric function, **tangent** $\left(\frac{y}{x}\right)$, is defined for all real angles θ except for angles with terminal sides on the y-axis. This is because the x-coordinate of any point on the y-axis equals zero, which cannot be used to create the ratio $\frac{y}{x}$. Thus, tangent is a function of all real angles, except for $90°, 270°$, and so on (generally, except for angles of the form $\boldsymbol{90° + k \cdot 180°}$, where $\boldsymbol{k}$ is an integer.)

- Notice that after dividing both sides of the Pythagorean equation $\boldsymbol{x^2 + y^2 = r^2}$ by $\boldsymbol{r^2}$, we have

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

Since $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$, we obtain the following **Pythagorean Identity**:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

- Also, observe that as long as $x \neq 0$, the quatient of the first two ratios gives us the third ratio:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{\cancel{r}} \cdot \frac{\cancel{r}}{x} = \frac{y}{x} = \tan\theta.$$

Thus, we have the identity

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

for all angles θ in the domain of the tangent.

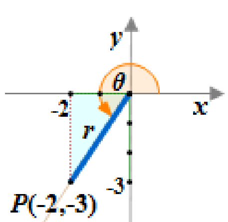
Example 3 ► **Evaluating Trigonometric Functions of any Angle in Standard Position**

Find the exact value of the three primary trigonometric functions of an angle θ in standard position whose terminal side contains the point

a. $P(-2, -3)$ **b.** $P(0,1)$

Solution ►

a. To ilustrate the situation, lets sketch the least positive angle θ in standard position with the point $P(-2, -3)$ on its terminal side.



To find values of the three trigonometric functions, first, we will determine the length of r:

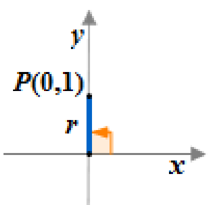
$$r = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

Now, we can state the exact values of the three trigonometric functions:

$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{13}} = \boldsymbol{\frac{-3\sqrt{13}}{13}}$$

$$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = \boldsymbol{\frac{-2\sqrt{13}}{13}}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-2} = \boldsymbol{\frac{3}{2}}$$



b. Since $x = 0,\ y = 1,\ r = \sqrt{0^2 + 1^2} = 1,$ then

$$\sin\theta = \frac{y}{r} = \frac{1}{1} = \mathbf{1}$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = \mathbf{0}$$

we can't divide by zero!

$$\tan\theta = \frac{y}{x} = \frac{1}{0} = \textbf{undefined}$$

Notice that the measure of the least positive angle θ in standard position with the point $P(0,1)$ on its terminal side is 90°. Therefore, we have

$$\sin 90° = 1, \quad \cos 90° = 0, \quad \tan 90° = \text{undefined}$$

The values of trigonometric functions of other commonly used quadrantal angles, such as $0°, 180°, 270°$, and $360°$, can be found similarly as in *Example 3b*. These values are summarized in the table below.

Table 2.1 Function Values of Quadrantal Angles

function \ θ =	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	−1	0
$\cos\theta$	1	0	−1	0	1
$\tan\theta$	0	undefined	0	undefined	0

Example 4 ► **Evaluating Trigonometric Functions Using Basic Identities**

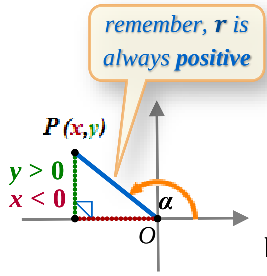
Knowing that $\cos\alpha = -\frac{3}{4}$ and the angle α is in quadrant II, find

a. $\sin\alpha$ **b.** $\tan\alpha$

Solution ► **a.** To find the value of $\sin\alpha$, we can use the Pythagorean Identity $\sin^2\alpha + \cos^2\alpha = 1$. After substituting $\cos\alpha = -\frac{3}{4}$, we have

$$\sin^2\alpha + \left(-\frac{3}{4}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{9}{16} = \frac{7}{16}$$



$$\sin\alpha = \pm\sqrt{\frac{7}{16}} = \pm\frac{\sqrt{7}}{4}$$

Since α is in in the second quadrant, $\sin\theta = \frac{y}{r}$ must be positive (as $y > 0$ in QII), so

$$\sin\alpha = \boldsymbol{\frac{\sqrt{7}}{4}}.$$

b. To find the value of $\tan\alpha$, since we already know the value of $\sin\alpha$, we can use the identity $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$. After substituting values $\sin\alpha = \frac{\sqrt{7}}{4}$ and $\cos\alpha = -\frac{3}{4}$, we obtain

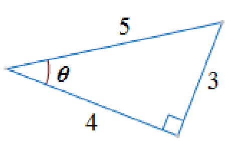
$$\tan\alpha = \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = \frac{\sqrt{7}}{4}\cdot\left(-\frac{4}{3}\right) = -\boldsymbol{\frac{\sqrt{7}}{3}}.$$

To confirm that the sign of $\tan\alpha = \frac{y}{x}$ in the second quadrant is indeed negative, observe that $y > 0$ and $x < 0$ in QII.

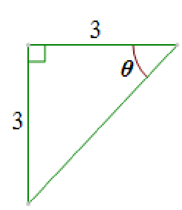
T.2 Exercises

Find the ***exact values*** *of the three trigonometric functions for the indicated angle θ. Rationalize denominators when applicable.*

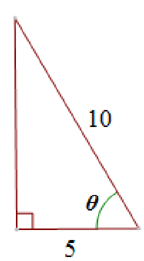
1.



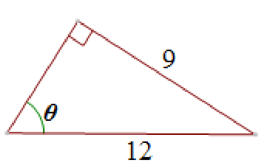
2.



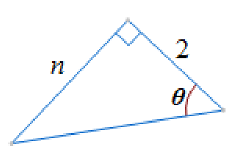
3.



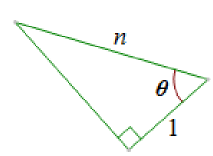
4.



5.



6.



Sketch an angle θ in standard position such that θ has the least positive measure, and the given point is on the terminal side of θ. Then find the values of the three trigonometric functions for each angle. Rationalize denominators when applicable.

7. $(-3,4)$ **8.** $(-4,-3)$ **9.** $(5,-12)$ **10.** $(0,3)$ **11.** $(-4,0)$

12. $(1,\sqrt{3})$ **13.** $(3,5)$ **14.** $(0,-8)$ **15.** $(-2\sqrt{3},-2)$ **16.** $(5,0)$

17. If the terminal side of an angle θ is in quadrant III, what is the sign of each of the trigonometric function values of θ?

Suppose that the point (x, y) is in the indicated quadrant. Decide whether the given ratio is ***positive*** *or* ***negative****.*

18. $Q\text{I},\ \frac{y}{x}$ **19.** $Q\text{II},\ \frac{y}{x}$ **20.** $Q\text{II},\ \frac{y}{r}$ **21.** $Q\text{III},\ \frac{x}{r}$ **22.** $Q\text{IV},\ \frac{y}{x}$

23. $Q\text{III},\ \frac{y}{x}$ **24.** $Q\text{IV},\ \frac{y}{r}$ **25.** $Q\text{I},\ \frac{y}{r}$ **26.** $Q\text{IV},\ \frac{x}{r}$ **27.** $Q\text{II},\ \frac{x}{r}$

Use the definition of trigonometric functions in terms of ***x****,* ***y****, and* ***r*** *to determine each value. If it is undefined, say so.*

28. $\sin 90°$ **29.** $\cos 0°$ **30.** $\tan 180°$ **31.** $\cos 180°$ **32.** $\tan 270°$

33. $\cos 270°$ **34.** $\sin 270°$ **35.** $\cos 90°$ **36.** $\sin 0°$ **37.** $\tan 90°$

Use basic identities to determine values of the remaining two trigonometric functions of the angle satisfying given conditions. Rationalize denominators when applicable.

38. $\sin\alpha = \frac{\sqrt{2}}{4};\ \alpha \in Q\text{II}$ **39.** $\sin\beta = -\frac{2}{3};\ \beta \in Q\text{III}$ **40.** $\cos\theta = \frac{2}{5};\ \theta \in Q\text{IV}$

T3 Evaluation of Trigonometric Functions

In the previous section, we defined sine, cosine, and tangent as functions of real angles. In this section, we will take interest in finding values of these functions for angles $\theta \in [0°,\ 360°)$. As shown before, one can find exact values of trigonometric functions of an angle θ with the aid of a right triangle with the acute angle θ and given side lengths, or by using coordinates of a given point on the terminal side of the angle θ in standard position. What if such data is not given? Then, one could consider approximating trigonometric function values by measuring sides of a right triangle with the desired angle θ and calculating corresponding ratios. However, this could easily prove to be a cumbersome process, with inaccurate results. Luckily, we can rely on calculators, which are programmed to return approximated values of the three primary trigonometric functions for any angle.

Attention: *In this section, any calculator instruction will refer to graphing calculators such as* ***TI-83*** *or* ***TI-84****.*

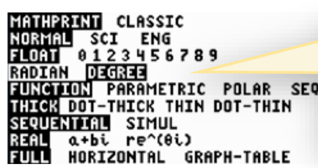
Example 1 ► **Evaluating Trigonometric Functions Using a Calculator**

Find each function value up to four decimal places.

a. $\sin 39°12'10"$ **b.** $\tan 102.6°$

Solution ►

a. Before entering the expression into the calculator, we need to check if the calculator is in degree mode by pressing the MODE key and highlighting the "degree" option, with the aid of arrows and the ENTER key.



When evaluating functions of angles in degrees, the calculator must be set to the **degree mode**.

To go back to the main screen, we press 2nd MODE, and now we can enter $\sin 39°12'10''$. The degree (°) and minute (′) signs can be found under the "ANGLE" list. To get the second (″) sign, we press ALPHA + .

Thus $\sin 39°12'10'' \approx \mathbf{0.6321}$ when rounded to four decimal places.

b. When evaluating trigonometric functions of angles in decimal degrees, it is not necessary to write the degree (°) sign when in degree mode. We simply key in TAN 102.6 ENTER to obtain $\tan 102.6° \approx \mathbf{-4.4737}$ when rounded to four decimal places.

Special Angles

It has already been discussed how to find the **exact values** of trigonometric functions of **quadrantal angles** using the definitions in terms of x, y, and r. *See Section T.2, Example 3.b, and Table 2.1.*

Are there any other angles for which the trigonometric functions can be evaluated exactly? Yes, we can find the exact values of trigonometric functions of any angle that can be modelled by a right triangle with known sides. For example, angles such as $30°, 45°$, or $60°$ can be modeled by half of a square or half of an equilateral triangle. In each triangle, the relations between the lengths of sides are easy to establish.

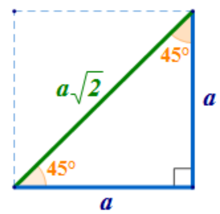


Figure 3.1

In the case of half a square *(see Figure 3.1)*, we obtain a right triangle with two acute angles of **45°**, and two equal sides of certain length $\boldsymbol{a}$.

Hence, by The Pythagorean Theorem, the diagonal $d = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \boldsymbol{a\sqrt{2}}$.

Summary: The sides of any $\mathbf{45° - 45° - 90°}$ triangle are in the relation $\boldsymbol{a - a - a\sqrt{2}}$.

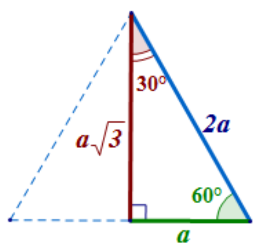


Figure 3.2

By dividing an equilateral triangle *(see Figure 3.1)* along its height, we obtain a right triangle with acute angles of **30°** and **60°**. If the length of the side of the original triangle is denoted by $\boldsymbol{2a}$, then the length of half a side is $\boldsymbol{a}$, and the length of the height can be calculated by applying The Pythagorean Theorem, $h = \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = \boldsymbol{a\sqrt{3}}$.

Summary: The sides of any $\mathbf{30° - 60° - 90°}$ triangle are in the relation $\boldsymbol{a - 2a - a\sqrt{3}}$.

Since the trigonometric ratios do not depend on the size of a triangle, for simplicity, we can assume that $a = 1$ and work with the following **special triangles**:

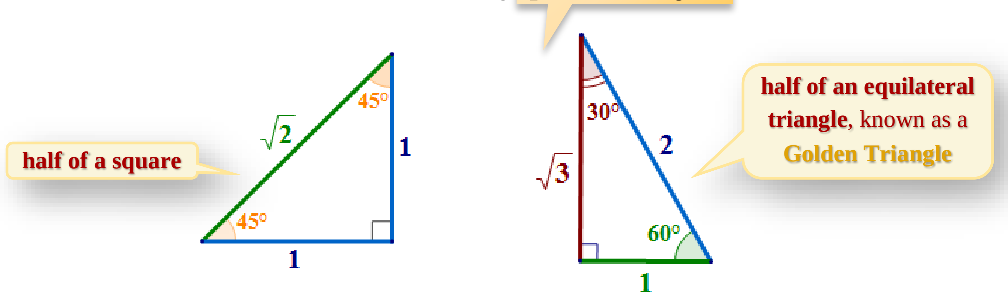


Figure 3.3

Special angles such as $\mathbf{30°}, \mathbf{45°}$, and $\mathbf{60°}$ are frequently seen in applications. We will often refer to the exact values of trigonometric functions of these angles. Special triangles give us a tool for finding those values.

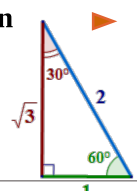
Advice: *Make sure that you can* ***recreate the special triangles*** *by taking half of a square or half of an equilateral triangle, anytime you wish to* ***recall the relations*** *between their sides.*

Example 2 ► **Finding Exact Values of Trigonometric Functions of Special Angles**

Find the *exact* value of each expression.

a. $\cos 60°$ **b.** $\tan 30°$ **c.** $\sin 45°$ **d.** $\tan 45°$

Solution ►



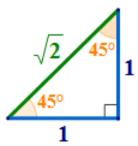
a. Refer to the $30° - 60° - 90°$ triangle and follow the SOH-CAH-TOA definition of sine:

$$\cos 60° = \frac{adj.}{hyp.} = \frac{1}{2}$$

b. Refer to the same triangle as above:

$$\tan 30° = \frac{opp.}{adj.} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

c. Refer to the $45° - 45° - 90°$ triangle:



$$\sin 45° = \frac{opp.}{hyp.} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

d. Refer to the $45° - 45° - 90°$ triangle:

$$\tan 45° = \frac{opp.}{adj.} = \frac{1}{1} = 1$$

The exact values of trigonometric functions of special angles are summarized in the table below.

Table 3.1 Function Values of Special Angles

function \ $\theta =$	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Observations:

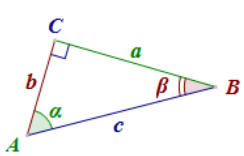
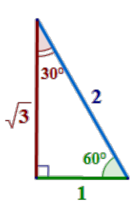


Figure 3.4

- Notice that $\sin 30° = \cos 60°$, $\sin 60° = \cos 30°$, and $\sin 45° = \cos 45°$. Is there any general rule to explain this fact? Lets look at a right triangle with acute angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (*see Figure 3.4*). Since the sum of angles in any triangle is $180°$ and $\angle C = 90°$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} = 90°$, therefore they are **complementary angles**. From the definition, we have $\sin\boldsymbol{\alpha} = \frac{a}{b} = \cos\boldsymbol{\beta}$. Since angle $\boldsymbol{\alpha}$ was chosen arbitrarily, this rule applies to any pair of acute complementary angles. It happens that this rule actually applies to all complementary angles. So we have the following claim:

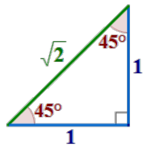
$\sin\boldsymbol{\alpha} = \cos(90° - \boldsymbol{\alpha})$

The **cofunctions** (like sine and cosine) of **complementary** angles are equal.



- Notice that $\tan 30° = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = \frac{1}{\tan 60°}$, or equivalently, $\tan 30° \cdot \tan 60° = 1$. This is because of the previously observed rules:

$$\tan\theta \cdot \tan(90° - \theta) = \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} = 1$$



In general, we have:

$$\tan\theta = \frac{1}{\tan(90° - \theta)}$$

- Observe that $\tan 45° = 1$. This is an easy, but very useful value to memorize.

Example 3 ► **Using the Cofunction Relationship**

Rewrite $\cos 75°$ in terms of the cofunction of the complementary angle.

Solution ► Since the complement of $75°$ is $90° - 75° = 15°$, then $\cos 75° = \mathbf{\sin 15°}$.

Reference Angles

Can we determine exact values of trigonometric functions of nonquadrantal angles that are larger than $90°$?

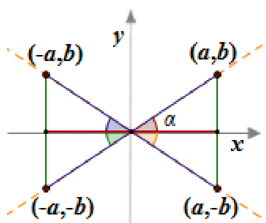
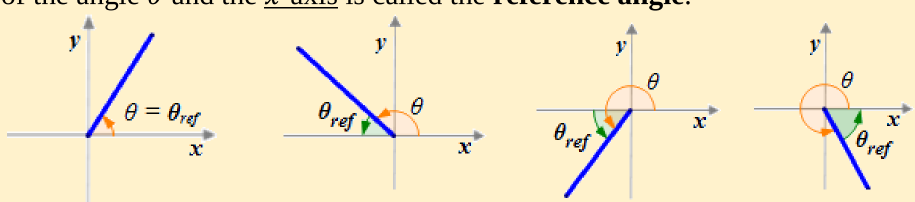


Figure 3.5

Assume that point (a, b) lies on the terminal side of acute angle α. By *Definition 2.2*, the values of trigonometric functions of angles with terminals containing points $(-a, b)$, $(-a, -b)$, and $(a, -b)$ are the same as the values of corresponding functions of the angle α, except for their signs.

Therefore, to find the value of a trigonometric function of any angle θ, it is enough to evaluate this function at the corresponding acute angle θ_{ref}, called the **reference angle**, and apply the sign appropriate to the quadrant of the terminal side of θ.

Definition 3.1 ► Let θ be an angle in standard position. The acute angle $\boldsymbol{\theta_{ref}}$ formed by the terminal side of the angle θ and the x-axis is called the **reference angle**.



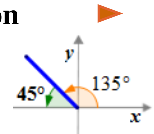
Attention: *Think of a* ***reference angle*** *as the smallest rotation of the terminal arm required to line it up with the* ***x-axis****.*

Example 4 ► **Finding the Reference Angle**

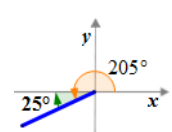
Find the **reference angle** for each of the given angles.

a. $40°$ **b.** $135°$ **c.** $210°$ **d.** $300°$

Solution ► **a.** Since $\mathbf{40°} \in QI$, this is already the reference angle.



b. Since $135° \in QII$, the reference angle equals $180° - 135° = \mathbf{45°}$.



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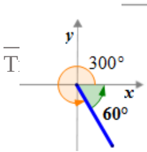
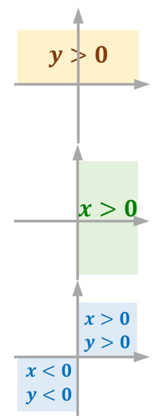


c. Since $205° \in QIII$, the reference angle equals $205° - 180° = \mathbf{25°}$.

d. Since $300° \in QIV$, the reference angle equals $360° - 300° = \mathbf{60°}$.

CAST Rule

Using the x, y, r definition of trigonometric functions, we can determine and summarize the signs of those functions in each of the quadrants.



Since $\sin\theta = \frac{y}{r}$ and r is positive, then the sign of the sine ratio is the same as the sign of the y-value. This means that the values of sine are positive only in quadrants where y is positive, thus in QI and QII.
Since $\cos\theta = \frac{x}{r}$ and r is positive, then the sign of the cosine ratio is the same as the sign of the x-value. This means that the values of cosine are positive only in quadrants where x is positive, thus in QI and QIV.
Since $\tan\theta = \frac{y}{x}$, then the values of the tangent ratio are positive only in quadrants where both x and y have the same signs, thus in QI and $QIII$.

Table 3.2 **Signs of Trigonometric Functions in Quadrants**

function \ $\theta \in$	QI	QII	$QIII$	QIV
$\sin\theta$	$+$	$+$	$-$	$-$
$\cos\theta$	$+$	$-$	$-$	$+$
$\tan\theta$	$+$	$-$	$+$	$-$

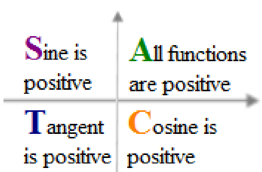


Figure 3.6

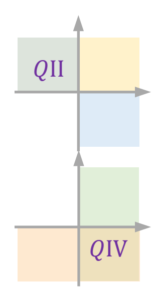
Since we will be making frequent decisions about signs of trigonometric function values, it is convenient to have an acronym helping us memorizing these signs in different quadrants. The first letters of the names of functions that are positive in particular quadrants, starting from the fourth quadrant and going counterclockwise, spells **CAST**, which is very helpful when working with trigonometric functions of any angles.

Example 5 ► **Identifying the Quadrant of an Angle**

Identify the quadrant or quadrants for each angle satisfying the given conditions.

a. $\sin\theta > 0;\ \tan\theta < 0$ **b.** $\cos\theta > 0;\ \sin\theta < 0$

Solution ►



a. Using **CAST**, we have $\sin\theta > 0$ in QI(**A**ll) and QII(**S**ine) and $\tan\theta < 0$ in QII and QIV. Therefore both conditions are met only in **quadrant II**.

b. $\cos\theta > 0$ in QI(**A**ll) and QIV(**C**osine) and $\sin\theta < 0$ in $QIII$ and QIV. Therefore both conditions are met only in **quadrant IV**.

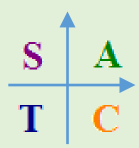
Example 6 ► **Identifying Signs of Trigonometric Functions of Any Angle**

Using the **CAST** rule, identify the sign of each function value.

a. $\cos 150°$ **b.** $\tan 225°$

Solution ► **a.** Since $150° \in Q\text{II}$ and cosine is negative in $Q\text{II}$, then $\cos 150°$ is **negative**.

b. Since $225° \in Q\text{III}$ and tangent is positive in $Q\text{III}$, then $\tan 225°$ is **positive**.



To find the exact value of a trigonometric function $\boldsymbol{T}$ of an angle $\boldsymbol{\theta}$ with the reference angle $\boldsymbol{\theta_{ref}}$ being a special angle, we follow the rule:

$$\boldsymbol{T(\theta) = \pm T(\theta_{ref})},$$

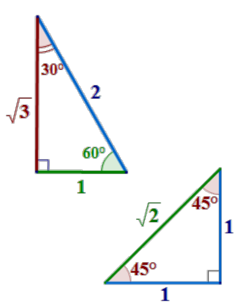
where the final sign is determined according to the quadrant of angle θ and the **CAST** rule.

Example 7 ► **Finding Exact Function Values Using Reference Angles**

Find the exact values of the following expressions.

a. $\sin 240°$ **b.** $\cos 315°$

Solution ►



a. The reference angle of $240°$ is $240° - 180° = 60°$. Since $240° \in Q\text{III}$ and sine in the third quadrant is **positive**, we have

$$\sin 240° = \sin 60° = \boldsymbol{\frac{\sqrt{3}}{2}}$$

b. The reference angle of $315°$ is $360° - 315° = 45°$. Since $315° \in Q\text{IV}$ and cosine in the fourth quadrant is **negative**, we have

$$\cos 315° = -\cos 45° = -\frac{1}{\sqrt{2}} = -\boldsymbol{\frac{\sqrt{2}}{2}}$$

Finding Special Angles in Various Quadrants when Given Trigonometric Function Value

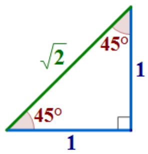
Now that it has been shown how to find exact values of trigonometric functions of angles that have a reference angle of one of the special angles ($\boldsymbol{30}°$, $\boldsymbol{45}°$, or $\boldsymbol{60}°$), we can work at reversing this process. Familiarity with values of trigonometric functions of the special angles, in combination with the idea of reference angles and quadrantal sign analysis, should help us in solving equations of the type $\boldsymbol{T(\theta) = exact\ value}$, where $\boldsymbol{T}$ represents any trigonometric function.

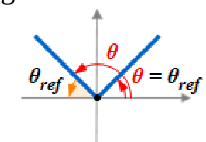
Example 8 ► **Finding Angles with a Given Exact Function Value, in Various Quadrants**

Find all angles θ satisfying the following conditions.

a. $\sin\theta = \frac{\sqrt{2}}{2}$; $\theta \in [0°, 180°)$ **b.** $\cos\theta = -\frac{1}{2}$; $\theta \in [0°, 360°)$

Solution ► **a.** Refering to the half of a square triangle, we recognize that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ represents the ratio of sine of $45°$. Thus, the reference angle $\theta_{ref} = 45°$. Moreover, we are searching for an angle θ from the interval $[0°, 180°)$ and we know that $\sin\theta > 0$. Therefore, θ must lie in the first or second quadrant and have the reference angle of $45°$. Each quadrant gives us one solution, as shown in the figure on the right.



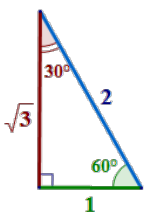


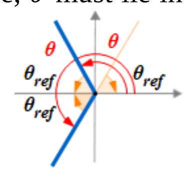
If θ is in the first quadrant, then $\theta = \theta_{ref} = 45°$.
If θ is in the second quadrant, then $\theta = 180° - 45° = 135°$.

So the solution set of the above problem is $\{\mathbf{45°}, \mathbf{135°}\}$.

here we can disregard the sign of the given value as we are interested in the reference angle only

b. Refering to the half of an equilateral triangle, we recognize that $\frac{1}{2}$ represents the ratio of cosine of $60°$. Thus, the reference angle $\theta_{ref} = 60°$. We are searching for an angle θ from the interval $[0°, 360°)$ and we know that $\cos\theta < 0$. Therefore, θ must lie in the second or third quadrant and have the reference angle of $60°$.





If θ is in the second quadrant, then $\theta = 180° - 60° = 120°$.
If θ is in the third quadrant, then $\theta = 180° + 60° = 240°$.

So the solution set of the above problem is $\{\mathbf{120°}, \mathbf{240°}\}$.

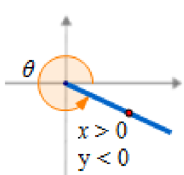
Finding Other Trigonometric Function Values

Example 9 ► **Finding Other Function Values Using a Known Value, Quadrant Analysis, and the x, y, r Definition of Trigonometric Ratios**

Find values of the remaining trigonometric functions of the angle satisfying the given conditions.

a. $\sin\theta = -\frac{7}{13}$; $\theta \in QIV$ **b.** $\tan\theta = \frac{15}{8}$; $\theta \in QIII$

Solution ► **a.** We know that $\sin\theta = -\frac{7}{13} = \frac{y}{r}$. Hence, the terminal side of angle $\theta \in QIV$ contains a point $P(x, y)$ satisfying the condition $\frac{y}{r} = -\frac{7}{13}$. Since r must be positive, we will assign $y = -7$ and $r = 13$, to model the situation. Using the Pythagorean equation and the fact that the x-coordinate of any point in the fourth quadrant is positive, we determine the corresponding x-value to be



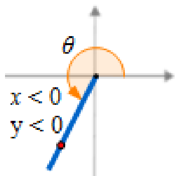
$$x = \sqrt{r^2 - y^2} = \sqrt{13^2 - (-7)^2} = \sqrt{169 - 49} = \sqrt{120} = 2\sqrt{30}.$$

Now, we are ready to state the remaining function values of angle θ:

$$\cos\theta = \frac{x}{r} = \frac{\mathbf{2\sqrt{30}}}{\mathbf{13}}$$

and

$$\tan\theta = \frac{y}{x} = \frac{-7}{2\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = \frac{\mathbf{-7\sqrt{30}}}{\mathbf{60}}.$$



b. We know that $\tan\theta = \frac{15}{8} = \frac{y}{x}$. Similarly as above, we would like to determine x, y, and r values that would model the situation. Since angle $\theta \in QIII$, both x and y values must be negative. So we assign $y = -15$ and $x = -8$. Therefore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Now, we are ready to state the remaining function values of angle θ:

$$\sin\theta = \frac{y}{r} = \boldsymbol{\frac{-15}{17}}$$

and

$$\cos\theta = \frac{x}{r} = \boldsymbol{\frac{-8}{17}}.$$

T.3 Exercises

*Use a calculator to **approximate** each value to **four** decimal places.*

1. $\sin 36°52'05''$ **2.** $\tan 57.125°$ **3.** $\cos 204°25'$

*Give the **exact** function value, **without** the aid of a calculator. Rationalize denominators when applicable.*

4. $\cos 30°$ **5.** $\sin 45°$ **6.** $\tan 60°$ **7.** $\sin 60°$

8. $\tan 30°$ **9.** $\cos 60°$ **10.** $\sin 30°$ **11.** $\tan 45°$

*Give the equivalent expression using the **cofunction** relationship.*

12. $\cos 50°$ **13.** $\sin 22.5°$ **14.** $\sin 10°$

*For each angle, find the **reference angle**.*

15. $98°$ **16.** $212°$ **17.** $13°$ **18.** $297°$ **19.** $186°$

*Identify the **quadrant** or **quadrants** for each angle satisfying the given conditions.*

20. $\cos\alpha > 0$ **21.** $\sin\beta < 0$ **22.** $\tan\gamma > 0$

23. $\sin\theta > 0; \cos\theta < 0$ **24.** $\cos\alpha < 0; \tan\alpha > 0$ **25.** $\sin\alpha < 0; \tan\alpha < 0$

*Identify the **sign** of each function value by quadrantal analysis.*

26. $\cos 74°$ **27.** $\sin 245°$ **28.** $\tan 129°$ **29.** $\sin 183°$

30. $\tan 298°$ **31.** $\cos 317°$ **32.** $\sin 285°$ **33.** $\tan 215°$

Using reference angles, quadrantal analysis, and special triangles, find the ***exact values*** *of the expressions. Rationalize denominators when applicable.*

34. $\cos 225°$ **35.** $\sin 120°$ **36.** $\tan 150°$ **37.** $\sin 150°$

38. $\tan 240°$ **39.** $\cos 210°$ **40.** $\sin 330°$ **41.** $\tan 225°$

Find all values of $\theta \in [0°, 360°)$ *satisfying the given condition.*

42. $\sin\theta = -\frac{1}{2}$ **43.** $\cos\theta = \frac{1}{2}$ **44.** $\tan\theta = -1$ **45.** $\sin\theta = \frac{\sqrt{3}}{2}$

46. $\tan\theta = \sqrt{3}$ **47.** $\cos\theta = -\frac{\sqrt{2}}{2}$ **48.** $\sin\theta = 0$ **49.** $\tan\theta = -\frac{\sqrt{3}}{3}$

Find values of the remaining trigonometric functions of the angle satisfying the given conditions.

50. $\sin\theta = \frac{\sqrt{5}}{7};\ \theta \in Q\text{II}$ **51.** $\cos\alpha = \frac{3}{5};\ \alpha \in Q\text{IV}$ **52.** $\tan\beta = \sqrt{3};\ \beta \in Q\text{III}$

T4 Applications of Right Angle Trigonometry

Solving Right Triangles

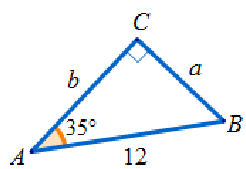
Geometry of right triangles has many applications in the real world. It is often used by carpenters, surveyors, engineers, navigators, scientists, astronomers, etc. Since many application problems can be modelled by a right triangle and trigonometric ratios allow us to find different parts of a right triangle, it is essential that we learn how to apply trigonometry to solve such triangles first.

Definition 4.1 ► To **solve a triangle** means to find the measures of all the unknown **sides** and **angles** of the triangle.

Example 1 ► **Solving a Right Triangle Given an Angle and a Side**

Given the information, solve triangle ABC, assuming that $\angle C = 90°$.

a.



b. $\angle B = 11.4°,\ b = 6\ cm$

Solution ► **a.** To find the length a, we want to relate it to the given length of 12 and the angle of 35°. Since a is opposite angle 35° and 12 is the length of the hypotenuse, we can use the ratio of sine:

$$\frac{a}{12} = \sin 35°$$

Then, after multiplying by 12, we have

$$a = 12 \sin 35° \simeq \mathbf{6.9}$$

round **lengths** to **one decimal place**

Attentin: To be more accurate, if possible, **use the given data** rather than the previously calculated ones, which are most likely already rounded.

Since we already have the value of a, the length b can be determined in two ways: by applying the Pythagorean Theorem, or by using the cosine ratio. For better accuracy, we will apply the cosine ratio:

$$\frac{b}{12} = \cos 35°$$

which gives

$$b = 12 \cos 35° \simeq \mathbf{9.8}$$

Finally, since the two acute angles are complementary, $\angle B = 90° - 35° = \mathbf{55}°$.

We have found the three missing measurements, $a \simeq 6.9$, $b \simeq 9.8$, and $\angle B = 55°$, so the triangle is solved.

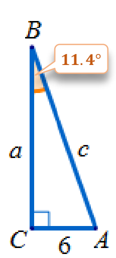


Figure 1

b. To visualize the situation, let's sketch a right triangle with $\angle B = 11.4°$ and $b = 6$ (see *Figure 1*). To find side a, we we would like to set up an equation that relates 6, a, and 11.4°. Since $b = 6$ is the opposite and a is the adjacent with respect to $\angle B = 11.4°$, we will use the ratio of tangent:

$$\tan 11.4° = \frac{6}{a}$$

To solve for a, we may want to multiply both sides of the equation by a and divide by $\tan 11.4°$. Observe that this will cause a and $\tan 11.4°$ to interchange (swap) their positions. So, we obtain

$$a = \frac{6}{\tan 11.4°} \simeq \mathbf{29.8}$$

To find side c, we will set up an equation that relates 6, c, and 11.4°. Since $b = 6$ is the opposite to $\angle B = 11.4°$ and c is the hypothenuse, the ratio of sine applies. So, we have

$$\sin 11.4° = \frac{6}{c}$$

Similarly as before, to solve for c, we can simply interchange the position of $\sin 11.4°$ and c to obtain

$$c = \frac{6}{\sin 11.4°} \simeq \mathbf{30.4}$$

Finally, $\angle A = 90° - 11.4° = \mathbf{78.6°}$, which completes the solution.

In summary, $\angle A = 78.6°$, $a \simeq 29.8$, and $c \simeq 30.4$.

Observation: Notice that after approximated length a was found, we could have used the Pythagorean Theoreom to find length c. However, this could decrease the accuracy of the result. For this reason, it is advised that we use the given rather than approximated data, if possible.

Finding an Angle Given a Trigonometric Function Value

So far we have been evaluating trigonometric functions for a given angle. Now, what if we wish to reverse this process and try to recover an angle that corresponds to a given trigonometric function value?

Example 2 ► **Finding an Angle Given a Trigonometric Function Value**

Find an angle θ, satisfying the given equation. *Round to one decimal place, if needed.*

a. $\sin\theta = 0.7508$ **b.** $\cos\theta = -0.5$

Solution ► **a.** Since 0.7508 is not a special value, we will not be able to find θ by relating the equation to a special triangle as we did in *Section T3, Example 8*. This time, we will need to rely on a calculator. To find θ, we want to "undo" the sine. The function that can "undo" the sine is called **arcsine**, or **inverse sine**, and it is often abbreviated by $\mathbf{sin^{-1}}$. By applying the $\sin^{-1}$ to both sides of the equation

$$\sin\theta = 0.7508,$$

we have

$$\cancel{\sin^{-1}}(\cancel{\sin}\,\theta) = \sin^{-1}(0.7508)$$

Since $\sin^{-1}$ "undoes" the sine function, we obtain

$$\theta = \sin^{-1} 0.7508 \simeq \mathbf{48.7}°$$

round **angles** to **one decimal place**

On most calculators, to find this value, we follow the sequence of keys:

2nd or INV or Shift , SIN , 0.7508, ENTER or =

b. In this example, the absolute value of cosine is a special value. This means that θ can be found by referring to the **golden triangle** properties and the **CAST** rule of signs as in *Section T3, Example 8b*. The other way of finding θ is via a calculator

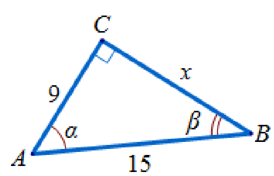
$$\theta = \cos^{-1}(-0.5) = \mathbf{120}°$$

Note: *Calculators are programed to return*
$\mathbf{sin^{-1}}$ *and* $\mathbf{tan^{-1}}$ *as angles from the interval* $[-\mathbf{90}°, \mathbf{90}°]$ *and*
$\mathbf{cos^{-1}}$ *as angles from the interval* $[\mathbf{0}°, \mathbf{180}°]$.
That implies that when looking for an obtuse angle, it is easier to work with $\mathbf{cos^{-1}}$*, if possible, as our calculator will return the actual angle. When using* $\mathbf{sin^{-1}}$ *or* $\mathbf{tan^{-1}}$*, we might need to search for a corresponding angle in the second quadrant on our own.*

More on Solving Right Triangles

Example 3 ► **Solving a Right Triangle Given Two Sides**

Solve the triangle.



Solution ► Since $\triangle ABC$ is a right triangle, to find the length x, we can use the Pythagorean Theorem.

$$x^2 + 9^2 = 15^2$$

so

$$\boldsymbol{x} = \sqrt{225 - 81} = \sqrt{144} = \mathbf{12}$$

To find the angle α, we can relate either $x = 12$, 9, and α, or 12, 15, and α. We will use the second triple and the ratio of sine. Thus, we have

$$\sin\alpha = \frac{12}{15},$$

therefore

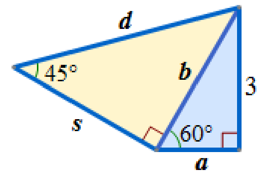
$$\boldsymbol{\alpha} = \sin^{-1}\frac{12}{15} \simeq \mathbf{53.1}°$$

Finally, $\boldsymbol{\beta} = 90° - \alpha \simeq 90° - 53.1° = \mathbf{36.9°}$.

In summary, $\alpha = 53.1°$, $\beta \simeq 36.9°$, and $x = 12$.

Example 4 ► **Using Relationships Between Sides of Special Triangles**

Find the **exact** value of each unknown in the figure.



Solution ► First, consider the blue right triangle. Since one of the acute angles is 60°, the other must be 30°. Thus the blue triangle represents half of an equilateral triangle with the side $\boldsymbol{b}$ and the height of 3 units. Using the relation $h = a\sqrt{3}$ between the height h and half a side a of an equilateral triangle, we obtain

$$\boldsymbol{a}\sqrt{3} = 3,$$

which gives us $\boldsymbol{a} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \boldsymbol{\sqrt{3}}$. Consequently, $\boldsymbol{b} = 2\boldsymbol{a} = \boldsymbol{2\sqrt{3}}$.

Now, considering the yellow right triangle, we observe that both acute angles are equal to 45° and therefore the triangle represents half of a square with the side $\boldsymbol{s} = \boldsymbol{b} = \boldsymbol{2\sqrt{3}}$.

Finally, using the relation between the diagonal and a side of a square, we have

$$\boldsymbol{d} = \boldsymbol{s}\sqrt{2} = 2\sqrt{3}\sqrt{2} = \boldsymbol{2\sqrt{6}}.$$

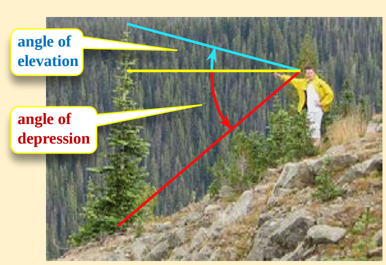
Angles of Elevation or Depression in Applications

The method of solving right triangles is widely adopted in solving many applied problems. One of the critical steps in the solution process is sketching a triangle that models the situation, and labeling the parts of this triangle correctly.

In trigonometry, many applied problems refer to angles of **elevation** or **depression**, or include some navigation terminology, such as **direction** or **bearing**.

Definition 4.2 ► **Angle of elevation** (or **inclination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **above** the horizontal line.

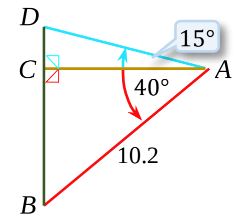
Angle of depression (or **declination**) is the acute angle formed by a **horizontal** line and the line of sight to an object **below** the horizontal line.



Example 5 ► **Applying Angles of Elevation or Depression**

Find the height of the tree in the picture given next to *Definition 4.2*, assuming that the observer sees the top of the tree at an angle of elevation of 15°, the base of the tree at an angle of depression of 40°, and the distance from the base of the tree to the observer's eyes is 10.2 meters.

Solution ► First, let's draw a diagram to model the situation, label the vertices, and place the given data. Then, observe that the height of the tree BD can be obtained as the sum of distances BC and CD.



BC can be found from $\triangle ABC$, by using the ratio of sine of 40°.
From the equation

$$\frac{BC}{10.2} = \sin 40°,$$

we have

$$BC = 10.2 \sin 40° \simeq \mathbf{6.56}$$

To calculate the length DC, we would need to have another piece of information about $\triangle ADC$ first. Notice that the side AC is common for the two triangles. This means that we can find it from $\triangle ABC$, and use it for $\triangle ADC$ in subsequent calculations.
From the equation

$$\frac{CA}{10.2} = \cos 40°,$$

we have

$$CA = 10.2 \cos 40° \simeq 7.8137$$

since we use this result in further calculations, four decimals of accuracy is advised

Now, employing tangent of 15° in $\triangle ADC$, we have

$$\frac{CD}{7.8137} = \tan 15°$$

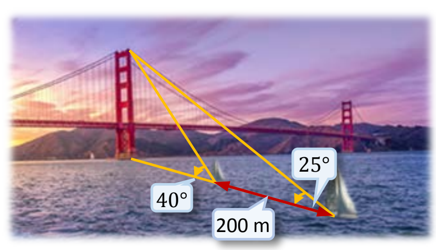
which gives us

$$CD = 7.8137 \cdot \tan 15° \simeq \mathbf{2.09}$$

Hence the height of the tree is $BC \simeq 6.56 + 2.09 = 8.65 \simeq \mathbf{8.7}$ meters.

Example 6 ► **Using Two Angles of Elevation at a Given Distance to Determine the Height**

When Ricky and Sonia were sailing their boat on a river, they observed the tip of a bridge tower at a 25° elevation angle. After sailing 200 meters closer to the tower, they noticed that the tip of the tower was visible at 40° elevation angle. Approximate the height of the tower to the nearest meter.



Solution ► To model the situation, let us draw the diagram and adopt the notation as in *Figure 2*. We look for height $\boldsymbol{h}$, which is a part of the two right triangles $\varDelta ABC$ and $\varDelta DBC$.

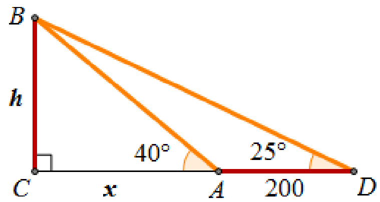


Figure 2

Since trigonometric ratios involve two sides of a triangle, and we already have length AD, a part of the side CD, it is reasonable to introduce another unknown, call it $\boldsymbol{x}$, to represent the remaining part CA. Then, applying the ratio of tangent to each of the right triangles, we produce the following system of equations:

$$\begin{cases} \dfrac{h}{x} = \tan 40° \\ \dfrac{h}{x+200} = \tan 25° \end{cases}$$

To solve the above system, we first solve each equation for h

$$\begin{cases} h \simeq 0.8391x \\ h \simeq 0.4663(x+200)\,, \end{cases}$$

substitute to the top equation

and then by equating the right sides, we obtain

$$0.8391x = 0.4663(x+200)$$

$$0.8391x - 0.4663x = 93.26$$

$$0.3728x = 93.26$$

$$x = \frac{93.26}{0.3728} \simeq 250.16$$

Therefore, $\boldsymbol{h} \simeq 0.8391 \cdot 250.16 \simeq \mathbf{210}\ \boldsymbol{m}$.

The height of the tower is approximately **210** meters.

Direction or Bearing in Applications

A large group of applied problems in trigonometry refer to **direction** or **bearing** to describe the location of an object, usually a plane or a ship. The idea comes from following the behaviour of a compass. The magnetic needle in a compass points North. Therefore, the location of an object is described as a clockwise deviation from the SOUTH-NORTH line.

There are two main ways of describing directions:

- One way is by stating the angle θ that starts from the North and opens clockwise until the line of sight of an object. For example, we can say that the point $\boldsymbol{B}$ is seen in the **direction** of 108° from the point $\boldsymbol{A}$, as in *Figure 2a*.
- Another way is by stating the acute angle formed by the South-North line and the line of sight. Such an angle starts either from the North (**N**) or the South (**S**) and opens either towards the East (**E**) or the West (**W**). For instance, the position of the point $\boldsymbol{B}$ in *Figure 2b* would be described as being at a **bearing** of **S**72°**E** (*read:* South 72° towards the East) from the point $\boldsymbol{A}$.

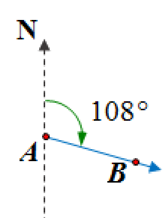


Figure 2a

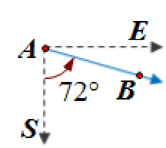


Figure 2b

This, for example, means that:

the direction of 195° can be seen as the bearing **S**15°**W** and the direction of 290° means the same as **N**70°**W**.

Example 7 ► **Using Direction in Applications Involving Navigation**

An airplane flying at a speed of 400 mi/hr flies from a point A in the direction of 153° for one hour and then flies in the direction of 63° for another hour.

a. How long will it take the plane to get back to the point A?

b. What is the direction that the plane needs to fly in order to get back to the point A?

Solution ►

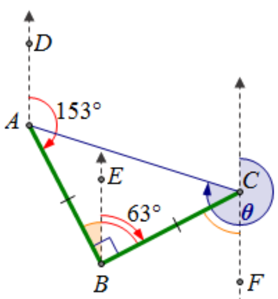


Figure 3

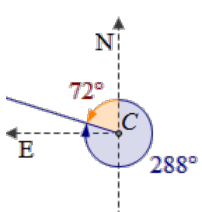
a. First, let's draw a diagram modeling the situation. Assume the notation as in *Figure 3*. Since the plane flies at 153° and the South-North lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ are parallel, by the property of interior angles, we have $\angle ABD = 180° - 153° = 27°$. This in turn gives us $\angle ABC = \angle ABE + \angle EBC = 27° + 63° = 90°$. So the $\triangle ABC$ is right angled with $\angle B = 90°$ and the two legs of length $AB = BC = 400\ mi$. This means that the $\triangle ABC$ is in fact a special triangle of the type $45° - 45° - 90°$.

Therefore $AC = AB\sqrt{2} = 400\sqrt{2} \simeq 565.7\ mi$.

Now, solving the well-known motion formula $R \cdot T = D$ for the time T, we have

$$T = \frac{D}{R} \simeq \frac{\cancel{400}\sqrt{2}}{\cancel{400}} = \sqrt{2} \simeq 1.4142\ hr \simeq \mathbf{1\ hr\ 25\ min}$$

Thus, it will take the plane approximately 1 hour and 25 minutes to return to the starting point A.



b. To direct the plane back to the starting point, we need to find angle θ, marked in blue, rotating clockwise from the North to the ray $\overrightarrow{CA}$. By the property of alternating angles, we know that $\angle FCB = 63°$. We also know that $\angle BCA = 45°$, as $\angle ABD$ is the "half of a square" special triangle. Therefore,

$$\theta = 180° + 63° + 45° = \mathbf{288}°.$$

Thus, to get back to the point A, the plane should fly in the direction of 288°. Notice that this direction can also be stated as **N**72°**W**.

T.4 Exercises

Using a calculator, find an angle θ satisfying the given equation. Leave your answer in decimal degrees rounded to the nearest tenth of a degree if needed.

1. $\sin\theta = 0.7906$

2. $\cos\theta = 0.7906$

3. $\tan\theta = 2.5302$

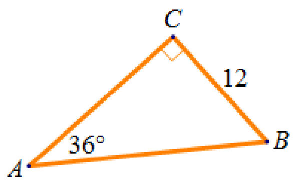
4. $\cos\theta = -0.75$

5. $\tan\theta = \sqrt{3}$

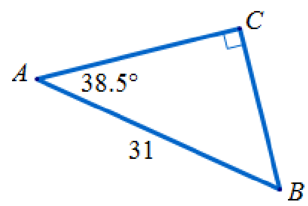
6. $\sin\theta = \frac{3}{4}$

Given the data, solve each triangle ABC with $\angle C = 90°$.

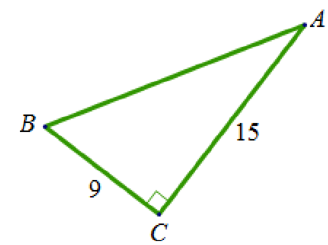
7.



8.



9.



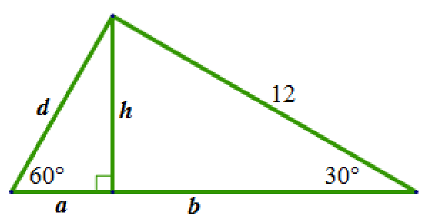
10. $\angle A = 42°,\ b = 17$

11. $a = 9.45,\ c = 9.81$

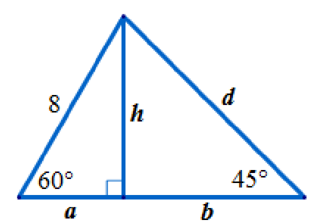
12. $\angle B = 63°12',\ b = 19.1$

*Find the **exact** value of each unknown in the figure.*

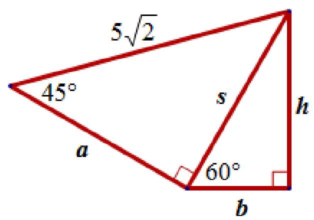
13.



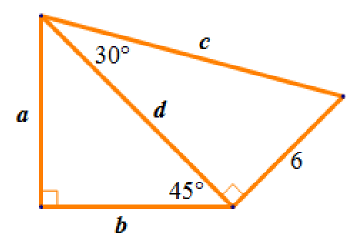
14.



15.



16.



17. A circle of radius 8 centimeters is inscribed in a regular hexagon. Find the exact perimeter of the hexagon.

18. A regular pentagon is inscribed in a circle with 10 meters diameter. To the nearest centimeter, find the perimeter of the pentagon.

19. A 25 meters long supporting rope connects the top of a 23 meters high mast of a sailboat with the deck of the boat. To the nearest degree, find the angle between the rope and the mast.

20. A 16 meters long guy wire is attached to the top of a utility pole. The angle between the guy wire and the ground is 54°. To the nearest tenth of a meter, how tall is the pole?

21. From the top of a 52 m high cliff, the angle of depression to a boat is $4°15'$. To the nearest meter, how far is the boat from the base of the cliff?

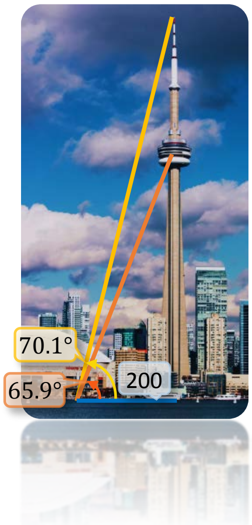
22. A spotlight reflector mounted to a ceiling of a 3.5 meters high hall is directed onto a piece of art displayed 1.5 meters above the floor. To the nearest degree, what angle of depression should be used to direct the light onto the piece of art if the reflector is 3.8 meters away from it?

23. To determine the height of the Eiffel Tower, a 1.8 meters tall tourist standing 50 meters from the center of the base of the tower measures the angle of elevation to the top of the tower to be 81°. Using this information, determine the height of the Eiffel Tower to the nearest meter.

24. To the nearest meter, find the height of an isosceles triangle with 25.2 meters long base and $35°40'$ angle by the base.

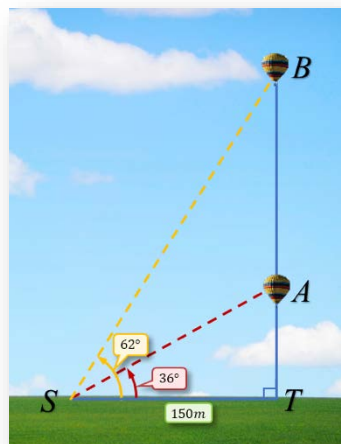
25. A plane flies 700 kilometers at a bearing of $\mathbf{N}56°\mathbf{E}$ and then 850 kilometers at a bearing of $\mathbf{S}34°\mathbf{E}$. How far and in what direction is the plane from the starting point? Round the answers to the nearest kilometer and the nearest degree.

26. A plane flies at 420 km/h for 30 minutes in the direction of $142°$. Then, it changes its direction to $232°$ and flies for 45 minutes. To the nearest kilometer, how far is the plane at that time from the starting point? To the nearest degree, in what direction should the plane fly to come back to the starting point?

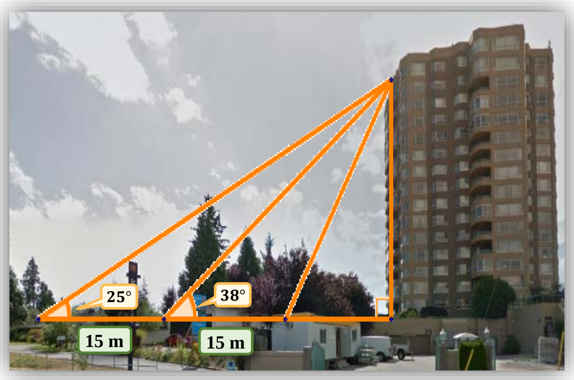


27. Standing 200 meters from the base of the CN Tower, a tourist sees the pinnacle of the tower at $70.1°$ elevation angle. The tower has a built-in restaurant as in the accompanying picture. The tourist can see this restaurant at $65.9°$ elevation angle. To the nearest meter, how tall is the CN Tower, including its pinnacle? How high above the ground is the restaurant?

28. A hot air balloon rises vertically at a constant rate, as shown in the accompanying figure. A hundred fifty meters away from the balloon's lift-off place, a spectator notices the balloon at $36°$ angle of elevation. A minute later, the spectator records that the angle of elevation of the balloon is $62°$. To the nearest meter per second, determine the rate of the balloon.



29. Two people observe an eagle nest on a tall tree in a park. One person sees the nest at the angle of elevation of $60°$ while the other at the angle of elevation of $75°$. If the people are 25 meters apart from each other and the tree is between them, determine the altitude at which the nest is situated. Round your answer to the nearest tenth of a meter.



30. A person approaching a tall building records the angle of elevation to the top of the building to be $32°$. Fifteen meters closer to the building, this angle becomes $40°$. To the nearest meter, how tall is the building? What would the angle of elevation be in another 15 meters?

31. Suppose that the length of the shadow of The Palace of Culture and Science in Warsaw increases by 15.5 meters when the angle of elevation of the sun decreases from $48°$ to $46°$. Based on this information, determine the height of the palace. Round your answer to the nearest meter.

32. A police officer observes a road from 150 meters distance as in the accompanying diagram. A car moving on the road covers the distance between two chosen by the officer points, A and B, in 1.5 seconds. If the angles between the lines of sight to points *A* and *B* and the line perpendicular to the observed road are respectively 34.1° and 20.3°, what was the speed of the car? State your answer in kilometers per hour rounded up to one decimal.

T5 The Law of Sines and Cosines and Its Applications

The concepts of solving triangles developed in *Section T4* can be extended to all triangles. A triangle that is not right-angled is called an **oblique triangle**. Many application problems involve solving oblique triangles. Yet, we can not use the SOH-CAH-TOA rules when solving those triangles since **SOH-CAH-TOA** definitions **apply only to right triangles!** So, we need to search for other rules that will allow us to solve oblique triangles.

The Sine Law

Observe that all triangles can be classified with respect to the size of their angles as **acute** (with all acute angles), **right** (with one right angle), or **obtuse** (with one obtuse angle). Therefore, oblique triangles are either acute or obtuse.

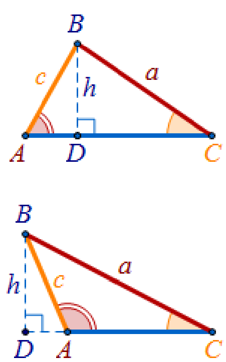


Figure 1

Let's consider both cases of an oblique $\triangle ABC$, as in *Figure 1*. In each case, let's drop the height h from vertex B onto the line $\overleftrightarrow{AC}$, meeting this line at point D. This way, we obtain two more right triangles, $\triangle ADB$ with hypotenuse c, and $\triangle BDC$ with hypotenuse a. Applying the ratio of sine to both of these triangles, we have:

$$\sin\angle A = \frac{h}{c}, \text{ so } h = c\sin\angle A$$

and

$$\sin\angle C = \frac{h}{a}, \text{ so } h = a\sin\angle C.$$

Thus,

$$a\sin\angle C = c\sin\angle A,$$

and we obtain

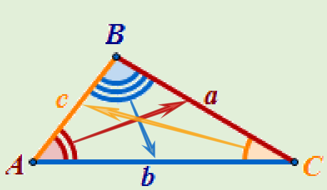
$$\frac{a}{\sin\angle A} = \frac{c}{\sin\angle C}.$$

Similarly, by dropping heights from the other two vertices, we can show that

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} \text{ and } \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}.$$

This result is known as the law of sines.

The Sine Law ► In any triangle ABC, the lengths of the **sides are proportional to the sines of the opposite angles**. This fact can be expressed in any of the following, equivalent forms:



$$\frac{a}{b} = \frac{\sin\angle A}{\sin\angle B}, \quad \frac{b}{c} = \frac{\sin\angle B}{\sin\angle C}, \quad \frac{c}{a} = \frac{\sin\angle C}{\sin\angle A}$$

or

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}$$

or

$$\frac{\sin\angle A}{a} = \frac{\sin\angle B}{b} = \frac{\sin\angle C}{c}$$

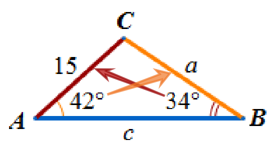
Observation: *As with any other proportion, to solve for one variable, we need to know the three remaining values. Notice that when using the Sine Law proportions, the three known values must include **one pair of opposite data**: a side and its opposite angle.*

Example 1 ► **Solving Oblique Triangles with the Aid of The Sine Law**

Given the information, solve each triangle ABC.

a. $\angle A = 42°,\ \angle B = 34°,\ b = 15$ **b.** $\angle A = 35°,\ a = 12,\ b = 9$

Solution ►



a. First, we will sketch a triangle ABC that models the given data. Since the sum of angles in any triangle equals 180°, we have

$$\angle \boldsymbol{C} = 180° - 42° - 34° = \mathbf{104°}.$$

Then, to find length a, we will use the pair $(a, \angle A)$ of opposite data, side a and $\angle A$, and the given pair $(b, \angle B)$. From the Sine Law proportion, we have

$$\frac{a}{\sin 42°} = \frac{15}{\sin 34°},$$

which gives

$$\boldsymbol{a} = \frac{15 \cdot \sin 42°}{\sin 34°} \simeq \mathbf{17.9}$$

To find length c, we will use the pair $(c, \angle C)$ and the given pair of opposite data $(b, \angle B)$. From the Sine Law proportion, we have

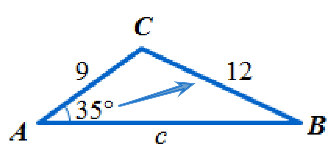
$$\frac{c}{\sin 104°} = \frac{15}{\sin 34°},$$

which gives

for easier calculations, keep the unknown in the numerator

$$\boldsymbol{c} = \frac{15 \cdot \sin 104°}{\sin 34°} \simeq \mathbf{26}$$

So the triangle is solved.



b. As before, we will start by sketching a triangle ABC that models the given data. Using the pair $(9, \angle B)$ and the given pair of opposite data $(12, 35°)$, we can set up a proportion

$$\frac{\sin \angle B}{9} = \frac{\sin 35°}{12}.$$

Then, solving it for $\sin \angle B$, we have

$$\sin \angle B = \frac{9 \cdot \sin 35°}{12} \simeq 0.4302,$$

which, after applying the inverse sine function, gives us

$$\angle \boldsymbol{B} \simeq \mathbf{25.5°}$$

Now, we are ready to find $\angle \boldsymbol{C} = 180° - 35° - 25.5° = \mathbf{119.5°}$, and finally, from the proportion

$$\frac{c}{\sin 119.5°} = \frac{12}{\sin 35°},$$

we have

$$\boldsymbol{c} = \frac{12 \cdot \sin 119.5°}{\sin 35°} \simeq \mathbf{18.2}$$

Thus, the triangle is solved.

Ambiguous Case

Observe that the size of one angle and the length of two sides does not always determine a unique triangle. For example, there are two different triangles that can be constructed with $\angle A = 35°,\ a = 9,\ b = 12$.

Such a situation is called an **ambiguous case**. It occurs when the opposite side to the given angle is shorter than the other given side but long enough to complete the construction of an oblique triangle, as illustrated in *Figure 2*.

In application problems, if the given information does not determine a unique triangle, both possibilities should be considered in order for the solution to be complete.

On the other hand, not every set of data allows for the construction of a triangle. For example (see *Figure 3*), if $\angle A = 35°,\ a = 5,\ b = 12$, the side a is too short to complete a triangle, or if $a = 2,\ b = 3,\ c = 6$, the sum of lengths of a and b is smaller than the length of c, which makes impossible to construct a triangle fitting the data.

Note that in any triangle, the **sum of lengths of any two sides** is always **bigger than the length of the third side**.

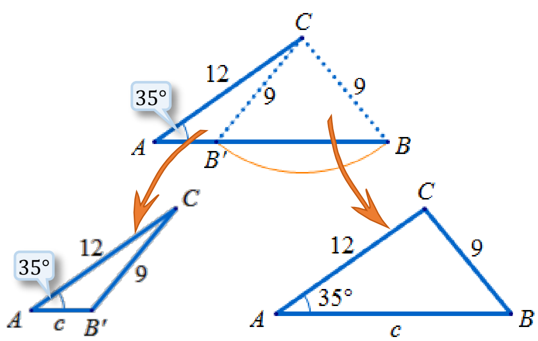


Figure 2

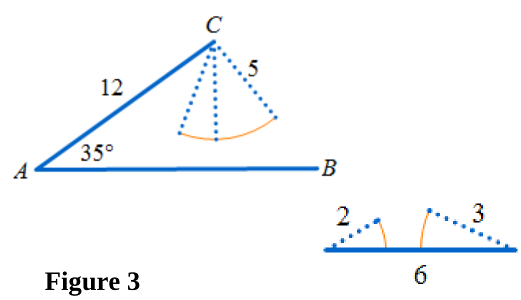
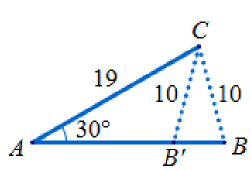


Figure 3

Example 2 ► **Using the Sine Law in an Ambiguous Case**

Solve triangle ABC, knowing that $\angle A = 30°,\ a = 10,\ b = 16$.

Solution ►



When sketching a diagram, we notice that there are two possible triangles, $\triangle ABC$ and $\triangle AB'C$, complying with the given information. $\triangle ABC$ can be solved in the same way as the triangle in *Example 1b*. In particular, one can calculate that in $\triangle ABC$, we have $\angle \boldsymbol{B} \simeq \mathbf{71.8}°,\ \angle \boldsymbol{C} \simeq \mathbf{78.2}°$, and $\boldsymbol{c} \simeq \mathbf{19.6}$.

Let's see how to solve $\triangle AB'C$ then. As before, to find $\angle B'$, we will use the proportion

$$\frac{\sin \angle B'}{19} = \frac{\sin 30°}{10},$$

which gives us $\sin\angle B' = \frac{19 \cdot \sin 30°}{10} = 0.95$. However, when applying the inverse sine function to the number 0.95, a calculator returns the approximate angle of 71.8°. Yet, we know that angle B' is obtuse. So, we should look for an angle in the second quadrant, with the reference angle of 71.8°. Therefore, $\angle \boldsymbol{B'} = 180° - 71.8° = \mathbf{108.2}°$.

Now, $\angle \boldsymbol{C} = 180° - 30° - 108.2° = \mathbf{41.8}°$

and finally, from the proportion

$$\frac{c}{\sin 41.8°} = \frac{10}{\sin 30°},$$

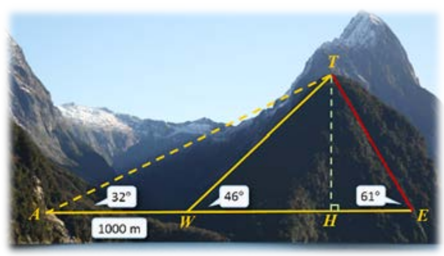
we have

$$\boldsymbol{c} = \frac{10 \cdot \sin 41.8°}{\sin 30°} \simeq \mathbf{13.3}$$

Thus, $\triangle AB'C$ is solved.

Example 3 ► **Solving an Application Problem Using the Sine Law**

Refer to the accompanying diagram. Round all your answers to the nearest tenth of a meter.



From a distance of 1000 meters from the west base of the mountain, the top of the mountain is visible at 32° angle of elevation. At the west base, the average slope of the mountain is estimated to be 46°.

a. Determine the distance WT from the west base to the top of the mountain.

b. What is the distance ET from the east base to the top of the mountain, if the average slope of the mountain there is 61°?

c. Find the height HT of the mountain.

Solution ► **a.** To find distance WT, consider $\triangle AWT$. Observe that one can easily find the remaining angles of this triangle, as shown below:

$$\angle AWT = 180° - 46° = 134°$$ supplementary angles

and

$$\angle ATW = 180° - 32° - 134° = 14°$$ sum of angles in a $\triangle$

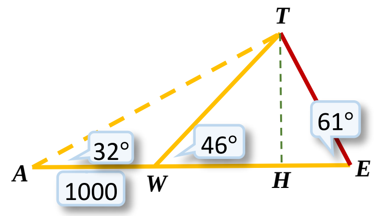
Therefore, applying the law of sines, we have

$$\frac{WT}{\sin 32°} = \frac{1000}{\sin 14°},$$

which gives

$$\boldsymbol{WT} = \frac{1000 \sin 32°}{\sin 14°} \simeq \mathbf{2190.5}\ \boldsymbol{m}.$$

b. To find distance ET, we can apply the law of sines to $\triangle WET$ using the pair $(2190.5, 61°)$. From the equation



$$\frac{ET}{\sin 46°} = \frac{2190.5}{\sin 61°},$$

we have

$$\boldsymbol{ET} = \frac{2190.5 \sin 46°}{\sin 61°} \simeq \mathbf{1801.6}\ \boldsymbol{m}.$$

c. To find the height HT of the mountain, we can use the right triangle WHT. By the definition of sine, we have

$$\frac{HT}{2190.5} = \sin 46°,$$

so $\boldsymbol{HT} = 2190.5 \sin 46° \simeq \mathbf{1575.7}\ \boldsymbol{m}$.

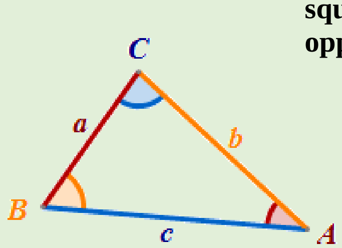
The Cosine Law

The above examples show how the **Sine Law** can help in solving oblique triangles when one **pair of opposite data** is given. However, the Sine Law is not enough to solve a triangle if the given information is

- the length of the **three sides** (but no angles), or
- the length of **two sides** and the **enclosed angle**.

Both of the above cases can be solved with the use of another property of a triangle, called the Cosine Law.

The Cosine Law ► In any triangle ABC, **the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the opposite angle**.



$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos \angle A}$$

$$\boldsymbol{b^2 = a^2 + c^2 - 2ac \cos \angle B}$$

$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos \angle C}$$

note the opposite side and angle

Observation: *If the angle of interest in any of the above equations is right, since* $\cos 90° = 0$, *the equation becomes Pythagorean. So the* ***Cosine Law*** *can be seen as an* ***extension of the Pythagorean Theorem***.

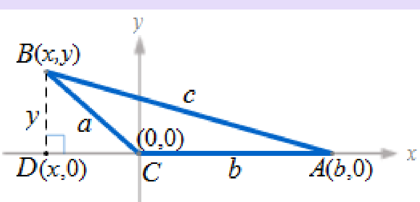


Figure 3

To derive this law, let's place an oblique triangle ABC in the system of coordinates so that vertex C is at the origin, side AC lies along the positive x-axis, and vertex B is above the x-axis, as in *Figure 3*.

Thus $C = (0,0)$ and $A = (b, 0)$. Suppose point B has coordinates (x, y). By *Definition 2.2*, we have

$$\sin \angle C = \frac{y}{a} \quad \text{and} \quad \cos \angle C = \frac{x}{a},$$

which gives us

$$y = a \sin \angle C \quad \text{and} \quad x = a \cos \angle C\,.$$

Let $D = (x, 0)$ be the perpendicular projection of the vertex B onto the x-axis. After applying the Pythagorean equation to the right triangle ABD, with $\angle D = 90°$, we obtain

here we use the Pythagorean identity developed in *Section T2*:

$$\mathbf{\sin^2 \theta + \cos^2 \theta = 1}$$

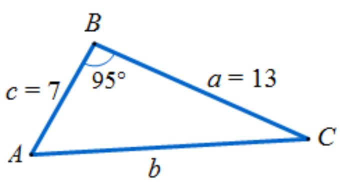
$$\begin{aligned}
\boldsymbol{c^2} &= y^2 + (b - x)^2 \\
&= (a \sin \angle C)^2 + (b - a \cos \angle C)^2 \\
&= a^2 \sin^2 \angle C + b^2 - 2ab \cos \angle C + a^2 \cos^2 \angle C \\
&= a^2 (\sin^2 \angle C + \cos^2 \angle C) + b^2 - 2ab \cos \angle C \\
&= \boldsymbol{a^2 + b^2 - 2ab \cos \angle C}
\end{aligned}$$

Similarly, by placing the vertices A or B at the origin, one can develop the remaining two forms of the Cosine Law.

Example 4 ► **Solving Oblique Triangles Given Two Sides and the Enclosed Angle**

Solve triangle ABC, given that $\angle B = 95°$, $a = 13$, and $c = 7$.

Solution ►



First, we will sketch an oblique triangle ABC to model the situation. Since there is no pair of opposite data given, we cannot use the law of sines. However, applying the law of cosines with respect to side b and $\angle B$ allows for finding the length b. From

$$b^2 = 13^2 + 7^2 - 2 \cdot 13 \cdot 7 \cos 95° \simeq 233.86,$$

we have $\boldsymbol{b \simeq 15.3}$.

watch the order of operations here!

Now, since we already have the pair of opposite data $(15.3, 95°)$, we can apply the law of sines to find, for example, $\angle C$. From the proportion

$$\frac{\sin \angle C}{7} = \frac{\sin 95°}{15.3},$$

we have

$$\sin \angle C = \frac{7 \cdot \sin 95°}{15.3} \simeq 0.4558,$$

thus $\angle \boldsymbol{C} = \sin^{-1} 0.4558 \simeq \mathbf{27.1}°$.
Finally, $\angle \boldsymbol{A} = 180° - 95° - 27.1° = \mathbf{57.9}°$ and the triangle is solved.

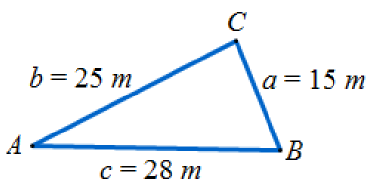
When applying the law of cosines in the above example, there was no other choice but to start with the pair of opposite data $(b, \angle B)$. However, in the case of three given sides, one could apply the law of cosines corresponding to any pair of opposite data. Is there any preference as to which pair to start with? Actually, yes. Observe that after using the law of cosines, we often use the **law of sines** to complete the solution since the **calculations are usually easier** to perform this way. Unfortunately, when solving a sine proportion for an obtuse angle, one would need to change the angle obtained from a calculator to its supplementary one. This is because calculators are programmed to return angles from the first quadrant when applying $\sin^{-1}$ to positive ratios. If we look for an obtuse angle, we need to employ the fact that $\sin \alpha = \sin(180° - \alpha)$ and take the supplement of the calculator's answer. To avoid this ambiguity, it is recommended to **apply the cosine law** to the pair of the **longest side and largest angle** first. This will guarantee that the law of sines will be used to find only acute angles and thus it will not cause ambiguity.

Recommendations:
- ***apply the Cosine Law only when it is absolutely necessary*** *(SAS or SSS)*
- ***apply the Cosine Law to find the largest angle first****, if applicable*

Example 5 ► **Solving Oblique Triangles Given Three Sides**

Solve triangle ABC, given that $a = 15\ m$, $b = 25\ m$, and $c = 28\ m$.

Solution ►



First, we will sketch a triangle ABC to model the situation. As before, there is no pair of opposite data given, so we cannot use the law of sines. So, we will apply the law of cosines with respect to the pair $(28, \angle C)$, as the side $c = 28$ is the longest. To solve the equation

$$28^2 = 15^2 + 25^2 - 2 \cdot 15 \cdot 25 \cos \angle C$$

watch the order of operations when solving for cosine

for $\angle C$, we will first solve it for $\cos \angle C$, and have

$$\cos \angle C = \frac{28^2 - 15^2 - 25^2}{-2 \cdot 15 \cdot 25} = \frac{-66}{-750} = 0.088,$$

which, after applying $\cos^{-1}$, gives $\boldsymbol{\angle C \simeq 85°}$.

Since now we already have the pair of opposite data $(28, 85°)$, we can apply the law of sines to find, for example, $\angle A$. From the proportion

$$\frac{\sin \angle A}{15} = \frac{\sin 85°}{28},$$

we have

$$\sin \angle A = \frac{15 \cdot \sin 85°}{28} \simeq 0.5337,$$

thus $\boldsymbol{\angle A} = \sin^{-1} 0.5337 \simeq \mathbf{32.3}°$.
Finally, $\boldsymbol{\angle B} = 180° - 85° - 32.3° = \mathbf{62.7}°$ and the triangle is solved.

Example 6 ► **Solving an Application Problem Using the Cosine Law**

Two planes leave an airport at the same time and fly in different directions. Plane A flies in the direction of 155° at 390 km/h and plane B flies in the direction of 260° at 415 km/h. To the nearest kilometer, how far apart are the planes after two hours?

Solution ► As usual, we start the solution by sketching a diagram appropriate to the situation. Assume the notation as in *Figure 3*.

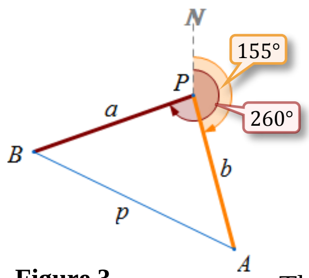


Figure 3

Since plane A flies at 390 km/h for two hours, we can find the distance

$$b = 2 \cdot 390 = 780\ km.$$

Similarly, since plane B flies at 415 km/h for two hours, we have

$$a = 2 \cdot 415 = 830\ km.$$

The measure of the enclosed angle APB can be obtained as a difference between the given directions. So we have

$$\angle APB = 260° - 155° = 105°.$$

Now, we are ready to apply the law of cosines in reference to the pair $(p, 105°)$. From the equation

$$p^2 = 830^2 + 780^2 - 2 \cdot 830 \cdot 780 \cos 105°,$$

we have $p \simeq \sqrt{1632418.9} \simeq 1278\ km.$

So we know that after two hours, the two planes are about **1278 kilometers** apart.

Area of a Triangle

The method used to derive the law of sines can also be used to derive a handy formula for finding the area of a triangle, without knowing its height.

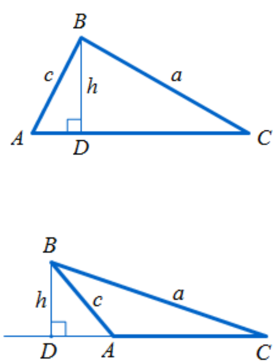


Figure 4

Let ABC be a triangle with height h dropped from the vertex B onto the line $\overleftrightarrow{AC}$, meeting $\overleftrightarrow{AC}$ at the point D, as shown in *Figure 4*. Using the right $\triangle ABD$, we have

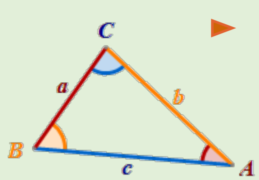
$$\sin \angle A = \frac{h}{c},$$

and equivalently $h = c \sin \angle A$, which after substituting into the well known formula for area of a triangle $[\boldsymbol{ABC}] = \frac{1}{2}\boldsymbol{bh}$, gives us

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle \boldsymbol{A}$$

Starting the proof with dropping a height from a different vertex would produce two more versions of this formula, as stated below.

The Sine Formula for Area of a Triangle

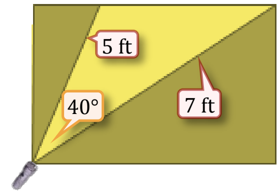


The **area** $[\boldsymbol{ABC}]$ of a triangle ABC can be calculated by taking **half of a product of the lengths of two sides and the sine of the enclosed angle.** We have

$$[\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{bc}\ \mathbf{sin}\ \angle\boldsymbol{A}, \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ac}\ \mathbf{sin}\ \angle\boldsymbol{B}, \quad \text{or} \quad [\boldsymbol{ABC}] = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ab}\ \mathbf{sin}\ \angle\boldsymbol{C}.$$

Example 7 ► **Finding Area of a Triangle Given Two Sides and the Enclosed Angle**

In a search for her lost earing, Irene used a flashlight to illuminate part of the floor under her bed. If the flashlight emitted the light at 40° angle and the length of the outside rays of light was 5 ft and 7 ft as indicated in the accompanying diagram, how many square feet of the floor was illuminated?



Solution ► We start with sketching an appropriate diagram. Assume the notation as in *Figure 5*.

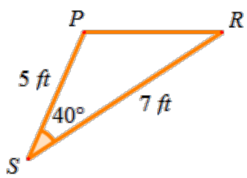


Figure 5

From the sine formula for area of a triangle, we have

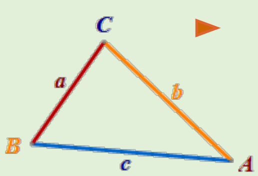
$$[PRS] = \frac{1}{2} \cdot 5 \cdot 7 \sin 40° \simeq 11.2 \text{ ft}^2.$$

The area of the illuminated part of the floor under the bed was about **11 square feet**.

Heron's Formula

The law of cosines can be used to derive a formula for the area of a triangle when only the lengths of the three sides are known. This formula is known as Heron's formula (as mentioned in *Section RD1*), named after the Greek mathematician Heron of Alexandria.

Heron's Formula for Area of a Triangle



► The **area** $[\boldsymbol{ABC}]$ of a triangle ABC with sides $a, b, c,$ and **semiperimeter** $\boldsymbol{s} = \frac{\boldsymbol{a+b+c}}{\boldsymbol{2}}$ can be calculated using the formula

$$[\boldsymbol{ABC}] = \sqrt{\boldsymbol{s(s-a)(s-b)(s-c)}}$$

Example 8 ► **Finding Area of a Triangle Given Three Sides**

The city of Abbotsford plans to convert a triangular lot into public parking. In square meters, what would the area of the parking be if the three sides of the lot are 45 m, 57 m, and 60 m long?

Solution ► To find the area of the triangular lot with given sides, we would like to use Heron's Formula. For this reason, we first calculate the semiperimeter

$$s = \frac{45 + 57 + 60}{2} = 81.$$

Then, the area equals

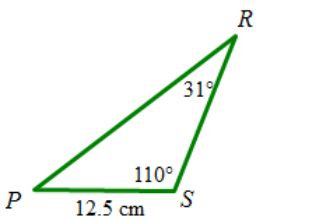
$$\sqrt{81(81-45)(81-57)(81-60)} = \sqrt{1469664} \simeq 1212.3\ m^2.$$

Thus, the area of the parking lot would be approximately **1212 square meters**.

T.5 Exercises

Use the law of sines to solve each triangle.

1.



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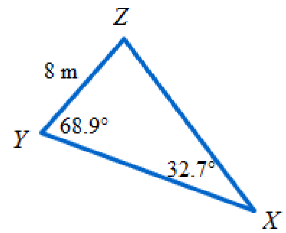
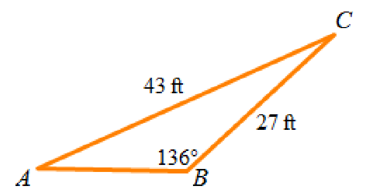


3.



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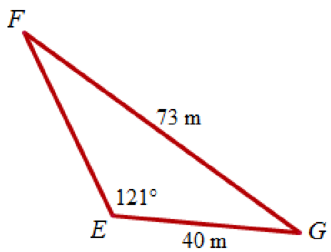

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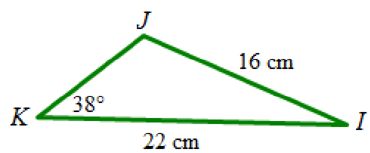

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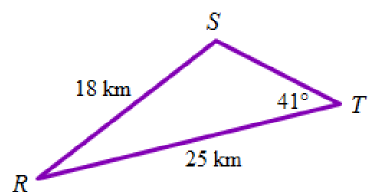


7. $\angle A = 30°,\ \angle B = 30°,\ a = 10$

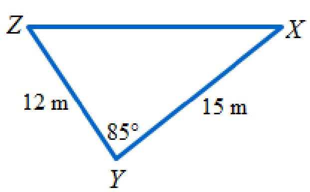
8. $\angle A = 150°,\ \angle C = 20°,\ a = 200$

9. $\angle C = 145°,\ b = 4,\ c = 14$

10. $\angle A = 110°15',\ a = 48,\ b = 16$

Use the law of cosines to solve each triangle.

11.



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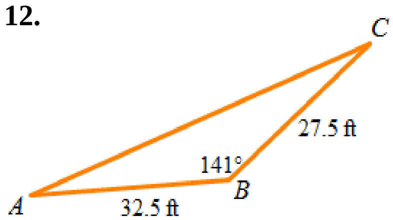

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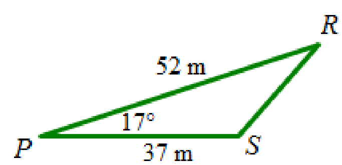

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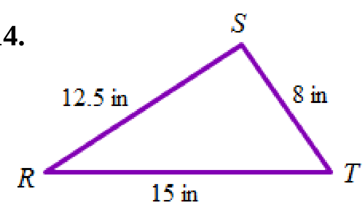

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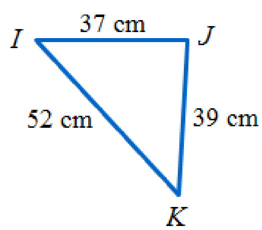
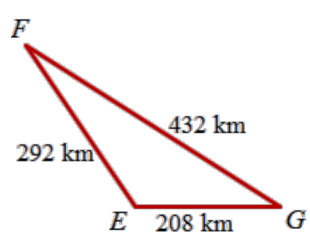


16.



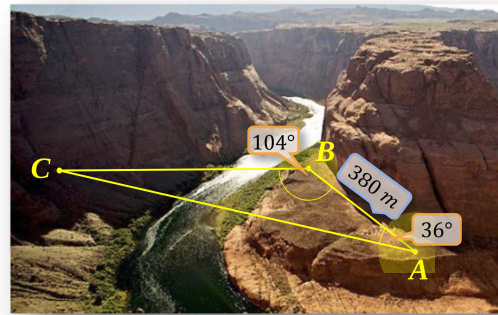
17. $\angle C = 60°,\ a = 3,\ b = 10$

18. $\angle B = 112°,\ a = 23,\ c = 31$

19. $a = 2,\ b = 3,\ c = 4$

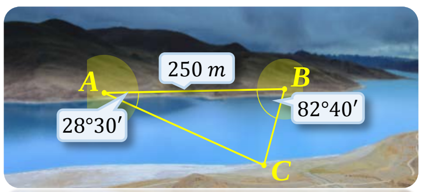
20. $a = 34,\ b = 12,\ c = 17.5$

21. In a triangle ABC, $\angle A$ is twice as large as $\angle B$. Does this mean that side a is twice as long as side b?



Use the appropriate law to solve each application problem.

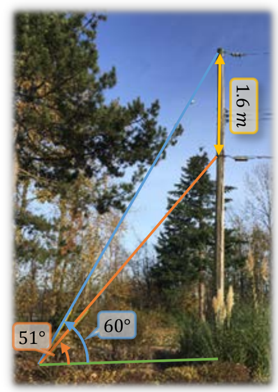
22. To approximate the distance across the Colorado River Canyon at the Horseshoe Bend, a hiker designates three points, A, B, and C, as in the accompanying figure. Then, he records the following measurements: $AB = 380$ meters, $\angle CAB = 36°$ and $\angle ABC = 104°$. How far is from B to C?



23. To find the width of a river, Peter designates three spots: A and B along one side of the river 250 meters apart from each other, and C, on the opposite side of the river (*see the accompanying figure*). Then, he finds that $\angle A = 28°30'$, and $\angle B = 82°40'$. To the nearest meter, what is the width of the river?

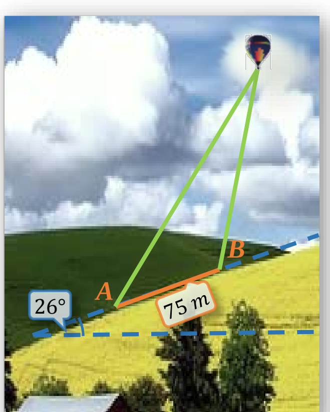
24. The captain of a ship sailing south spotted a castle tower at the distance of approximately 8 kilometers and the bearing of $\boldsymbol{S}47.5°\boldsymbol{E}$. In half an hour, the bearing of the tower was $\boldsymbol{N}35.7°\boldsymbol{E}$. What was the speed of the ship in km/h?

25. The captain of a ship sailing south saw a lighthouse at the bearing of $\boldsymbol{N}52.5°\boldsymbol{W}$. In 4 kilometers, the bearing of the lighthouse was $\boldsymbol{N}35.8°\boldsymbol{E}$. To the nearest tenth of a kilometer, how far was the ship from the lighthouse at each location?



26. Sam and Dan started sailing their boats at the same time and from the same spot. Sam followed the bearing of N12°W while Dan directed his boat at N5°E. After 3 hours, Sam was exactly west of Dan. If both sailors were 4 kilometers away from each other at that time, determine the distance sailed by Sam. *Round your answer to the nearest meter.*

27. A pole is anchored to the ground by two metal cables, as shown in the accompanying figure. The angles of inclination of the two cables are 51° and 60° respectively. Approximately how long is the top cable if the bottom one is attached to the pole 1.6 meters lower than the top one? *Round your answer to the nearest tenth of a meter.*



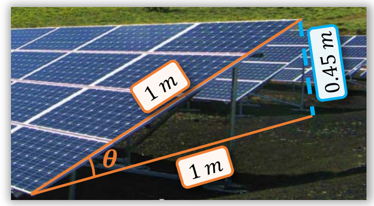
28. Two forest rangers were observing the forest from different lookout towers. At a certain moment, they spotted a group of lost hikers. The ranger on tower A saw the hikers at the direction of 46.7° and ranger on tower B saw the hikers at the direction of 315.8°. If tower A was 3.25 kilometers west of tower B, how far were the hikers from tower A? *Round your answer to the nearest hundredth of a kilometer.*

29. A hot-air balloon rises above a hill that inclines at 26°, as indicated in the accompanying diagram. Two spectators positioned on the hill at points A and B (*refer to the diagram*) observe the movement of the balloon. They notice that at a particular moment, the angle of elevation of the balloon from point A is 64° and from point B is 73°. If the spectators are 75 meters from each other, how far is the balloon from each of them? *Round your answers to the nearest meter.*

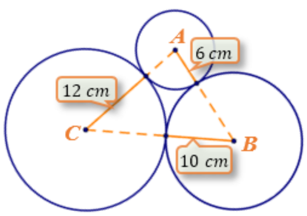
30. To the nearest centimeter, how long is the chord subtending a central angle of 25° in a circle of radius 30 cm?

31. An airplane takes off from city A and flies in the direction of $32°15'$ to city B, which is 500 km from A. After an hour of layover, the plane is heading in the direction of $137°25'$ to reach city C, which is 740 km from A. How far and in what direction should the plane fly to go back to city A?

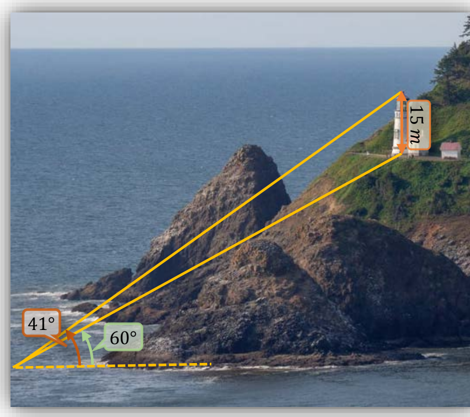
32. Find the area of a triangular hang-glider with two 7.5-meter sides that enclose the angle of 142°. *Round your answer to the nearest tenth of a square meter.*



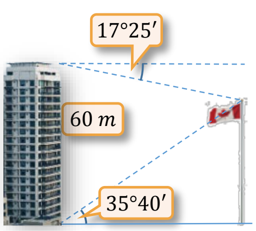
33. One meter wide solar panels were installed on a flat surface by tilting them up at an angle θ, as shown in the accompanying figure. If the distance between the top corner of a panel in the flat and tilted position is 0.45 meters, determine the measure of angle θ.



34. Three pipes with centres at points A, B, and C are tangent to each other. A perpendicular cross-section of the arrangement is shown in the accompanying figure. To the nearest tenth of a degree, determine the angles of triangle ABC, if the radii of the pipes are 6 cm, 10 cm, and 12 cm, respectively.



35. A 15-meters tall lighthouse is standing on a cliff. A person observing the lighthouse from a boat approaching the shore notices that the angle of elevation to the top of the lighthouse is 41° and to the bottom is 36°. Disregarding the person's height, estimate the height of the cliff.

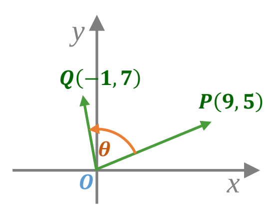


36. The top of a flag pole is visible from the top of a 60 meters high building at $17°25'$ angle of depression. From the bottom of this building, the tip of the flag pole can be seen at $35°40'$ angle of elevation. To the nearest centimeter, how tall is the flag pole?

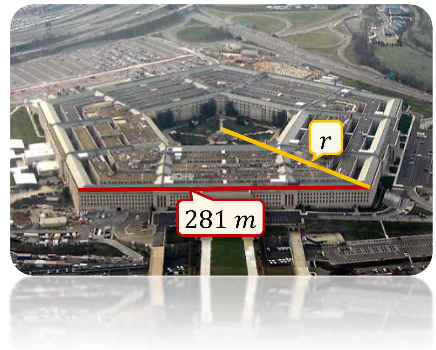
37. Find the area of a triangular parcel having two sides of lengths 51.4 m and 62.1 m, and 48.7° angle between them.

38. A city plans to pave a triangular area with sides of length 82 meters, 78 meters, and 112 meters. A pallet of bricks chosen for the job can cover 10 square meters of area. How many pallets should be ordered?

39. Suppose points P and Q are located respectively at $(9, 5)$ and $(-1, 7)$. If point O is the origin of the Cartesian coordinate system, determine the angle between vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. *Round your answer to the nearest degree.*



40. The building of The Pentagon in Washington D.C. is in a shape of a regular pentagon with about 281 meters long side, as shown in the accompanying figure. To the nearest meter, determine the radius of the **circumcircle** of this pentagon (*the circle that passes through all the vertices of the polygon*).



41. The locations A, B, and C of three FM radio transmitters form a triangle with sides $AB = 75$ m, $BC = 85$ m, and $AC = 90$ m. The transmitters at A, B, and C have a circular range of radius 35 m, 40 m, and 50 m, correspondingly. Assuming that no area can receive a signal from more than one transmitter, determine the area of the ABC triangle that does not receive any signal from any of the three FM radio transmitters. Round your answer to the nearest tenth of a square meter.

Attributions

Sequences and Series

In everyday life, we can observe sequences or series of events in many contexts. For instance, we line up to enter a store in a sequence, we make a sequence of mortgage payments, or we observe a series of events that lead to a particular outcome. In this section, we will consider mathematical definitions for sequences and series, and explore some applications of these concepts.

S.1 Sequences and Series

Think of a **sequence of numbers** as an **ordered list of numbers**. For example, the waiting time, in minutes, of each person standing in line to Tim Horton's to be served

$$0, 1, 2, 2, 3, 5, 5, 5, 7, 9, 10, 11$$

or the number of bacteria in a colony after each hour, if the colony starts with one bacteria and each bacteria divides into two every hour

$$1, 2, 4, 8, 16, 32, \dots, 2^{n-1}, \dots$$

The first example illustrates a finite sequence, while the second example shows an infinite sequence. Notice that numbers listed in a sequence, called **terms**, can repeat, like in the first example, or they can follow a certain pattern, like in the second example. If we can recognize the pattern of the listed terms, it is convenient to state it as a general rule by listing the n-th term. The sequence of numbers in our second example shows consecutive powers of two, starting with 2^0, so the n-th term of this sequence is 2^{n-1}.

Sequences as Functions

Formally, the definition of sequence can be stated by using the terminology of functions.

Definition 1.1 ► An **infinite sequence** is a function whose **domain** is the set of all **natural numbers**.
A **finite sequence** is a function whose **domain** is the first n natural numbers $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{n}\}$.
The **terms** (or elements) of a sequence are the function values, the entries of the ordered list of numbers.
The **general term** of a sequence is its n-th term.

Notation: Customarily, sequence functions assume names such as a, b, c, rather than f, g, h.
If the name of a sequence function is a, than the function values (the **terms** of the sequence) are denoted $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \dots$ rather than $a(1), a(2), a(3), \dots$.
The **index** $\boldsymbol{k}$ in the notation $\boldsymbol{a_k}$ indicates the position of the term in the sequence.
$\boldsymbol{a_n}$ denotes the **general term** of the sequence and $\{\boldsymbol{a_n}\}$ represents the entire sequence.

Example 1 ► **Finding Terms of a Sequence When Given the General Term**

Given the sequence $a_n = \frac{n-1}{n+1}$, find the following

a. the first four terms of $\{a_n\}$ **b.** the 12-th term a_{12}

Solution ► **a.** To find the first four terms of the given sequence, we evaluate a_n for $n = 1,2,3,4$.

We have
$$a_1 = \frac{1-1}{1+1} = 0$$
$$a_2 = \frac{2-1}{2+1} = \frac{1}{3}$$
$$a_3 = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$
$$a_4 = \frac{4-1}{4+1} = \frac{3}{5}$$

so the first four terms are $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}$.

b. The twelfth term is $a_{12} = \frac{12-1}{12+1} = \frac{11}{13}$.

Example 2 ► **Finding the General Term of a Sequence**

Determine the expression for the general term a_n of the sequence

a. $3, 9, 27, 81, \dots$ **b.** $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots$

Solution ► **a.** Observe that all terms of the given sequence are powers of 3.

We have
$$a_1 = 3 = 3^1$$
$$a_2 = 9 = 3^2$$
$$a_3 = 27 = 3^3$$
$$a_4 = 81 = 3^4, \text{ and so on.}$$

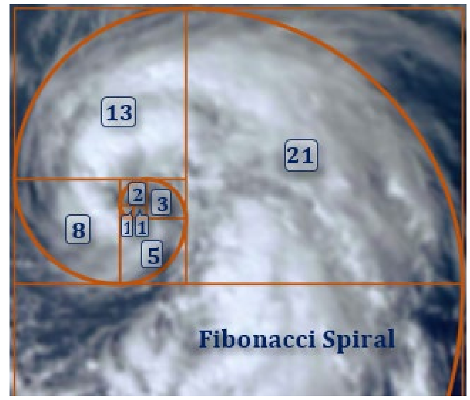
Notice that in each term, the exponent of 3 is the same as the index of the term. The above pattern suggests the candidate $a_n = 3^n$ for the general term of this sequence. To convince oneself that this is indeed the general term, one may want to generate the given terms with the aid of the developed formula $a_n = 3^n$. If the generated terms match the given ones, the formula is correct.

b. Here, observe that the signs of the given terms alter, starting with the negative sign. While building a formula for the general term, to accommodate for this change in signs, we may want to use the factor of $(-1)^n$. This is because
$$(-1)^n = \begin{cases} -1, & \text{for } n = 1, 3, 5, \dots \\ 1\ , & \text{for } n = 2, 4, 6, \dots \end{cases}$$
Then we observe that all terms may be seen as fractions with the numerator equal to 1 and denominator matching the index of the term,

$$a_1 = -1 = (-1)^1 \frac{1}{1}$$
$$a_2 = \frac{1}{2} = (-1)^2 \frac{1}{2}$$
$$a_3 = -\frac{1}{3} = (-1)^3 \frac{1}{3}$$
$$a_4 = \frac{1}{4} = (-1)^4 \frac{1}{4}$$
$$a_5 = -\frac{1}{5} = (-1)^5 \frac{1}{5}, \text{ and so on.}$$

The above pattern suggests that the formula $a_n = (-1)^n \frac{1}{n}$ would work for the general term of this sequence. As before, please convince yourself that this is indeed the general term of the sequence by generating the given terms with the aid of the suggested formula.

Sometimes it is difficult to describe a sequence by stating the explicit formula for its general term. For example, in the case of the **Fibonacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, one can observe the rule of obtaining the next term by adding the previous two terms (for terms after the second term), but it would be very difficult to come up with an explicit formula for the general term a_n. Yet the Fibonacci sequence can be defined through the following equations $a_1 = a_2 = 1$ and $a_n = a_{n-2} + a_{n-1}$, for $n \geq 3$. Notice that the n-th term is not given explicitly but it can be found as long as the previous terms are known. In such a case we say that the sequence is **defined recursively**.



Definition 1.2 ► A sequence is defined **recursively** if
- the initial term or terms are given, and
- the n-th term is defined by a formula that refers to the *preceding* terms.

Example 3 ► **Finding Terms of a Sequence Given Recursively**

Find the first 5 terms of the sequence given by the conditions
$a_1 = 1,\ a_2 = 2$, and $a_n = 2a_{n-1} + a_{n-2}$, for $n \geq 3$.

Solution ► The first two terms are given, $a_1 = \mathbf{1}$, $a_2 = \mathbf{2}$. To find the third term, we substitute $n = 3$ into the recursive formula, to obtain

$$a_3 = 2a_{3-1} + a_{3-2} = 2a_2 + a_1 = 2 \cdot \mathbf{2} + \mathbf{1} = \mathbf{5}$$

Similarly
$$a_4 = 2a_{4-1} - a_{4-2} = 2a_3 - a_2 = 2 \cdot \mathbf{5} + \mathbf{2} = \mathbf{12}$$

and
$$a_5 = 2a_{5-1} - a_{5-2} = 2a_4 - a_3 = 2 \cdot \mathbf{12} + \mathbf{5} = \mathbf{29}$$

So the first five terms of this sequence are: **1**, **2**, **5**, **12**, and **29**.

Example 4 ► **Using Sequences in Application Problems**

Peter borrowed \$6000. To pay off this debt, the lender requests monthly payments of \$600 and 1% interest of the unpaid balance from the previous month. If his first payment is due one month from the date of borrowing, find

a. the total number of payments needed to pay off the debt,
b. the sequence of his first four payments,
c. the general term of the sequence of payments,
d. the last payment.

Solution ► **a.** Since Peter pays \$600 off his \$6000 principal each time, the total number of payments is $\frac{6000}{600} = \mathbf{10}$.

b. Let $a_1, a_2, \dots, a_{10}$ be the sequence of Peter's payments.
After the first month, Peter pays $a_1 = \$600 + 0.01 \cdot \$6000 = \mathbf{\$660}$ and the remaining balance becomes $\$6000 - \$600 = \$5400$.
Then, Peter's second payment is $a_2 = \$600 + 0.01 \cdot \$5400 = \mathbf{\$654}$ and the remaining balance becomes $\$5400 - \$600 = \$4800$.
The third payment is equal to $a_3 = \$600 + 0.01 \cdot \$4800 = \mathbf{\$648}$ and the remaining balance becomes $\$4800 - \$600 = \$4200$.
Finally, the fourth payment is $a_4 = \$600 + 0.01 \cdot \$4200 = \mathbf{\$642}$ with the remaining balance of $\$4200 - \$600 = \$3600$.

So the sequence of Peter's first four payments is $\mathbf{\$660, \$654, \$648, \$642.}$

c. Notice that the terms of the above sequence diminish by 6.
We have
$$a_1 = 660 = 660 - \mathbf{0} \cdot 6$$
$$a_2 = 654 = 660 - \mathbf{1} \cdot 6$$
$$a_3 = 648 = 660 - \mathbf{2} \cdot 6$$
$$a_4 = 642 = 660 - \mathbf{3} \cdot 6, \text{ and so on.}$$

Since the blue coefficient by "6" is one lower than the index of the term, we can write the general term as $a_n = 660 - (\boldsymbol{n} - \mathbf{1}) \cdot 6$, which after simplifying can take the form $a_n = 660 - 6n + 6 = \mathbf{666 - 6n}$.

d. Since there are 10 payments, the last one equals to $a_{10} = 666 - 6 \cdot 10 = \$\mathbf{606}$.

Series and Summation Notation

Often, we take interest in finding sums of terms of a sequence. For instance, in *Example 4*, we might be interested in finding the total amount paid in the first four months $\$550 + \$545 + \$540 + \$535$, or the total cost of borrowing $\$550 + \$545 + \dots + \$505$. The terms of a sequence connected by the operation of addition create an expression called a **series**.

Note: *The word "**series**" is both singular and plural.*

Definition 1.3 ► A **series** is the sum of terms of a finite or infinite sequence, before evaluation.
The value of a **finite series** can always be determined because addition of a finite number of values can always be performed.
The value of an **infinite series** may not exist. For example, $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots = 1$ but $1 + 2 + \dots + n + \dots = DNE$ (doesn't exist).

Series involve writing sums of many terms, which is often cumbersome. To write such sums in compact form, we use summation notation referred to as **sigma notation**, where the Greek letter $\boldsymbol{\Sigma}$ (sigma) is used to represent the operation of adding all the terms of a sequence. For example, the finite series $1^2 + 2^2 + 3^2 + \dots + 10^2$ can be recorded in sigma notation as

$$\sum_{i=1}^{10} i^2 \quad \text{or} \quad \sum\nolimits_{i=1}^{10} i^2$$

Here, the letter i is called the **index of summation** and takes integral values from 1 to 10. The expression i^2 (the general term of the corresponding sequence) generates the terms being added. The number 1 is the lower limit of the summation, and the number 10 is the upper limit of the summation. We read "the sum from $i = 1$ to 10 of i^2." To find this sum, we replace the letter i in i^2 with 1, 2, 3, …, 10, and add the resulting terms.

Note: *Any letter can be used for the index of summation; however, the most commonly used letters are i,j,k,m,n.*

A finite series with an unknown number of terms, such as $1 + 2 + \cdots + n$, can be recorded as

$$\sum\nolimits_{i=1}^{n} i$$

Here, since the last term equals to n, the value of the overall sum is an expression in terms of n, rather than a specific number.

An infinite series, such as $0.3 + 0.03 + 0.003 + \cdots$ can be recorded as

$$\sum\nolimits_{i=1}^{\infty} \frac{3}{10^i}$$

In this case, the series can be evaluated and its sum equals to $0.333\ldots = \frac{1}{3}$.

Example 5 ► **Evaluating Finite Series Given in Sigma Notation**

Evaluate the sum.

a. $\sum_{i=1}^{5}(2i + 1)$ **b.** $\sum_{k=1}^{6}(-1)^k \frac{1}{k}$

Solution ►

a. $\sum_{i=0}^{5}(2i + 1) = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1)$

$= 1 + 3 + 5 + 7 + 9 = \mathbf{25}$

b. $\sum_{k=1}^{6}(-1)^k \frac{1}{k} = (-1)^1 \frac{1}{1} + (-1)^2 \frac{1}{2} + (-1)^3 \frac{1}{3} + (-1)^4 \frac{1}{4} + (-1)^5 \frac{1}{5} + (-1)^6 \frac{1}{6}$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{-60+30-20+15-12+10}{60} = -\mathbf{\frac{37}{60}}$

Example 6 ► **Writing Series in Sigma Notation**

Write the given series using sigma notation.

a. $5 + 7 + 9 + \cdots + 47 + 49$ **b.** $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$

Solution ►

a. Observe that the series consists of a sequence of odd integers, from 5 to 49. An odd integer can be represented by the expression $2n + 1$ or $2n - 1$. The first expression assumes a value of 5 for $n = 2$, and a value of 49 for $n = 24$. Therefore, the general term of the series could be written as $a_n = 2n + 1$, for $n = 2,3, \ldots, 24$. Hence, the series might be written in the form

$$\sum_{i=2}^{24}(2i+1)$$

Using the second expression, $2n-1$, the general term $a_n = 2n-1$ would work for $n = 3,4,\dots,25$. Hence, the series might also be written in the form

$$\sum_{i=3}^{25}(2i-1)$$

The sequence can also be written with the index of summation set to start with 1. Then we would have

$$\sum_{i=1}^{23}(2i+3)$$

Check that all of the above sigma expressions produce the same series.

b. In this infinite series $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots$ the signs of consecutive terms alter. To accommodate for the change of signs, we may want to use a factor of $(-1)^n$ or $(-1)^{n+1}$, depending on the sign of the first term. Since the first term is positive, we use the factor of $(-1)^{n+1}$ that equals to 1 for $n = 1$. In addition, the terms consist of fractions with constant numerators equal to 1 and denominators equal to consecutive even numbers that could be represented by $2n$. Hence, the series might be written in the form

$$\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{2i}$$

Notice that by renaming the index of summation to, for example, $k = i-1$, the series takes the form

$$\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2(k+1)}$$

Check on your own that both of the above sigma expressions produce the same series.

Observation: *Series in sigma notation can be written in many different yet equivalent forms. This is because the starting value of the index of summation is arbitrary. Commonly, we start at 1, or 0, unless other values make the general term formula simpler.*

Example 7 ► **Adjusting the Index of Summation**

In each series, change index j to index k that starts at 1.

a. $\sum_{j=2}^{7}(-1)^{j-1}j^3$ **b.** $\sum_{j=0}^{\infty}3^{2j-1}$

Solution ► **a.** If $k = 1$ when $j = 2$, then $j - k = 1$, or equivalently $j = k+1$. In this relation, the upper limit $j = 7$ corresponds to $k = 6$. So by substitution, we obtain

$$\sum_{j=2}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{6}(-1)^{k+1-1}(k+1)^3 = \boldsymbol{\sum_{k=1}^{6}(-1)^{k}(k+1)^3}$$

b. If $k = 1$ when $j = 0$, then $k - j = 1$, or equivalently $j = k - 1$. By substitution, we have

$$\sum_{j=0}^{\infty} 3^{2j-1} = \sum_{k=1}^{\infty} 3^{2(k-1)-1} = \sum_{k=1}^{\infty} \mathbf{3^{2k-3}}$$

Example 8 ► **Using Series in Application Problems**

In reference to Example 4 of this section:

Peter borrowed \$6000. To pay off this debt, the lender requests monthly payments of \$300 and 1% interest of the unpaid balance from the previous month. If his first payment is due one month from the date of borrowing, find

a. the sequence $\{b_n\}$, where b_n represents the remaining balance before the n-th payment,
b. the total interest paid by Peter.

Solution ►

a. As indicated in the solution to *Example 4b,* the sequence of monthly balances before the n-th payment is $6000, 5400, 4800, \ldots, 600$. Since the balance decreases each month by 600, the general term of this sequence is

$$\boldsymbol{b_n} = 6000 - (n-1)600 = \mathbf{6600 - 600}\boldsymbol{n}.$$

b. Since Peter pays 1% on the unpaid balance b_n each month and the number of payments is 20, the total interest paid can be represented by the series

$$\sum_{k=1}^{10} (0.01 \cdot b_k) = \sum_{k=1}^{10} [0.01 \cdot (6600 - 600k)] = \sum_{k=1}^{10} (66 - 6k)$$

$$= 60 + 54 + 48 + 42 + 36 + 30 + 24 + 18 + 12 + 6 = 330$$

Therefore, Peter paid the total interest of **\$330**.

Arithmetic Mean

When calculating the final mark in a course, we often take an average of a sequence of marks we received on assignments, quizzes, or tests. We do this by adding all the marks and dividing the sum into the number of marks used. This average gives us some information about the overall performance on the particular task.

We are often interested in finding averages in many other life situations. For example, we might like to know the average number of calories consumed per day, the average lifetime of particular appliances, the average salary of a particular profession, the average yield from a crop, the average height of a Christmas tree, etc.
In mathematics, the commonly used term "**average**" is called the **arithmetic mean** or just **mean**. If a mean is calculated for a large set of numbers, it is convenient to write it using summation notation.

Definition 1.4 ► The **arithmetic mean (average)** of the numbers $\boldsymbol{x_1, x_2, x_3, \dots, x_n}$, often denoted $\boldsymbol{\bar{x}}$, is given by the formula

$$\boldsymbol{\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

Example 9 ► **Calculating the Arithmetic Mean**

a. Find the arithmetic mean of $-12, 3, 0, 4, -2, 10$.

b. Evaluate the mean $\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5}$.

Solution ► **a.** The arithmetic mean equals

$$\frac{-12 + 3 + 0 + 4 + (-2) + 10}{6} = \frac{3}{6} = \boldsymbol{\frac{1}{2}}$$

b. To evaluate this mean by may want to write out all the terms first. We have

$$\bar{x} = \frac{\sum_{i=1}^{5} i^2}{5} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2}{5} = \frac{1 + 4 + 9 + 16 + 25}{5} = \frac{55}{5} = \mathbf{11}$$

S.1 Exercises

Find the ***first four terms*** *and the* ***10-th*** *term of each infinite sequence whose n-th term is given.*

1. $a_n = 2n - 3$

2. $a_n = \frac{n+2}{n}$

3. $a_n = (-1)^{n-1} 3n$

4. $a_n = 1 - \frac{1}{n}$

5. $a_n = \frac{(-1)^n}{n^2}$

6. $a_n = (n+1)(2n-3)$

7. $a_n = (-1)^n (n-2)^2$

8. $a_n = \frac{1}{n(n+1)}$

9. $a_n = \frac{(-1)^{n+1}}{2n-1}$

Write a formula for the n-th term of each sequence.

10. $2, 5, 8, 11, \dots$

11. $1, -1, 1, -1, \dots$

12. $\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \dots$

13. $3, 9, 27, 81, \dots$

14. $15, 10, 5, 0, \dots$

15. $6, 9, 12, 15, \dots$

16. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{25}, \dots$

17. $1, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \dots$

18. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

Find the first five terms of each infinite sequence given by a recursion formula.

19. $a_n = 2a_{n-1} + 5,\ a_1 = -3$

20. $a_n = 1 - \frac{1}{a_{n-1}},\ a_1 = 2$

21. $a_n = 2a_{n-1} + a_{n-2},\ a_1 = 1, a_2 = 2$

22. $a_n = (a_{n-1})^2 - 1,\ a_1 = 2$

Solve each applied problem by writing the first few terms of a sequence.

23. Lucy borrowed \$2600. To pay off her debt, she agreed to make monthly payments of \$200 and 2% interest on the unpaid balance from the previous month. If her first payment is due one month from the date of borrowing, find her first five payments and the remaining balance at the end of that period.

24. This year, Max was hired by a company that offered him a salary of $24{,}800 + 800(n-1)$ dollars per year at the beginning of the n-th year.

a) Write a sequence representing Max's salary for the first 5 years of work.
b) What is his salary increase per year?
c) Write the first five terms of a sequence of the percent increase in his yearly salary. Round the percentages up to two decimal places.
d) How much would he earn during the tenth year of work?

25. Suppose a penalty for not returning a book to the library on time is 50 cents for the first day plus 30 cents for each additional day after the due date. Write a formula for the penalty p_n on returning the book n days after the due day. What is the penalty for returning the book two weeks late?

26. It is estimated that the value of a one-year-old passenger car depreciates at a 10% annual rate. Corina bought a one-year-old Honda Accord in 2018 for \$19,600.

a. Applying the 10% depreciation rule, determine the value of this car in the years 2019 through 2022. *Round these values to the nearest dollar.*
b. Write a formula for the sequence p_n representing the value of this car n years after 2018.
c. Using the formula from part (**b**), approximate the value of this car in 2030.

27. It is advised that the chlorine level in a swimming pool water is kept between 1 and 3 ppm (parts per million). Assume that the chlorine level decreases by approximately 25% per day.

a. If the chlorine was initially at $a_0 = 3$ ppm level and no chlorine was added, construct a sequence a_n that expresses the amount of chlorine present in the pool water after n days.
b. In how many days the chlorine needs to be added to the pool because its level drops below 1 ppm?

Evaluate each sum.

28. $\sum_{i=1}^{5}(i+2)$

29. $\sum_{i=1}^{10} 5$

30. $\sum_{i=1}^{8}(-1)^i i$

31. $\sum_{i=1}^{4}(-1)^i(2i-1)$

32. $\sum_{i=1}^{6}(i^2-1)$

33. $\sum_{i=3}^{7} 2^i$

34. $\sum_{i=0}^{5}(i-1)\,i$

35. $\sum_{i=2}^{5}\frac{(-1)^i}{i}$

36. $\sum_{i=0}^{7}(i-2)\,(i-3)$

Write each series using sigma notation.

37. $2+4+6+8+10+12$

38. $3-6+9-12+15-18$

39. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots+\frac{1}{50}$

40. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots-\frac{1}{100}$

41. $1+8+27+64+\cdots$

42. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$

In each series change index ***n*** *to index* ***m*** *that starts at* ***1****.*

43. $\sum_{n=0}^{9}(3n-1)$

44. $\sum_{n=3}^{7}2^{n-2}$

45. $\sum_{n=2}^{6}\frac{n}{n+2}$

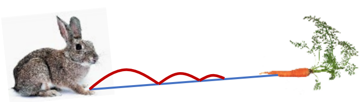
46. $\sum_{n=0}^{\infty}(-1)^{n+1}n$

47. $\sum_{n=2}^{\infty}(-1)^{n}(n-1)$

48. $\sum_{n=2}^{\infty}\frac{1}{n^2}$

Use a series to model the situation in each of the following problems.

49. A rabbit sees a carrot 2 meters away from him. Since he suspects that this could be a trap, he moves slowly towards the carrot by jumping one-third of the way and looking around to determine if it is safe to get closer. He repeats this strategy several times, each time jumping one-third of the remaining distance. For the first three jumps, calculate the length of each jump. Observe the pattern and then, using summation notation, write a series representing the total distance the rabbit has moved after six such jumps.



50. Sonia plans to put \$2000 at the beginning of each year into an account that pays 6% interest compounded annually. Using summation notation, write a series that represents Sonia's predicted savings in this account at the end of the

a. third year;

b. tenth year.

Find the ***arithmetic mean*** *of each sequence, or evaluate the mean written in sigma notation.*

51. $10, 12, 8, 0, 2, 19, 23, 6$

52. $5, -9, 8, 2, -4, 7, 5$

53. $\bar{x}=\frac{\sum_{i=-5}^{5} i}{11}$

54. $\bar{x}=\frac{\sum_{i=1}^{8} 2^{i}}{8}$

S.2 Arithmetic Sequences and Series

Some sequences consist of random values while other sequences follow a certain pattern that is used to arrive at the sequence's terms. In this section, we will take interest in sequences that follow the pattern of having a constant difference between their consecutive terms. Such sequences are called **arithmetic**. For example, the sequence $3, 5, 7, 9, \ldots$ is arithmetic because its consecutive terms always differ by 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **arithmetic** if the difference $d = a_{n+1} - a_n$ between any consecutive terms of the sequence is constantly the same.

The **general term** of an **arithmetic sequence** is given by the formula

$$a_n = a_1 + (n-1)d$$

The difference d is referred to as **the common difference** of the sequence.

Similarly to functions, sequences can be visualized by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $3, 5, 7, 9, \ldots$. Notice that the common difference of 2 makes the graph linear in nature. This is because the slope between the consecutive points of the graph is always the same. Generally, any arithmetic sequence follows a linear pattern with the **slope** being the **common difference** and the ***y*-intercept** being the **first term diminished by the common difference**, as illustrated by *Figure 1*. This means that we should be able to write the general term of an arithmetic sequence by following the slope-intercept equation of a line. Using this strategy, we obtain

$$a_n = \textit{slope} \cdot n + (\textit{y-intercept}) = dn + (a_1 - d) = a_1 + dn - d$$
$$= a_1 + (n-1)d$$

Particularly, the general term of the sequence $3, 5, 7, 9, \ldots$ is equal to $a_n = 3 + (n-1)2$, or equivalently to $a_n = 2n + 1$.

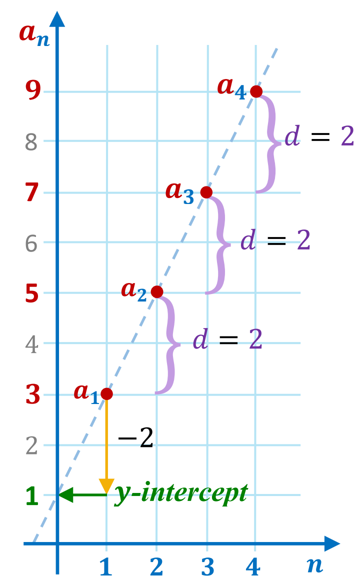


Figure 1

Example 1 ► **Identifying Arithmetic Sequences and Writing its General Term**

Determine whether the given sequence $\{a_n\}$ is arithmetic. If it is, then write a formula for the general term of the sequence.

a. $2{,}4{,}8{,}16, \ldots$ **b.** $3{,}1, -1, -3, \ldots$

Solution ►

a. Since the differences between consecutive terms, $a_2 - a_1 = 4 - 2 = 2$ and $a_3 - a_2 = 8 - 4 = 4$, are not the same, the sequence is not arithmetic.

b. Here, the differences between consecutive terms are constantly equal to -2, so the sequence is arithmetic with $a_1 = 3$, and the common difference $d = -2$. Therefore, using the formula for the general term $a_n = a_1 + (n-1)d$, we have

$$a_n = 3 + (n-1)(-2) = 3 - 2n + 2 = \mathbf{-2n + 5}.$$

Example 2 ► **Finding Terms of an Arithmetic Sequence**

Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 10-th term a_{10}.

a. $a_n = 12 - 3n$ **b.** $a_1 = 3,\ d = 5$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_{\mathbf{1}} = 12 - 3 \cdot \mathbf{1} = 9$$
$$a_{\mathbf{2}} = 12 - 3 \cdot \mathbf{2} = 6$$
$$a_{\mathbf{3}} = 12 - 3 \cdot \mathbf{3} = 3$$
$$a_{\mathbf{4}} = 12 - 3 \cdot \mathbf{4} = 0$$
$$a_{\mathbf{5}} = 12 - 3 \cdot \mathbf{5} = -3$$

So, the first five terms are $\mathbf{9}, \mathbf{6}, \mathbf{3}, \mathbf{0}$, and $-\mathbf{3}$.

The 10-th term equals $a_{\mathbf{10}} = 12 - 3 \cdot \mathbf{10} = -\mathbf{18}$.

b. To find the first five terms of an arithmetic sequence with $\boldsymbol{a_1} = \mathbf{3}$, $\boldsymbol{d} = \mathbf{5}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d} = \mathbf{3} + (\boldsymbol{n} - \mathbf{1})\mathbf{5},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us
$$a_{\mathbf{1}} = 3 + \mathbf{0} \cdot 5 = 3$$
$$a_{\mathbf{2}} = 3 + \mathbf{1} \cdot 5 = 8$$
$$a_{\mathbf{3}} = 3 + \mathbf{2} \cdot 5 = 13$$
$$a_{\mathbf{4}} = 3 + \mathbf{3} \cdot 5 = 18$$
$$a_{\mathbf{5}} = 3 + \mathbf{4} \cdot 5 = 23$$

So, the first five terms are $\mathbf{3}, \mathbf{8}, \mathbf{13}, \mathbf{18}$, and $\mathbf{23}$.

The 10-th term equals $a_{\mathbf{10}} = 3 + 9 \cdot 5 = \mathbf{48}$.

Example 3 ► **Finding the Number of Terms in a Finite Arithmetic Sequence**

Determine the number of terms in the arithmetic sequence $1,5,9,13, \dots, 45$.

Solution ► Notice that the common difference d of this sequence is $5 - 1 = 4$ and the first term $a_1 = 1$. Therefore the n-th term $a_n = 1 + (n-1)4 = 4n - 3$. Since the last term is 45, we can set up the equation

$$a_n = 4n - 3 = 45, \quad \text{and solve it for } n.$$

This gives us
$$4n = 48,$$

and finally
$$n = 12.$$

So, there are 12 terms in the given sequence.

Example 4 **Finding Missing Terms of an Arithmetic Sequence**

Given the information, determine the values of the indicated terms of an arithmetic sequence.

a. $a_5 = 2$ and $a_7 = 8$; find a_6

b. $a_3 = 5$ and $a_{10} = -9$; find a_1 and a_{15}

Solution

a. Let d be the common difference of the given sequence. Since $a_7 = a_6 + d$ and $a_6 = a_5 + d$, then $a_7 = a_5 + 2d$. Hence, $2d = a_7 - a_5$, which gives

$$d = \frac{a_7 - a_5}{2} = \frac{8-2}{2} = 3.$$

Therefore,

$$\boldsymbol{a_6} = a_5 + d = 2 + 3 = \mathbf{5}.$$

Remark: *An **arithmetic mean** of two quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\frac{\boldsymbol{a+b}}{\mathbf{2}}$. Notice that $\boldsymbol{a_6} = 5 = \frac{2+8}{2} = \frac{\boldsymbol{a_5+a_7}}{\mathbf{2}}$, so $\boldsymbol{a_6}$ is indeed the arithmetic mean of $\boldsymbol{a_5}$ and $\boldsymbol{a_7}$. Generally, for any $n > 1$, we have*

$$\boldsymbol{a_n} = a_{n-1} + d = \frac{2a_{n-1} + 2d}{2} = \frac{a_{n-1} + (a_{n-1} + 2d)}{2} = \frac{\boldsymbol{a_{n-1} + a_{n+1}}}{\mathbf{2}},$$

so every term (except for the first one) of an arithmetic sequence is the arithmetic mean of its adjacent terms.

b. As before, let d be the common difference of the given sequence. Using the general term formula $a_n = a_1 + (n-1)d$ for $n = 10$ and $n = 3$, we can set up a system of two equations in two variables, d and a_1:

$$\begin{cases} -9 = a_1 + 9d & (1) \\ 5 = a_1 + 2d & (2) \end{cases}$$

To solve this system, we can subtract the two equations side by side, obtaining

$$-14 = 7d,$$

which gives

$$d = -2.$$

After substitution to equation (2), we have $5 = a_1 + 2 \cdot 2$, which allows us to find the value a_1:

$$\boldsymbol{a_1} = 5 - 4 = \mathbf{1}.$$

To find the value of a_{15}, we substitute $a_1 = 1, d = -2$, and $n = 15$ to the formula $a_n = a_1 + (n-1)d$ to obtain

$$\boldsymbol{a_{15}} = 1 + (15-1)(-2) = 2 - 28 = \mathbf{-26}.$$

Partial Sums

Sometimes, we are interested in evaluating the sum of the first n terms of a sequence. For example, we might be interested in finding a formula for the sum $S_n = 1 + 2 + \cdots n$ of the first n consecutive natural numbers. To do this, we can write this sum in increasing and decreasing order, as below.

$$\begin{aligned} S_n &= \boxed{1} + \boxed{\quad 2 \quad} + \cdots + \boxed{(n-1)} + \boxed{n} \\ S_n &= \boxed{n} + \boxed{(n-1)} + \cdots + \boxed{\quad 2 \quad} + \boxed{1} \end{aligned}$$

Now, observe that the sum of terms in each column is always $(n + 1)$, and there are n columns. Therefore, after adding the two equations side by side, we obtain:

$$2S_n = n(n+1),$$

which in turn gives us a very useful formula

$$S_n = \frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}} \qquad (1)$$

for the sum of the first n consecutive natural numbers.

Figure 2 shows us a geometrical interpretation of this formula, for $n = 6$. For example, to find the area of the shape composed of blocks of heights from 1 to 6, we cut the shape at half the height and rearrange it to obtain a rectangle of length $6 + 1 = 7$ and height $\frac{6}{2} = 3$. This way, the area of the original shape equals to the area of the 7 by 3 rectangle, which according to equation (1), is calculated as $\frac{6(6+1)}{2} = \frac{6}{2} \cdot (6+1) = 3 \cdot 7 = 21$.

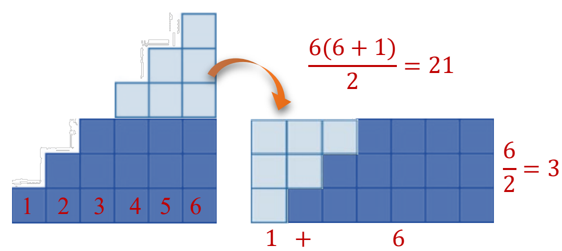


Figure 2

Formally, a partial sum of any sequence is defined as follows:

Definition 1.2 ► Let $\{a_n\}$ be a sequence and $a_1 + a_2 + \cdots + a_n + \cdots$ be its associated series.
The ***n*-th partial sum**, denoted $\boldsymbol{S_n}$, of the sequence (or the series) is the sum

$$\boldsymbol{a_1 + a_2 + \cdots + a_n}.$$

The overall sum of the entire series can be denoted by $\boldsymbol{S_\infty}$.
The **partial sums** on its own create a **sequence** $\{\boldsymbol{S_n}\}$.

Observation:

$$\boldsymbol{S_1 = a_1}$$
$$\boldsymbol{a_n = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = S_n - S_{n-1}}$$

To find the partial sum $\boldsymbol{S_n}$ of the first n terms of an **arithmetic sequence**, as before, we write it in increasing and decreasing order of terms and then add the resulting equations side by side.

$$\begin{aligned} \boldsymbol{S_n} &= \boxed{\boldsymbol{a_1}} + \boxed{\boldsymbol{(a_1 + d)}} + \boxed{\boldsymbol{(a_1 + 2d)}} + \cdots + \boxed{\boldsymbol{(a_1 + (n-1)d)}} \\ \boldsymbol{S_n} &= \boxed{\boldsymbol{a_n}} + \boxed{\boldsymbol{(a_n - d)}} + \boxed{\boldsymbol{(a_n - 2d)}} + \cdots + \boxed{\boldsymbol{(a_n - (n-1)d)}} \end{aligned}$$

each column adds to $\boldsymbol{a_1 + a_n}$ and there are $\boldsymbol{n}$ columns

So, we obtain $\qquad \boldsymbol{2S_n = n(a_1 + a_n)},$

which gives us

$$\boldsymbol{S_n = \frac{n(a_1 + a_n)}{2}} \qquad (2)$$

Notice that by substituting of the general term $a_n = a_1 + (n-1)d$ into the above formula, we can express the partial sum S_n in terms of the first term a_1 and the common difference d, as follows:

$$\boldsymbol{S_n = \frac{n(2a_1 + (n-1)d)}{2} \overset{\textit{or equivalently}}{=} \frac{n}{2}(2a_1 + (n-1)d)} \qquad (3)$$

Example 5 ► **Finding a Partial Sum of an Arithmetic Sequence**

a. Find the sum of the first 100 consecutive natural numbers.
b. Find S_{20}, for the sequence $-10, -5, 0, 5, \dots$.
c. Evaluate the sum $2 + (-1) + (-4) + \cdots + (-25)$.

Solution ► **a.** Using the formula (1) for $n = 100$, we have

$$\boldsymbol{S_{100}} = \frac{100 \cdot (100+1)}{2} = 50 \cdot 101 = \mathbf{5050}.$$

So the sum of the first 100 consecutive natural numbers is 5050.

b. To find S_{20}, we can use either formula (2) or formula (3). We are given $n = 20$ and $a_1 = -10$. To use formula (2) it is enough to calculate a_{20}. Since $d = 5$, we have

$$\boldsymbol{a_{20}} = a_1 + 19d = -10 + 19 \cdot 5 = \mathbf{85},$$

which gives us

$$\boldsymbol{S_{20}} = \frac{20(-10+85)}{2} = 10 \cdot 75 = \mathbf{750}.$$

Alternatively, using formula (3), we also have

$$\boldsymbol{S_{20}} = \frac{20}{2}(2(-10) + 19 \cdot 5) = 10(-20 + 95) = 10 \cdot 75 = \mathbf{750}.$$

c. This time, we are given $a_1 = 2$ and $a_n = -25$, but we need to figure out the number of terms n. To do this, we can use the n-th term formula $a_1 + (n-1)d$ and equal it to -25. Since $d = -1 - 2 = -3$, then we have

$$2 + (n-1)(-3) = -25,$$

which becomes

$$(n-1) = \frac{-27}{-3}$$

and finally

$$n = 10.$$

Now, using formula (2), we evaluate the requested sum to be

$$\boldsymbol{S_{10}} = \frac{10\big(2 + (-25)\big)}{2} = 5 \cdot (-23) = \mathbf{-115}.$$

As we saw in the beginning of this section, an **arithmetic sequence** is **linear** in nature and, as such, it can be identified by the formula $a_n = dn + b$, where $n \in \mathbb{N}$, $d, b \in \mathbb{R}$, and $b = a_1 - d$. This means that the n-th partial sum $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_n$ of the associated arithmetic series can be written as

$$\sum_{i=1}^{n}(di + b),$$

and otherwise; each such sum represents the n-th partial sum S_n of an arithmetic series with the first term $d + b$ and the common difference d. Therefore, the above sum can be evaluated with the aid of formula (2), as shown in the next example.

Example 6 ► **Evaluating Finite Arithmetic Series Given in Sigma Notation**

Evaluate the sum $\sum_{i=1}^{16}(2i - 1)$.

Solution ► First, notice that the sum $\sum_{i=1}^{16}(2i - 1)$ represents S_{16} of an arithmetic series with the general term $a_n = 2n - 1$. Since $a_1 = 2 \cdot 1 - 1 = 1$ and $a_{16} = 2 \cdot 16 - 1 = 31$, then applying formula (2), we have

$$\sum_{i=1}^{16}(2i - 1) = \frac{16(1 + 31)}{2} = 8 \cdot 32 = \mathbf{256}.$$

Example 7 ► **Using Arithmetic Sequences and Series in Application Problems**

A worker is stacking wooden logs in layers. Each layer contains three logs less than the layer below it. There are two logs in the top layer, five logs in the layer below, and so on. If there are 7 layers in the stack, determine

a. the number of logs in the bottom layer;
b. the number of logs in the entire stack.

Solution ►

a. First, we observe that the number of logs in consecutive layers, starting from the top, can be expressed by an arithmetic sequence with $a_1 = 2$ and $d = 3$. Since we look for the number of logs in the seventh layer, we use $n = 7$ and the formula

$$a_n = 2 + (n - 1)3 = 3n - 1.$$

This gives us $a_7 = 3 \cdot 7 - 1 = \mathbf{20}$.
Therefore, there are 20 wooden logs in the bottom layer.

b. To find the total number of logs in the stack, we can evaluate the 7-th partial sum $\sum_{i=1}^{7}(3i - 1)$. Using formula (2), we have

$$\sum_{i=1}^{7}(3i - 1) = \frac{7(2 + 20)}{2} = 7 \cdot 11 = \mathbf{77}.$$

So, the entire stack consists of 77 wooden logs.

S.2 Exercises

True or False?

1. The sequence $3, 1, -1, -3, \dots$ is an arithmetic sequence.

2. The common difference for $2, 4, 2, 4, 2, 4, \dots$ is 2.

3. The series $\sum_{i=1}^{12}(3 + 2i)$ is an arithmetic series.

4. The n-th partial sum S_n of any series can be calculated according to the formula $S_n = \frac{n(a_1+a_n)}{2}$.

Write a formula for the n-th term of each arithmetic sequence.

5. $1, 3, 5, 7, 9, \dots$

6. $0, 6, 12, 18, 24, \dots$

7. $-4, -2, 0, 2, 4, \dots$

8. $5, 1, -3, -7, -11, \dots$

9. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \dots$

10. $1, \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \dots$

Given the information, write out the first five terms of the arithmetic sequence $\{a_n\}$. Then, find the 12-th term a_{12}.

11. $a_n = 3 + (n-1)(-2)$

12. $a_n = 3 + 5n$

13. $a_1 = -8,\ d = 4$

14. $a_1 = 5,\ d = -2$

15. $a_1 = 10,\ a_2 = 8$

16. $a_1 = -7,\ a_2 = 3$

Find the number of terms in each arithmetic sequence.

17. $3, 5, 7, 9, \dots, 31$

18. $0, 5, 10, 15, \dots, 55$

19. $4, 1, -2, \dots, -32$

20. $-3, -7, -11, \dots, -39$

21. $-2, -\frac{3}{2}, -1, -\frac{1}{2}, \dots, 5$

22. $\frac{3}{4}, 3, \frac{21}{4}, \dots, 12$

Given the information, find the indicated term of each arithmetic sequence.

23. $a_2 = 5,\ d = 3;\quad a_8$

24. $a_3 = -4,\ a_4 = -6;\quad a_{20}$

25. $1, 5, 9, 13, \dots;\quad a_{50}$

26. $6, 3, 0, -3, \dots;\quad a_{25}$

27. $a_1 = -8,\ a_9 = -64;\quad a_{10}$

28. $a_1 = 6,\ a_{18} = 74;\quad a_{20}$

29. $a_8 = 28,\ a_{12} = 40;\quad a_1$

30. $a_{10} = -37,\ a_{12} = -45;\quad a_2$

Given the arithmetic sequence, evaluate the indicated partial sum.

31. $a_n = 3n - 8;\quad S_{12}$

32. $a_n = 2 - 3n;\quad S_{16}$

33. $6, 3, 0, -3, \dots;\quad S_9$

34. $1, 6, 11, 16, \dots;\quad S_{15}$

35. $a_1 = 4,\ d = 3;\quad S_{10}$

36. $a_1 = 6,\ a_4 = -2;\quad S_{19}$

Use a formula for S_n to evaluate each series.

37. $1+2+3+\cdots+25$

38. $2+4+6+\cdots+50$

39. $\sum_{i=1}^{17} 3i$

40. $\sum_{i=1}^{22}(5i+4)$

41. $\sum_{i=1}^{15}\left(\frac{1}{2}i+1\right)$

42. $\sum_{i=1}^{20}(4i-7)$

43. $\sum_{i=1}^{25}(-3-2i)$

44. $\sum_{i=1}^{13}\left(\frac{1}{4}+\frac{3}{4}i\right)$

Solve each problem.

45. The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (*12-sided figure*).

46. Parents of a newborn made a resolution to open an account and save some money for the child's education. They plan to deposit \$500 on the first birthday, \$600 on the second birthday, \$700 on the third birthday, and so on until the child's 18th birthday. According to this plan, how much money (disregarding the interest) would be gathered in this account just after the child's 18th birthday?

47. Donovan signed up to a 14-lesson driving course. His first lesson was 15 minutes long, and each subsequent lesson was 5 minutes longer than the lesson before.

a. How long was his 14th lesson?

b. Overall, how long was Donovan's driving training?

48. Suppose the seating in an auditorium is arranged in rows that are increasing in length. If the first row consists of 12 chairs and each consecutive row has 2 more chairs than the previous one, how many seats are in 12th row? What is the total number of seats in all 12 rows?

49. Karissa plans to do her math homework in the Math Centre, where she can get help if stuck with a question. It took her 40 minutes to do the first-week homework. She needed 7 minutes longer during the second week, and she predicted that, on average, she would need about 7 minutes longer each subsequent week than the week before.

a. How much time should she reserve for doing her math homework during the 13th week of the semester?

b. What would be her overall amount of time spent in the Math Centre doing the math homework during the whole 13-week semester?

S.3 Geometric Sequences and Series

In the previous section, we studied sequences where each term was obtained by adding a constant number to the previous term. In this section, we will take interest in sequences where each term is obtained by multiplying the previous term by a constant number. Such sequences are called **geometric**. For example, the sequence $1, 2, 4, 8, \ldots$ is geometric because each term is multiplied by 2 to obtain the next term. Equivallently, the ratios between consecutive terms of this sequence are always 2.

Definition 2.1 ► A sequence $\{a_n\}$ is called **geometric** if the quotient $r = \frac{a_{n+1}}{a_n}$ of any consecutive terms of the sequence is constantly the same.

The **general term** of a **geometric sequence** is given by the formula

$$a_n = a_1 r^{n-1}$$

The quotient r is referred to as **the common ratio** of the sequence.

Similarly as in the previous section, we can be visualize geometric sequences by plotting their values in a system of coordinates. For instance, *Figure 1* presents the graph of the sequence $1,2,4,8, \ldots$. The common ratio of 2 causes each cosecutive point of the graph to be plotted twice as high as the previous one, and the slope between the n-th and $(n+1)$-st point to be exactly equal to the value of a_n. Generally, the **slope** between the n-th and $(n+1)$-st point of any geometric sequence **is proportional** to the value of a_n. This property characterises exponential functions. Hence, geometric sequences are **exponential** in nature. To develop the formula for the general term, we observe the pattern

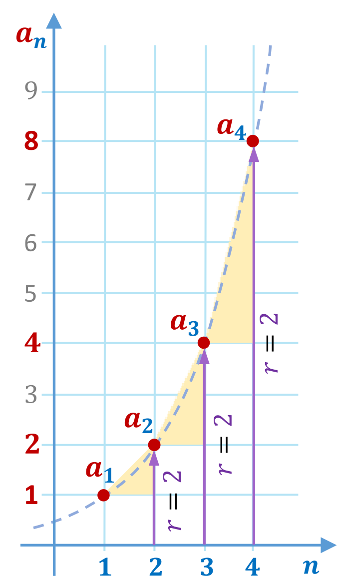


Figure 1

$$a_1 = a_1$$
$$a_2 = a_1 r$$
$$a_3 = a_1 r^2$$
$$a_4 = a_1 r^3$$
$$\vdots$$

so

$$\boxed{a_n = a_1 r^{n-1}}$$

This is the **general term** of a geometric sequence

Particularly, the general term of the sequence $1, 2, 4, 8, \ldots$ is equal to $a_n = 2^{n-1}$, because $a_1 = 1$ and $r = 2$ (the ratios of consecutive terms are constantly equal to 2).

Note: *To find the common ratio of a geometric sequence, divide any of its terms by the preceeding term.*

Example 1 ► **Identifying Geometric Sequences and Writing Their General Terms**

Determine whether the given sequence $\{a_n\}$ is geometric. If it is, then write a formula for the general term of the sequence.

a. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$ **b.** $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots$

Solution ► **a.** After calculating ratios of terms by their peceding terms, we notice that they are always equal to $-\frac{1}{2}$. Indeed, $\frac{a_2}{a_1} = \frac{-\frac{1}{4}}{\frac{1}{3}} = -\frac{1}{2}, \frac{a_3}{a_2} = \frac{\frac{1}{8}}{-\frac{1}{4}} = -\frac{1}{2}$, and so on. Therefore, the given sequence is geometric with $a_1 = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$.

To find its general term, we follow the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_n} = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} = \boldsymbol{\frac{(-1)^{n-1}}{2^n}}.$$

b. Here, the ratios of terms by their peceding terms, $\frac{a_2}{a_1} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$ and $\frac{a_3}{a_2} = \frac{\frac{1}{9}}{\frac{1}{6}} = \frac{2}{3}$, are not the same. So the sequence is not geometric.

Example 2 ► Finding Terms of a Gometric Sequence

Given the information, write out the first five terms of the geometric sequence $\{a_n\}$. Then, find the 8-th term a_8.

a. $a_n = 5(-2)^{n-1}$ **b.** $a_1 = 3,\ r = \frac{2}{3}$

Solution ► **a.** To find the first five terms of this sequence, we evaluate a_n for $n = 1,2,3,4,5$.

$$a_1 = 5(-2)^0 = 5$$
$$a_2 = 5(-2)^1 = -10$$
$$a_3 = 5(-2)^2 = 20$$
$$a_4 = 5(-2)^3 = -40$$
$$a_5 = 5(-2)^4 = 80$$

So, the first five terms are $\mathbf{5}, \mathbf{-10}, \mathbf{20}, \mathbf{-40}$, and $\mathbf{80}$.

The 8-th term equals $a_8 = 5(-2)^7 = \mathbf{-640}$.

b. To find the first five terms of a geometric sequence with $\boldsymbol{a_1} = \mathbf{3}, \boldsymbol{r} = \frac{2}{3}$, we substitute these values into the general term formula

$$\boldsymbol{a_n} = \boldsymbol{a_1 r^{n-1}} = \mathbf{3}\left(\frac{2}{3}\right)^{\boldsymbol{n-1}},$$

and then evaluate it for $n = 1,2,3,4,5$.

This gives us $a_1 = 3\left(\frac{2}{3}\right)^0 = 3$

$$a_2 = 3\left(\frac{2}{3}\right)^1 = 2$$
$$a_3 = 3\left(\frac{2}{3}\right)^2 = \frac{4}{3}$$
$$a_4 = 3\left(\frac{2}{3}\right)^3 = \frac{8}{9}$$
$$a_5 = 3\left(\frac{2}{3}\right)^4 = \frac{16}{27}$$

So, the first five terms are $\mathbf{3}, \mathbf{2}, \frac{\mathbf{4}}{\mathbf{3}}, \frac{\mathbf{8}}{\mathbf{9}}$, and $\frac{\mathbf{16}}{\mathbf{27}}$.

The 8-th term equals $a_8 = 3\left(\frac{2}{3}\right)^7 = \frac{\mathbf{128}}{\mathbf{6561}}$.

Example 3 ► **Finding the Number of Terms in a Finite Geometric Sequence**

Determine the number of terms in the geometric sequence $1, -3, 9, -27, \dots, 729$.

Solution ► Since the common ratio r of this sequence is -3 and the first term $a_1 = 1$, then the n-th term $a_n = (-3)^{n-1}$. Since the last term is 729, we can set up the equation

$$a_n = (-3)^{n-1} = 729,$$

which can be written as

$$(-3)^{n-1} = (-3)^6$$

This equation holds if $\quad n - 1 = 6,$

which gives us $\quad n = 7.$

So, there are 7 terms in the given sequence.

Example 4 ► **Finding Missing Terms of a Geometric Sequence**

Given the information, determine the values of the indicated terms of a geometric sequence.

a. $a_3 = 5$ and $a_6 = -135$; find a_1 and a_8

b. $a_3 = 200$ and $a_5 = 50$; find a_4 if $a_4 > 0$

Solution ► **a.** As before, let r be the common ratio of the given sequence. Using the general term formula $a_n = a_1 r^{n-1}$ for $n = 6$ and $n = 3$, we can set up a system of two equations in two variables, r and a_1:

$$\begin{cases} -135 = a_1 r^5 \\ \quad 5 = a_1 r^2 \end{cases}$$

To solve this system, let's divide the two equations side by side, obtaining

$$-27 = r^3,$$

which gives us

$$r = -3.$$

Substituting this value to equation (2), we have $5 = a_1 \cdot (-3)^2$, which gives us

$$\boldsymbol{a_1} = \frac{\mathbf{5}}{\mathbf{9}}.$$

To find value a_8, we substitute $a_1 = -\frac{5}{2}, r = -3$, and $n = 8$ to the formula $a_n = a_1 r^{n-1}$. This gives us

$$\boldsymbol{a_8} = \frac{5}{9}(-3)^7 = -\mathbf{1215}.$$

b. Let r be the common ratio of the given sequence. Since $a_5 = a_4 r$ and $a_4 = a_3 r$, then $a_5 = a_3 r^2$. Hence, $r^2 = \frac{a_5}{a_3}$. Therefore,

$$r = \pm\sqrt{\frac{a_5}{a_3}} = \pm\sqrt{\frac{50}{200}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}. \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Since $a_3, a_4 > 0$ and $a_4 = a_3 r$, we choose the positive r-value. So we have

$$\boldsymbol{a_4} = a_3 r = 200\left(\frac{1}{2}\right) = \mathbf{100}.$$

Remark: *A **geometric mean** of two quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\sqrt{\boldsymbol{ab}}$. Notice that $\boldsymbol{a_4} = 100 = \sqrt{50 \cdot 200} = \sqrt{\boldsymbol{a_3} \cdot \boldsymbol{a_5}}$, so $\boldsymbol{a_4}$ is indeed the geometric mean of $\boldsymbol{a_3}$ and $\boldsymbol{a_5}$. Generally, for any $n > 1$, we have*

$$\boldsymbol{a_n} = a_{n-1} r = \sqrt{a_{n-1}^2 r^2} = \sqrt{a_{n-1} \cdot a_{n-1} r^2} = \sqrt{\boldsymbol{a_{n-1}} \cdot \boldsymbol{a_{n+1}}},$$

so every term (except for the first one) of a geometric sequence is the geometric mean of its adjacent terms.

Partial Sums

Similarly as with arithmetic sequences, we might be interested in evaluating the sum of the first n terms of a geometric sequence.

To find partial sum $\boldsymbol{S_n}$ of the first n terms of a **geometric sequence**, we line up formulas for $\boldsymbol{S_n}$ and $-\boldsymbol{rS_n}$ as shown below and then add the resulting equations side by side.

$$\begin{array}{rcl} \boldsymbol{S_n} & = & \boldsymbol{a_1} \boxed{+\,\boldsymbol{a_1 r}} \boxed{+\,\boldsymbol{a_1 r^2}} + \cdots \boxed{+\,\boldsymbol{a_1 r^{n-1}}} \\ -\boldsymbol{rS_n} & = & \phantom{\boldsymbol{a_1}} \boxed{-\boldsymbol{a_1 r}} \boxed{-\,\boldsymbol{a_1 r^2}} - \cdots \boxed{-\,\boldsymbol{a_1 r^{n-1}}} - \boldsymbol{a_1 r^n} \\ \hline (\mathbf{1} - \boldsymbol{r})\boldsymbol{S_n} & = & \boldsymbol{a_1} - \boldsymbol{a_1 r^n}, \end{array}$$

the terms of inside columns add to zero, so they cancel each other out

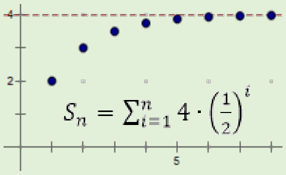
So, we obtain $(\mathbf{1} - \boldsymbol{r})\boldsymbol{S_n} = \boldsymbol{a_1} - \boldsymbol{a_1 r^n}$,

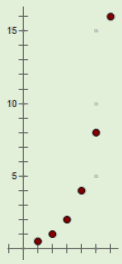
which gives us

$$\boxed{\boldsymbol{S_n} = \frac{\boldsymbol{a_1}(\mathbf{1} - \boldsymbol{r^n})}{\mathbf{1} - \boldsymbol{r}},} \qquad (3)$$

as long as $r \neq 1$.

Observe that





$S_n = \sum_{i=1}^{n} \frac{1}{2} \cdot (2)^i$

If $|\boldsymbol{r}| < \mathbf{1}$, then the value of r^n gets closer and closer to zero for larger and larger n (we write: $r^n \to 0$ for $n \to \infty$). This means that the sum of all infinitely many terms of such a sequence **exists** and is equal to

$$\boldsymbol{S_\infty} = \frac{\boldsymbol{a_1}}{\mathbf{1} - \boldsymbol{r}} \qquad (4)$$

If $|r| > 1$, then the value of $|r^n|$ grows without bound for larger and larger n. Therefore, the sum S_∞ of all terms of such a sequence does not have a finite value. We say that such a sum does not exist.

If $|r| = 1$, then the sum S_∞ becomes $a_1 + a_1 + a_1 + \cdots$, or $a_1 - a_1 + a_1 - \cdots$. Neither of these sums has a finite value, unless $a_1 = 0$.

Hence overall, if $|\boldsymbol{r}| \geq \mathbf{1}$, then the sum $\boldsymbol{S_\infty}$ of a nonzero geometric sequence **does not exist**.

Example 5 ► **Finding a Partial Sum of a Geometric Sequence**

a. Find S_6, for the geometric sequence with $a_1 = 0.5$ and $r = 0.1$.

b. Evaluate the sum $1 - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \cdots - \left(\frac{3}{4}\right)^9$.

Solution ► **a.** Using formula (3) for $n = 6$, $a_1 = 0.5$ and $r = 0.1$, we calculate

$$\boldsymbol{S_6} = \frac{0.5(1 - 0.1^6)}{1 - 0.1} = \frac{0.5 \cdot 0.999999}{0.9} = \mathbf{0.555555}.$$

b. First, we observe that the given series is geometric with $a_1 = 1$ and $r = -\frac{3}{4}$. Equating the formula for the general term to the last term of the sum

$$a_1 r^{n-1} = \left(-\frac{3}{4}\right)^{n-1} = -\left(\frac{3}{4}\right)^9 = \left(-\frac{3}{4}\right)^9$$

and comparing the exponents,

$$n - 1 = 9$$

allows us to find the number of terms $n = \mathbf{10}$.

Now, we are ready to calculate the sum of the given series

$$\boldsymbol{S_{10}} = \frac{1\left(1 - \left(-\frac{3}{4}\right)^{10}\right)}{1 - \left(-\frac{3}{4}\right)} \cong \mathbf{0.539249}$$

Example 6 ► **Evaluating Infinite Geometric Series**

Decide wether or not the overall sum S_∞ of each geometric series exists and if it does, evaluate it.

a. $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \cdots$ **b.** $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i$

Solution ►

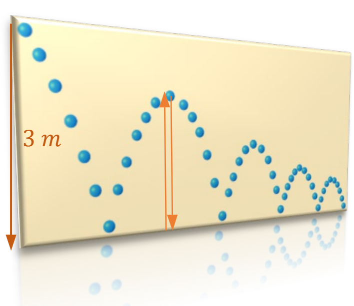
a. Since the common ratio of this series is $|r| = \left|\frac{-\frac{9}{2}}{3}\right| = \left|-\frac{3}{2}\right| = \frac{3}{2} > 1$, then the sum S_∞ does not exist.

b. This time, $|r| = \frac{2}{3} < 1$, so the sum S_∞ exists and can be calculated by following the formula (4). Using $a_1 = 3$ and $r = \frac{2}{3}$, we have

$$S_\infty = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = \mathbf{9}$$

So, $\sum_{i=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^i = \mathbf{9}$.

Example 7 ► **Using Geometric Sequences and Series in Application Problems**



A ball is dropped from a window that is 3 meters above the ground. Suppose the ball always rebounds to $\frac{3}{4}$ of its previous height.

a. To the nearest centimeter, determine the hight that the ball can attain (the rebound height) after the third bounce.
b. Find a formula for the n-th rebound height of the ball.
c. Assuming that the ball bounces forever, what is the total vertical distance travelled by the ball?

Solution ►

a. Let h_n represents the ball's rebound height after the n-th bounce, where $n \in \mathbb{N}$. Since the ball rebounds $\frac{3}{4}$ of the previous height, we have

$$h_1 = 3 \cdot \left(\frac{3}{4}\right)$$

$$h_2 = h_1 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^2$$

$$h_3 = h_2 \cdot \left(\frac{3}{4}\right) = 3 \cdot \left(\frac{3}{4}\right)^3 \simeq 1.266\ m \simeq 127\ cm$$

After the third bounce, the ball will rebound approximately **127 centimeters**.

b. Notice that the formulas developed in solution to *Example 4a* follow the pattern

$$\boldsymbol{h_n = 3 \cdot \left(\frac{3}{4}\right)^n}$$

So, this is the formula for the rebound height of the ball after its n-th bounce.

c. Let $h_0 = 3$ represent the vertical distance before the first bounce. To find the total verical distance D travelled by the ball, we add the vertical distance h_0 before the first bounce and twice the vertical distances h_n after each bounce. So, we have

$$D = h_0 + \sum_{n=1}^{\infty} h_n = 3 + \sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{4}\right)^n$$

Applying the formula $\frac{a_1}{1-r}$ for the infinite sum of a geometric series, we calculate

$$D = 3 + \frac{\frac{9}{4}}{1 - \frac{3}{4}} = 3 + \frac{\frac{9}{4}}{\frac{1}{4}} = 3 + \frac{9}{\cancel{4}} \cdot \frac{\cancel{4}}{1} = 3 + 9 = 12\ m$$

Thus, the total vertical distance travelled by the ball is **12 meters**.

S.3 Exercises

True or False?

1. The sequence $3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$ is a geometric sequence.

2. The common ratio for $0.05, 0.0505, 0.050505, \ldots$ is 0.05 .

3. The series $\sum_{i=1}^{7}(3 \cdot 2^i)$ is a geometric series.

4. The n-th partial sum S_n of any finite geometric series exists and it can be evaluated by using the formula $S_n = \frac{a_1(1-r^n)}{1-r}$.

Identify whether or not the given sequence is geometric. If it is, write a formula for its n-th term.

5. $0, 3, 9, 27, \ldots$

6. $1, 5, 25, 125, \ldots$

7. $-9, 3, -1, \frac{1}{3}, \ldots$

8. $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \ldots$

9. $1, -1, 1, -1, \ldots$

10. $0.9, 0.09, 0.009, 0.0009, \ldots$

11. $81, -27, 9, -3, \ldots$

12. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

13. $-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}, -\frac{16}{125}, \ldots$

Given the information, write out the first four terms of the geometyric sequence $\{a_n\}$. Then, find the 8-th term a_8.

14. $a_n = 3 \cdot 2^{n-1}$

15. $a_n = (-2)^{-n}$

16. $a_1 = 6,\ r = \frac{1}{3}$

17. $a_1 = 5,\ r = -1$

18. $a_1 = \frac{1}{3},\ a_2 = -\frac{1}{6}$

19. $a_1 = 100,\ a_2 = 10$

Find the number of terms in each geometric sequence.

20. $1, 2, 4, \dots, 1024$

21. $20, 10, 5, \dots, \frac{5}{128}$

22. $-4, 2, -1, \dots, \frac{1}{32}$

23. $3, -1, \frac{1}{3}, \dots, \frac{1}{243}$

24. $6, -2, \frac{2}{3}, \dots, -\frac{2}{81}$

25. $-24, 12, -6, \dots, -\frac{3}{32}$

Given the information, find the indicated term of each geometric sequence.

26. $a_2 = 40,\ r = 0.1;\quad a_5$

27. $a_3 = 4,\ a_4 = -8;\quad a_{10}$

28. $2, -2, 2, -2, \dots;\quad a_{50}$

29. $-4, 2, -1, \dots;\quad a_{12}$

30. $a_1 = 6,\ a_4 = -\frac{2}{9};\quad a_8$

31. $a_1 = \frac{1}{9},\ a_6 = 27;\quad a_9$

32. $a_3 = \frac{1}{2},\ a_7 = \frac{1}{32};\quad a_4$ if $a_4 > 0$

33. $a_5 = 48,\ a_8 = -384;\quad a_{10}$

Given the geometric sequence, evaluate the indicated partial sum. Round your answer to three decimal places, if needed.

34. $a_n = 5\left(\frac{2}{3}\right)^{n-1};\quad S_6$

35. $a_n = -2\left(\frac{1}{4}\right)^{n-1};\quad S_{10}$

36. $2, 6, 18, \dots;\quad S_8$

37. $6, 3, \frac{3}{2}, \dots;\quad S_{12}$

38. $1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \cdots + \left(\frac{1}{5}\right)^5$

39. $1 - 3 + 3^2 - \cdots - 3^9$

40. $\sum_{i=1}^{7} 2(1.05)^{i-1}$

41. $\sum_{i=1}^{10} 3(2)^{i-1}$

Decide wether or not the infinite sum S_∞ of each geometric series exists and if it does, evaluate it.

42. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

43. $1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots$

44. $1 + 1.02 + 1.02^2 + 1.02^3 + \cdots$

45. $1 + 0.8 + 0.8^2 + 0.8^3 + \cdots$

46. $\sum_{i=1}^{\infty}(0.6)^{i-1}$

47. $\sum_{i=1}^{\infty}\frac{2}{5}(1.1)^{i-1}$

48. $\sum_{i=1}^{\infty} 2\left(\frac{4}{3}\right)^{i}$

49. $\sum_{i=1}^{\infty} 2\left(-\frac{3}{4}\right)^{i}$

Solve each problem.

50. Suppose that you got an offer to work for a company that pays \$42,000 for the first year of work and a 3% increase in the previous year salary for each consecutive year of employment. What would your salary be in the 15th year of your career?

51. Johny asked his parents to pay him for helping them with yard work for the next two weeks. He asked to be paid 10 cents for the first day and double the daily wage for each consecutive day of his work. If the parents agree to this arrangement, how much will Johny be paid on the 14th day of work? How much will he earn in total for the two-week work?

52. Suppose you deposit \$2000 at the beginning of each year for 35 years into an account that pays the interest of 6% compounded annually. How much would you have in this account at the end of the thirty-fifth year?

53. Matilda's parents decided to save some money for her future education. Each year on Matilda's birthday, they are going to deposit \$1000 into an account that pays 3.5% interest compounded annually. If the first deposit is made on her first birthday, determine the amount of money in the account on her 18^{th} birthday, just before the 18th deposit is made.

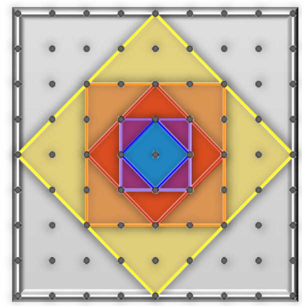
54. A ball is dropped from 2 meters above the ground. With each bounce, the ball comes back to 80% of its previous height.

a. To the nearest millimeter, how high does the ball bounce just after the fifth bounce?

b. To the nearest centimeter, what is the total vertical distance that the ball has travelled when it hits the ground for the tenth time?

55. Suppose that a ball always rebounds $\frac{3}{5}$ of its previous height. The ball was dropped from a height of 3 meters. Approximate the total vertical distance travelled by the ball before it comes to rest.

56. Suppose an infinite sequence of squares is constructed as follows: The first square, s_1, is a **unit square** (*a square whose sides are one unit in length*). The second square, s_2, is constructed by connecting the midpoints of the sides of the first square (*see the yellow square in the accompanying figure*). Generally, the $(n+1)^{\text{st}}$ square, s_{n+1}, is constructed by connecting the midpoints of the previous square, s_n. What is the sum of areas of the infinite sequence of squares $\{s_n\}$?



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Additional Functions, Conic Sections, and Nonlinear Systems

Relations and functions are an essential part of mathematics as they allow to describe interactions between two or more variable quantities. In this chapter, we will give a quick overview of the commonly used functions, their properties such as domain and range, and some of their transformations, particularly translations. Some of these functions (i.e., linear, quadratic, square root, and reciprocal functions) were already discussed in detail in *Sections G1*, *Q3*, *RD1*, and *RT5*. In *Section C1*, we will explore some additional functions, such as absolute value or greatest integer functions, as well as functions of the form $\frac{1}{f(x)}$ or $|f(x)|$.

Aside from new functions, we will discuss equations and graphs of commonly used relations such as circles, ellipses, and hyperbolas. These relations are known as conic sections as their graphs have the shape of a curve formed by the intersection of a cone and a plane. Conic sections are geometric representations of quadratic equations in two variables and as such, they include parabolas. Thus, studying conic sections is an extension of studying parabolas. When working with conic sections, we are often in need of finding intersection points of given curves. Thus, at the end of this chapter, we will discuss solving systems of nonlinear equations as well as nonlinear inequalities.

C1 Properties and Graphs of Additional Functions

The graphs of some **basic functions**, such as

$$f(x) = x^2,\ \ f(x) = |x|, f(x) = \sqrt{x}, \text{ or } f(x) = \frac{1}{x},$$

were already presented throughout this text. Knowing the shapes of the graphs of these functions is very useful for graphing related functions, such as $g(x) = |x| - 2$ or $f(x) = \sqrt{x+1}$. In *Section Q3*, we observed that the graph of function $g(x) = (x-p)^2 + q$ could be obtained by translating a graph of the basic parabola p units horizontally and q units vertically. This observation applies to any function $f(x)$.

To graph a function $\boldsymbol{f(x-a)+b}$, it is enough to translate the graph of $\boldsymbol{f(x)}$ by $\boldsymbol{a}$ **units horizontally** and $\boldsymbol{b}$ **units vertically**.

Examine the relations between the defining formula of a function and its graph in the following examples.

Basic Functions and Their Translations

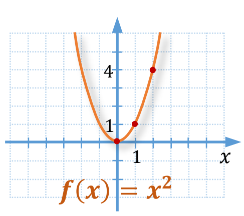


Figure 1.1a

Parabola $f(x) = x^2$

Recall the shape of the graph of the basic parabola $f(x) = x^2$, as in *Figure 1.1a*. The domain of this function is $\mathbb{R}$, and the range is the interval $[0, \infty)$.

The graph of the basic parabola can be used to graph other quadratic functions such as $\boldsymbol{g(x) = (x-3)^2 - 1}$. Function g

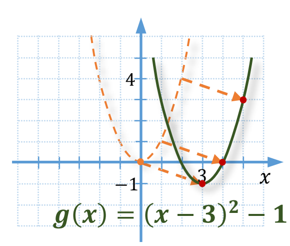


Figure 1.1b

can be graphed by translating the basic parabola **3 units to the right** and **1 unit down**, as in *Figure 1.1b*.

Observe that under this translation,

- the vertex $(0,0)$ of the basic parabola is moved to the **vertex** $(\mathbf{3}, -\mathbf{1})$ of function g;
- the **domain** of function g remains unchanged, and it is still $\mathbb{R}$;
- the **range** of function g is the interval $[-\mathbf{1}, \infty)$ as a result of the translation of the range $[0, \infty)$ of the basic parabola by 1 unit down.

Absolute Value $f(x) = |x|$

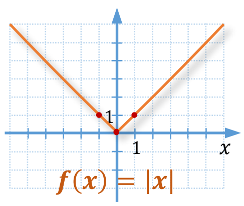


Figure 1.2a

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

Using a table of values, we can graph the basic absolute value function, $f(x) = |x|$, as in *Figure 1.2a*. The domain of this function is $\mathbb{R}$, and the range is the interval $[0, \infty)$. Similarly as in the case of the basic parabola, the lowest point, called the vertex, is at $(0,0)$.

The graph of the basic absolute value function can be used to graph other absolute value functions such as $\boldsymbol{g(x) = |x+1| + 2}$. Function g can be graphed by translating function $f(x) = |x|$ by **1 unit to the left** and **2 units up**, as in *Figure 1.2b*.

x	$g(x)$
-3	4
-2	3
-1	2
0	3
1	4

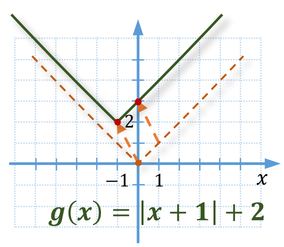


Figure 1.2b

Observe that under this translation,

- the vertex $(0,0)$ of function f is moved to the **vertex** $(-\mathbf{1}, \mathbf{2})$ of function g;
- the **domain** of function g remains unchanged and it is still $\mathbb{R}$;
- the **range** of function g is the interval $[\mathbf{2}, \infty)$, as a result of the translation of the range $[0, \infty)$ of function f by 2 units up.

Square Root $f(x) = \sqrt{x}$

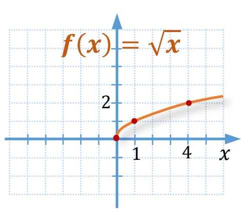


Figure 1.3a

x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

Using a table of values, we can graph the basic square root function, $f(x) = \sqrt{x}$, as in *Figure 1.3a*. The domain of this function is the interval $[0, \infty)$, and the range is also $[0, \infty)$. The curve starts at the origin $(0,0)$.

The graph of the basic square root function can be used to graph other square root functions such as $\boldsymbol{g(x) = \sqrt{x+1} - 2}$. Function g can be graphed by translating function $f(x) = \sqrt{x}$ by **1 unit to the left** and **2 units down**, as in *Figure 1.3b*.

x	$g(x)$
-2	2
-1	1
0	0
1	1
2	2

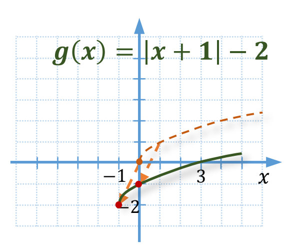


Figure 1.3b

Observe that under this translation,

- the initial point $(0,0)$ of function f is moved to the **initial point** $(-\mathbf{1}, \mathbf{2})$ of function g;
- the **domain** of function g is moved to $[-\mathbf{1}, \infty)$, by subtracting 1 from all domain values $[0, \infty)$ of function f;
- the **range** of function g is moved to $[-\mathbf{2}, \infty)$, by subtracting 2 from all range values $[0, \infty)$ of function f.

Cubic $f(x) = x^3$

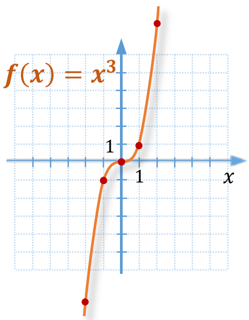


Figure 1.4a

x	$f(x)$
-2	-8
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	1
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8

x	$g(x)$
-2	2
-1	1
0	0
1	1
2	2

Using a table of values, we can graph the basic cubic function, $f(x) = x^3$, as in *Figure 1.4a*. The domain and range of this function are both $\mathbb{R}$. The curve is symmetric about the origin $(0,0)$.

The graph of the basic cubic function can be used to graph other cubic functions such as $\boldsymbol{g(x) = (x-3)^3 - 2}$. Function g can be graphed by translating function $f(x) = x^3$ by **3 units to the right** and **2 units down**, as in *Figure 1.4b*.

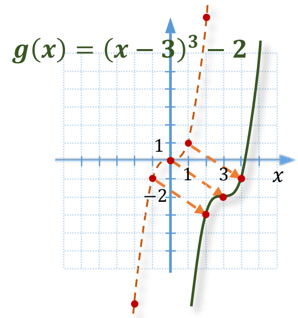


Figure 1.4b

Observe that under this translation,

- the central point $(0,0)$ of function f is moved to the **central point** $\boldsymbol{(3,-2)}$ of function g;
- the **domain** of function g remains unchanged, and it is still $\mathbb{R}$;
- the **range** of function g remains unchanged, and it is still $\mathbb{R}$.

Reciprocal $f(x) = \frac{1}{x}$

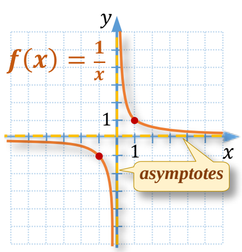


Figure 1.5a

x	$f(x)$
-4	$-\frac{1}{4}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	*undefined*
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

x	$g(x)$
0	$\frac{1}{2}$
1	0
$\frac{3}{2}$	-1
2	*undefined*
$\frac{5}{2}$	3
3	2
4	$\frac{3}{2}$

Using a table of values, we can graph the basic reciprocal function, $f(x) = \frac{1}{x}$, as in *Figure 1.5a*. The domain and range of this function is the set of all real numbers except for zero, $\mathbb{R} \setminus \{0\}$. The graph consists of two curves that are approaching two asymptotes, the horizontal asymptote $y = 0$ and the vertical asymptote $x = 0$.

The graph of the basic reciprocal function can be used to graph other reciprocal functions such as $g(x) = \frac{1}{x-2} + 1$. Function g can be graphed by translating function $f(x) = \frac{1}{x}$ by 2 units to the right and 1 unit up, as in *Figure 1.5b*.

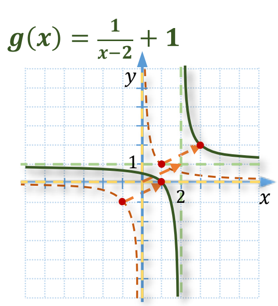


Figure 1.5b

Observe that under this translation,

- the **horizontal asymptote** of function f is moved **1 unit up**, and the **vertical asymptote** of function f is moved **2 units to the right**;
- the **domain** of function g is moved to $\mathbb{R} \setminus \{\mathbf{2}\}$, by adding 2 to all domain values $\mathbb{R} \setminus \{0\}$ of function f;
- the **range** of function g is moved to $\mathbb{R} \setminus \{\mathbf{1}\}$, by adding 1 to all range values $\mathbb{R} \setminus \{0\}$ of function f.

Greatest Integer $f(x) = [\![x]\!]$

Definition 1.1 ► The greatest integer, denoted $[\![x]\!]$, of a real number x is the **greatest integer that does not exceed this number** x. For example,

$$[\![0.9]\!] = 0, \qquad [\![1]\!] = 1, \qquad [\![1.1]\!] = 1, \qquad [\![1.9]\!] = 1$$

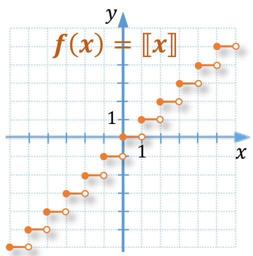


Figure 1.6a

x	$f(x)$
-1.5	-2
-1.1	-2
-1	-1
-0.5	-1
-0.1	-1
0	0
0.5	0
0.9	0
1	1
1.2	1
2	2

x	$g(x)$
-0.5	1
0	2
0.5	2
1	3
1.5	3

Using a table of values, we can graph the basic greatest integer function, $f(x) = [\![x]\!]$, as in *Figure 1.6a*. The domain of this function is the set of real numbers $\mathbb{R}$ while the range is the set of integers $\mathbb{Z}$. The graph consists of infinitely many half-open segments that line up along the diagonal, $y = x$.

The graph of the basic greatest integer function can be used to graph other greatest integer functions such as $\boldsymbol{g(x) = [\![x]\!] + 2}$. Function g can be graphed by translating function $f(x) = [\![x]\!]$ by 1 unit up, as in *Figure 1.6b*.

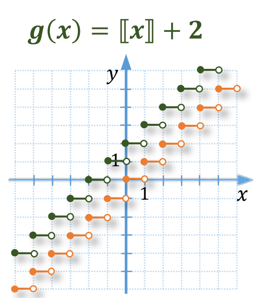


Figure 1.6b

Observe that under this translation,

- the segments of the graph g line up along the line $y = x + 2$;
- the **domain** of function g remains unchanged, and it is still $\mathbb{R}$;
- the **range** of function g remains unchanged, and it is still the set of all integers $\mathbb{Z}$;

Other Transformations of Basic Functions

Aside from translating, graphs can be transformed by flipping them along x- or y-axis, or stretching or shrinking (dilating) in different directions. In the next two examples, observe the relation between the defining formula of a function and the graph transformation of the corresponding basic function.

Example 1 ► **Graphing Functions and Identifying Transformations**

Graph each function. State the transformation(s) of the corresponding basic function that would result in the obtained graph. Then, describe the main properties of the function, such as domain, range, vertex, asymptotes, and symmetry, if applicable.

a. $f(x) = -\frac{1}{x}$ **b.** $f(x) = 2[\![x]\!]$ **d.** $f(x) = \frac{1}{2}|x| - 3$

Solution ► **a.** To graph $f(x) = -\frac{1}{x}$, we first observe that this is a modified reciprocal function. So, we expect that the graph might have some asymptotes.

Since we cannot divide by zero, $x = 0$ does not belong to the domain of this function. This suggests that the graph may have a vertical asymptote, $x = 0$. Also, since the numerator of the fraction $-\frac{1}{x}$ is never equal to zero, then function $f(x) = -\frac{1}{x}$ would never assume the value of zero. So, zero is out of the range of this function. This suggests that the graph may have a horizontal asymptote, $y = 0$.

After graphing the two asymptotes and plotting a few points of the graph, we obtain the final graph, as in *Figure 1.7*.

x	$f(x)$
-2	$\frac{1}{2}$
-1	1
$-\frac{1}{2}$	2
0	*undefined*
$\frac{1}{2}$	-2
1	-1
2	$-\frac{1}{2}$

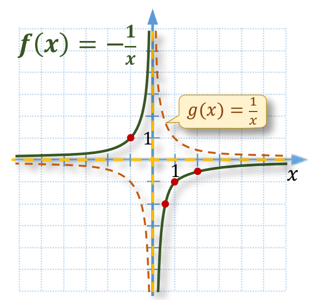


Figure 1.7

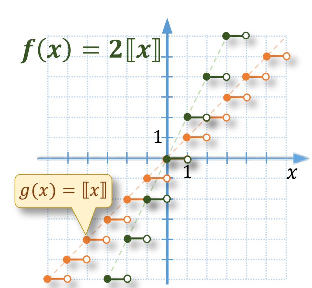
Notice that the graph of function $f(x) = -\frac{1}{x}$ could be obtained by **reflecting** the graph of the basic reciprocal function, $g(x) = \frac{1}{x}$, in the ***x*-axis**.

Function f has the following properties:
Domain: $\mathbb{R} \setminus \{\mathbf{0}\}$
Range: $\mathbb{R} \setminus \{\mathbf{0}\}$
Equations of asymptotes: $\boldsymbol{x = 0}$, $\boldsymbol{y = 0}$
Symmetry: The graph is **symmetrical with respect to the origin**.

b. To graph $f(x) = 2[\![x]\!]$, first, we observe that this is a modified greatest integer function. So, we expect that the graph will consist of half-open segments that line up along a certain line.

Notice that for every x, the value of function f is obtained by multiplying the corresponding value of function $g(x) = [\![x]\!]$ by the factor of 2. Since the segments of the graph of the basic greatest integer function line up along the line $y = x$, we may predict that the segments of the graph of function $f(x) = 2[\![x]\!]$ would line up along the line $y = 2x$. This can be confirmed by calculating and plotting a sufficient number of points, as below.

x	$f(x)$
-0.5	-2
0	0
0.5	0
1	2
1.9	2
2	4



Notice that the graph of function $f(x) = 2[\![x]\!]$ could be obtained by **stretching** the graph of the basic greatest integer function, $g(x) = [\![x]\!]$, in $\boldsymbol{y}$**-axis** by a factor of **2**.

Function f has the following properties:
Domain: $\mathbb{R}$
Range: $\mathbb{Z}$
The segments of the graph line up along the line $y = 2x$.

c. To graph $f(x) = \frac{1}{2}|x| - 3$, first, we observe that this is a modified absolute value function. So, we expect a "V" shape for its graph.

Notice that for every x, the value of function f is obtained by multiplying the corresponding value of the basic absolute value function $g(x) = |x|$ by a factor of $\frac{1}{2}$, and then subtracting 3. Observe how these operations impact the vertex $(0,0)$ of the basic "V" shape. Since the y-value of the vertex is zero, multiplying it by a factor $\frac{1}{2}$ does not change its position. However, subtracting 3 from the y-value of zero causes the vertex to move to $(0, -3)$.

After plotting the vertex and a few more points, as computed in the table below, we obtain the final graph, as illustrated in *Figure 1.8*.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\mathbf{-2}$	$\frac{1}{2}$
$\mathbf{-1}$	1
$\mathbf{-\frac{1}{2}}$	2
$\mathbf{0}$	$undefined$
$\mathbf{\frac{1}{2}}$	-2
$\mathbf{1}$	-1
$\mathbf{2}$	$-\frac{1}{2}$

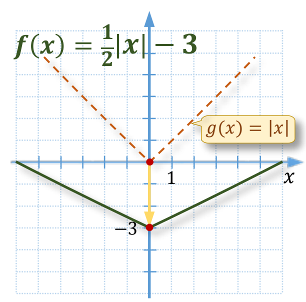


Figure 1.8

Notice that the obtained shape is wider than the shape of the basic absolute value graph. This is because the slopes of the linear sections of the graph are half as steep. So, the function $f(x) = \frac{1}{2}|x| - 3$ could be obtained by

- **compressing** the graph of the basic absolute value function, $g(x) = |x|$, in $\boldsymbol{y}$**-axis** by a factor of $\frac{1}{2}$, and then
- **translating** the resulting graph by **3 units down**.

Function f has the following properties:
Domain: $\mathbb{R}$
Range: $[\mathbf{-3}, \infty)$
Vertex: $(\mathbf{0}, \mathbf{-3})$
Symmetry: The graph is **symmetrical with respect to the** $\boldsymbol{y}$**-axis**.

Generally, to graph a function $\boldsymbol{kf(x)}$, it is enough to **dilate** the graph of $\boldsymbol{f(x)}$, $\boldsymbol{k}$ times in **y-axis**. This dilation is a

- **stretching**, if $|\boldsymbol{k}| > \mathbf{1}$
- **compressing**, if $\mathbf{0} < |\boldsymbol{k}| < \mathbf{1}$
- **flipping** over the $\boldsymbol{x}$**-axis**, if $\boldsymbol{k} = \mathbf{-1}$

Functions of the form $\frac{1}{f(x)}$ or $|f(x)|$

Consider the graphs of a linear function, $f(x) = x + 2$, and its reciprocal, $g(x) = \frac{1}{x+2}$, as illustrated in *Figure 1.9*.

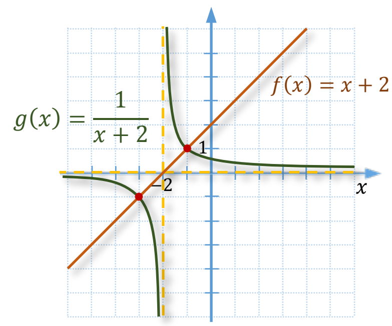


Figure 1.9

Notice that:

- the reciprocal function (in green) has its vertical asymptote at the x-intercept of the linear function (in orange);
- the horizontal asymptote of the reciprocal function $g(x) = \frac{1}{x+2}$ is the x-axis, $y = 0$;
- the points with the y-coordinate equal to 1 or -1 are common for both functions;
- the reciprocal of values close to zero are far away from zero while the reciprocals of values that are far away from zero are close to zero;
- the values of the reciprocal function are of the same sign as the corresponding values of the linear function.

$\boldsymbol{f(x) \to \frac{1}{f(x)}}$

Generally, the graph of the reciprocal of a linear function, $\boldsymbol{g(x) = \frac{1}{ax+b}}$, has the **$x$-axis** as its **horizontal asymptote** and $\boldsymbol{y = -\frac{b}{a}}$ as its **vertical asymptote**.

Now, consider the graphs of the quadratic function $f(x) = x^2 + x - 2$ and the absolute value of this function $g(x) = |x^2 + x - 2|$, as illustrated in *Figure 1.10*.

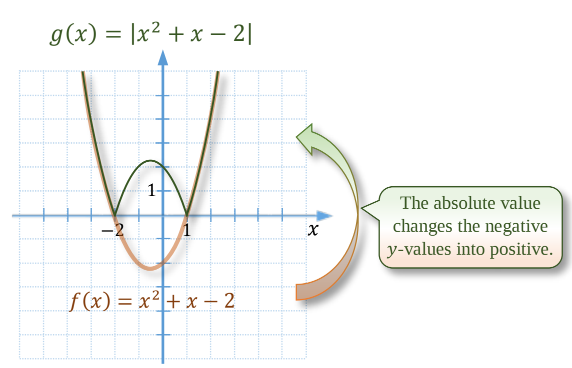


Figure 1.10

Notice that the absolute value function g (in green) follows the original function f (in orange) wherever function f assumes positive values. Otherwise, function g assumes opposite values to function f. So, the graph of the absolute value function g can be obtained by flipping the negative section (section below the x-axis) of the graph of the quadratic function f over the x-axis and leaving the positive sections (above the x-axis) unchanged.

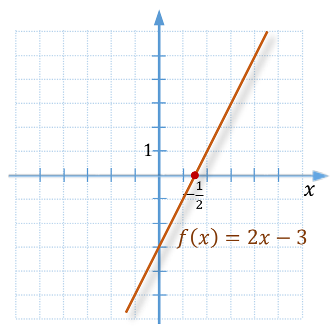
$\boldsymbol{f(x) \to |f(x)|}$

Generally, the graph of the absolute value of any given function $f(x)$, $\boldsymbol{g(x) = |f(x)|}$, can be obtained by **flipping the section(s)** of the graph of the original function f **below the x-axis over the x-axis** and leaving the section(s) above the x-axis unchanged.

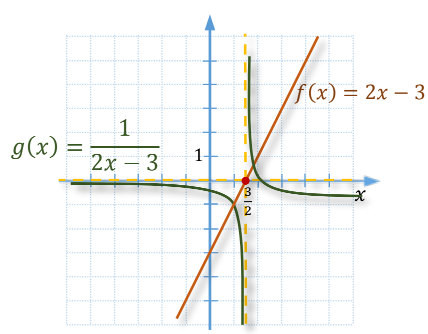
Example 2 ► **Graphing the Reciprocal of a Linear Function**

Using the graph of $f(x) = 2x - 3$, graph function $g(x) = \frac{1}{f(x)} = \frac{1}{2x+3}$. Determine the x-intercept of function f and the equation of the vertical asymptote of function g.

Solution ► First, we graph function $f(x) = 2x - 3$ as below.



Then, we plot a few 'reciprocal' points. For example, since point $(0, -3)$ belongs to function f, then point $\left(0, -\frac{1}{3}\right)$ must belong to function g. Notice that points $(1, -1)$ and $(2,1)$ are common to both functions, as the reciprocals of -1 and 1 are the same numbers -1 and 1. The graph of function g arises by joining the obtained 'reciprocal' points, as illustrated below.



The equation of the vertical asymptote of the graph of function g is $x = \frac{3}{2}$, and it crosses the x-axis at the x-intercept of function f, which is $\left(\frac{3}{2}, 0\right)$.

Example 3 ► **Graphing the Absolute Value of a Given Function**

Using the graph of $f(x) = 2x - x^2$, graph function $g(x) = |f(x)| = |2x - x^2|$.

Solution ► To graph function $g(x) = |2x - x^2|$, we may graph function $f(x) = 2x - x^2$ first. Since $2x - x^2 = x(2 - x)$ then the x-intercepts of this parabola are at $x = 0$ and $x = 2$. The first coordinate of the vertex is the average of the two intercepts, so it is 1. Since $f(1) = 1$, then the parabola has its vertex at the point (1,1). So, the graph of function f can be obtained by connecting the two intercepts and the vertex with a parabolic curve. See the orange graph in *Figure 1.11*.

Since $|f(x)| = \begin{cases} f(x) & if \quad f(x) > 0 \\ -f(x) & if \quad f(x) < 0 \end{cases}$, then the graph of function $g(x) = |f(x)|$ is obtained by

- following the (orange) graph of f for the parts where this graph is above the x-axis and
- flipping the parts of the orange graph that lie below the x-axis over the x-axis, as illustrated in green in *Figure 1.11*.

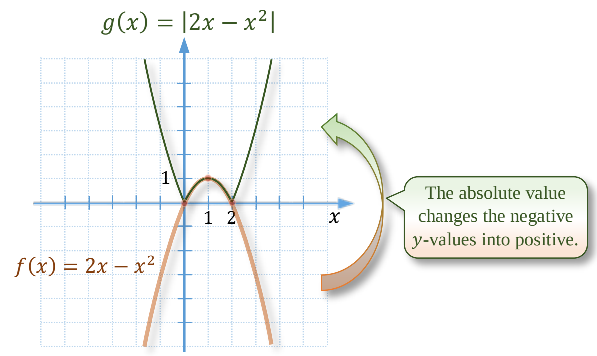


Figure 1.11

Step Function in Applications

The greatest integer function, $[\![x]\!]$, is an example of a larger class of functions, called **step functions**.

Definition 1.2 ► A **step function** is a function whose graph consists of a series of horizontal line segments with jumps in-between them. The line segments can be half-open, open, or closed.

A step function is a constant function on given intervals. However, the value of this function is different for each interval. For example, the function defined as follows:

$f(x) = 1$ for all x-values from the interval $[-2, 1)$,

$f(x) = 2$ for all x-values from the interval $[1, 3)$,

$f(x) = 4$ for all x-values from the interval $[3, 4]$, is a step function with a staircase-like graph, as illustrated in *Figure 1.12*. Such function can be defined with the use of a **piecewise notation**, as below.

$$f(x) = \begin{cases} 0, & if \quad -2 \le x < 1 \\ 2, & if \quad 1 \le x < 3 \\ 4, & if \quad 3 \le x \le 4 \end{cases}$$

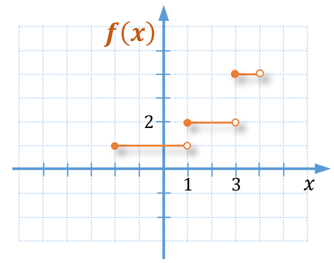


Figure 1.12

Step functions are used in many areas of life, particularly in business. For example, utilities or taxes are often billed according to a step function.

Example 4 ► **Finding a Step Function that Models a Parking Charge**

Suppose the charge for parking a car at a hospital parking lot is \$5 for the first hour or its portion and \$3 for each additional hour or its portion, up to \$14 per day. Let $C(t)$ represent the charge for parking a car for t hours. Graph $C(t)$ for t in the interval $(0, 6]$. Then, using piecewise notation, state the formula for the graphed function.

Solution ► To graph $C(t)$, we may create a table of values first. Observe that

t	$C(t)$
0.5	5
1	5
1.5	$5+3=8$
2	$5+3=8$
2.5	$5+2\cdot 3=11$
3	$5+2\cdot 3=11$
3.5	$5+3\cdot 3=14$
4	$5+3\cdot 3=14$
5	14
6	14

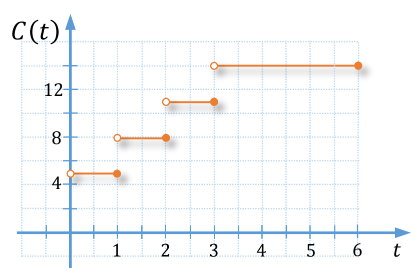
$C(t) = 5$ for all t-values from the interval $(0, 1]$,

$C(t) = 8$ for all t-values from the interval $(1, 2]$,

$C(t) = 11$ for all t-values from the interval $(2, 3]$, and

$C(t) = 14$ for all t-values from the interval $(3, 6]$.

So, we graph $C(t)$ as below.



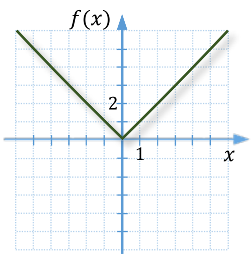
Using piecewise notation, function C can be written as

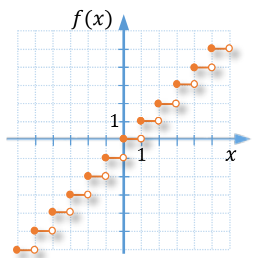
$$C(t) = \begin{cases} 5, & if \quad 0 < t \le 1 \\ 8, & if \quad 1 < t \le 2 \\ 11, & if \quad 2 < t \le 3 \\ 14, & if \quad 3 < t \le 6 \end{cases}$$

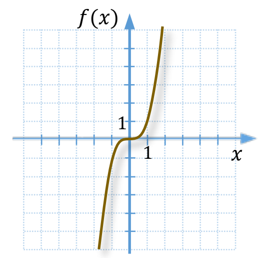
C.1 Exercises

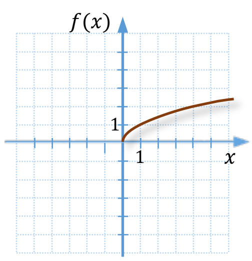
1. Match the name of the basic function provided in **a.-f.** with the corresponding graph in **I-VI**. Then, give the equation of this function and state its domain and range.

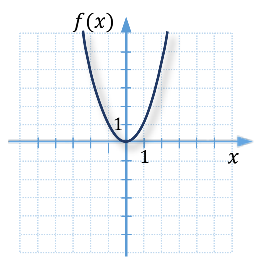
a. quadratic **b.** cubic **c.** square root
d. absolute value **e.** greatest integer **f.** reciprocal

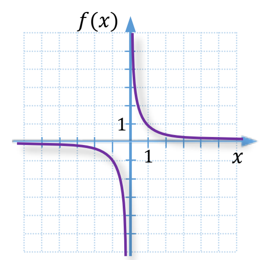
I


II


III


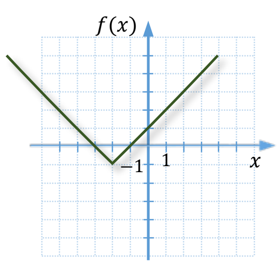
IV


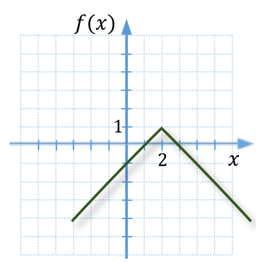
V


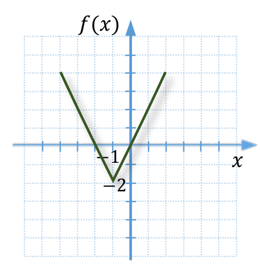
VI


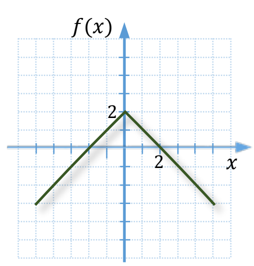
2. Match each absolute value function given in **a.-d.** with its graph in **I-IV**.

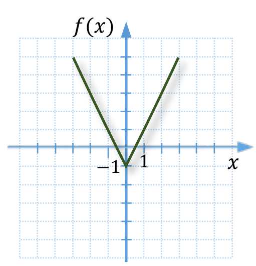
a. $f(x) = -|x-1| + 2$ **b.** $f(x) = 2|x| - 1$ **c.** $f(x) = -|x-2| + 1$
d. $f(x) = |x+2| - 1$ **e.** $f(x) = -|x| + 2$ **f.** $f(x) = 2|x+1| - 2$

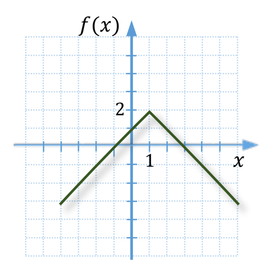
I


II


III


IV


V


VI


3. How is the graph of $f(x) = \frac{1}{x-5} + 3$ obtained from the graph of $g(x) = \frac{1}{x}$?

4. How is the graph of $f(x) = \sqrt{x+4} - 1$ obtained from the graph of $g(x) = \sqrt{x}$?

Graph each function. Give the domain and range. For rational functions, give the equations of their asymptotes.

5. $f(x) = |x+2|$

6. $f(x) = |x-3|$

7. $f(x) = \sqrt{x} + 2$

8. $f(x) = \sqrt{x} - 3$

9. $f(x) = \frac{1}{x} - 2$

10. $f(x) = \frac{1}{x} + 1$

11. $f(x) = -\frac{2}{x-1}$

12. $f(x) = \frac{1}{x+3} - 2$

13. $f(x) = -\sqrt{x+3}$

14. $f(x) = -(x+2)^3 + 1$

15. $f(x) = 2(x+3)^2 - 4$

16. $f(x) = 2|x+1| - 3$

Evaluate each expression.

17. $[\![2.1]\!]$

18. $[\![-2.1]\!]$

19. $-[\![2.1]\!]$

20. $-[\![-1.9]\!]$

Graph each function.

21. $f(x) = -[\![x]\!]$

22. $f(x) = [\![x]\!] - 2$

23. $f(x) = [\![x+3]\!]$

For each function $f(x)$, graph its reciprocal $g(x) = \frac{1}{f(x)}$. Determine the x-intercept of function f and the equation of the vertical asymptote of function g.

24. $f(x) = -x$

25. $f(x) = \frac{1}{2}x - 2$

26. $f(x) = -2x + 1$

27. $f(x) = x + 3$

28. $f(x) = -x + 2$

29. $f(x) = 4x - 3$

For each function $f(x)$, graph its absolute value $g(x) = |f(x)|$.

30. $f(x) = x^2 - 4$

31. $f(x) = (2-x)(x+3)$

32. $f(x) = 2x^2 + 3x$

33. $f(x) = (2x+1)(x-3)$

34. $f(x) = -2x^2 - 5x$

35. $f(x) = x^2 + 3x - 4$

Solve each problem.

36. A rental company charges $C(n)$ dollars for renting a specific construction tool for n hours. Suppose the charge is calculated by the formula

$$C(n) = 5\left[\!\!\left[\frac{n}{4}\right]\!\!\right] + 20.$$

a. Find the charge for renting the tool for 7.5 hours.
b. Find the charge for renting the tool for 10 hours.

37. The postage rate for sending an international letter depends on the letter's weight. Suppose the postage costs \$2.50 for a letter up to 30 grams, \$3.50 for a letter over 30 grams up to 50 grams, \$6.25 for a letter over 50 grams up to 100 grams, and \$10 for a letter over 100 grams. Let $C(m)$ represent the postage cost of a letter weighing m grams. Graph $C(m)$ for m in the interval $(0, 200]$. Then, using piecewise notation, state the formula for the graphed function.

38. Suppose a taxicab charges 5\$ for the first kilometer and \$2 for each additional kilometer or its portion. Let $C(n)$ represent the charge for an n kilometers long ride in this taxicab. Graph $C(n)$ for n in the interval $(0, 5]$. Then, using piecewise notation, state the formula for the graphed function.

39. The salary of a furniture store salesperson is often based on the size of the sales he or she can make during a given month. Suppose that the salesperson is paid \$2000 base salary for sales below \$10,000; \$2800 for sales of at least \$10,000 but below \$15,000; \$3800 for sales of at least \$15,000 but below \$20,000; and \$5000 for sales of at least \$20,000. Let $P(s)$ represent the salary received for the monthly sales of s dollars. Graph $P(s)$ for s in the interval $[0,25000]$. Then, using piecewise notation, state the formula for the graphed function.

C2 Equations and Graphs of Conic Sections

In this section, we give an overview of the main properties of the curves called **conic sections**. Geometrically, these curves can be defined as intersections of a plane with a double cone. These intersections can take the shape of a point, a line, two intersecting lines, a circle, an ellipse, a parabola, or a hyperbola, depending on the position of the plane with respect to the cone.

Conic sections play an important role in mathematics, physics, astronomy, and other sciences, including medicine. For instance, planets, comets, and satellites move along conic pathways. Radio telescopes are built with the use of parabolic dishes while reflecting telescopes often contain hyperbolic mirrors. Conic sections are present in both analyzing and constructing many important structures in our world.

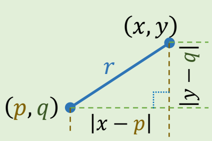
Since lines and parabolas were already discussed in previous chapters, this section will focus on circles, ellipses, and hyperbolas.

Circles

A circle is a conic section formed by the intersection of a cone and a plane parallel to the base of the cone. In coordinate geometry, a circle is defined as follows.

Definition 2.1 ► A **circle** with a fixed **centre** and the **radius** of length $\boldsymbol{r}$ is the set of all points in a plane that lie at the **constant distance** $\boldsymbol{r}$ from this centre.

Equation of a Circle in Standard Form



A **circle** with **centre** $(\boldsymbol{p}, \boldsymbol{q})$ and **radius** $\boldsymbol{r}$ is given by the equation:

$$(\boldsymbol{x}-\boldsymbol{p})^2+(\boldsymbol{y}-\boldsymbol{q})^2=\boldsymbol{r}^2$$

In particular, the equation of a circle centered at the origin and with radius $\boldsymbol{r}$ takes the form

$$\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{r}^2$$

Proof: ► Suppose a point (x, y) belongs to the circle with centre (p, q) and radius r. By *Definition 2.1*, the distance between this point and the centre is equal to r. Using the distance formula that was developed in *Section RD3*, we have

$$r=\sqrt{(x-p)^2+(y-q)^2}$$

Hence, after squaring both sides of this equation, we obtain the equation of the circle:

$$r^2=(x-p)^2+(y-q)^2$$

Example 1 ► **Finding an Equation of a Circle and Graphing It**

Find an equation of the circle with radius 2 and center at $(0,1)$ and graph it.

Solution By substituting $p = 0$, $q = 1$, and $r = 2$ into the standard form of the equation of a circle, we obtain

$$x^2 + (y - 1)^2 = 4$$

To graph this circle, we plot the centre $(0,1)$ first, and then plot points that are 2 units apart in the four main directions, East, West, North, and South. The circle passes through these four points, as in *Figure 2.1*.

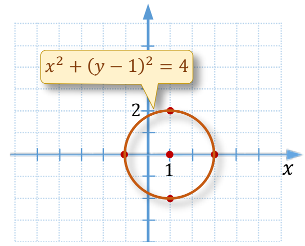


Figure 2.1

Example 2 **Graphing a Circle Given Its Equation**

Identify the center and radius of each circle. Then graph it and state the domain and range of the relation.

a. $x^2 + y^2 = 7$ **b.** $(x - 3)^2 + (y + 2)^2 = 6.25$

c. $x^2 + 4x + y^2 - 2y = 4$

Solution

a. The equation can be written as $(x - 0)^2 + (y - 0)^2 = \left(\sqrt{7}\right)^2$. So, the **centre** of this circle is at $(\mathbf{0}, \mathbf{0})$, and the length of the **radius** is $\sqrt{\mathbf{7}}$. The graph is shown in *Figure 2.2a*.

By projecting the graph onto the x-axis, we observe that the **domain** of this relation is $\left[-\sqrt{\mathbf{7}}, \sqrt{\mathbf{7}}\right]$. Similarly, by projecting the graph onto the y-axis, we obtain the **range**, which is also $\left[-\sqrt{\mathbf{7}}, \sqrt{\mathbf{7}}\right]$.

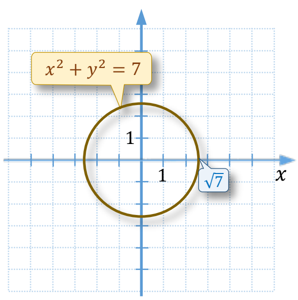


Figure 2.2a

b. The **centre** of this circle is at $(\mathbf{3}, -\mathbf{2})$ and the length of the **radius** is $\sqrt{6.25} = \mathbf{2.5}$. The graph is shown in *Figure 2.2b*. The **domain** of the relation is $[\mathbf{0.5}, \mathbf{5.5}]$, and the **range** is $[-\mathbf{4.5}, \mathbf{0.5}]$.

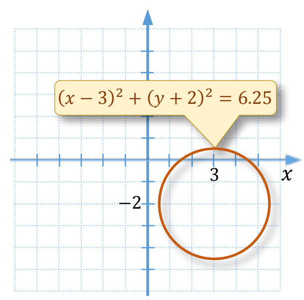


Figure 2.2b

c. The given equation is not in standard form. To rewrite it in standard form, we apply the completing the square procedure to the x-terms and to the y-terms.

$$\begin{aligned} x^2 + 4x + y^2 - 2y &= 4 \\ (x + 2)^2 - 4 + (y - 1)^2 - 1 &= 4 \\ (x + 2)^2 + (y - 1)^2 &= 9 \\ (x + 2)^2 + (y - 1)^2 &= 3^2 \end{aligned}$$

So, the **centre** of this circle is at $(-\mathbf{2}, \mathbf{1})$ and the length of the **radius** is $\mathbf{3}$. The graph is shown in *Figure 2.2c*. The **domain** of the relation is $[-\mathbf{5}, \mathbf{1}]$ and the **range** is $[-\mathbf{2}, \mathbf{4}]$.

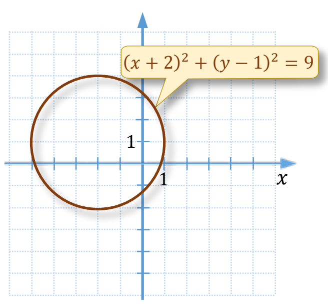
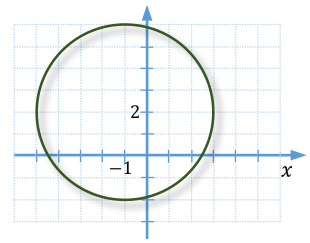


Figure 2.2c

Example 3 ► **Finding Equation of a Circle Given Its Graph**

Determine the equation of the circle shown in the graph.



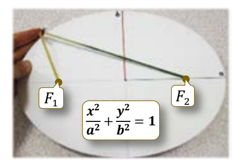
Solution ► Reading from the graph, the centre of the circle is at $(-1,2)$ and the radius is 4. So the equation of this circle is

$$(x+1)^2+(y-2)^2=4^2$$

Ellipses

A conic section formed by the intersection of a cone and a plane slanted to the base but not parallel to the side of the cone is called an ellipse. In coordinate geometry, an ellipse is defined as follows.

Definition 2.2 ► An **ellipse** is the set of points in a plane with a constant *sum* of distances from two fixed points. These fixed points are called **foci** (*singular:* **focus**). The point halfway between the two foci is called the **center** of the ellipse.

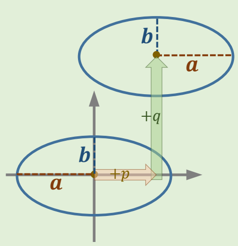


An ellipse has an interesting property of reflection.

Reflecting Property of an Ellipse

When a ray of light or sound emanating from one focus of an ellipse bounces off the ellipse, it passes through the other focus.

Equation of an Ellipse in Standard Form



An **ellipse** with its **centre** at the origin, **radius along the x-axis** (r_x) of length a, and **radius along the y-axis** (r_y) of length b is given by the equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

An **ellipse** with its **centre** at the point (p,q), **radius along the x-axis** (r_x) of length a, and **radius along the y-axis** (r_y) of length b is given by the equation:

$$\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1$$

Note: A circle is a special case of an ellipse, where $a=b=r$.

Example 4 ► **Graphing an Ellipse Given Its Equation**

Identify the center and the two radii of each ellipse. Then graph it and state the domain and range of the relation.

a. $9x^2 + y^2 = 9$ **b.** $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{4} = 1$

Solution ►

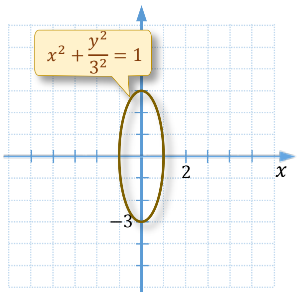


Figure 2.3a

a. First, we may want to change the equation to its standard form. This can be done by dividing both sides of the given equation by 9, to make the right side equal to 1. So, we obtain

$$x^2 + \frac{y^2}{9} = 1$$

or equivalently,

$$x^2 + \frac{y^2}{3^2} = 1$$

Hence, the **centre** of this ellipse is at $(\mathbf{0}, \mathbf{0})$, and the two **radii** are $\boldsymbol{r_x = 1}$ and $\boldsymbol{r_y = 3}$. Thus, we graph this ellipse as in *Figure 2.3a*. The **domain** of the relation is $[-\mathbf{1}, \mathbf{1}]$ and the **range** is $[-\mathbf{3}, \mathbf{3}]$.

b. The given equation can be written as

$$\frac{(x-1)^2}{4^2} + \frac{(y+2)^2}{2^2} = 1$$

So, the **centre** of this ellipse is at $(\mathbf{1}, -\mathbf{2})$ and the two **radii** are $\boldsymbol{r_x = 4}$ and $\boldsymbol{r_y = 2}$. The graph is shown in *Figure 2.3b*. The **domain** of the relation is $[-\mathbf{3}, \mathbf{5}]$, and the **range** is $[-\mathbf{4}, \mathbf{0}]$.

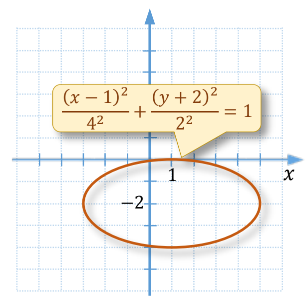
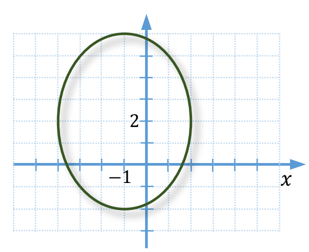


Figure 2.3b

Example 5 ► **Finding Equation of an Ellipse Given Its Graph**

Give the equation of the ellipse shown in the accompanying graph.



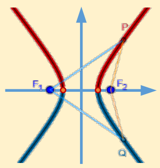
Solution ► Reading from the graph, the centre of the ellipse is at $(-1,2)$, the radius r_x equals 3, and the radius r_y equals 4. So, the equation of this ellipse is

$$\frac{(x+1)^2}{3^2} + \frac{(y-2)^2}{4^2} = 1$$

Hyperbolas

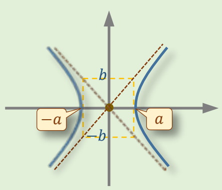
A conic section formed by the intersection of a cone and a plane perpendicular to the base of the cone is called a hyperbola. In coordinate geometry, a hyperbola is defined as follows.

Definition 2.2 ►



A **hyperbola** is the set of points in a plane with a constant absolute value of the *difference* of distances from two fixed points. These fixed points are called **foci** (*singular:* **focus**). The point halfway between the two foci is the **center** of the hyperbola. The graph of a hyperbola consists of two branches and has two axes of symmetry. The axis of symmetry that passes through the foci is called the **transverse** axis. The intercepts of the hyperbola and its transverse are the **vertices** of the hyperbola. The line passing through the centre of the hyperbola and perpendicular to the transverse is the other axis of symmetry, called the **conjugate** axis.

Equation of a Hyperbola in Standard Form

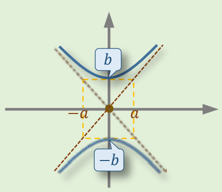


A **hyperbola** with its **centre** at the origin, **transverse axis** on the ***x*-axis**, and **vertices** at $(-a, 0)$ and $(a, 0)$ is given by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

A **hyperbola** with its **centre** at (p, q), **horizontal transverse axis**, and **vertices** at $(-a, 0)$ and $(a, 0)$ is given by the equation:

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = 1$$



A **hyperbola** with its **centre** at the origin, **transverse axis** on the ***y*-axis**, and **vertices** at $(0, -b)$ and $(0, b)$ is given by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

A **hyperbola** with its **centre** at (p, q), **vertical transverse axis**, and **vertices** at $(0, -b)$ and $(0, b)$ is given by the equation:

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = -1$$

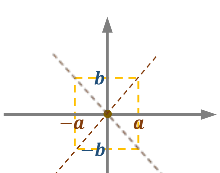


Figure 2.4a

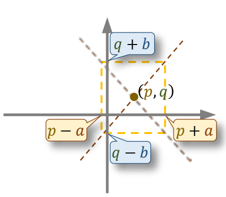


Figure 2.4b

Fundamental Rectangle and Asymptotes of a Hyperbola

The graph of a hyperbola given by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ is based on a rectangle formed by the lines $x = \pm a$ and $y = \pm b$. This rectangle is called the **fundamental rectangle** (see *Figure 2.4*). The extensions of the diagonals of the fundamental rectangle are the **asymptotes** of the hyperbola. Their equations are $y = \pm \frac{b}{a} x$.

Generally, the **fundamental rectangle** of a hyperbola given by the equation

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = \pm 1$$

is formed by the lines $x = p \pm a$ and $y = q \pm b$. The extensions of the diagonals of this rectangle are the **asymptotes** of the hyperbola.

Example 6 ► Graphing a Hyperbola Given Its Equation

Determine the center, transverse axis, and vertices of each hyperbola. Graph the fundamental rectangle and asymptotes of the hyperbola. Then, graph the hyperbola and state its domain and range.

a. $9x^2 - 4y^2 = 36$ **b.** $(x-2)^2 - \frac{(y+1)^2}{4} = -1$

Solution ►

a. First, we may want to change the equation to its standard form. This can be done by dividing both sides of the given equation by 36, to make the right side equal to 1. So, we obtain

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

or equivalently

$$\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$$

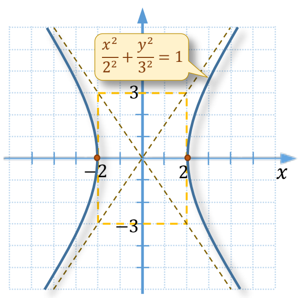


Figure 2.5

Hence, the **centre** of this hyperbola is at $(\mathbf{0}, \mathbf{0})$, and the transverse axis is on the ***x*-axis**. Thus, the vertices of the hyperbola are $(\mathbf{-2}, \mathbf{0})$ and $(\mathbf{2}, \mathbf{0})$.

The **fundamental rectangle** is centered at the origin, and it spans 2 units horizontally apart from the centre and 3 units vertically apart from the centre, as in *Figure 2.5*. The **asymptotes** pass through the opposite vertices of the fundamental rectangle. The final graph consists of two branches. Each of them passes through the corresponding vertex and is shaped by the asymptotes, as shown in *Figure 2.5*.

The **domain** of the relation is $(-\infty, \mathbf{-2}] \cup [\mathbf{2}, \infty)$ and the **range** is $\mathbb{R}$.

b. The equation can be written as

$$\frac{(x-2)^2}{1^2} - \frac{(y+1)^2}{2^2} = -1$$

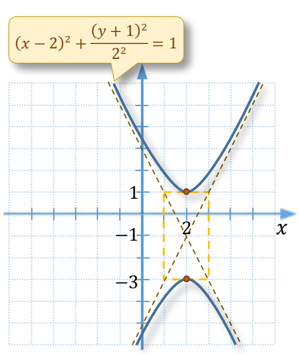


Figure 2.6

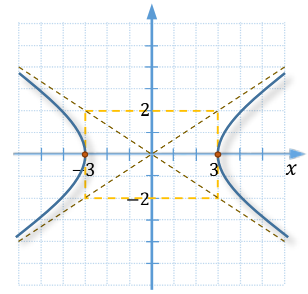
The **centre** of this hyperbola is at $(\mathbf{2}, \mathbf{-1})$. The -1 on the right side of this equation indicates that the **transverse axis is vertical**. Thus, the vertices of the hyperbola are 2 units vertically apart from the centre. So, they are $(\mathbf{2}, \mathbf{-3})$ and $(\mathbf{2}, \mathbf{1})$.

The **fundamental rectangle** is centered at $(\mathbf{2}, \mathbf{-1})$ and it spans 1 unit horizontally apart from the centre and 2 units vertically apart from the centre, as in *Figure 2.6*. The **asymptotes** pass through the opposite vertices of the fundamental box. The final graph consists of two branches. Each of them passes through the corresponding vertex and is shaped by the asymptotes, as shown in *Figure 2.6*.

The **domain** of the relation is $\mathbb{R}$, and the **range** is $(-\infty, \mathbf{-3}] \cup [\mathbf{1}, \infty)$.

Example 7 ► **Finding the Equation of a Hyperbola Given Its Graph**

Give the equation of a hyperbola shown in the accompanying graph.



Solution ► Reading from the graph, the centre of the hyperbola is at $(0,0)$, the transverse axis is the x-axis, and the vertices are $(-3,0)$ and $(3,0)$. The fundamental rectangle spans 2 units vertically apart from the centre. So, we substitute $p = 0$, $q = 0$, $a = 3$, and $b = 2$ to the standard equation of a hyperbola. Thus the equation is

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$$

Generalized Square Root Functions $f(x) = \sqrt{g(x)}$ for Quadratic Functions $g(x)$

Conic sections are relations but usually not functions. However, we could consider parts of conic sections that are already functions. For example, when solving the equation of a circle

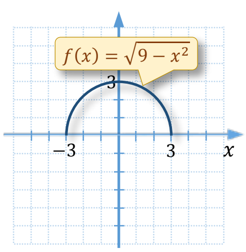
$$x^2 + y^2 = 9$$

for y, we obtain

$$y^2 = 9 - x^2$$

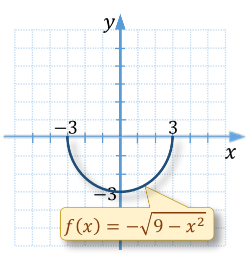
$$|y| = \sqrt{9 - x^2}$$

$$y = \pm\sqrt{9 - x^2}.$$



So, the graph of this circle can be obtained by graphing the two functions: $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$.

Since the equation $\boldsymbol{y = \sqrt{9 - x^2}}$ describes all the points of the circle with a nonnegative y-coordinate, its graph must be the **top half of the circle** centered at the origin and with the radius of length 3. So, the domain of this function is $[-3,3]$ and the range is $[0,3]$.



Likewise, since the equation $\boldsymbol{y = -\sqrt{9 - x^2}}$ describes all the points of the circle with a nonpositive y-coordinate, its graph must be the **bottom half of the circle** centered at the origin and with the radius of length 3. Thus, the domain of this function is $[-3,3]$ and the range is $[-3,0]$.

Note: Notice that the function $f(x) = \sqrt{9 - x^2}$ is a composition of the square root function and the quadratic function $g(x) = 9 - x^2$. One could prove that the graph of the square root of any quadratic function is the top half of one of the conic sections. Similarly, the graph of the negative square root of any quadratic function is the bottom half of one of the conic sections.

Example 8 ► **Graphing Generalized Square Root Functions**

Graph each function. Give its domain and range.

a. $f(x) = 1 - \sqrt{16 - x^2}$ **b.** $f(x) = 2\sqrt{1 + \frac{x^2}{9}}$

Solution ► **a.** To recognize the shape of the graph of function *f*, let us rearrange its equation first.

$$y = 1 - \sqrt{16 - x^2}$$

$$y - 1 = -\sqrt{16 - x^2}$$

$$(y - 1)^2 = \left(-\sqrt{16 - x^2}\right)^2$$

$$(y - 1)^2 = 16 - x^2$$

$$x^2 + (y - 1)^2 = 16$$

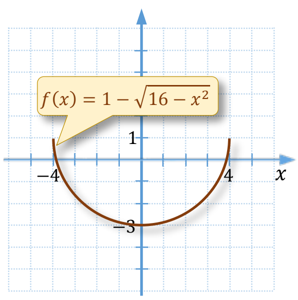


Figure 2.7

The resulting equation represents a circle with its centre at $(0,1)$ and a radius of 4. So, the graph of $f(x) = 1 - \sqrt{16 - x^2}$ must be part of this circle. Since $y - 1 = -\sqrt{16 - x^2} \leq 0$, then $y \leq 1$. Thus the graph of function f is the **bottom half** of this **circle**, as shown in *Figure 2.7*.
So, the **domain** of function f is $[\mathbf{-4}, \mathbf{4}]$ and the **range** is $[\mathbf{-3}, \mathbf{1}]$.

b. To recognize the shape of the graph of function *f*, let us rearrange its equation first.

$$y = 2\sqrt{1 + \frac{x^2}{9}}$$

$$\frac{y}{2} = \sqrt{1 + \frac{x^2}{9}}$$

$$\left(\frac{y}{2}\right)^2 = 1 + \frac{x^2}{9}$$

$$-1 = \frac{x^2}{9} - \left(\frac{y}{2}\right)^2$$

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} = -1$$

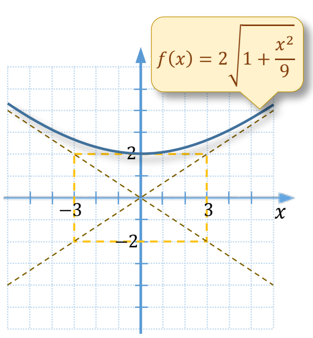


Figure 2.8

The resulting equation represents a hyperbola centered at the origin, with a vertical transverse axis. Its fundamental rectangle spans horizontally 3 units and vertically 2 units from the centre. Since the graph of $f(x) = 2\sqrt{1 + \frac{x^2}{9}}$ must be a part of this hyperbola and the values $f(x)$ are nonnegative, then its graph is the **top half** of this **hyperbola**, as shown in *Figure 2.8*.
So, the **domain** of function f is $\mathbb{R}$, and the **range** is $[\mathbf{2}, \infty]$.

C.2 Exercises

True or false.

1. A circle is a set of points, where the center is one of these points.

2. If the foci of an ellipse coincide, then the ellipse is a circle.

3. The x-intercepts of $\frac{x^2}{9}+\frac{y^2}{4}=1$ are $(-9,0)$ and $(9,0)$.

4. The graph of $2x^2+y^2=1$ is an ellipse.

5. The y-intercepts of $x^2+\frac{y^2}{3}=1$ are $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},0\right)$.

6. The graph of $y^2=1-x^2$ is a hyperbola centered at the origin.

7. The transverse axis of the hyperbola $-y^2=1-x^2$ is the x-axis.

Find the equation of a circle satisfying the given conditions.

8. centre at $(-1,-2)$; radius 1

9. centre at $(3,1)$; radius $\sqrt{3}$

10. centre at $(2,-1)$; diameter 6

11. centre at $(-2,2)$; diameter 5

Find the ***center*** *and* ***radius*** *of each circle.*

12. $x^2+y^2+4x+6y+9=0$

13. $x^2+y^2-8x-10y+5=0$

14. $x^2+y^2+6x-16=0$

15. $x^2+y^2-12x+12=0$

16. $2x^2+2y^2+20y+10=0$

17. $3x^2+3y^2-12y-24=0$

Identify the ***center*** *and* ***radius*** *of each circle. Then graph the relation and state its* ***domain*** *and* ***range****.*

18. $x^2+(y-1)^2=16$

19. $(x+1)^2+y^2=2.25$

20. $(x-2)^2+(y+3)^2=4$

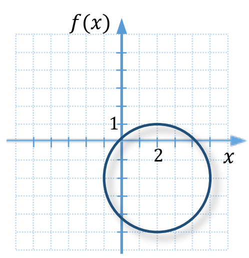
21. $(x+3)^2+(y-2)^2=9$

22. $x^2+y^2+2x+2y-23=0$

23. $x^2+y^2+4x+2y+1=0$

Use the given graph to determine the equation of the ***circle****.*

24.



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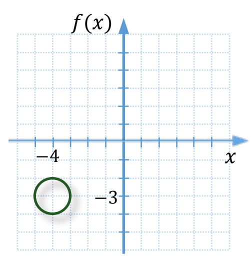

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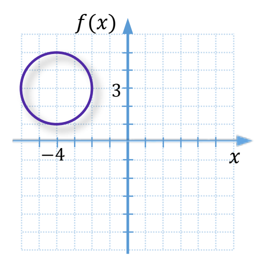
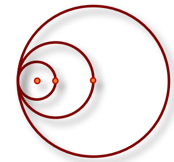


27. The equation of the smallest circle shown is $x^2 + y^2 = r^2$. What is the equation of the largest circle?



*Identify the **center** and the horizontal (r_x) and vertical (r_y) **radii** of each ellipse. Then graph the relation and state its **domain** and **range**.*

28. $\frac{x^2}{4} + (y-1)^2 = 1$

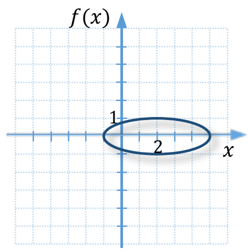
29. $(x+1)^2 + \frac{y^2}{9} = 1$

30. $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$

31. $\frac{(x-4)^2}{4} + \frac{(y-2)^2}{9} = 1$

*Use the given graph to determine the equation of the **ellipse**.*

32.



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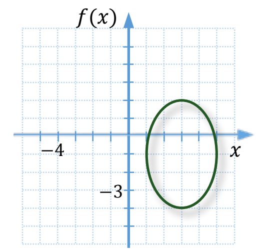

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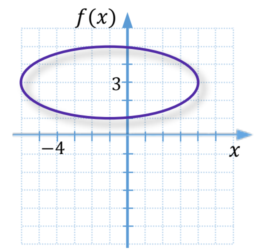


*Identify the **center** and the **transverse axis** of each hyperbola. Then graph the **fundamental box** and the **hyperbola**. State the **domain** and **range** of the relation.*

35. $\frac{x^2}{4} - (y-1)^2 = 1$

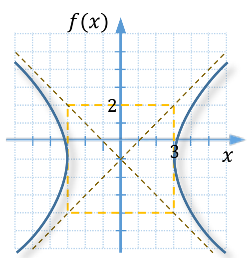
36. $(x+1)^2 + \frac{y^2}{9} = -1$

37. $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{4} = -1$

38. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{9} = 1$

*Use the given graph to determine the equation of the **hyperbola**.*

39.



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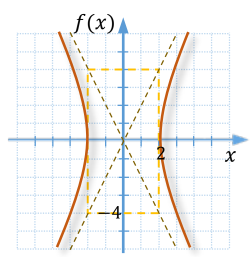

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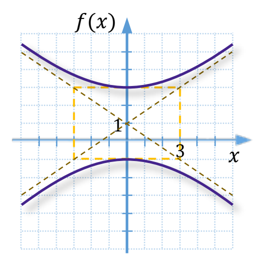


42. Match each equation with its graph.

a. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

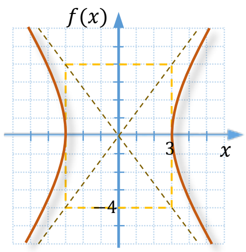
c. $\frac{x^2}{9} - \frac{y^2}{16} = -1$

d. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

e. $\frac{x^2}{9} + \frac{y^2}{9} = 1$

f. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

I



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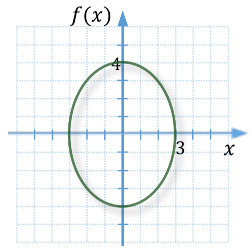
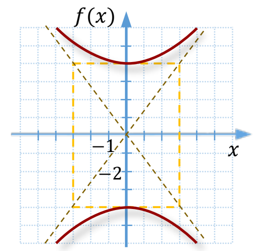


III



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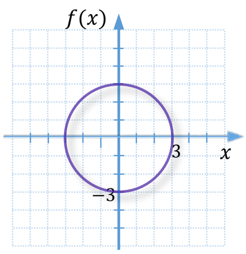
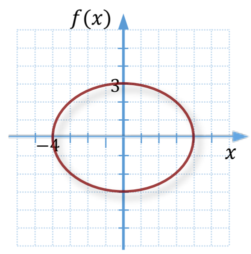
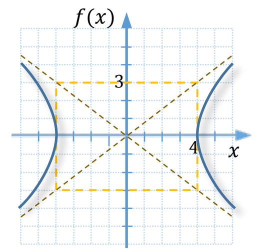


V



VI



Graph each ***generalized square root*** *function. Give the* ***domain*** *and* ***range****.*

43. $f(x) = \sqrt{4 - x^2}$

44. $f(x) = -\sqrt{25 - x^2}$

45. $f(x) = -2\sqrt{1 - \frac{x^2}{9}}$

46. $f(x) = 3\sqrt{1 - \frac{x^2}{4}}$

47. $\frac{y}{3} = \sqrt{x^2 - 1}$

48. $\frac{y}{2} = -\sqrt{1 + \frac{x^2}{9}}$

Solve each problem.

49. The arch under a bridge has the shape of the upper half of an ellipse, as illustrated in the accompanying figure. Assuming that the ellipse is modelled by $25x^2 + 144y^2 = 3600$, where x and y are in meters, find the width and height of the arch (*above the yellow line*).

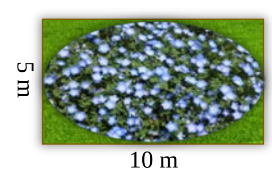
50. Suppose a network service outage affects the area within a 10-km radius of the service provider centre.

a. If the service provider centre is situated 7 km west and 3 km north of the city hall, find an equation of the circle that represents the boundary of the outage with respect to the location of the city hall.

b. Will customers of a caffee located 2 km east and 1 km south of the city hall be affected by the network outage?

51. From a distance, the sides of a cooling tower look like a portion of the branches of the hyperbola with equation $25x^2 - 9y^2 = 22500$, where x and y are in meters. What is the diameter of this cooling tower at its narrowest place?

52. A city plans to create an elliptical flower bed that fits in a rectangular space with the dimensions of 5 meters by 10 meters, as indicated in the accompanying figure.



a. Using the formula $A = \pi ab$ for the area of an ellipse with radii a and b, find the area of the largest such elliptic flower bed. Round your answer to the nearest tenth of a square meter.

b. Assuming that each square meter of this flower bed is filled with 25 plants, approximate the number of plants in the entire flower bed.

C3 Nonlinear Systems of Equations and Inequalities

In *Section E1*, we discussed methods of solving systems of two linear equations. Recall that solutions to such systems are the intercepts of the two lines. So, we could have either zero, or one, or infinitely many solutions.

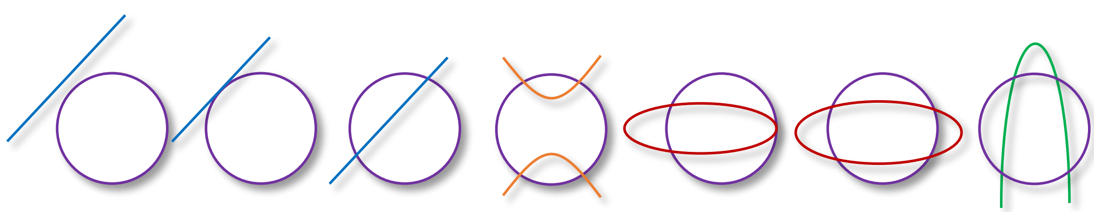
In this section, we will consider systems of two equations that are not necessarily linear. In particular, we will focus on solving systems composed of equations of conic sections. Since the solutions to such systems can be seen as the intercepts of the curves represented by the equations, we may expect a different number of solutions. For example, a circle may intercept an ellipse in 0, 1, 2, 3, or 4 points. We encourage the reader to visualise these situations by drawing a circle and an ellipse in various positions.

Aside from discussing methods of solving nonlinear systems of equations, we will also graph solutions of nonlinear systems of inequalities, using similar techniques as presented in *Section G4*, where solutions to linear inequalities were graphed.

Nonlinear Systems of Equations

Definition 3.1 ► A **nonlinear system of equations** is a system of equations containing at least one equation that is not linear.

When solving a nonlinear system of two equations, it is useful to predict the possible number of solutions by visualising the shapes and position of the graphs of these equations. For example, the number of solutions to a system of two equations representing conic sections can be determined by observing the number of intercept points of the two curves. Here are some possible situations.



0 solutions   1 solution   2 solutions   4 solutions   3 solutions   4 solutions   4 solutions

To solve a nonlinear system of two equations, we may use any of the algebraic methods discussed in *Section E1*, the substitution or the elimination method, whichever makes the calculations easier.

Example 1 ► **Solving Nonlinear Systems of Two Equations by Substitution**

Solve each system of equations.

a. $\begin{cases} xy = 4 \\ 4y + x = 8 \end{cases}$

b. $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$

Solution ► **a.** The system $\begin{cases} xy = 4 \\ 4y + x = 8 \end{cases}$ consists of a reciprocal function (which is a hyperbola) and a line. So, we may expect 0, 1, or 2 solutions. To solve this system, we may want to solve the second equation for x,

$$x = -4y + 8,$$

and then substituting the resulting expression into the first equation. So, we have

$$(8 - 4y)y = 4 \qquad / -4$$
$$8y - 4y^2 - 4 = 0 \qquad / \cdot(-1)$$
$$4y^2 - 8y + 4 = 0 \qquad / \div 4$$
$$y^2 - 2y + 1 = 0$$
$$(y - 1)^2 = 0$$
$$y = 1$$

Then, using the substitution equation, we calculate

$$x = -4 \cdot 1 + 8 = 4.$$

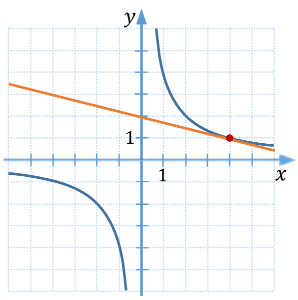


Figure 3.1

So, the solution set consists of one point, $(\mathbf{4}, \mathbf{1})$, as illustrated in *Figure 3.1*.

b. The system $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$ consists of a circle and a line. So, we may expect 0, 1, or 2 solutions. To solve this system, we may want to solve the second equation, for example for x,

$$x = y + 1,$$

and then substitute the resulting expression into the first equation. So, we have

$$(y + 1)^2 + y^2 = 9$$
$$y^2 + 2y + 1 + y^2 = 9$$
$$2y^2 + 2y - 8 = 0 \qquad / \div 2$$
$$y^2 + y - 4 = 0$$
$$y = \frac{-1 \pm \sqrt{1 + 4 \cdot 4}}{2} = \frac{-1 \pm \sqrt{17}}{2}$$

Then, using substitution, we calculate

$$x = y + 1 = \frac{-1 \pm \sqrt{17}}{2} + 1 = \frac{-1 \pm \sqrt{17} + 2}{2} = \frac{1 \pm \sqrt{17}}{2}.$$

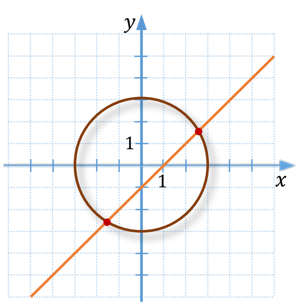


Figure 3.2

So, the solution set consists of two points, $\left(\frac{\mathbf{1}-\sqrt{\mathbf{17}}}{\mathbf{2}}, \frac{\mathbf{-1}-\sqrt{\mathbf{17}}}{\mathbf{2}}\right)$ and $\left(\frac{\mathbf{1}+\sqrt{\mathbf{17}}}{\mathbf{2}}, \frac{\mathbf{-1}+\sqrt{\mathbf{17}}}{\mathbf{2}}\right)$, as illustrated in *Figure 3.2*.
Their approximations are $(-1.56, -2.56)$ and $(2.56, 1.56)$.

Example 2 ► **Solving Nonlinear Systems of Two Equations by Elimination**

Solve the system of equations $\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases}$ using elimination.

Solution ► The system consists of a circle and a hyperbola, so we may expect up to four solutions. To solve it, we can start by eliminating the y-variable by adding the two equations, side by side.

$$+\begin{cases} x^2 + y^2 = 9 \\ 2x^2 - y^2 = -6 \end{cases}$$

$$3x^2 = 3 \qquad / \div 3$$

$$x^2 = 1$$

$$x = \pm 1$$

Then, by substituting the obtained x-values into the first equation, we can find the corresponding y-values. So, if $x = 1$, we have

$$1^2 + y^2 = 9 \qquad / -1$$

$$y^2 = 8$$

$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Similarly, if $x = -1$, we have

$$(-1)^2 + y^2 = 9 \qquad / -1$$

$$y^2 = 8$$

$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

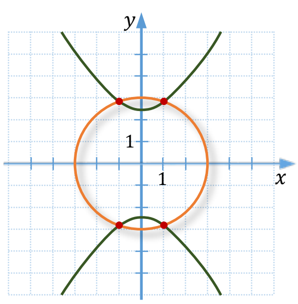


Figure 3.3

So, the solution set is $\left\{\left(\mathbf{1}, \mathbf{2}\sqrt{\mathbf{2}}\right), \left(\mathbf{1}, -\mathbf{2}\sqrt{\mathbf{2}}\right), \left(-\mathbf{1}, \mathbf{2}\sqrt{\mathbf{2}}\right), \left(\mathbf{1}, -\mathbf{2}\sqrt{\mathbf{2}}\right)\right\}$. These are the four intersection points of the two curves, circle and hyperbola, as illustrated in *Figure 3.3*.

Nonlinear systems of equations appear in many application problems, especially in the field of geometry, physics, astronomy, astrophysics, engineering, etc.

Example 3 ► **Nonlinear Systems of Two Equations in Applied Problems**

Suppose the side area of a 250 cm^3 cylindrical can is 200 square centimeters. What are the dimensions of this can?

Solution ► Using the formulas for the volume, $V = \pi r^2 h$, and side area, $A = 2\pi r h$, of a cylinder with radius r and height h, we can set up the system of two equations,

$$\begin{cases} \pi r^2 h = 250 \\ 2\pi r h = 200 \end{cases} \qquad / \div 2$$

To solve this system, first, we may want to divide the second equation by 2 and then divide the two equations, side by side. So, we obtain

$$\begin{cases} \underline{\cancel{\pi} r^{\cancel{2}} \cancel{h}} = \underline{250} \\ \cancel{\pi}\cancel{r}\cancel{h} = 100 \end{cases}$$

$$r = 2.5$$

After substituting this value into the equation $2\pi r h = 200$, we can find the corresponding h-value:

$$2\pi(2.5)h = 200 \qquad / \div 5\pi$$

$$h = \frac{200}{5\pi} \approx 12.7$$

So, the can should have a **radius** of **2.5 cm** and a **height** of about **12.7 cm**.

Nonlinear Systems of Inequalities

In *Section G.4* we discussed graphical solutions to linear inequalities and systems of linear inequalities in two variables. Nonlinear inequalities in two variables and systems of such inequalities can be solved using similar graphic techniques.

Example 4 ► **Graphing Solutions to a Nonlinear Inequality**

Graph the solution set of each inequality.

a. $y \geq (x-2)^2 - 3$ **b.** $9x^2 - 4y^2 < 36$

Solution ► **a.** First, we graph the related equation of the parabola $y = (x-2)^2 - 3$, using a solid curve. So, we plot the vertex $(2, -3)$ and follow the shape of the basic parabola $y = x^2$, with arms directed upwards.

This parabola separates the plane into two regions, the one above the parabola and the one below the parabola. The inequality $\boldsymbol{y} \geq (x-2)^2 - 3$ indicates that the y-values of the solution points are **above** the parabola $y = (x-2)^2 - 3$. To confirm this observation, we may want to pick a **test point** outside of the parabola and check whether or not it satisfies the inequality. For example, the point $(2,0)$ makes the inequality

$$0 \geq (2-2)^2 - 3 = -3$$

a true statement. Thus, the point $(2,0)$ is one of the solutions of the inequality $y \geq (x-2)^2 - 3$ and so are the points of the whole region containing $(2,0)$. To illustrate the solution set of the given inequality, we shade this region, as in *Figure 3.4*. Thus, the solution set consists of all points above the parabola, including the parabola itself.

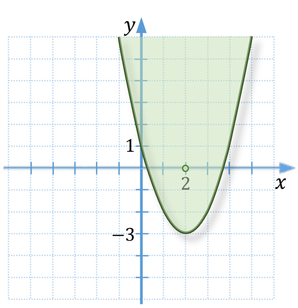


Figure 3.4

b. The related equation, $9x^2 - 4y^2 = 36$ represents a hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ centered at the origin, with a horizontal transverse axis, and with a fundamental rectangle that stretches 2 units horizontally and 3 units vertically apart from the centre. Since the inequality is strong ($<$), we graph this hyperbola using a dashed line. This indicates that the points on the hyperbola are not among the solutions of the inequality.

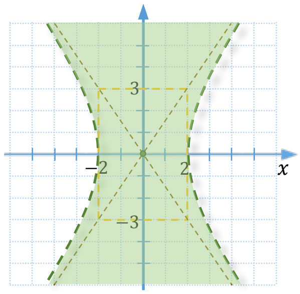


Figure 3.5

To decide which region should be shaded as the set of solutions for the given inequality, we can pick a test point that is easy to calculate, for instance $(0,0)$. Since

$$9 \cdot 0^2 - 4 \cdot 0^2 = 0 < 36$$

is a true statement, the point $(0,0)$ is one of the solutions of the inequality $9x^2 - 4y^2 < 36$, and so are the points of the whole region containing $(0,0)$. Thus, the solution set consists of all the points shaded in green (see *Figure 3.5*), but not the points of the hyperbola itself.

Note: If we chose a test point that does not satisfy the inequality, then the solution set is the region that does not contain this test point.

Example 5 ► **Graphing Solutions to a System of Nonlinear Inequalities**

Graph the solution set of each system of inequalities.

a. $\begin{cases} x^2 + y < 4 \\ y - x \geq 2 \end{cases}$ **b.** $\begin{cases} \frac{x^2}{9} + \frac{y^2}{16} \leq 1 \\ x^2 - y^2 \geq -1 \end{cases}$

Solution ►

a. Observe that the first inequality, $y < -x^2 + 4$, represents the sets of points **below the parabola** $y = -x^2 + 4$. We will shade it in blue, as in *Figure 3.6*.

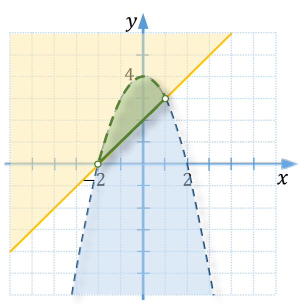


Figure 3.6

The second inequality, $y \geq x + 2$, represents the sets of points **above the line** $y = x + 2$, including the points on the line. We will shade it in yellow, as in *Figure 3.6*.

Thus, the solution set of the system of these inequalities is the **intersection of the blue and yellow region**, as illustrated in green in *Figure 3.6*. The top boundary of the green region is marked by a **dashed line** as these points do not belong to the solution set, and the bottom boundary is marked by a **solid line**, indicating that these points are among the solutions to the system. Also, since the intersection points of the two curves do not satisfy the first inequality, they are not solutions to the system. So, we mark them with **hollow circles**.

b. Observe that the first inequality, $\frac{x^2}{9} + \frac{y^2}{16} \leq 1$, represents the sets of points **inside the ellipse** $\frac{x^2}{9} + \frac{y^2}{16} = 1$, including the points on the ellipse. This can be confirmed by testing a point, for instance $(0,0)$. Since $\frac{0^2}{9} + \frac{0^2}{16} = 0 \leq 1$ is a true statement, then the region containing the origin is the solution set to this inequality. We will shade it in blue, as in *Figure 3.7*.

The second inequality, $x^2 - y^2 \geq -1$, represents the sets of points **outside the hyperbola** $x^2 - y^2 \geq -1$, including the points on the curve. Again, we can confirm

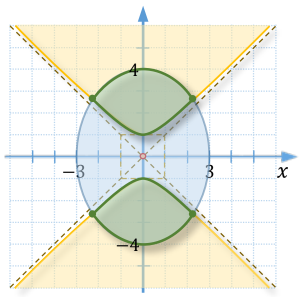


Figure 3.6

this by testing the $(0,0)$ point. Since $0^2 - 0^2 = 0 \geq -1$ is a false statement, then the solution set to this inequality is outside the region containing the origin. We will shade it in yellow, as in *Figure 3.7*.

Thus, the solution set of the system of these inequalities is the **intersection of the blue and yellow region**, as illustrated in green in *Figure 3.7*. The boundary of the green region is marked by a **solid line** as these points satisfy both inequalities and therefore are among the solutions of the system. Since the intersection points of the two curves also satisfy both inequalities, they are solutions of the system as well. So, we mark them using **filled-in circles**.

C.3 Exercises

*In problem 1-10, **sketch a diagram** illustrating the described curves with the given number of intersection points. (There may be more than one way to do this.)*

1. a line and a circle; one intercept

2. a line and a hyperbola; no intercepts

3. a line and a hyperbola; two intercepts

4. a circle and an ellipse; four intercepts

5. a circle and an ellipse; three intercepts

6. a parabola and a hyperbola; one intercept

7. a parabola and a hyperbola; two intercepts

8. an ellipse and an ellipse; two intercepts

9. an ellipse and a hyperbola; no intercepts

10. an ellipse and a parabola; four intercepts

11. Give the maximum number of points at which the following pairs of graphs can intersect.

a. a line and an ellipse
b. a line and a parabola
c. two different ellipses
d. two different circles with centers at the origin
e. two hyperbolas with centers at the origin

Solve each system.

12. $\begin{cases} y = x^2 + 6x \\ y = 4x \end{cases}$

13. $\begin{cases} y = x^2 + 8x + 16 \\ x - y = -4 \end{cases}$

14. $\begin{cases} xy = 12 \\ x + y = 8 \end{cases}$

15. $\begin{cases} xy = -5 \\ 2x + y = 3 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 2 \\ 2x + y = 1 \end{cases}$

17. $\begin{cases} 2x^2 + 4y^2 = 4 \\ x = 4y \end{cases}$

18. $\begin{cases} x^2 + y^2 = 4 \\ y = x^2 - 2 \end{cases}$

19. $\begin{cases} x^2 + y^2 = 9 \\ y = 3 - x^2 \end{cases}$

20. $\begin{cases} x^2 + y^2 = 4 \\ x + y = 3 \end{cases}$

21. $\begin{cases} x^2 - 2y^2 = 1 \\ x = 2y \end{cases}$

22. $\begin{cases} 3x^2 + 2y^2 = 12 \\ x^2 + 3y^2 = 4 \end{cases}$

23. $\begin{cases} 2x^2 + 3y^2 = 6 \\ x^2 + 3y^2 = 3 \end{cases}$

24. $\begin{cases} (x-4)^2 + y^2 = 4 \\ (x+2)^2 + y^2 = 16 \end{cases}$

25. $\begin{cases} 4x^2 + y^2 = 30 \\ 5x^2 - y^2 = 15 \end{cases}$

26. $\begin{cases} \frac{x^2}{9} - y^2 = -1 \\ \frac{x^2}{16} - \frac{y^2}{4} = 1 \end{cases}$

Solve each problem by using a nonlinear system.

27. Suppose the perimeter of a 60 m^2 rectangular room is 31 meters. What are the dimensions of this room?

28. A company producing navigation compasses observed that the cost C (in hundreds of dollars) of producing n hundred compasses could be calculated by using the formula

$$C = 3n^2 + 30n + 35,$$

while the revenue R (in hundreds of dollars) from the sale of n hundred compasses follows the relation

$$52n^2 - 5R = 0.$$

How many compasses need to be produced and sold so that the company **breaks even**? (*Breaking even means that the revenue covers exactly the cost of the production. There is no profit but also no loss for the company.*)

For questions 29-34, decide whether the statement is True or False.

29. A nonlinear system of equations can have up to four solutions.

30. The solution set of the inequality $x^2 + \frac{y^2}{25} \geq 1$ consists of points outside of the ellipse $x^2 + \frac{y^2}{25} = 1$, including the points of the ellipse.

31. The solution set of the inequality $x^2 + \frac{y^2}{25} < 1$ consists of points inside the ellipse $x^2 + \frac{y^2}{25} = 1$.

32. The intersection points of the curves $x^2 + y^2 = 5$ and $x - y = 3$ belong to the solution set of the system $\begin{cases} x^2 + y^2 \geq 5 \\ x - y > 3 \end{cases}$

33. The solution set of the inequality $y \geq x^2 + 3$ consists of points above or on the parabola $y = x^2 + 3$.

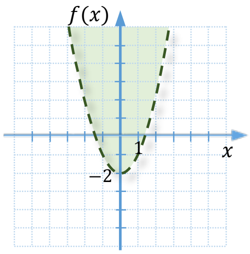
34. The solution set of the inequality $y < x^2 + 3$ consists of points below or on the parabola $y = x^2 + 3$.

35. Fill in each blank with the appropriate response.

The graph of the system $\begin{cases} x^2 + y^2 < 16 \\ y > -x \end{cases}$ consists of all points $\underset{(outside/inside)}{\underline{\hspace{4cm}}}$ the circle $x^2 + y^2 = 16$ and $\underset{(above/below)}{\underline{\hspace{3cm}}}$ the line $y = -x$.

36. Match each inequality with the graph of its solution set.

a. $y \geq x^2 - 2$ **b.** $y \leq x^2 - 2$ **c.** $y > x^2 - 2$ **d.** $y < x^2 - 2$

I


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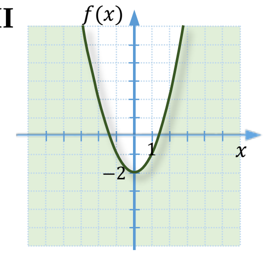

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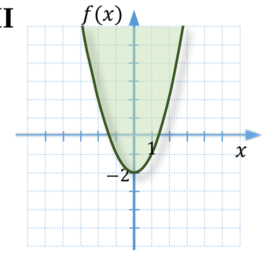

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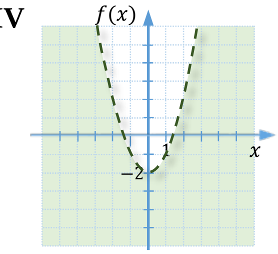
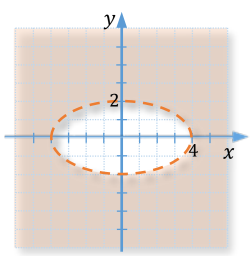


37. Match each inequality with the graph of its solution set.

a. $\frac{x^2}{16} + \frac{y^2}{4} < 1$ **b.** $\frac{x^2}{16} + \frac{y^2}{4} \leq 1$ **c.** $\frac{x^2}{16} + \frac{y^2}{4} \geq 1$ **d.** $\frac{x^2}{16} + \frac{y^2}{4} > 1$

I


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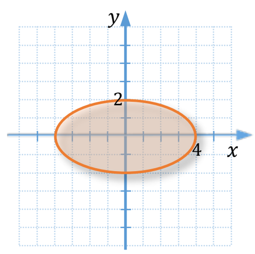

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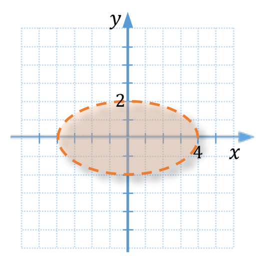

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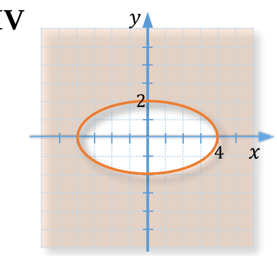


Graph each nonlinear inequality.

38. $x^2 + y^2 > 9$

39. $(x-1)^2 + (y+2)^2 \leq 16$

40. $y < 2x^2 - 6x$

41. $9x^2 + 4y^2 \geq 36$

42. $4x^2 - y^2 > 16$

43. $y \leq \frac{1}{2}(x+3)^2$

44. $x^2 + 9y^2 < 36$

45. $x^2 - 4 \geq -4y^2$

46. $y \geq x^2 - 8x + 12$

Graph the solution set to each nonlinear system of inequalities.

47. $\begin{cases} x^2 + y^2 < 16 \\ y > -2x \end{cases}$

48. $\begin{cases} y > x^2 - 4 \\ y < -x^2 + 3 \end{cases}$

49. $\begin{cases} x^2 + y^2 \geq 4 \\ x \geq 0 \end{cases}$

50. $\begin{cases} x^2 + y^2 \geq 1 \\ x^2 - 4y^2 \leq 16 \end{cases}$

51. $\begin{cases} x^2 + y^2 < 4 \\ y \geq x^2 + 3 \end{cases}$

52. $\begin{cases} x^2 + 16y^2 > 16 \\ 4x^2 + 9y^2 < 36 \end{cases}$

53. $\begin{cases} x^2 + y^2 \leq 4 \\ (x-2)^2 + y^2 \leq 4 \end{cases}$

54. $\begin{cases} x^2 - y^2 \leq 4 \\ x^2 + y^2 \leq 9 \end{cases}$

55. $\begin{cases} x^2 - y^2 \geq -4 \\ y < 1 - x^2 \end{cases}$

56. Is it possible for a system of nonlinear inequalities to have a single point as its solution set? If so, give an example of such a system. If not, explain why not.

Attributions

Review - ANSWERS

R1 Exercises

1. true **3.** true **5.** false

7. $\{1,2,3,4,5,6,7,8\}$ **9.** $\{2,4,6,\dots\}$ **11.** $\{4,5,6,7,8\}$

13. Answers may vary. Examples of correct answers: $\{n \in \mathbb{W} \,|\, n < 6\}$, $\{n \in \mathbb{Z} \,|\, 0 \le n \le 5\}$

15. $\{x \in \mathbb{R} | x > -3\}$, or $\{x | x > -3\}$

17. Answers may vary. An example of a correct answer: $\{n \in \mathbb{Z} | n = 3k, k \in \mathbb{Z}\}$

19. $\in$ **21.** $\not\subset$ **23.** $\notin$ **25.** $\in$ **27.** $=$

29. **a.** $\sqrt{16}$; **b.** $0, \sqrt{16}$; **c.** $0, \sqrt{16}$ **d.** $0.999\dots$, -5.001, 0, $5\frac{3}{4}$, $1.4\overline{05}$, $\frac{7}{8}$, $\sqrt{16}$,

e. $\sqrt{2}$, $9.010010001\dots$ **f.** $0.999\dots$, -5.001, 0, $5\frac{3}{4}$, $1.4\overline{05}$, $\frac{7}{8}$, $\sqrt{2}$, $\sqrt{16}$, $9.010010001\dots$

31. $1.\overline{02} = \frac{101}{99} \in \mathbb{Q}$ **33.** $2.0\overline{125} = \frac{4021}{1998} \in \mathbb{Q}$

35. $5.22\overline{54} = \frac{1437}{275} \in \mathbb{Q}$

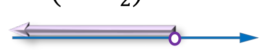
R2 Exercises

1. $-6 < -3$ **3.** $17 \ge x$ **5.** $2x + 3 \ne 0$ **7.** $2 < x < 5$

9. $-2 \le 2x < 6$

11. $[-4, \infty)$

13. $\left(-\infty, \frac{5}{2}\right)$



15. $(0, 6)$

17. $[-5, 16)$

19. -7 **21.** 8 **23.** -12

25. $<$ **27.** $=$ **29.** $\le$ **31.** 25

33. $\frac{1}{6}$ **35.** $|y - 5|$ **37.** $\{-5, 5\}$ **39.** $\{a - 5, a + 5\}$

R3 Exercises

1. true	**3.** true	**5.** false	**7.** true
9. false	**11.** $7 \cdot (5 \cdot 2)$	**13.** $-a$	**15.** 1
17. 0	**19.** $\frac{7}{20}$	**21.** $-\frac{1}{2}$	**23.** $-6x$
25. -8	**27.** -10	**29.** 18	**31.** $-\frac{41}{24}$ or $-1\frac{17}{24}$
33. $-\frac{101}{24}$ or $-4\frac{5}{24}$	**35.** -61	**37.** $-x + y$	**39.** $16x + 8y - 10$
41. $-x$	**43.** $-13x^2 + 12x$	**45.** $2\sqrt{x} - 4$	**47.** -2
49. $-25x^2$	**51.** $a + 7$	**53.** $-3x + 5$	**55.** $33a - 10$
57. $66x + 14$	**59.** $-12x + 615$	**61.** -8	**63.** 24
65. $\frac{8}{9}$	**67.** no	**69.** no	**71.** $29 \cdot 100 = 2900$
73. $\frac{8}{5} \cdot 30 = 48$	**75.** $9 \cdot 5 + 2 - (8 \cdot 3 + 1) = 22$		

Linear Equations - ANSWERS

L1 Exercises

1. true **3.** false **5.** true **7.** expression

9. equation **11.** linear **13.** not linear **15.** linear

17. yes **19.** No **21.** $\frac{5}{6}$ **23.** -2

25. -1 **27.** $\mathbb{R}$; identity **29.** $\emptyset$; contradiction **31.** $-\frac{2}{3}$

33. -6 **35.** $\frac{13}{66}$ **37.** -1 **39.** -12

41. 3 **43.** $\frac{5}{32}$ **45.** $\frac{145}{23}$ **47.** 2500

L2 Exercise

1. A and C

3. $r = \frac{I}{Pt}$

5. $m = \frac{E}{c^2}$

7. $b = 2A - a$

9. $l = \frac{P-2w}{2}$ or $l = \frac{P}{2} - w$

11. $\pi = \frac{S}{rs+r^2}$

13. $C = \frac{5}{9}(F - 32)$

15. $p = 2Q + q$

17. $q = \frac{T-B}{Bt}$

19. $R = \frac{d}{1-st}$

21. **a.** $C(n) = 1.9n + 3.2$
b. $\$22.20$
c. 15 km

23. **a.** 5 ml
b. $d = \frac{c(a+12)}{a}$
c. 75 mg

25. **a.** $C = 1060d$
b. 7420

27. $L = \frac{A}{W}$

29. **a.** $t = \frac{I}{Pr}$
b. 3 year

31. **a.** $k = 120;\ N = 120P$
b. 75,778,800 bottles

33. 67 g **35.** ~5 cm **37.** \$3000

39. The area would decrease by 25%

41. ~105 barrels

L3 Exercises

1. $x - 7$

3. $\frac{1}{2}(x + y)$

5. $x^2 - y^2$

7. $n + (n + 1) + (n + 2) = 30$

9. $\frac{3x}{10}$

11. $0.03x - 100$

13. $x^2 - x$

15. 8

17. 238 and 239

19. 86.9%

21. $11, 13, 15$

23. 33, 35, 37

25. \$1850

27. ~ 158700

29. \$9000 at 3%
\$42000 at 6.5%

31. \$6000 at 4.5%
\$8000 at 5.25%

33. \$4800

35. $39°, 63°, 78°$

37. 8 ft by 16 ft

39. 7 nickels; 9 dimes

41. 12 kg of pecans
18 kg of cashews

43. 126 tickets for adults; 52 tickets for children

45. 8.22 \$/kg

47. 20 grams

49. 375 ml

51. 40 ml

53. **a.** 1 hr 51 min; **b.** 1 hr 23 min

55. 6 km

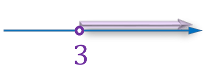
57. 1 min 12 sec

L4 Exercises

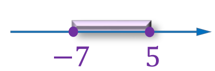
1. $[2,3]$

3. $(-\infty, 4)$

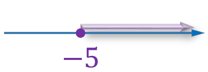
5. $x > 3$



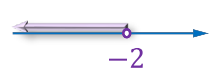
7. $-7 \le x \le 5$



9. $[-5, \infty)$



11. $(-\infty, -2)$



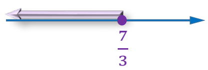
13. $(-4,1)$

15. $(-5,-2]$

17. yes

19. yes

21. $\left(-\infty,\frac{7}{3}\right]$

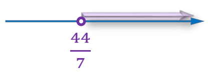


23. $(15,\infty)$

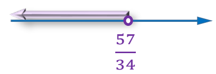
25. $[-9,\infty)$

27. $(-\infty,\infty)=\mathbb{R}$

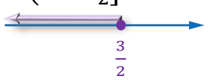
29. $\left(\frac{44}{7},\infty\right)$



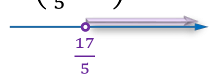
31. $\left(-\infty,\frac{57}{34}\right)$



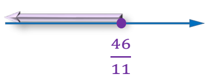
33. $\left(-\infty,\frac{3}{2}\right]$



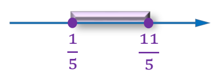
35. $\left(\frac{17}{5},\infty\right)$



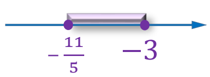
37. $\left(-\infty,\frac{46}{11}\right]$



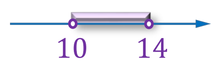
39. $\left[\frac{1}{5},\frac{11}{5}\right]$



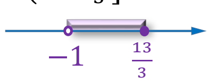
41. $\left[-\frac{11}{3},-3\right]$



43. $(10,14)$



45. $\left(-1,\frac{13}{3}\right]$



47. $x+5>12$
$x\in(7,\infty)$

49. $3x+2\geq 8$
$x\in[2,\infty)$

51. $\frac{1}{2}(x+3)\leq 12$
$x\in(-\infty,21]$

53. $-6<2x<8$
$x\in(-3,4)$

55. at least 87%

57. up to 112 days

59. between -5°C and 20°C

61. more than \$30,000

63. 65 cheques

L5 Exercises

1. $\{1,\ 3\}$

3. $\{1,\ 3,\ 5\}$

5. $\emptyset$

7. $\{5\}$

9. $[1,3]$

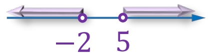
11. $(0,7]$

13. $(-\infty,\infty)$

15. $\{1\}$

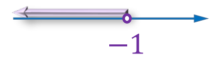
17. $(-2, \infty)$

19. $(-\infty, -2) \cup (5, \infty)$



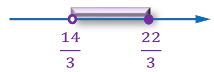
21. $(-2, 1)$

23. $(-\infty, -1)$



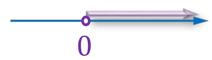
25. $[6, \infty)$

27. $\left(\frac{14}{3}, \frac{22}{3}\right]$

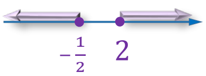


29. $(-\infty, \infty) = \mathbb{R}$

31. $(0, \infty)$



33. $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$



35. $[-4, 7)$

37. 8.5 to 11.5 hr/day

39. at least 14 and at most 24

41. **a.** $\{Education, Humanities, Nursing, Veterinary\ Medicine\}$
b. $\{Nursing\}$
c. $\{Education, Humanities, Business, Mathematics, Dentistry, Veterinary\ Medicine\}$
d. $\{Business, Mathematics, Dentistry\}$

L6 Exercises

1. $2x^2$

3. $\frac{5}{|y|}$

5. $7x^4|y|^3$

7. $\frac{x^2}{|y|}$

9. $\frac{x^2}{2}$

11. $(x-1)^2$

13. **a.** $\emptyset$ **b.** 1 **c.** 2

15. $\{-4, 4\}$

17. $\{-5, 11\}$

19. $\left\{0, \frac{10}{3}\right\}$

21. $\{-28, 16\}$

23. no solution

25. $\{-2, 2\}$

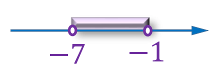
27. $\{-7, 8\}$

29. $\left\{-\frac{3}{5}, 5\right\}$

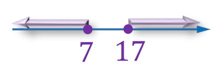
31. $\left\{-\frac{40}{3}, -\frac{20}{7}\right\}$

33. $\left\{\frac{20}{17}, \frac{40}{13}\right\}$

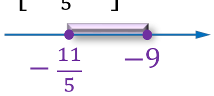
35. $(-7, -1)$



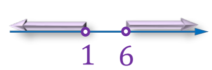
37. $(-\infty, 7] \cup [17, \infty)$



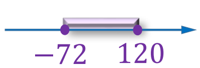
39. $\left[-\frac{11}{5}, 1\right]$



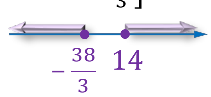
41. $(-\infty, 1) \cup (6, \infty)$



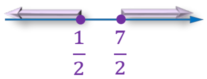
43. $[-72, 120]$



45. $\left(-\infty, -\frac{38}{3}\right] \cup [14, \infty)$



47. $\left(-\infty, \frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$



49. $[-5, -3]$

51. $\{-6\}$

53. $\emptyset$

55. $\mathbb{R}$

57. **a.** $|M - 370| \leq 50$
b. $M \in [320, 420]$

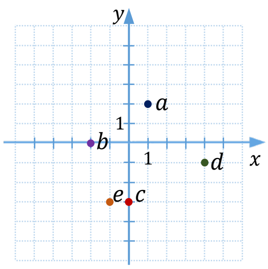
59. $|C - 1200| \leq 100$
$C \in [1100, 1300]$

61. $|r - 85| \leq 25$

Graphs and Linear Functions - ANSWERS

G1 Exercises

1.

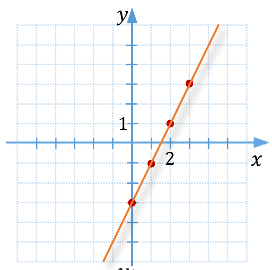


3. yes

5. no

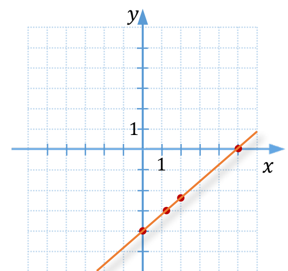
7.

x	y
-3	3
0	2
3	1
6	0



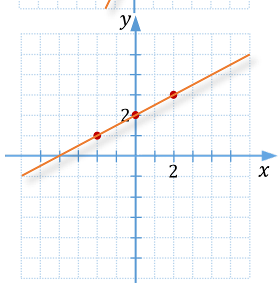
9.

x	y
0	-4
5	0
2	$-\frac{12}{5}$
$\frac{5}{4}$	-3



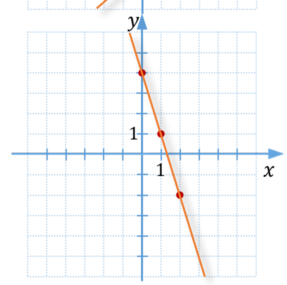
11.

x	y
0	2
2	3
-2	1



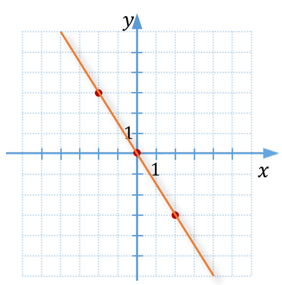
13.

x	y
0	4
1	1
2	-2



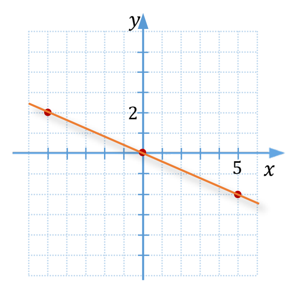
15.

x	y
-2	3
0	0
2	-3



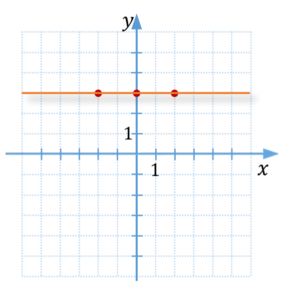
17.

x	y
0	0
5	-2
-5	2



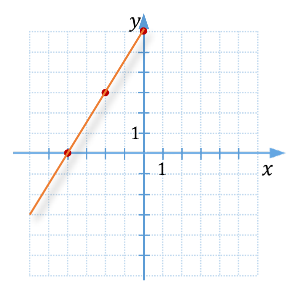
19.

x	y
-2	3
0	3
2	3



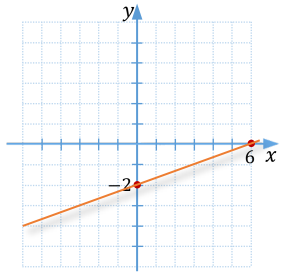
21.

x	y
0	6
-2	3
-4	0



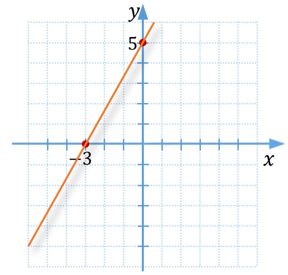
23.

x	y
6	0
0	-2



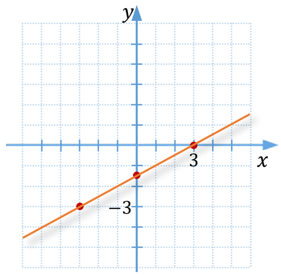
25.

x	y
-3	0
0	5



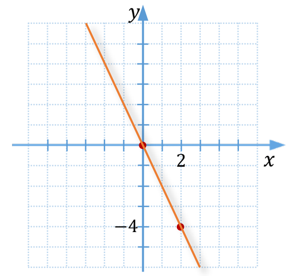
27.

x	y
3	0
0	$-\frac{3}{2}$
-3	-3

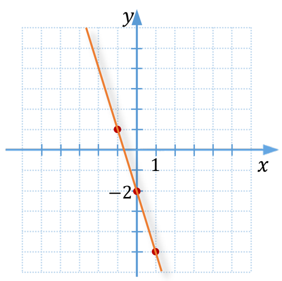


29.

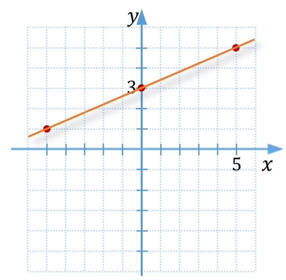
x	y
0	0
2	-4



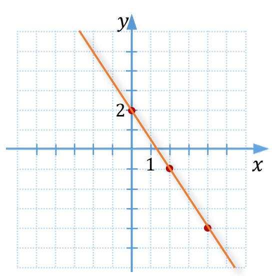
31. y-int. $= 2$
slope $= -3$



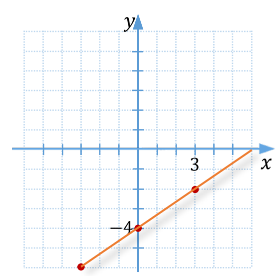
33. y-int. $= 3$
slope $= \frac{2}{5}$



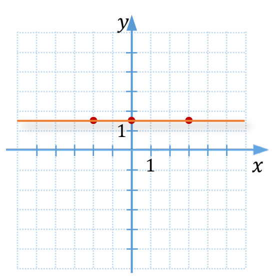
35. y-int. $= 2$
slope $= -\frac{3}{2}$



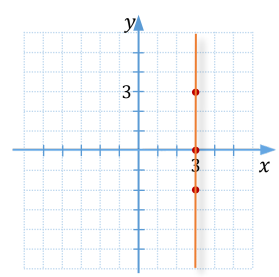
37. y-int. $= -4$
slope $= \frac{2}{3}$



39. y-int. $= \frac{3}{2}$
slope $= 0$



41. y-int. $= none$
slope $= undefined$



43. $\left(\frac{3}{2}, 0\right)$

45. $\left(-\frac{9}{2}, 8\right)$

47. $\left(\frac{11}{20}, -\frac{17}{12}\right)$

49. $(3, -4)$

51. $(3,10)$

G2 Exercises

1. $-\frac{1}{3}$

3. 4

5. $\frac{1}{2}$

7. $\frac{4}{5}$

9. undefined

11. -1

13. $\frac{4}{9}$

15. $y = -3x - 5$

17. $y = -\frac{2}{5}x + \frac{14}{5}$

19. $y = -1$

21. $\frac{1}{2}$

23. $\frac{2}{3}$

25. $-\frac{5}{3}$

27. 0

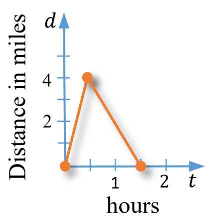
29. 3

31. $a - C, b - A, c - D, d - B$

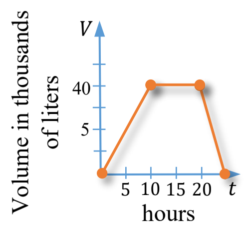
33. For the first 4 years, the pay raise was 0 $\%/year$.

35. On average, between 6 and 16 years old boys grow 6.7 $cm/year$.

37.



39.



41. 375 km/hr

43. perpendicular

45. parallel

47. neither

49. perpendicular

51. not collinear

G3 Exercises

1. $x + 2y = -14$

3. $4x - 5y = 20$

5. $4x + 6y = -9$

7. $y = \frac{1}{6}x - \frac{5}{6}$

9. $y = \frac{4}{5}x - 2$

11. $y = \frac{4}{5}x - 2$

13. $y = \frac{1}{4}x + 2$

15. $y = -x + 3$

17. $y = \frac{1}{2}x + \frac{7}{2}$
$x - 2y = -7$

19. $y = \frac{3}{2}x - 1$
$3x - 2y = 2$

21. $y = -x + 3$
$x + y = 3$

23. $y = -\frac{7}{6}x + \frac{4}{3}$
$7x + 6y = 8$

25. $y = \frac{5}{4}x - \frac{1}{3}$
$15x - 12y = 4$

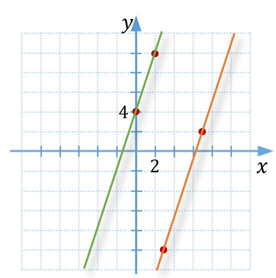
27. $y = 7$

29. $x = -1$

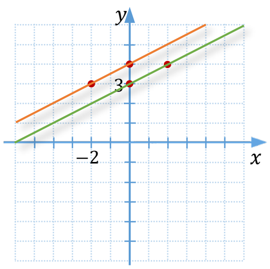
31. $y = 6$

33. $x = -\frac{3}{4}$

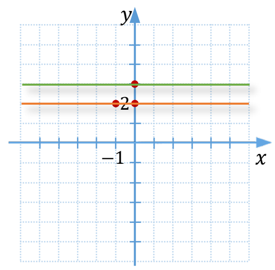
35. $3x - y = 19$



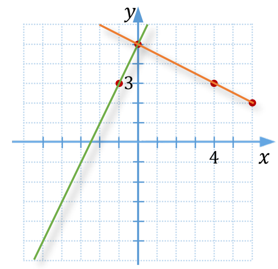
37. $x - 2y = -8$



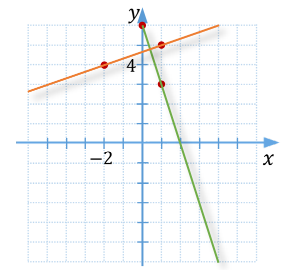
39. $y = 2$



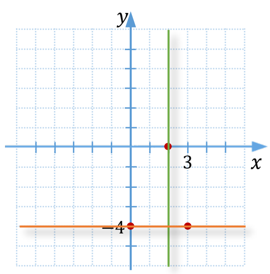
41. $x + 2y = 10$



43. $x - 3y = -14$



45. $y = -4$



47. $C = 49.95n + 80$; $679.40

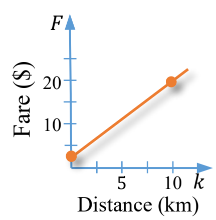
49. **a.** $C = 23d + 60$; **b.** 6 days

51. $N = \frac{17}{3}t + 8$

53. **a.** $C = 800y - 1581200$;
b. The slope of 800 indicates that the annual tuition and fees for out-of-state students at Oxford University was increasing by 800$/year between 2007 and 2016.
c. $36400

55. $A = 180t + 2000$

57. **a.** $F = 1.75k + 2.5$
b.



c. the charge per kilometer
d. 12 km

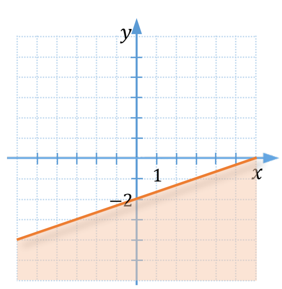
G4 Exercises

1. yes; yes

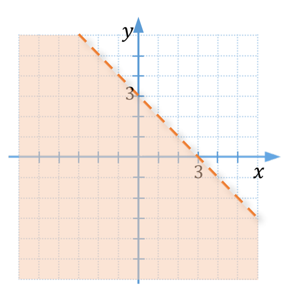
3. no; yes

5. a. – II; b. – IV; c. – I; d. – III;

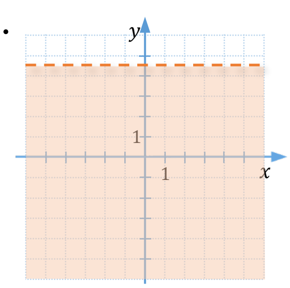
7.



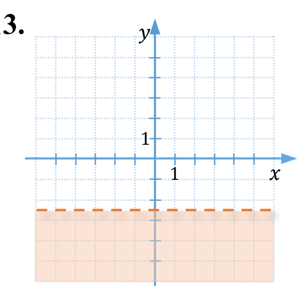
9.



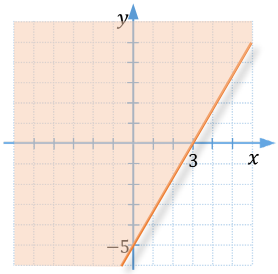
11.



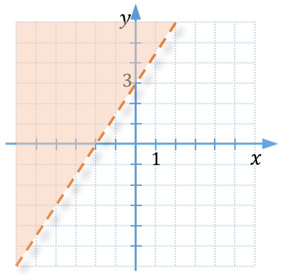
13.



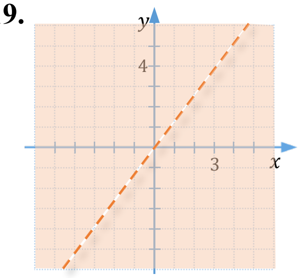
15.



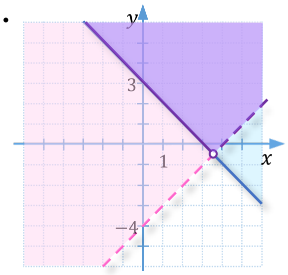
17.



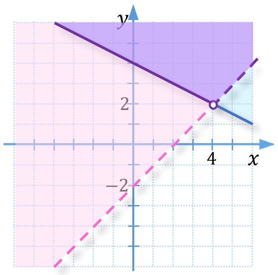
19.



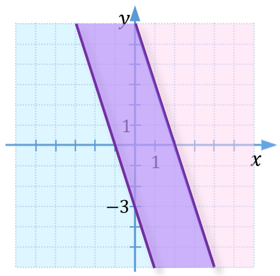
21.



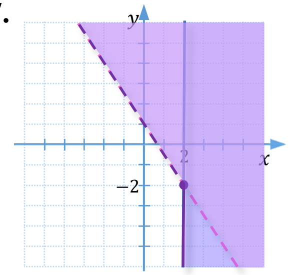
23.



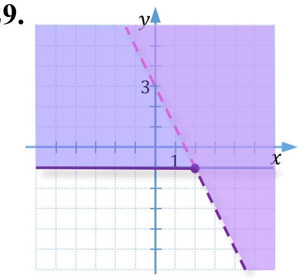
25.



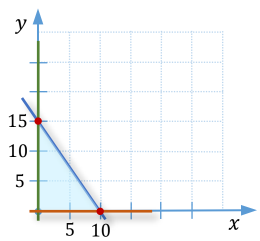
27.



29.



31.



G5 Exercises

1. not a function
domain $= \{0, 2\}$
range $= \{2, 3, 4\}$

3. function
domain $= \{2, 3, 4, 5\}$
range $= \{2, 3, 4, 5\}$

5. not a function
domain $= \{a, b\}$
range $= \{2, 4, 5\}$

7. function
domain $= \{a, b, c\}$
range $= \{2, 4\}$

9. not a function
domain $= \{0, 1\}$
range $= \{-2, -1, 1, 2\}$

11. function
domain $= \{3,6,9,12\}$
range $= \{1, 2\}$

13. function
domain $= \mathbb{R}$
range $= [0,\infty)$

15. function
domain $= \mathbb{R}$
range $= \mathbb{R}$

17. not a function
domain $= \mathbb{R}$
range $= [-4,4]$

19. not a function
domain $= \mathbb{R}$
range $= \mathbb{R}$

21. function
domain $= \mathbb{R}$

23. function
domain $= \mathbb{R}$

25. not a function
domain $= \mathbb{R}$

27. not a function
domain $= [0,\infty)$

29. function
domain $= [0,\infty)$

31. function
domain $= \mathbb{R} \setminus \{-5\}$

33. function
domain $= \mathbb{R} \setminus \{2\}$

35. not a function
domain $= \mathbb{R}$

37. not a function
domain $= \mathbb{R}$

39. function
domain $= \mathbb{R}$

41. not a function
domain $= [-2,2]$

G6 Exercises

1. **a.** 2 **b.** 3

3. **a.** 1 **b.** $\{-1,0\}$ **5.** **a.** 4 **b.** 2 **7.** **a.** -1 **b.** $\{-5,1\}$ **9.** $f(1) = 2$

11. $g(-1) = -4$ **13.** $f(p) = -3p + 5$ **15.** $g(-x) = -x^2 - 2x - 1$

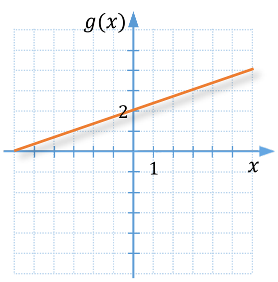
17. $f(a+1) = -3a + 2$ **19.** $g(x-1) = -x^2 + 4x - 4$

21. $f(2+h) = -3h - 1$ **23.** $g(a+h) = -a^2 - 2ah - h^2 + 2a + 2h - 1$

25. $f(3) - g(3) = 0$ **27.** $3g(x) + f(x) = -3x^2 + 3x + 2$

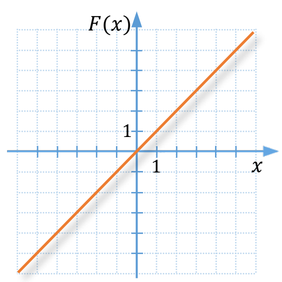
29. line; 4; $-2x + 6$; 4; $(1,4)$

31.



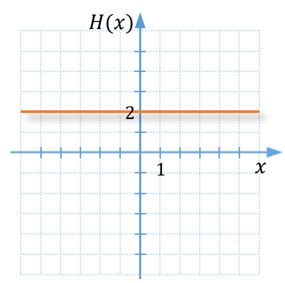
domain $= \mathbb{R}$
range $= \mathbb{R}$

33.



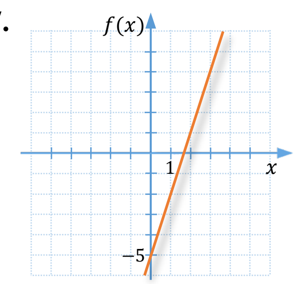
domain $= \mathbb{R}$
range $= \mathbb{R}$

35.



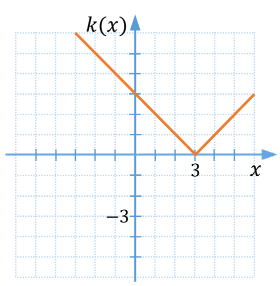
domain $= \mathbb{R}$
range $= \{2\}$

37.



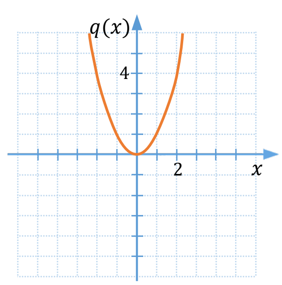
domain $= \mathbb{R}$
range $= \mathbb{R}$

39.



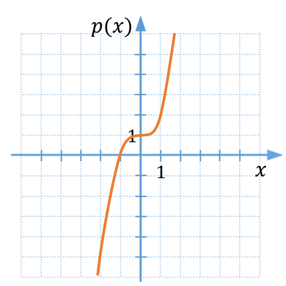
domain $= \mathbb{R}$
range $= [0, \infty)$

41.



domain $= \mathbb{R}$
range $= [0, \infty)$

43.



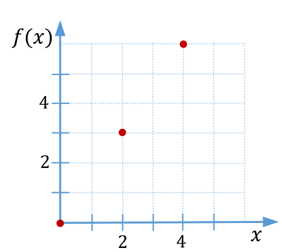
domain $= \mathbb{R}$
range $= \mathbb{R}$

45. **a.**

x	$f(x)$
0	0
2	3.00
4	6.00

b. $f(x) = 1.5x$

c.



47. **a.** $C(d) = 24.6d + 18.8$ **b.** $C(4) = 117.20$; The cost of renting the car for 4 days is \$117.20.
c. $d = 7$

49. **a.** $t \in [0,20]$; $f(t) \in [0,600]$ **b.** 5 minutes; 10 minutes **c.** 600 meters
d. $f(15) = 300$; In 15 minutes, the person is 300 meters from home.

51. The height of water in the bathtub decreases quickly, then remains constant, and finally increases slowly until it reaches half of the original height.

Systems of Linear Equations - ANSWERS

E1 Exercises

1. system

3. Consistent

5. inconsistent

7. empty

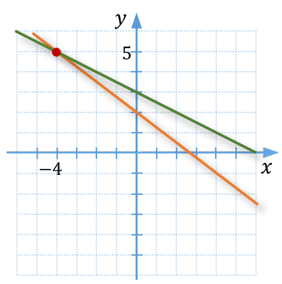
9. one

11. opposite

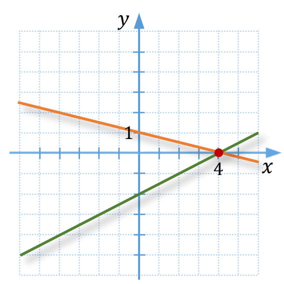
13. yes

15. no

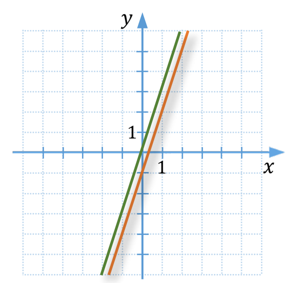
17. $(-4,5)$
consistent;
independent



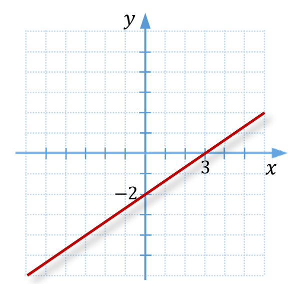
19. $(4,0)$
consistent;
independent



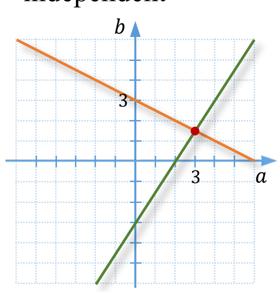
21. no solution;
inconsistent;
independent



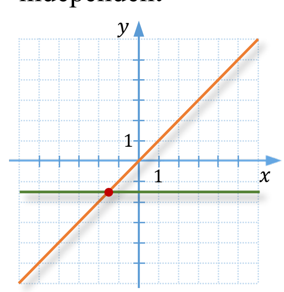
23. $\{(x,y)|2x-3y=6\}$
consistent;
dependent



25. $\left(3,\frac{3}{2}\right)$
consistent;
independent



27. $\left(-\frac{3}{2},-\frac{3}{2}\right)$
consistent;
independent



29. $(-1,-1)$

31. $\left(\frac{7}{3},\frac{1}{3}\right)$

33. no solution;
parallel lines

35. $(5,1)$

37. $(4,6)$

39. $(4.2,-4.4)$

41. $(12,8)$

43. $(4,-1)$

45. $(-1,1)$

47. infinitely many
solutions; same line

49. $\left(\frac{140}{13},-\frac{50}{13}\right)$

51. $\left(\frac{10}{21},\frac{11}{14}\right)$

53. There are infinitely many other solutions. Any point satisfying the equation $5x+3y=14$ is a solution. For example, $(-2,8)$ is another solution.

55. $\begin{cases} y = -\frac{2}{3}x + \frac{1}{3} \\ y = -\frac{2}{3}x + \frac{1}{3} \end{cases}$, infinitely many solutions

57. $(-3, 2)$

59. $(9, 4)$

61. infinitely many solutions: $\{(x, y) | x + 2y = 48\}$

63. $\left(\frac{2}{3}, \frac{1}{3}\right)$

65. $\left(-5, -\frac{5}{3}\right)$

67. $(0,\ 4)$

69. $\left(\frac{1}{a}, \frac{1}{b}\right)$

71. $\left(-\frac{3}{5a}, \frac{7}{5}\right)$

73. $[0, 30)$

75. 2017

77. since 2006; ~7%

E2 Exercises

1. 8 liters

3. $9.25n$

5. $r + c; r - c$

7. $69°, 21°$

9. base = 154 cm; heigth = 77 cm

11. 35 km

13. 13 gold; 11 silver; 9 bronze

15. 113 espressos; 339 cappuccinos

17. 322 adult tickets; 283 youth tickets

19. 2.49 $/egg salad sandwich; 3.99 $/meat sandwich

21. 416 $/week in New York; 340 $/week in Paris

23. $2300 at 3.25%; $2500 at 2.75%

25. $3200 at 3.7%; $2800 at 8.2%

27. 9 L of 4% brine; 3 L of 20% brine

29. $66\frac{2}{3}$ g of cottage cheese; 40 g of vanilla yogurt

31. 5 loonies; 9 quarters

33. houseboat: 12 km/h; current: 3 km/h

35. plane: 275 km/h; wind: 25 km/h

37. plane: 315 km/h; wind: 45 km/h

39. 480 km

41. 0.7 L

Polynomials and Polynomial Functions - ANSWERS

P1 Exercises

1. yes
3. no
5. $4;\ 1$
7. $2;\ \sqrt{2}$
9. $-\frac{2}{5}x^3 + 3x^2 - x + 5;\ 3;\ -\frac{2}{5}$
11. $x^5 + 8x^4 + 2x^3 - 3x;\ 5;\ 1$
13. $3q^4 + q^2 - 2q + 1;\ 4;\ 3$
15. first degree binomial
17. zero degree monomial
19. seventh degree monomial
21. -8
23. -12
25. -5
27. $2a - 3$
29. -21
31. $6a - 9$
33. $-x + 13y$
35. $4xy + 3x$
37. $6p^3 - 3p^2 + p + 2$
39. $3m + 11$
41. $-x - 4$
43. $-5x^2 + 4y^2 - 11z^2$
45. $-4x^2 - 3x - 5$
47. $5r^6 - r^5 - 7r^2 + 5$
49. $-5a^4 - 6a^3 + 9a^2 - 11$
51. $5x^2y^2 - 7y^3 + 17xy$
53. $-z^2 + x + 4m$
55. $10z^2 - 16z$
57. **a.** $(f+g)(x) = 8x - 8$ **b.** $(f-g)(x) = 2x - 4$
59. **a.** $(f+g)(x) = -2x^2 - 3x + 1$ **b.** $(f-g)(x) = 8x^2 - 7x - 1$
61. **a.** $(f+g)(x) = -6x^{2n} - 2x^n - 1$ **b.** $(f-g)(x) = 10x^{2n} - 4x^n + 7$
63. $(P-Q)(-2) = -1$
65. $(R-Q)(0) = -7$
67. $(P+Q)(a) = a^2 + 2a + 1$
69. $(P+R)(2k) = 4k^2 + 2k - 6$
71. ~9.3 cm
73. **a.** $R(n) = 56n$ **b.** $P(n) = 24n - 1500$ **c.** $P(100) = 900$;
The profit from selling 100 dresses is \$900.

P2 Exercises

1. **a.** no; $x^2 \cdot x^4 = x^6$ **b.** no; $-2x^2$ is in the simplest form **c.** yes **d.** yes **e.** no; $(a^2)^3 = a^6$
f. no; $4^5 \cdot 4^2 = 4^7$ **g.** no; $\frac{6^5}{3^2} = 2^5 \cdot 3^3$ **h.** no; $xy^0 = x$ **i.** yes
3. $-8y^8$
5. $14x^3y^8$
7. $-27x^6y^3$
9. $\frac{-5x^3}{y^2}$

11. $\frac{64a^6}{b^2}$ **13.** $\frac{-125p^3}{q^9}$ **15.** $12a^5b^5$ **17.** $\frac{16y}{x^3}$

19. $64x^{18}y^6$ **21.** x^{2n-1} **23.** 5^{2ab} **25.** $-2x^2$

27. $x^{a^2-b^2}$ **29.** $-16x^7y^4$ **31.** $-6x^2+10x$ **33.** $-12x^5y+9x^4y^2$

35. $15k^4-10k^3+20k^2$ **37.** x^2+x-30 **39.** $6x^2+5x-6$

41. $6u^4-8u^3-30u^2$ **43.** $6x^3-7x^2-13x+15$

45. $6m^4-13m^2n^2+5n^4$ **47.** a^2-4b^2 **49.** $a^2-4ab+4b^2$

51. y^3+27 **53.** $2x^4-4x^3y-x^2y^2+3xy^3-2y^4$ **55.** true

57. true **59.** false; $(2-1)^3 \neq 2^3-1^3$ **61.** $25x^2-16$

63. $\frac{1}{4}x^2-9y^2$ **65.** x^4-49y^6 **67.** $0.64a^2+0.16ab+0.04b^2$

69. x^2-6x+9 **71.** $25x^2-60xy+36y^2$ **73.** $4n^2-\frac{4}{3}n+\frac{1}{9}$

75. $x^8y^4+6x^4y^2+9$ **77.** $4x^4-12x^2y^3+9y^6$

79. $8a^5+40a^4b+50a^4b^2$ **81.** $x^4-x^2y^2$

83. x^4-1 **85.** $a^4-2a^2b^2+b^4$ **87.** $4x^2+12xy+9y^2-25$

89. $4k^2=12k+4hk-6h+h^2+9$ **91.** $x^{4a}-y^{4b}$

93. $101\cdot 99=(100+1)(100-1)=10000-1=9999$

95. $505\cdot 495=(500+5)(500-5)=250000-25=249975$

97. x^2-x-12 **99.** $(fg)(x)=15x^2-28x+12$

101. $(fg)(x)=-3x^4+8x^3+22x^2-45x$ **103.** $(PR)(x)=x^3-2x^2-4x+8$

105. $(PQ)(a)=2a^3-8a$ **107.** $(PQ)(3)=30$

109. $(QR)(x)=2x^2-4x$ **111.** $(QR)(a+1)=2a^2-2$

113. $P(2a+3)=4a^2+12a+5$ **115.** $4x^3-40x^2+96x$

P3 Exercises

1. False; When dividing powers with the same bases, we subtract exponents. So, the quotient will be a fourth-degree polynomial.

3. $4x^2-3x+1$ **5.** $2xy-6$ **7.** $-3a^3+5a^2-4a$ **9.** $8-\frac{9}{x}+\frac{3}{2x^2}$

11. $\frac{2b}{a}+\frac{5}{3}+\frac{3c}{a}$ **13.** $y+5$ **15.** $t-4$ R -21

17. $2a^2-a+2$ R 6 **19.** $2z^2-4z+1$ R -10 **21.** $3x+1$ R $-3x-7$

23. $3k^2+4k+1$ **25.** $\frac{5}{4}t+1$ R -5 **27.** p^2+p+1

29. y^3-2y^2+4y-8 R 32 **31.** $Q(x)=2x^2-x+6;\ R(x)=4$

33. $\left(\frac{f}{g}\right)(x)=3x-2;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\{0\}$ **35.** $\left(\frac{f}{g}\right)(x)=x-6;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\{-6\}$

37. $\left(\frac{f}{g}\right)(x)=x+1;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\left\{\frac{3}{2}\right\}$ **39.** $\left(\frac{f}{g}\right)(x)=4x^2-10x+25;\quad D_{\frac{f}{g}}=\mathbb{R}\setminus\left\{-\frac{5}{2}\right\}$

41. $\left(\frac{R}{Q}\right)(x)=\frac{x-2}{2x}$ **43.** $\left(\frac{R}{P}\right)(x)=\frac{1}{x+2},\ x\neq 2$ **45.** $\left(\frac{R}{Q}\right)(0)=DNE$

47. $\left(\frac{R}{P}\right)(-2)=DNE$ **49.** $\left(\frac{P}{R}\right)(a)=a+2$ **51.** $\frac{1}{2}\left(\frac{Q}{R}\right)(x)=\frac{x}{x-2}$

53. **a.** $L=3x-2$ **b.** 10 m

P4 Exercises

1. $(f\circ g)(1)=18$ **3.** $(f\circ g)(x)=x^2-10x+27$ **5.** $(f\circ h)(-1)=27$

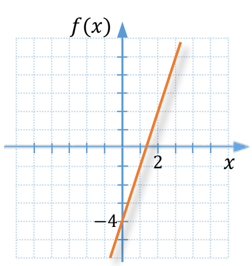
7. $(f\circ h)(x)=4x^2-12x+11$ **9.** $(h\circ g)(-2)=11$

11. $(h\circ g)(x)=-2x+7$ **13.** $(f\circ f)(2)=38$ **15.** $(h\circ h)(x)=4x-9$

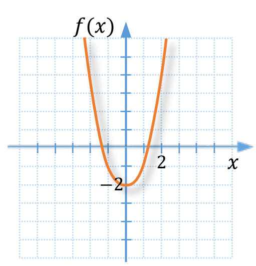
17. $(g\circ f)(x)=30.48x$ computes the number of centimeters in x feet

19. **a.** $r=\frac{C}{2\pi}$ **b.** $A=\frac{C^2}{4\pi}$ **c.** $A(6\pi)=9\pi$

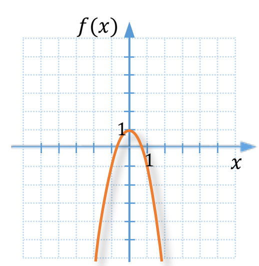
21. No. It is 40.5% off. To find the new price we use composition of functions $(f\circ g)(x)$ where $f(x)=.85x$ and $g(x)=.7x$. So, the discount is $x-f\big(g(x)\big)=x-.85\cdot .7x=(1-.595)x=.405x$. Thus, the dress was overall discounted by 40.5%.

23.


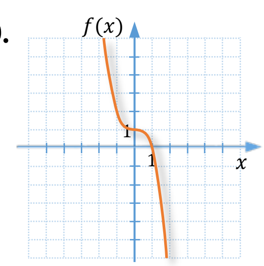
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

25.


Domain: $\mathbb{R}$
Range: $[-2,\infty)$

27.


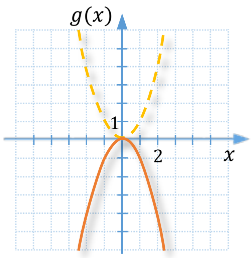
Domain: $\mathbb{R}$
Range: $(-\infty,1]$

29.


Domain: $\mathbb{R}$
Range: $\mathbb{R}$

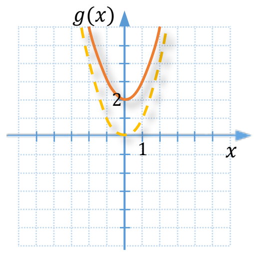
A20

31.



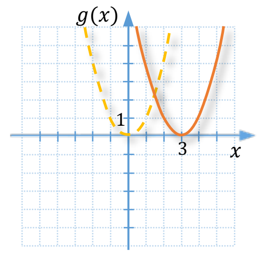
symmetry in x-axis

33.



translation: 2 up

35.



translation: 3 to the right

Factoring - ANSWERS

F1 Exercises

1. false
3. Both are correct but the second one is preferable as the binomial factor has integral coefficients.
5. $7a^3b^5$
7. $x(x-3)$
9. $(x-2y)$
11. $x^{-4}(x+2)^{-2}$
13. $8k(k^2+3)$
15. $-6a^2(6a^2-a-3)$
17. $5x^2y^2(y-2x)$
19. $(a-2)(y^2-3)$
21. $2n(n-2)$
23. $(4x-y)(4x+1)$
25. $-(p-3)(p^2-10p+19)$
27. $k^{-4}(k^2+2)$
29. $-p^{-5}(2p^3-p^2-3)$
31. $-x^{-2}y^{-3}(2xy-5)$
33. $(a^2-7)(2a+1)$
35. $-(xy+3)(x-2)$
37. $(x^2-y)(x-y)$
39. $-(y-3)(x^2+z^2)$
41. $(x-6)(y+3)$
43. $(x^2-a)(y^2-b)$
45. $(x^n+1)(y-3)$
47. $2(s+1)(3r-7)$
49. $x(x-1)(x^3+x^2-1)$
51. no, as $(2xy^2-4)$ can still be factored to $2(xy^2-2)$
53. $p=\frac{2M}{q+r}$
55. $y=\frac{x}{3-w}$
57. $A=(4-\pi)x^2$
59. $A=(\pi-1)r^2$

F2 Exercises

1. no
3. All of them; however, the preferable answer is $-(2x-3)(x+5)$.
5. $x-3$
7. $x-5y$
9. $(x+3)(x+4)$
11. $(y+8)(y-6)$
13. not factorable
15. $(m-7)(m-8)$
17. $-(n+9)(n-2)$
19. $(x-2y)(x-3y)$
21. $-(x+3)(x-7)$
23. $n^2(n+2)(n-15)$
25. $-2(x-10)(x-4)$
27. $y(x^2+12)(x^2-5)$
29. $-5(t^{13}+8)(t^{13}-1)$
31. $-n(n^4+16)(n^4-3)$
33. $\pm 12, \pm 13, \pm 15, \pm 20, \pm 37$
35. $3x-4$
37. $3x-5$
39. $(2y+1)(3y-2)$
41. $(6t-1)(t-6)$
43. $(6n+5)(7n-5)$
45. $-2(2x-3)(3x+5)$
47. $(6x+5y)(3x+2y)$
49. $-(2n+5)(4n-3)$
51. $2x^2(2x-1)(x+3)$
53. $(9xy-4)(xy+1)$
55. $(2p^2-7q)^2$
57. $(2a+9)(a+5)$
59. $\pm 3, \pm 4, \pm 11, \pm 17, \pm 28, \pm 59$
61. $(3x+2)$ feet

F3 Exercises

1. difference of squares
3. neither
5. difference of cubes
7. difference of squares
9. perfect square
11. difference of cubes
13. $25x^2 + 100 = 25(x^2 + 4)$; The sum of squares is factorable in integral coefficients only if the two terms have a common factor.
15. $(x + y)(x - y)$
17. $(x - y)(x^2 + xy + y^2)$
19. $(2z - 1)^2$
21. not factorable
23. $(5 - y)(25 + 5y + y^2)$
25. $(n + 10m)^2$
27. $(3a^2 + 5b^3)(3a^2 - 5b^3)$
29. $(p^2 - 4q)(p^4 + 4p^2q + 16q^2)$
31. $(7p + 2)^2$
33. $r^2(r + 3)(r - 3)$
35. $\frac{1}{8}(1 - 2a)(1 + 2a + 4a^2)$ or $\left(\frac{1}{2} - a\right)\left(\frac{1}{4} + \frac{1}{2}a + a^2\right)$
37. not factorable
39. $x^2(4x^2 + 11y^2)(4x^2 - 11y^2)$
41. $-(ab + 5c^2)(a^2b^2 - 5abc^2 + 25c^4)$
43. $(3a^4 - 8b)^2$
45. $(x + 8)(x - 6)$
47. $2t(t - 4)(t^2 + 4t + 16)$
49. $(x^n + 3)^2$
51. $(4z^2 + 1)(2z + 1)(2z - 1)$
53. $5(3x^2 + 15x + 25)$
55. $0.01(5z - 7)^2$ or $(0.5z - 0.7)^2$
57. $-3y(2x - y)$
59. $4(3x^2 + 4)$
61. $2(x - 5a)^2$
63. $(y + 6 + 3a)(y + 6 - 3a)$
65. $(m + 2)(m^2 - 2m + 4)(m - 1)(m^2 + m + 1)$
67. $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$
69. $(x^2 + 1)(x + 3)(x - 3)$
71. $(a + b + 3)(a - b - 3)$
72. $z(3xy + 4z)(xy + 7z)$
75. $(x^2 + 1)(x + 1)(x - 1)^3$
77. $c(c^w + 1)^2$

F4 Exercises

1. true
3. false
5. false
7. $x \in \{-4, 1\}$
9. $x \in \left\{-\frac{4}{5}, -\frac{1}{3}\right\}$
11. $x \in \{-6, -3\}$
13. $x \in \left\{-\frac{7}{2}, 1\right\}$
15. $x \in \{-6, 0\}$
17. $x \in \{4\}$
19. $x \in \left\{\frac{5}{2}\right\}$
21. $x \in \{-8, 4\}$
23. $x \in \left\{\frac{1}{3}, 3\right\}$
25. $x \in \left\{-2, \frac{8}{9}\right\}$
27. $x \in \{0, 6\}$
29. $x \in \{-4, 2\}$
31. $x \in \{1, 5\}$

33. $x \in \left\{-\frac{15}{8}, -1\right\}$ **35.** $x \in \{-5, 0, 3\}$ **37.** $x \in \left\{-\frac{8}{5}, 0, \frac{8}{5}\right\}$ **39.** $x \in \{-5, -1, 1, 5\}$

41. $x \in \{0, 2, 4\}$ **43.** $x \in \{-3, -1, 3\}$ **45.** $x \in \left\{-2, -\frac{2}{5}, 2\right\}$

47. 3; $\{-3,0,3\}$; Do not divide by x as x can be equal to zero. Also, $\sqrt{x^2} = |x|$ so in the last step, we should obtain $x = \pm 3$. The safest way to solve polynomial equations is by factoring and using the zero-product property.

49. $x \in \left\{\frac{1}{2}, 7\right\}$ **51.** $x \in \left\{-3, \frac{7}{3}\right\}$ **53.** $s = \frac{5-2p}{r+3}$ **55.** $r = \frac{R}{E-1}$

57. $t = \frac{4}{c+2}$ **59.** 8 seconds **61.** -12 or 13

63. width = 9 cm; length = 16 cm **65.** width = 7 m; height = 10 m

67. 7 m by 12 m **69.** 2 cm **71.** 9 in

Rational Expressions and Functions - ANSWERS

RT1 Exercises

1. true **3.** true **5.** true **7.** false

9. false **11.** false **13.** $\frac{1}{64}$ **15.** $\frac{1}{512}$

17. $-\frac{125}{81}$ **19.** $\frac{3}{8}$ **21.** $-\frac{14}{x^{11}}$ **23.** $-\frac{36}{x^{12n}}$

25. $-\frac{4}{x^3}$ **27.** $3n^4m^2$ **29.** $\frac{3x^2}{2y^2}$ **31.** $-\frac{b^{15}}{27a^6}$

33. $\frac{27}{8x^9y^3}$ **35.** $\frac{x^{10}y^5}{5^{10}}$ **37.** $\frac{64}{x^{24}y^{12}}$ **39.** $\frac{4k^5}{m^2}$

41. $-\frac{5^3y^3}{x^{30}}$ **43.** $-\frac{1}{3^8x^8y^8}$ **45.** $\frac{1}{5^{a^2}}$ **47.** $3n^x$

49. x^{b+5} **51.** $2.6 \cdot 10^{10}$ **53.** $1.05 \cdot 10^{-8}$ **55.** $670{,}000{,}000$

57. $2{,}000{,}000{,}000{,}000$ **59.** $1048576 = 1.05 \cdot 10^6$ **61.** $1.3338 \cdot 10^{-10}$

63. $5 \cdot 10^{-5}$ **65.** $2.5 \cdot 10^7$ **67.** $1.25 \cdot 10^3$ **69.** 18,108.11 \$/person

71. $1.59 \cdot 10^7$ ft³/min; $3.816 \cdot 10^8$ ft³/day **73.** 81 times

RT2 Exercises

1. false **3.** true **5.** $f(-1) = \frac{1}{3},\ f(0) = 0,\ f(2) = undefined$

7. $f(-1) = \frac{1}{2},\ f(0) = \frac{1}{3},\ f(2) = undefined$ **9.** $6;\ D = \mathbb{R} \setminus \{6\};\ D = (-\infty, 6) \cup (6, \infty)$

11. $\frac{4}{5};\ D = \mathbb{R} \setminus \left\{\frac{4}{5}\right\};\ D = \left(-\infty, \frac{4}{5}\right) \cup \left(\frac{4}{5}, \infty\right)$ **13.** none; $D = \mathbb{R}$; $D = (-\infty, \infty)$

15. $-7, -5;\ D = \mathbb{R} \setminus \{-7, -5\};\ D = (-\infty, -7) \cup (-7, -5) \cup (-5, \infty)$

17. **b.**, **d.**, and **e.** are equivallent to -1 **19.** $\frac{8a^2}{b^2}$ **21.** -1

23. 1 **25.** $\frac{4x-5}{7}$ **27.** $\frac{y-3}{y+3}$ **29.** $\frac{6}{7}$

31. $-\frac{m+5}{4}$ **33.** $\frac{t+5}{t-5}$ **35.** $\frac{x-8}{x+4}$ **37.** $\frac{x^2+xy+y^2}{x+y}$

39. $10ab^2$ **41.** $\frac{3}{2y^4}$ **43.** $\frac{10}{9a^2}$ **45.** $-\frac{y+5}{2y}$

47. $(2a-1)(3a-8)$ **49.** $\frac{x^2-16}{x(x+3)}$ **51.** $\frac{1}{b(b-1)}$ **53.** $\frac{x(3x+2)}{(3x+1)(3x-2)}$

55. $\frac{a^2+ab+b^2}{a-b}$ **57.** $\frac{x^2+4x+16}{(x+4)^2}$ **59.** $\frac{1}{2x+3y}$ **61.** $-\frac{7x+3}{7}$

63. $\frac{15}{y^2}$ **65.** $\frac{2b}{a+2b}$ **67.** $\frac{x-6}{x+5}$

69. $f(x)\cdot g(x)=\frac{2(x-4)}{(x+1)^2}$; $f(x)\div g(x)=\frac{x-4}{2x^2}$

71. $f(x)\cdot g(x)=-(x-3)^2$; $f(x)\div g(x)=-\frac{(x-4)^2}{(x+3)^2}$

RT3 Exercises

1. **a.** 18; **b.** 18 **3.** 36; $\frac{41}{36}$ **5.** 240; $\frac{221}{240}$ **7.** $72a^5b^4$

9. $x(x+2)(x-2)$ **11.** $(x-1)^2$ **13.** $y(x+y)(x-y)$ **15.** $(x+1)^2(x-5)$

17. $(x-3)^2(2x+1)(x-1)$ **19.** $6x^3(x+2)^2(x-2)$

21. true; $\frac{1}{3-x}$ is opposite to $\frac{1}{x-3}$ **23.** false; $\frac{3}{4}+\frac{x}{5}=\frac{3\cdot 5+4x}{20}=\frac{4x+15}{20}$

25. $\frac{8}{a+1}$ **27.** $\frac{3n-3}{n-2}$ **29.** $\frac{1}{a+7}$ **31.** a^2+ab+b^2

33. $\frac{2x^2-x+14}{(x+3)(x-4)}$ **35.** $\frac{(x+y)^2}{(x+y)(x-y)}$ **37.** $\frac{y-34}{20(y+2)}$ **39.** $\frac{4y+17}{y^2-4}$

41. $\frac{x(3x+19)}{(x-4)(x-2)(x+3)}$ **43.** $\frac{3y^2+7y+14}{(2y-5)(y+2)(y-1)}$ **45.** $\frac{2x^2-13x+7}{(x+3)(x-3)(x-1)}$ **47.** $\frac{-y}{(y+3)(y-1)}$

49. $\frac{-(14y^2+3y-3)}{(2y+1)(2y-1)}$ **51.** $\frac{6+x^2}{3x^3}$ **53.** $\frac{x-14}{(x+1)(x-4)}$ **55.** $\frac{-(2x^2+5x-2)}{(x+2)(x+1)}$

57. $(f+g)(x)=\frac{x^2+x+8}{(x+2)(x-3)}$; $(f-g)(x)=\frac{x^2-7x-8}{(x+2)(x-3)}$

59. $(f+g)(x)=\frac{3x^2-2x+3}{(x-1)^2(x+3)}$; $(f-g)(x)=\frac{3x^2-4x-3}{(x-1)^2(x+3)}$

61. every 12^{th} day **63.** $\frac{100(P_1-P_0)}{P_0}$

RT4 Exercises

1. $\frac{5}{16}$

3. $-\frac{111}{160}$

5. xy^2

7. $\frac{a-1}{4a+1}$

9. $\frac{-9(x-4)}{2(x+3)}$

11. $\frac{2y-x}{2y+x}$

13. $\frac{a^2(b-3)}{b^2(a-1)}$

15. $\frac{-(2x+y)}{x}$

17. $\frac{n-3}{n}$

19. $\frac{1}{a(a-h)}$

21. $\frac{4}{5}$

23. $\frac{a+b}{ab}$

25. $\frac{(x-3)(x+1)}{x^2+x-1}$

27. $\frac{-ab(a-b)}{a^2-ab+b^2}$

29. The expressions $\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}}$ and $\frac{x+y}{x^2+y^2}$ are **not** equivalent, as if we assume for example that $x = 1$ and $y = 2$, the first expression results in $\frac{5}{6}$ while the second results in $\frac{3}{5}$. Notice that the powers with negative exponents can't be 'shifted to a different level' due to the addition in the numerator and denominator. Only powers that are factors of the numerator or denominator can be 'shifted to a different level' to change the sign of their exponents.

31. $\frac{x+1}{3x}$

33. $\frac{n}{n+1}$

35. $\frac{-2(2a-h)}{a^2(a+h)^2}$

37. $\frac{1}{(a-2)(a+h-2)}$

39. $\frac{-3x-2}{x-2}$

RT5 Exercises

1. $\mathbb{R}$

3. $\mathbb{R} \setminus \{-4,11\}$

5. $\mathbb{R} \setminus \{-5,5,7\}$

7. $x = \frac{17}{2}$

9. $x \in \{-8,-1\}$

11. $r = 2$

13. $r = 30$

15. $y = 3$

17. $x = -5$

19. $x \in \{-3, 1\}$

21. $y = -3$

23. $k = \frac{5}{4}$

25. $y = 4$

27. $x = \frac{1}{5}$

29. $x = \frac{31}{5}$

31. $x = -2$

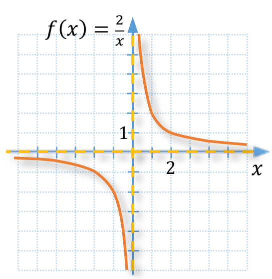
33. $x = 2$

35. $x \in \left\{-\frac{1}{3}, 5\right\}$

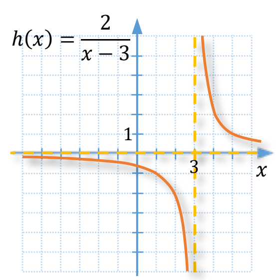
37. $x \in \left\{-\frac{5}{2}, 3\right\}$

39. $x \in \{-2, 6\}$

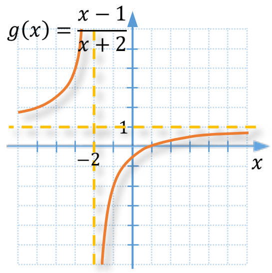
41. $D = \mathbb{R} \setminus \{0\}$; range $= \mathbb{R} \setminus \{0\}$;
VA: $x = 0$; HA: $y = 0$



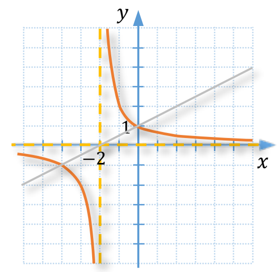
43. $D = \mathbb{R} \setminus \{3\}$; range $= \mathbb{R} \setminus \{0\}$;
VA: $x = 3$; HA: $y = 0$



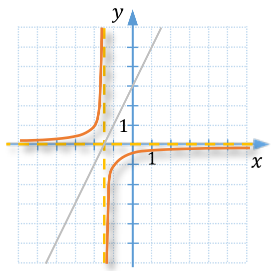
45. $D = \mathbb{R} \setminus \{-2\}$; range $= \mathbb{R} \setminus \{1\}$;
VA: $x = -2$; HA: $y = 1$



47. $g(x) = \frac{2}{x+2}$
VA: $x = -2$; HA: $y = 0$



49. $g(x) = \frac{-1}{2x+3}$
VA: $x = -\frac{3}{2}$; HA: $y = 0$



51. $x \in \left\{-1, \frac{1}{2}\right\}$

53. **a.** $D(10) = 0.9$ If a smoker is 10 times more likely to die of lung cancer than a non-smoker, then 90% of deaths is caused by smoking.

b. $x = 2$

c. The incidence rate is 0 if a smoker is as likely to die of lung cancer as a nonsmoker.

RT6 Exercises

1. $q = 15$

3. factorization of k

5. $a = \frac{F}{m}$

7. $d_1 = \frac{W_1 d_2}{W_2}$

9. $t = \frac{2s}{v_1 + v_2}$

11. $R = \frac{r_1 r_2}{r_1 + r_2}$

13. $q = \frac{fp}{p-f}$

15. $v = \frac{PVt}{Tp}$

17. $b = \frac{2A}{h} - a$ or $b = \frac{2A - ah}{h}$

19. $s = \frac{Rg}{g-R}$

21. $n = \frac{IE}{E - Ir}$

23. $r = \frac{Re}{E-e}$

25. $R = \frac{V}{\pi h^2} + \frac{h}{3}$ or $R = \frac{3V + \pi h^3}{3\pi h^2}$

27. $h = \frac{2R^2 g}{V^2} - R$ or $h = \frac{2R^2 g - V^2 R}{V^2}$

29. 12.375 kg

31. 77 km

33. ~1142 zebras

35. ~155 white-tailed eagles

37. $PR = 6; PS = 3; SR = 4.2$

39. ~17.8 km/h

41. 4.8 km/h

43. 50 km/h

45. 1275 km

A28

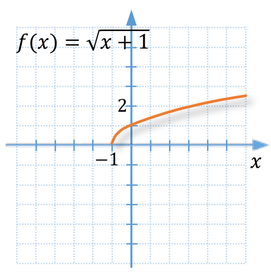
47. 2 km	**49.** $\frac{4(x+y)}{xy}$	**51.** 15 hr	**53.** 24 hr
55. 2450 people	**57.** 20 km	**59.** 12 hours	**61.** 1.4 m
63. 2651 km	**65.** ~1802 N		

Radicals and Radical Functions - ANSWERS

RD1 Exercises

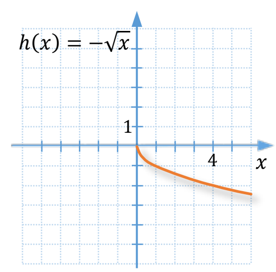
1. 7	**3.** not a real number	**5.** 0.04	**7.** 4
9. 0.2	**11.** $\frac{1}{0.03}$	**13.** 0.2	**15.** not a real number
17. a. negative **b.** not a real number **c.** 0		**19.** 15	**21.** $\lvert x\rvert$
23. $9\lvert x\rvert$	**25.** $\lvert x+3\rvert$	**27.** $\lvert x-2\rvert$	**29.** -5
31. $-5a$	**33.** $5\lvert x\rvert$	**35.** $y-3$	**37.** $\lvert 2a-b\rvert$
39. $\lvert a+1\rvert^3$	**41.** $-k^5$	**43.** 18.708	**45.** 1.710
47. 8	**49.** 11	**51.** 50	**53.** 14 m by 7 m; 42 m

55. $D=[-1,\infty)$
range $=[0,\infty)$



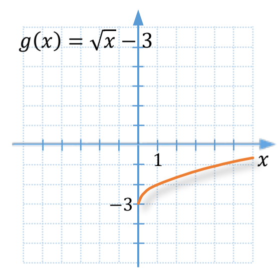
Translation: 1 step to the left

57. $D=[0,\infty)$
range $=(-\infty,0]$



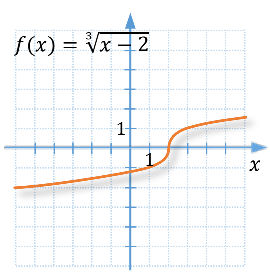
Reflection in x-axis

59. $D=[0,\infty)$
range $=[-3,\infty)$



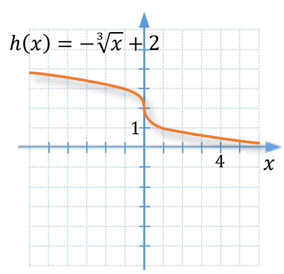
Translation: 3 steps down

61. $D=\mathbb{R}$
range $=\mathbb{R}$



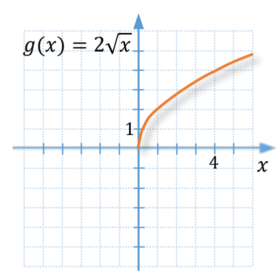
Translation: 2 steps to the right

63. $D=[0,\infty)$
range $=(-\infty,0]$

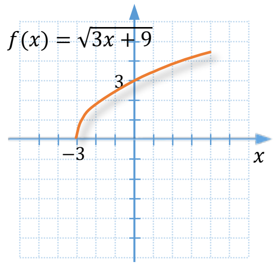


Reflection in x-axis

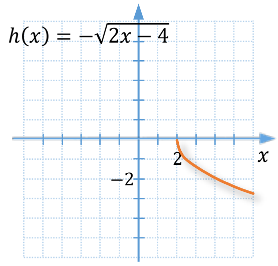
65. $D=[0,\infty)$
range $=[0,\infty)$



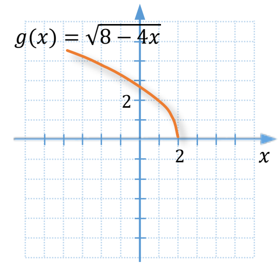
67. $D = [-3, \infty)$
range $= [0, \infty)$



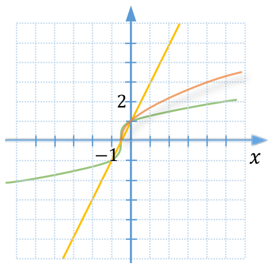
69. $D = [2, \infty)$
range $= (-\infty, 0]$



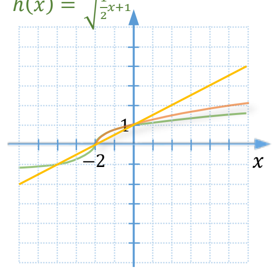
71. $D = (-\infty, 2]$
range $= [0, \infty)$



73. $f(x) = 2x + 1$
$g(x) = \sqrt{2x+1}$
$h(x) = \sqrt[3]{2x+1}$



75. $f(x) = \frac{1}{2}x + 1$
$g(x) = \sqrt{\frac{1}{2}x+1}$
$h(x) = \sqrt[3]{\frac{1}{2}x+1}$



77. ~186 cm **79.** 3.25 m **81.** $700\sqrt{15}$ m^2

RD2 Exercises

1. **a.-B.; b.-A.; c.-C.; d.-F.; e.-D.; f.-E.** **3.** 2 **5.** -343

7. $-\frac{1}{10}$ **9.** $\frac{8}{27}$ **11.** not a real number **13.** -2

15. $5^{\frac{1}{2}}$ **17.** x^3 **19.** $4x^2$ **21.** $5x^{-\frac{5}{2}}$

23. 32 **25.** $\sqrt[5]{x^3}$ **27.** $\sqrt[3]{9}$ **29.** $\frac{2}{\sqrt{x}}$

31. $3^{\frac{7}{8}}$ **33.** $2^{\frac{3}{4}}$ **35.** $5^{\frac{5}{4}}$ **37.** $x^{\frac{1}{2}} \cdot y^{\frac{10}{3}}$

39. $\frac{x^{\frac{5}{9}}}{y^{\frac{1}{2}}}$ **41.** $5x^{\frac{4}{15}}$ **43.** $\sqrt[3]{x}$ **45.** y^{-3} or $\frac{1}{y^3}$

47. $\sqrt[3]{9}$ **49.** $2y^2$ **51.** $2x^2\sqrt[3]{2y^2}$ **53.** $2x\sqrt{y}$

55. $\sqrt[6]{5^5}$

57. $\sqrt[6]{9a^5}$

59. $x\sqrt{x}$

61. $\frac{\sqrt{x}}{x^2}$ or $\frac{1}{x\sqrt{x}}$

63. $\frac{2}{\sqrt[12]{x^5}}$

65. $\sqrt[12]{xy}$

67. $\sqrt[24]{x}$

69. $\sqrt[8]{x^3}$

71. To treat an equation as an identity, the equation must be true for all variable values in the domain. The fact that the equation is true for specific values does not guarantee that it is true for all values of x and y. A counterexample: Let $x = y = 2$. Then $\sqrt[n]{2^n + 2^n} = \sqrt[n]{2 \cdot 2^n} = 2\sqrt[n]{2} \neq 2 + 2 = 4$.

73. 30 beats per minute

RD3 Exercises

1. 5

3. $3\sqrt{2}$

5. $30\sqrt{3}$

7. $3x^4\sqrt{2}$

9. $4x^3y\sqrt{6xy}$

11. $2x^2$

13. $3\sqrt{2}$

15. $\sqrt{6}$

17. $2b\sqrt{b}$

19. $4x\sqrt{y}$

21. 2

23. $2a\sqrt[3]{b}$

25. $12x^2y^4\sqrt{y}$

27. $-5a^2b^3c^4$

29. $\frac{m^2n^5}{2}$

31. $a^3b^3\sqrt{7a}$

33. $2x^2y^3\sqrt[5]{2x^2}$

35. $-3a^3b^2\sqrt[4]{2a^3b^2}$

37. $\frac{4}{7}$

39. $\frac{11}{y}$

41. $\frac{3a\sqrt[3]{a^2}}{4}$

43. $\frac{2x^3}{yz^4}$

45. $\sqrt{6}$

47. $-x^2\sqrt{x}$

49. $\frac{-\sqrt{xy}}{x^2y}$

51. $\frac{x^2\sqrt[6]{x}}{yz^2}$

53. This is not correct as the radical of a sum is not the sum of radicals. We can simplify it by factoring the radicand: $\sqrt{x^3 + x^2} = \sqrt{x^2(x+1)} = |x|\sqrt{x+1}$

55. $\sqrt[10]{x^7}$

57. $2\sqrt[15]{2^4}$ or $2\sqrt[15]{16}$

59. $\sqrt[4]{x}$

61. $\frac{\sqrt[15]{2^7a^{11}}}{a}$

63. $\sqrt[6]{2x^5}$

65. $\sqrt[12]{x^{11}}$

67. $\sqrt{6}$

69. $\sqrt{n^2 - 9}$

71. $2\sqrt{31}$

73. $2\sqrt{5}$

75. $\frac{\sqrt{41}}{7}$

77. $2\sqrt{38}$

79. $\sqrt{p^2 + q^2}$

81. ~7.05 meters

83. $(-4, 0)$ and $(4, 0)$

85. 30 m

RD4 Exercises

1. No. The equation must be true for all $x \geq 0$.
3. $7\sqrt{3}$
5. $13y\sqrt{3x}$
7. $14\sqrt{2} + 2\sqrt{3}$
9. $11\sqrt[3]{2}$
11. $(1 + 6a)\sqrt{5a}$
13. $(4x - 6)\sqrt{x}$ or $2(2x - 3)\sqrt{x}$
15. $24\sqrt{2x}$
17. $(x + 1)\sqrt[3]{6x}$
19. $-8n\sqrt{2}$
21. $(6ab^2 - 9ab)\sqrt{ab}$ or $3ab(2b - 3)\sqrt{ab}$
23. $5x\sqrt[4]{xy}$
25. $-x\sqrt[3]{2x} + \sqrt{2}$
27. $\sqrt{x + 3}$
29 $(5 - x)\sqrt{x - 1}$
31. $\frac{3\sqrt{3}}{4}$
33. $\frac{4a\sqrt[4]{a}}{9}$
35. Error: cannot add unlike radicals (see line 3). Correct solution: $2\sqrt{2} + 2\sqrt[3]{2} = 2\left(\sqrt{2} + \sqrt[3]{2}\right)$
37. $3\sqrt{5} - 10$
39. $9 - 2\sqrt{5}$
41. -6
43. 1
45. -13
47. $30 - 10\sqrt{5}$
49. $a - 25b$
51. $9 + 6\sqrt{2}$
53. $38 + 12\sqrt{10}$
55. $22 - 13\sqrt{3}$
57. $\sqrt[3]{4y^2} - 4\sqrt[3]{2y} - 5$
59. 1
61. $(f + g)(x) = 13x\sqrt{5x}$; $(fg)(x) = 150x^3$
63. $\frac{\sqrt{10}}{4}$
65. $2\sqrt{6}$
67. $-\sqrt{5}$
69. $\frac{\sqrt{10y}}{8}$
71. $\frac{y\sqrt[3]{9x^2y}}{3x^2}$
73. $\sqrt[4]{pq^3}$
75. $\frac{6-\sqrt{2}}{2}$
77. $6 + 2\sqrt{6}$
79. $\frac{3\sqrt{5}-2\sqrt{3}}{11}$
81. $\sqrt{m} - 2$
83. $\frac{3+4\sqrt{3x}+4x}{3-4x}$
85. $\frac{2a+2\sqrt{ab}}{a-b}$
87. $1 - 2\sqrt{5}$
89 $\frac{2-9\sqrt{2}}{3}$
91. $\frac{6-2\sqrt{6p}}{3}$
93. Yes. $\frac{\sqrt{3}-1}{1+\sqrt{3}}$ after rationalization of the denominator becomes $2 - \sqrt{3}$.
95. $2\sqrt{3} \approx 3.5$ cm

RD5 Exercises

1. False, as the radicals do not contain a variable.
3. True, as the radical cannot be negative.
5. $x = \frac{39}{7}$
7. $x = \frac{2}{3}$
9. no solution
11. $x = -27$
13. $y = 19$
15. $a = \frac{1}{25}$
17. $r = 5$
19. $y = 18$

21. $x = 9$

23. $x \in \{-1, 3\}$

25. $y = 4$

27. $x = 5$

29. not correct, as $(8 - x)^2 = 64 - 16x + x^2$

31. $x = 2$

33. $p = 9$

35. No solution

37. $t = -1$

39. No solution

41. $n = 3$

43. $n = -2$

45. $a \in \{2, 6\}$

47. No solution

49. $m = 2$

51. $x \in \left\{-1, \frac{1}{3}\right\}$

53. $x \in \{1,9\}$

55. $x = \frac{4}{9}$

57. $k \in \{-2, -1\}$

59. $x \in \{-5, 5\}$

61. $a \in \left\{0, \frac{125}{4}\right\}$

63. $L = CZ^2$

65. $m = \frac{2K}{V^2}$

67. $F = \frac{Mm}{r^2}$

69. $C = \frac{1}{4\pi^2 F^2 L}$

71. $r = \frac{a}{4\pi^2 N^2}$

73. 189 cm

75. 22 m

Quadratic Equations and Functions - ANSWERS

Q1 Exercises

1. False

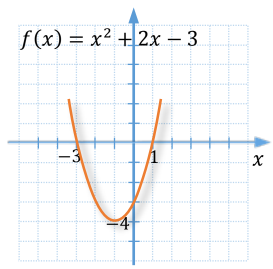
3. True

5. False

7. False

9. **a)**

x	$f(x)$
1	0
0	-3
-1	-4
-2	-3
-3	0



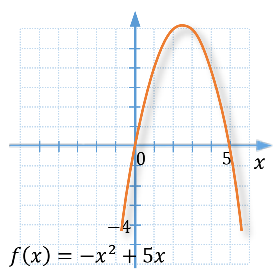
b) $(-3,0)$, $(1,0)$

c) $x \in \{-3,1\}$

The solutions are the first coordinates of the x-intercepts.

11. **a)**

x	$f(x)$
0	0
1	4
2	6
$\frac{5}{2}$	6.25
3	6
4	4
5	0



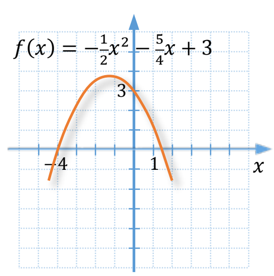
b) $(0,0)$, $(5,0)$

c) $x \in \{0,5\}$

The solutions are the first coordinates of the x-intercepts.

13. **a)**

x	$f(x)$
-4	0
-2	3.5
-1	3.75
0	3
1	1.25
2	-1.5



b) $(-4,0)$, $\left(\frac{3}{2}, 0\right)$

c) $x \in \left\{-4, \frac{3}{2}\right\}$

The solutions are the first coordinates of the x-intercepts.

15. $x \in \{-4\sqrt{2}, 4\sqrt{2}\}$

17. $n \in \{-2\sqrt{6}, 2\sqrt{6}\}$

19. $y \in \{-2\sqrt{10}, 2\sqrt{10}\}$

21. $x \in \{-7,1\}$

23. $t \in \left\{\frac{-2-2\sqrt{3}}{5}, \frac{-2+2\sqrt{3}}{5}\right\}$

25. $y \in \{-4 - 2\sqrt{11}, -4 + 2\sqrt{11}\}$

27. $y \in \left\{\frac{44}{5}, \frac{56}{5}\right\}$

29. No solution

31. $x \in \left\{\frac{1-3\sqrt{2}}{2}, \frac{1+3\sqrt{2}}{2}\right\}$

33. $y \in \{0,3\}$

35. $n = -2$

37. $y \in \left\{\frac{-7-\sqrt{53}}{2}, \frac{-7+\sqrt{53}}{2}\right\}$

39. No solution

41. $x \in \{6 - 2\sqrt{5}, 6 + 2\sqrt{5}\}$

43. $x \in \left\{\frac{-1-\sqrt{7}}{3}, \frac{-1+\sqrt{7}}{3}\right\}$

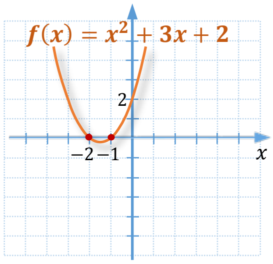
45. $x \in \left\{\frac{4-\sqrt{3}}{3}, \frac{4+\sqrt{3}}{3}\right\}$

47. $x \in \left\{\frac{2-\sqrt{3}}{3}, \frac{2+\sqrt{3}}{3}\right\}$

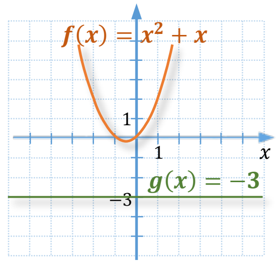
49. $x \in \left\{\frac{1-2\sqrt{19}}{5}, \frac{1+2\sqrt{19}}{5}\right\}$

51. $x \in \{1 - 2\sqrt{2}, 1 + 2\sqrt{2}\}$

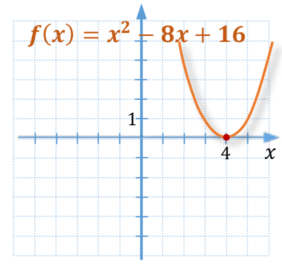
53. $x \in \{-2, -1\}$



55. No solution



57. $x = 4$



59. $a = 1 - \sqrt{5} \approx -1.24$, or $a = 1 + \sqrt{5} \approx 3.24$

61. $x = \frac{-5-\sqrt{11}}{2} \approx -4.16$, or $x = \frac{-5+\sqrt{11}}{2} \approx -0.84$

63. $y = \frac{-1-\sqrt{7}}{6} \approx -0.27$, or $y = \frac{-1+\sqrt{7}}{6} \approx 0.61$

65. $x = \frac{17-\sqrt{249}}{10} \approx 0.12$, or $x = \frac{17-\sqrt{249}}{10} \approx 3.28$

67. $x = \frac{5-\sqrt{7}}{6} \approx 0.39$, or $x = \frac{5+\sqrt{7}}{6} \approx 1.27$

69. 2 rational solutions; factoring possible

71. 2 real solutions; use quadratic formula

73. 1 double rational solution; factoring possible

75. $k = 25$

77. No, as the product of a rational and irrational number is irrational. This would contradict the fact that the quadratic equation has integral coefficients.

79. $x = 1 \pm \sqrt{10}$

81. $x = \frac{5 \pm 2\sqrt{6}}{2}$

83. $x \in \{-3, 2\}$

85. No solution

87. $x \in \left\{-\frac{3}{2}, 1\right\}$

89. $x = 5 \pm \sqrt{53}$

Q2 Exercises

1. The solution is incorrect as the question calls for the values of x not a.

3. $x \in \{-\sqrt{5}, -\sqrt{2}, \sqrt{2}, \sqrt{5}\}$

5. $x \in \left\{\frac{1}{4}, 16\right\}$

7. $a \in \{-1, 2\}$

9. $x = 9$

11. $x \in \{-1, 3, 1 - \sqrt{2}, 1 + \sqrt{2}\}$

13. $x = 8$

15. $u \in \left\{-\frac{8}{3}, -1\right\}$

17. $x \in \{-1 \pm \sqrt{2}, 3 \pm \sqrt{10}\}$

19. $r = \pm\sqrt{\frac{V}{\pi h}}$

21. $s = \pm\sqrt{\frac{3}{Vh}}$

23. $s = \pm\sqrt{\frac{kq_1q_2}{N}}$

25. $H = \pm\sqrt{\frac{703W}{I}}$

27. $r = \frac{-\pi h \pm \sqrt{\pi^2 h^2 + 2\pi A}}{2\pi}$

29. $a = \pm\frac{bt}{\sqrt{1-t^2}}$ **31.** $I = \frac{E \pm \sqrt{E^2-4PR}}{2R}$ **33.** $v = \frac{c\sqrt{m^2-m_0^2}}{m}$ **35.** $\frac{(r+R)\sqrt{pR}}{R}$

37. **a.** $r - c$ **b.** $r + c$ **39.** $7 + 2\sqrt{35},\ 10 + 2\sqrt{35},$ and $17 + 2\sqrt{35}$

41. 5 ft by 12 ft **43.** 10 ft 3 in **45.** 9 in by 13 in **47.** $10\sqrt{2}$ m by $5\sqrt{2}$ m

49. 1.5 ft **51.** 12 cm **53.** 7 cm by 13 cm **55.** 60 km/h

57. Skyhawk at 250 km/h; Mooney Bravo at 350 km/h **59.** ~10.8 km/h

61. 800 km/h and 840 km/h **63.** 7 hr 32 min

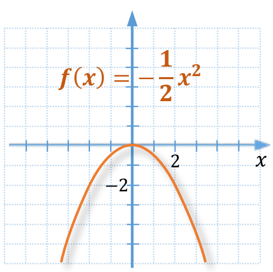
65. Helen: ~16 hr 31 min; Monica: ~15 hr 31 min **67.** smaller-diameter pipe: 2 hr; larger-diameter pipe: 3 hr

69. ~3.8 sec **71.** 4.2%

Q3 Exercises

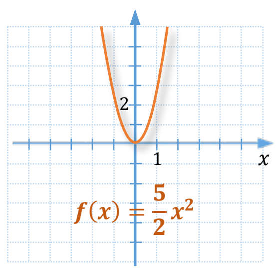
1. **a.-III; b.-I; c.-IV; d.-II** **3.** **a.-II; b.-III; c.-I; d.-IV**

5. wider; opens down



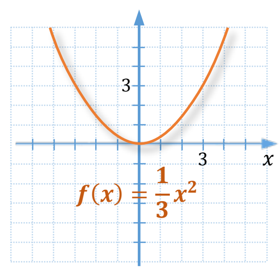
Range $= (-\infty, 0]$

7. narrower; opens up



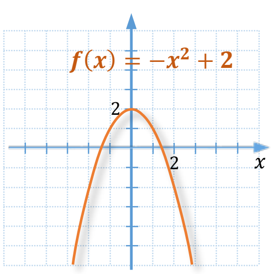
Range $= [0, \infty)$

9. narrower; opens up



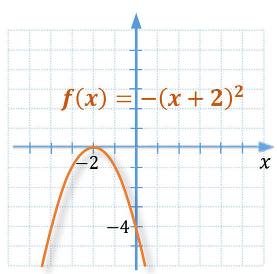
Range $= [0, \infty)$

11. S_x; shift 2 units up



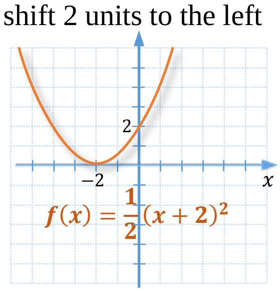
Domain $= \mathbb{R}$
Range $= (-\infty, 2]$
Axis of symmetry: $x = 0$

13. S_x; shift 2 units to the left



Domain $= \mathbb{R}$
Range $= (-\infty, 0]$
Axis of symmetry: $x = -2$

15. vertical dilation by $\frac{1}{2}$; shift 2 units to the left



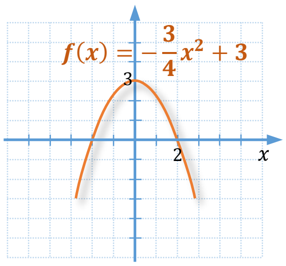
Domain $= \mathbb{R}$
Range $= [0, \infty)$
Axis of symmetry: $x = -2$

17. vertex = (0,3)

shape of $\frac{3}{4}x^2$; opens down

x-int.: $(-2,0)$, $(2,0)$

y-int.: $(0,3)$



S_x; vertical dilation by $\frac{3}{4}$;

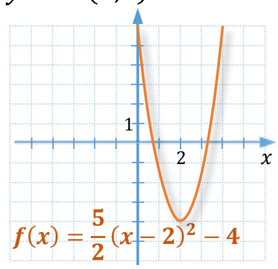
shift 3 units up

19. vertex = $(2,-4)$

shape of $\frac{5}{2}x^2$; opens up

x-int.: $\left(\frac{10-2\sqrt{10}}{5}, 0\right)$, $\left(\frac{10+2\sqrt{10}}{5}, 0\right)$

y-int.: $(0,6)$



vertical dilation by $\frac{5}{2}$;

shift 2 units to the right;

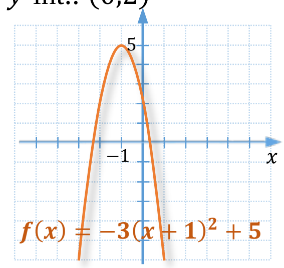
shift 4 units down

21. vertex = $(-1,5)$

shape of $3x^2$; opens down

x-int.: $\left(\frac{-3-\sqrt{15}}{3}, 0\right)$, $\left(\frac{-3+\sqrt{15}}{3}, 0\right)$

y-int.: $(0,2)$



S_x; vertical dilation by 3;

shift 1 unit to the left;

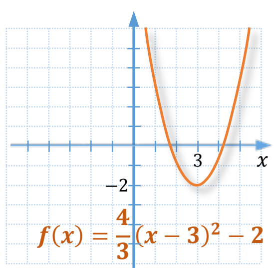
shift 5 units up

23. vertex = $(3,-2)$

shape of $\frac{4}{3}x^2$; opens up

x-int.: $(-2,0)$, $(2,0)$

y-int.: $(0,3)$

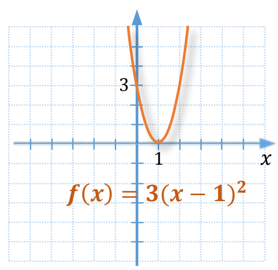


vertical dilation by $\frac{4}{3}$;

shift 3 units to the right;

shift 2 units down

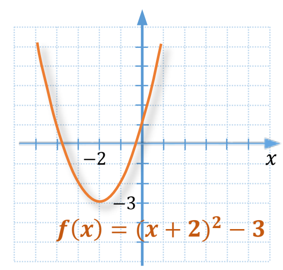
25. vertex = (1,0)



Minimum value = 0;

Range = $[0, \infty)$

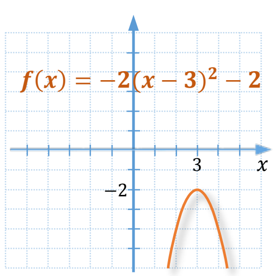
27. vertex = $(-2,-3)$



Minimum value = -3;

Range = $[-3, \infty)$

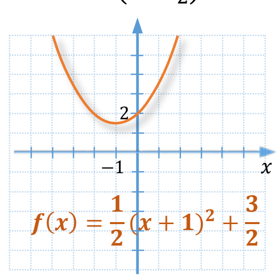
29. vertex = $(3,-2)$



Maximum value = -2;

Range = $(-\infty, -2]$

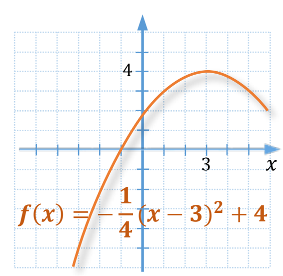
31. vertex = $\left(-1, \frac{3}{2}\right)$



Minimum value = $\frac{3}{2}$;

Range = $\left[\frac{3}{2}, \infty\right)$

33. vertex = (3,4)



Maximum value = 4;

Range = $(-\infty, 4]$

35. $f(x) = (x+3)^2 - 4$

37. $f(x) = 2(x-1)^2 - 5$

39. $f(x) = -3(x+2)^2 + 6$

Q4 Exercises

1. $f(x) = (x+3)^2 + 1;\ V(-3,1)$

3. $f(x) = \left(x+\frac{1}{2}\right)^2 - \frac{13}{4};\ V(-\frac{1}{2}, -\frac{13}{4})$

5. $f(x) = -\left(x-\frac{7}{2}\right)^2 + \frac{61}{4};\ V\left(\frac{7}{2}, \frac{61}{4}\right)$

7. $f(x) = -3(x-1)^2 + 15;\ V(1,15)$

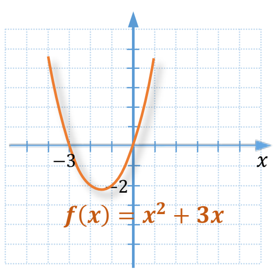
9. $f(x) = \frac{1}{2}(x+3)^2 - \frac{11}{2};\ V\left(-3, -\frac{11}{2}\right)$

11. $V\left(\frac{3}{2}, -\frac{11}{4}\right)$

13. $V(1,8)$

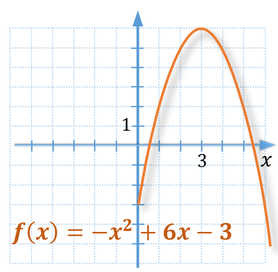
15. $V(-1,-23)$

17. $V\left(-\frac{3}{2}, -\frac{9}{4}\right)$; opens up
shape of x^2



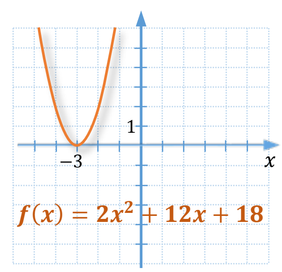
$D = \mathbb{R}$; Range $= \left[-\frac{9}{4}, \infty\right)$

19. $V(3,6)$; opens down
shape of x^2



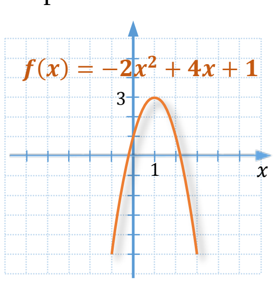
$D = \mathbb{R}$; Range $= (-\infty, 6]$

21. $V(-3,0)$; opens up
shape of $2x^2$



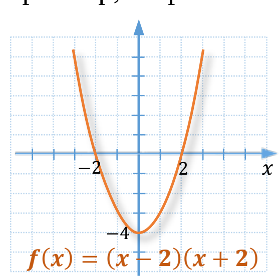
$D = \mathbb{R}$; Range $= [0, \infty)$

23. $V(1,3)$; opens down
shape of $2x^2$



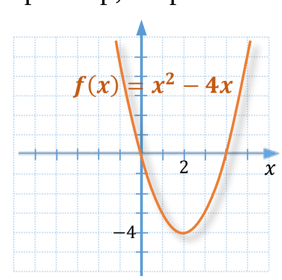
$D = \mathbb{R}$; Range $= (-\infty, 3]$

25. zeros: -2, 2; $V(0,-4)$;
opens up; shape of x^2



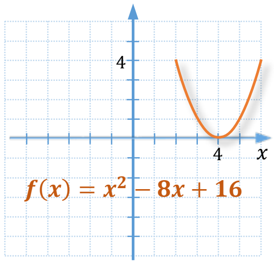
Minimum value $= -4$
Minimum occurs at $x = 0$

27. zeros: 0, 4; $V(2,-4)$;
opens up; shape of x^2



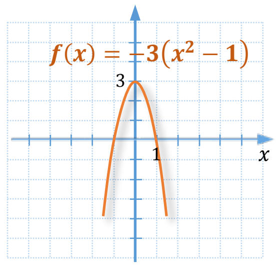
Minimum value $= -4$
Minimum occurs at $x = 2$

29. zero: 4; $V(4,0)$;
opens up; shape of x^2



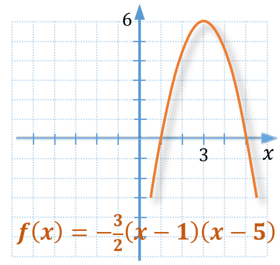
Minimum value = 0
Minimum occurs at $x = 4$

31. zero: -1, 1; $V(0,3)$;
opens down; shape of $3x^2$



Maximum value = 3
Maximum occurs at $x = 0$

33. zeros: 1, 5; $V(3,6)$;
opens down; shape of $\frac{3}{2}x^2$



Maximum value = 6
Maximum occurs at $x = 3$

35. $f(x) = x(5x-2)$

37. $f(x) = \frac{3}{4}(x-1)(x-4)$

39. $x(3x-1) = 0$

41. $(x-2)^2 = 0$

43. By observing the second coordinate of the vertex in combination with the opening. For example, the second coordinate "+'ve" with openning up means no x-intercepts while the second coordinate "+'ve" with openning down indicates 2 x-intercepts.

45. true

47. true

49. true

51. 30.625 m; 5 sec

53. 20; \$150

55. 16, 16

57. 4 m by 8 m

59. **a.** $P(n) = 60 - 2n$ **b.** $R(n) = (60-2n)n$ **c.** 15 **d.** 450\$

Q5 Exercises

1. false

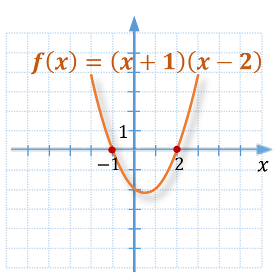
3. false

5. true

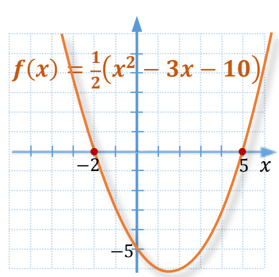
7. **a.** $[-1,3]$ **b.** $(-\infty,-1) \cup (3,\infty)$

9. **a.** $(-\infty,-1] \cup \{3\}$ **b.** $(-1,3) \cup (3,\infty)$

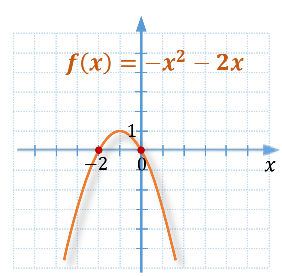
11. $x \in (-1,2)$



13. $x \in [-2,5]$



15. $x \in (-\infty,-2) \cup (0,\infty)$



17. $x \in (-4,5)$

19. $x \in (-\infty, -3] \cup [4, \infty)$

21. $x \in \left(-2, -\frac{1}{2}\right)$

23. $x = 2$

25. $x \in (-\infty, -5) \cup (4, \infty)$

27. $x \in (-\infty, -3] \cup [2, 5]$

29. $x \in \left[0, \frac{1}{2}\right] \cup [2, \infty)$

31. $x \in (-\infty, -1) \cup (0, \infty)$

33. $x \in \left(-3, \frac{1}{2}\right]$

35. $y \in (-5, -2)$

37. $x \in \left(-\infty, -\frac{7}{2}\right] \cup (-2, \infty)$

39. $t \in \left(-\infty, -\frac{15}{2}\right] \cup (-3, \infty)$

41. $y \in \left(-\infty, -\frac{5}{2}\right) \cup \left[-\frac{3}{2}, \infty\right)$

43. $x \in \left[0, \frac{1}{2}\right) \cup \left[\frac{5}{2}, \infty\right)$

45. $x \in (-\infty, -1) \cup \left(-1, \frac{3}{2}\right]$

47. $x \in \left(-\sqrt{5}, \sqrt{5}\right)$

49. $x \in (-2,3)$

51. $x \in (-\infty, -1] \cup [3, 4)$

53. $x \in (-\infty, -5) \cup (-5, -2) \cup (4, \infty)$

55. $\mathbb{R}$

57. $x = \frac{1}{2}$

59. $x = 0$

61. $(0\ sec, 2.47\ sec)$

63. $[9\ m, 12\ m]$

65. $R \in (0,20)$

Trigonometry - ANSWERS

T1 Exercises

1. $20.075°$

3. $274.304°$

5. $15.168°$

7. $18°0'45''$

9. $65°0'5''$

11. $175°23'58''$

13. $83°59'$

15. $33°50'$

17. $28°03'03''$

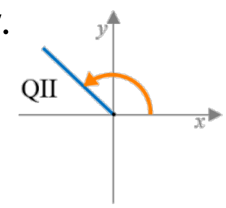
19. $60°, 150°$

21. $45°, 135°$

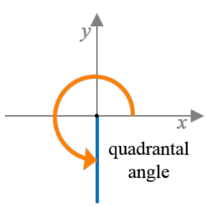
23. $74°30', 164°30'$

25. $180 - \theta°$

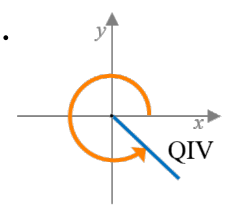
27.



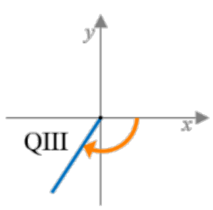
29.



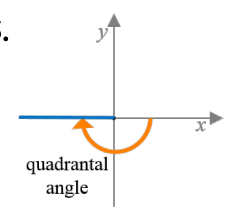
31.



33.



35.



37. $15°$

39. $135°$

41. $30° + k \cdot 360°$

43. $k \cdot 360°$

45. $\alpha° + k \cdot 360°$

47. $7.5°$

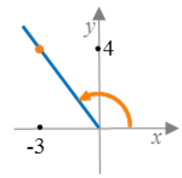
T2 Exercises

1. $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

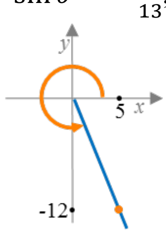
3. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\tan\theta = \sqrt{3}$

5. $\sin\theta = \frac{n}{\sqrt{n^2+4}}$, $\cos\theta = \frac{2}{\sqrt{n^2+4}}$, $\tan\theta = \frac{n}{2}$

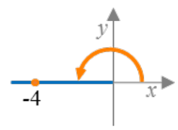
7. $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\tan\theta = -\frac{4}{3}$



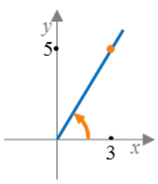
9. $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = -\frac{12}{5}$



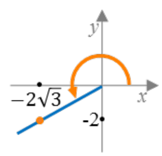
11. $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$



13. $\sin\theta = \frac{5\sqrt{34}}{34}$, $\cos\theta = \frac{3\sqrt{34}}{34}$, $\tan\theta = \frac{5}{3}$



15. $\sin\theta = -\frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$



17. sine and cosine is negative, tangent is positive

19. negative

21. negative

23. positive

25. positive

27. negative

29. 1

31. -1

33. 0

35. 0

37. $undefined$

39. $\cos\beta = -\frac{\sqrt{5}}{3}$

$\tan\beta = \frac{2\sqrt{5}}{5}$

T3 Exercises

1. 0.6000
3. -0.9106
5. $\frac{\sqrt{2}}{2}$
7. $\frac{\sqrt{3}}{2}$
9. $\frac{1}{2}$
11. 1
13. $\cos 67.5°$
15. $82°$
17. $13°$
19. $6°$
21. QIII and QIV
23. QII
25. QIV
27. negative
29. negative
31. positive
33. positive
35. $\frac{\sqrt{3}}{2}$
37. $\frac{1}{2}$
39. $-\frac{\sqrt{3}}{2}$
41. 1
43. $60°, 300°$
45. $60°, 120°$
47. $135°, 225°$
49. $150°, 330°$
51. $\sin\alpha = -\frac{4}{5}$
    $\tan\alpha = -\frac{4}{3}$

T4 Exercises

1. $52.2°$
3. $68.4°$
5. $60°$
7. $\angle B = 54°,\ b \simeq 16.5,\ c \simeq 20.4$
9. $\angle A \simeq 31.0°,\ \angle B \simeq 59.0°,\ c \simeq 17.5$
11. $\angle A \simeq 74.4°,\ \angle B \simeq 15.6°,\ b \simeq 2.6$
13. $a = 2\sqrt{3},\ b = 6\sqrt{3},\ d = 4\sqrt{3},\ h = 6$
15. $a = 5,\ b = \frac{5}{2},\ h = \frac{5\sqrt{3}}{2},\ s = 5$
17. $32\sqrt{3}$ cm
19. $23°$
21. 700 m
23. 317 m
25. 1101 km; direction of 107° (or **S**73°**E**)
27. 552 m; 447 m
29. 29.6 m
31. 237 m

T5 Exercises

1. $\angle P = 39°,\ p \simeq 15.3\ \text{cm},\ s \simeq 22.8\ \text{cm}$

3. $\angle A \simeq 25.9°,\ \angle C \simeq 18.1°,\ c \simeq 19.3\ \text{ft}$

5. $\angle I \simeq 19.8°,\ i \simeq 8.8\ \text{cm},\ \angle J \simeq 122.2$

7. $b = 10,\ \angle C = 120°,\ c \simeq 17.3$

9. $\angle A \simeq 25.6°,\ a \simeq 10.5,\ \angle B \simeq 9.4°$

11. $\angle X \simeq 40.6°,\ y \simeq 18.4\ \text{m},\ \angle Z \simeq 54.4°$

13. $p \simeq 19.8\ \text{m},\ \angle R \simeq 33.1°,\ \angle S \simeq 129.9°$

15. $\angle I \simeq 48.5°,\ \angle J \simeq 86.3°,\ \angle K \simeq 45.2°$

17. $\angle A \simeq 17°,\ \angle B \simeq 103°,\ c \simeq 8.9$

19. $\angle A \simeq 34.7°,\ \angle B \simeq 48.1°,\ \angle C \simeq 97.2°$

21. No, because the ratio of sines of angles is not the same as the ratios of those angles.
For instance, $\frac{\sin 90°}{\sin 45°} = \sqrt{2} \neq \frac{90°}{45°} = 2$.

23. 127 m

25. 8.1 km; 11.0 km

27. ~ 6.4 m

29. ~ 351 m from A; ~ 295 from B

31. ~ 777 km; direction: ~ 279°2′

33. ~26°

35. ~76 m

37. ~ 1199 m^2

39. ~ 69°

41. ~247.3 m^2

Sequences and Series - ANSWERS

S1 Exercises

1. $-1, 1, 3, 5, a_{10} = 17$

3. $3, -6, 9, -12, a_{10} = -30$

5. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, a_{10} = \frac{1}{100}$

7. $-1, 0, -1, 4, a_{10} = 64$

9. $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, a_{10} = -\frac{1}{19}$

11. $a_n = (-1)^{n+1}$ or $a_n = (-1)^{n-1}$

13. $a_n = 3^n$

15. $a_n = 3n + 3$

17. $a_n = \frac{(-1)^{n+1}}{n^3}$ or $a_n = \frac{(-1)^{n-1}}{n^3}$

19. $-3, -1, 3, 11, 27$

21. $1, 2, 5, 12, 29$

23. \$252, \$248, \$244, \$240, \$236; remaining balance = \$1600

25. $p_n = 0.2 + 0.3n$; $p_{14} = \$4.40$

27. **a.** $a_n = 3(.75)^n$
b. 4^{th} day

29. 50

31. 4

33. 248

35. $\frac{13}{60}$

37. $\sum_{i=1}^{6} 2i$

39. $\sum_{i=2}^{50} \frac{(-1)^i}{i}$

41. $\sum_{i=1}^{\infty} i^3$

43. $\sum_{m=1}^{10}(3m - 4)$

45. $\sum_{m=1}^{5} \frac{m+1}{m+3}$

47. $\sum_{m=1}^{\infty}(-1)^{m+1}m$

49. $\frac{2}{3}, \left(\frac{2}{3}\right)^2, \left(\frac{2}{3}\right)^3$; $\sum_{i=1}^{6}\left(\frac{2}{3}\right)^i$

51. 10

53. 0

S2 Exercises

1. true

3. true

5. $a_n = 2n - 1$

7. $a_n = 2n - 6$

9. $a_n = \frac{1}{2}n - \frac{5}{2}$

11. $3, 1, -1, -3, -5$; $a_{12} = -19$

13. $-8, -4, 0, 4, 8$; $a_{12} = 36$

15. $10, 8, 6, 4, 2$; $a_{12} = -12$

17. 15

19. 13

21. 17

23. $a_8 = 23$

25. $a_{50} = 197$

27. $a_{10} = -71$

29. $a_1 = 7$

31. $S_{12} = 138$

33. $S_9 = -54$ **35.** $S_{10} = 175$ **37.** 325 **39.** 459

41. 75 **43.** -725 **45.** 1800°

47. **a.** 1 hr 20 min **b.** 11 hr 5 min **49.** **a.** 2 hr 04 min **b.** 17 hr 46 min

S3 Exercises

1. true **3.** true **5.** not geometric **7.** yes; $a_n = \frac{(-1)^n}{3^{n-3}}$

9. yes; $a_n = (-1)^{n-1}$ **11.** yes; $a_n = \frac{(-1)^{n-1}}{3^{n-5}}$ **13.** yes; $a_n = -\frac{4^{n-2}}{5^{n-1}}$

15. $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$; $a_8 = \frac{1}{256}$ **17.** $5, -5, 5, -5$; $a_8 = -5$

19. $100, 10, 1, 0.1$; $a_8 = 0.00001$ **21.** 10 **23.** 7

25. 9 **27.** $a_{10} = 256$ **29.** $a_{12} = \frac{1}{512}$ **31.** a_9 =729

33. $a_{10} = -1536$ **35.** ~ -2.667 **37.** ~11.997 **39.** -14762

41. 3069 **43.** S_∞ doesn't exist **45.** $S_\infty = 5$ **47.** S_∞ doesn't exist

49. $S_\infty = -\frac{6}{7}$ **51.** \$819.20; \$1638.30 **53.** \$25,357.18 **55.** 12 meters

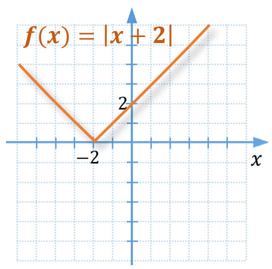
Additional Functions, Conic Sections, and Nonlinear Systems - ANSWERS

C1 Exercises

1. **a.-V**; $y = x^2$; $D = \mathbb{R}$; range $= [0, \infty)$
b.-III; $y = x^3$; $D = \mathbb{R}$; range $= \mathbb{R}$
c.-IV; $y = \sqrt{x}$; $D = [0, \infty)$; range $= [0, \infty)$
d.-I; $y = |x|$; $D = [0, \infty)$; range $= \mathbb{R}$
e.-II; $y = [\![x]\!]$; $D = \mathbb{R}$; range $= \mathbb{Z}$
f.-VI; $y = \frac{1}{x}$; $D = \mathbb{R} \setminus \{0\}$; range $= \mathbb{R} \setminus \{0\}$

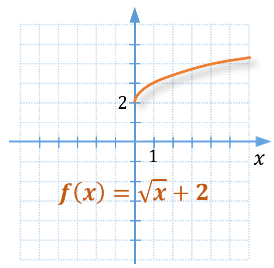
3. Translation: 5 units to the right, 3 units up

5.



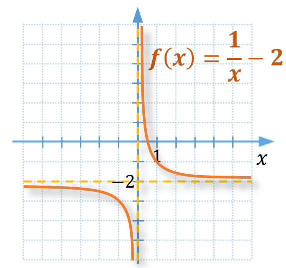
$D = \mathbb{R}$
range $= [0, \infty)$

7.



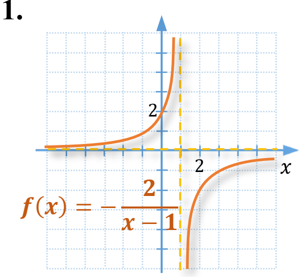
$D = [0, \infty)$
range $= [2, \infty)$

9.



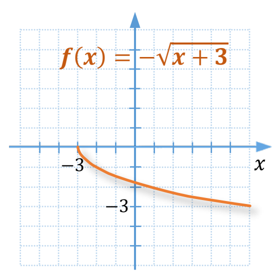
$D = \mathbb{R} \setminus \{0\}$
range $= \mathbb{R} \setminus \{-2\}$
VA: $x = 0$
HA: $y = -2$

11.



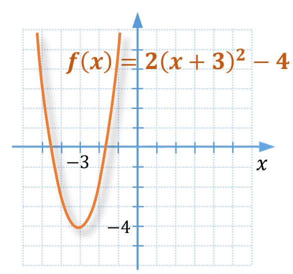
$D = \mathbb{R} \setminus \{0\}$
range $= \mathbb{R} \setminus \{-2\}$
VA: $x = 1$
HA: $y = 0$

13.



$D = [-3, \infty)$
range $= (-\infty, 0]$

15.

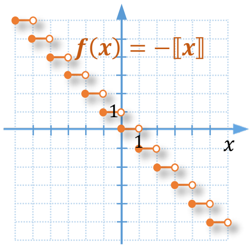


$D = \mathbb{R}$
range $= [-4, \infty)$

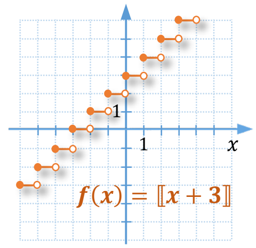
17. 2

19. -2

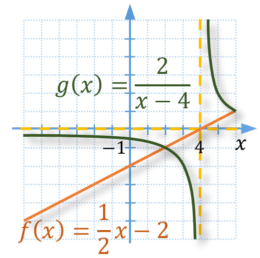
21.



23.



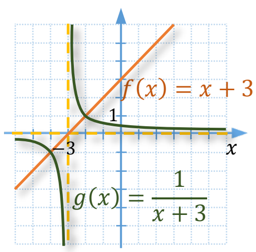
25.



x-int. of f: $(4,0)$

VA of g: $x = 4$

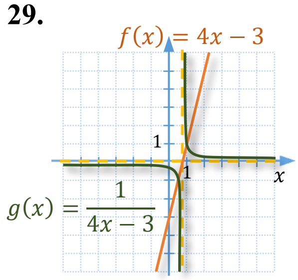
27.



x-int. of f: $(-3,0)$

VA of g: $x = -3$

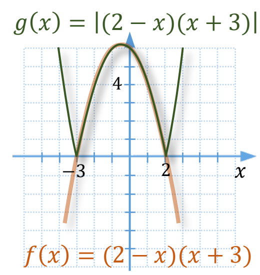
29.



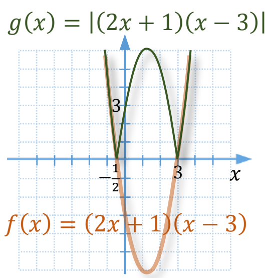
x-int. of f: $\left(\frac{3}{2}, 0\right)$

VA of g: $x = \frac{3}{2}$

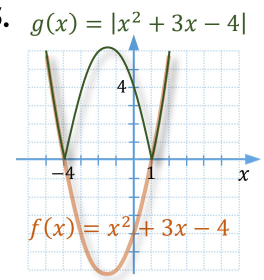
31.



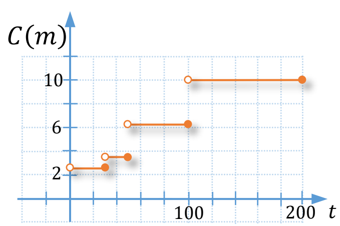
33.



35.

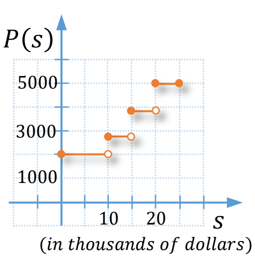


37.



$$C(t) = \begin{cases} 2.50, & if \quad 0 < m \le 30 \\ 3.50, & if \quad 30 < m \le 50 \\ 6.25, & if \quad 50 < m \le 100 \\ 10, & if \quad 100 < m \end{cases}$$

39.



$$P(s) = \begin{cases} 2000, & if \quad 0 \le s < 10000 \\ 2800, & if \quad 10000 \le s < 15000 \\ 3800, & if \quad 15000 \le s < 20000 \\ 5000, & if \quad 20000 \le s \end{cases}$$

C2 Exercises

1. false

3. false

5. false

7. true

9. $(x-3)^2+(y-1)^2=3$

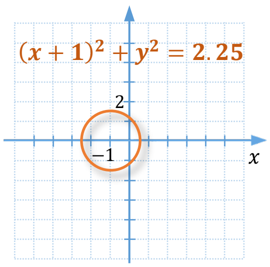
11. $(x+2)^2+(y-2)^2=\frac{25}{4}$

13. $C(4,5);\ r=6$

15. $C(6,0);\ r=2\sqrt{6}$

17. $C(0,2);\ r=2\sqrt{3}$

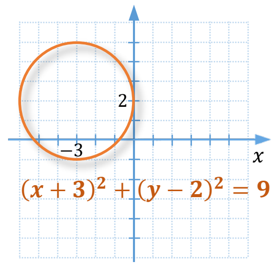
19. $C(-1,0);\ r=1.5$



$D=\left[-\frac{5}{2},\frac{1}{2}\right]$

$\text{range}=\left[-\frac{3}{2},\frac{3}{2}\right]$

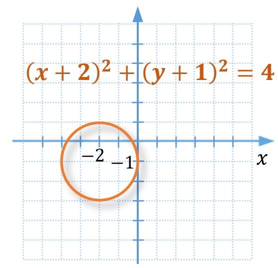
21. $C(-3,2);\ r=3$



$D=[-6,0]$

$\text{range}=[-1,5]$

23. $C(-2,-1);\ r=2$



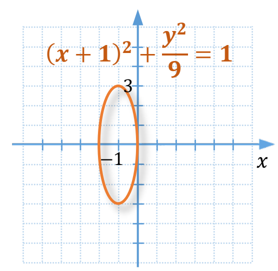
$D=[-4,0]$

$\text{range}=[-3,1]$

25. $(x+4)^2+(y+3)^2=1$

27. $(x-3r)^2+y^2=16r^2$

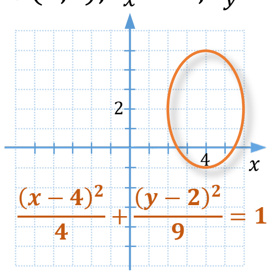
29. $C(-1,0);\ r_x=1;\ r_y=3$



$D=[-2,0]$

$\text{range}=[-3,3]$

31. $C(4,2);\ r_x=2;\ r_y=3$

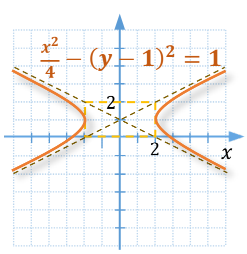


$D=[2,6]$

$\text{range}=[-1,5]$

33. $\frac{(x-3)^2}{4}+\frac{(y+2)^2}{9}=1$

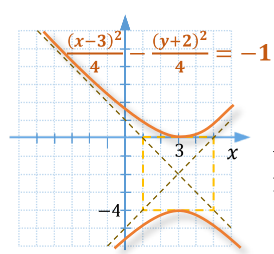
35. $C(0,1);\ y=0$



$D=(-\infty,-2]\cup[2,\infty)$

$\text{range}=\mathbb{R}$

37. $C(3,-2);\ x=3$



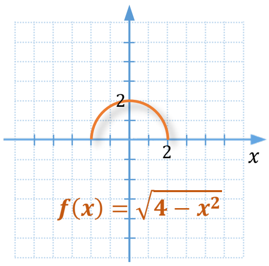
$D=\mathbb{R}$

$\text{range}=(-\infty,-1]\cup[3,\infty)$

39. $\frac{x^2}{9} - \frac{(y+1)^2}{9} = 1$

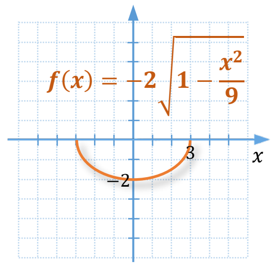
41. $\frac{x^2}{9} - \frac{(y-1)^2}{4} = -1$

43.



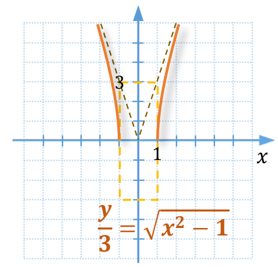
$D = [-2,2]$
range $= [0,2]$

45.



$D = [-3,3]$
range $= [-2,0]$

47.



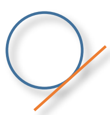
$D = (-\infty,-1] \cup [1,\infty)$
range $= [0,\infty)$

49. 24 m; 5 m

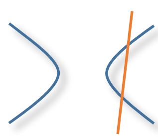
51. 60 m

C3 Exercises

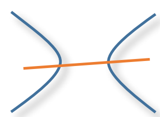
1.



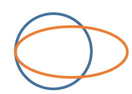
3.



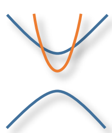
or



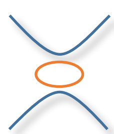
5.



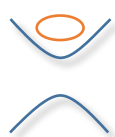
7.



9.



or



11. **a.** 2; **b.** 2; **c.** 4; **d.** 0; **e.** 4

13. $\{(-4,0),(-3,1)\}$

15. $\left\{(-1,5),\left(\frac{5}{2},-2\right)\right\}$

17. $\left\{\left(-\frac{4}{3},-\frac{1}{3}\right),\left(\frac{4}{3},\frac{1}{3}\right)\right\}$

19. $\{(-\sqrt{5},-2),(\sqrt{5},-2),(0,3)\}$

21. $\left\{\left(\sqrt{2},\frac{\sqrt{2}}{2}\right),\left(-\sqrt{2},-\frac{\sqrt{2}}{2}\right)\right\}$

23. $\{(-\sqrt{3},0),(\sqrt{3},0)\}$

25. $\{(-\sqrt{5},-\sqrt{10}),(-\sqrt{5},\sqrt{10}),(\sqrt{5},-\sqrt{10}),(\sqrt{5},\sqrt{10})\}$

27. 8 m by 7.5 m

29. false

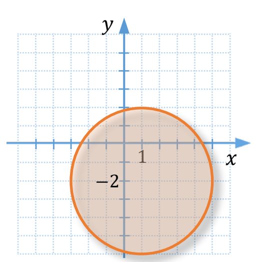
31. true

33. true

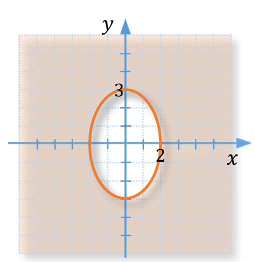
35. inside; above

37. **a.-III**; **b.-II**; **c.-IV**; **d.-I**

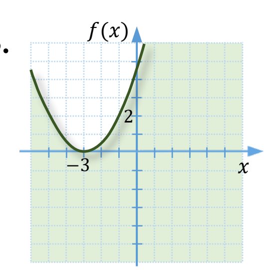
39.



41.



43.



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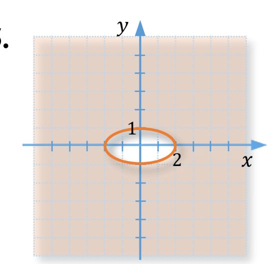

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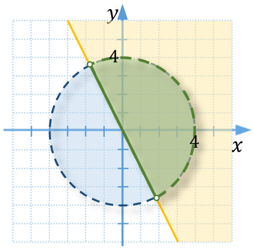

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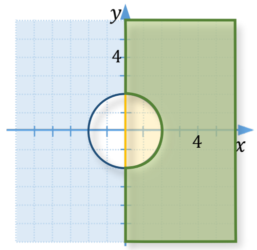

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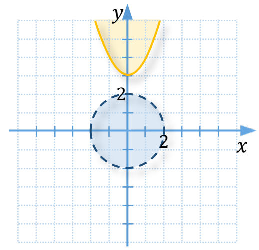
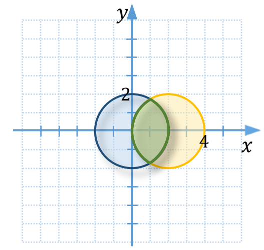


no solution

53.



55.