APPLICATIONS OF MAXWELL'S EQUATION

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About the Authors

John F. Cochran, Professor late of Simon Fraser University John Cochran was a charter faculty member in the Department of Physics at Simon Fraser University. He was highly influential in charting the course of the department over its first several decades. He served as Department Chair from 1968 through 1974, and as Dean of Science from 1981 through 1985. As a researcher, Professor Cochran was pivotal in developing an internationally recognized research program in magnetism and magnetic materials at SFU. He served as a mentor and collaborator in this area to faculty members who came later, including Professors Tony Arrott, Brett Heinrich and Erol Girt. In addition to his service, Professor Cochran was an excellent teacher. This textbook was widely used at SFU and elsewhere.

Bretislav Heinrich, Professor Emeritus Simon Fraser University Prof. Heinrich got his PhD degree at the Czechoslovakian Academy of Sciences. He came to the Physices Department at SFU as a post-doctoral fellow in 1969 and joined the research faculty also at SFU, a rare occurrence, eventually becoming a full professor in 1996. He has published over 250 papers in international Journals and 14 book chapters. He is also a co-Author of the book “Ultrathin Magnetic Structures”. 2005

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CHAPTER OVERVIEW

1: Maxwell’s Equations

1.1: Fundamental Postulates

The properties of ordinary matter are a consequence of the forces acting between charged particles. Extensive experimental investigations have established the following properties of electrical charges:

(1) There are two kinds of charges. These have been labeled positive charge and negative charge.

(2) Electrical charge is quantized. All particles so far observed carry charges which are integer multiples of the charge on an electron. In the MKS system of units, the charge on an electron is e = $-1.60\text{x}10^{-19}$ Coulombs. By definition, the electron carries a negative charge and a proton carries a positive charge; the charge on a proton is $+1.60\text{x}10^{-19}$ Coulombs. No one knows why charge comes in multiples of the electron charge.

(3) Equality of the positive and the negative charge quantum. The quantum of positive charge and the quantum of negative charge are equal to at least 1 part in 10^{20}. This has been determined from experiments designed to measure the net charge on neutral atoms.

(4) In any closed system charge is conserved. This means that the algebraic sum of all positive charges plus all negative charges does not change with time. This does not mean that individual charged particles are conserved. For example, a positron, which carries a positive charge of $1.60\text{x}10^{-19}$ Coulombs, can interact with an electron, which carries a negative charge of $1.60\text{x}10^{-19}$ Coulombs, in such a way that the electron and positron disappear and two neutral particles called photons are produced. The total charge before and after this transformation occurs remains exactly the same, namely zero. The individual charged particles have disappeared but the total charge has been conserved.

(5) Charges generate electric and magnetic fields. Charged particles set up a disturbance in space which can be described by two vector fields; an electric field, $\vec{E}$, and a magnetic field, $\vec{B}$. The units of the electric field are Volts/meter; the units of the magnetic field are Webers/m^2 . Since these are vector fields they are characterized by a direction and a magnitude. Each of these fields at any point in space can be described by its components along three mutually perpendicular axes. For example, with respect to a rectangular cartesian system of axes, xyz (see Figure (1.1.1)),

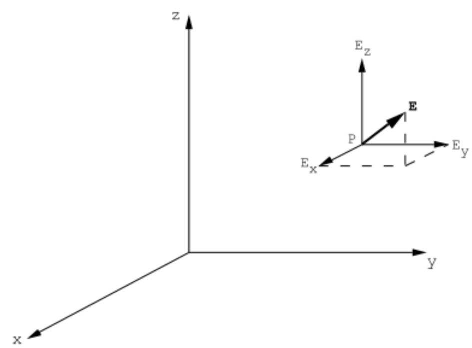


Figure 1.1.1: A cartesian co-ordinate system used to specify an electric vector field

the electric field can be resolved into the three components E_x(x,y,z,t), E_y(x,y,z,t), and E_z(x,y,z,t) where the magnitude of the electric field is given by $\mathrm{E}=\sqrt{\mathrm{E}_x^2+\mathrm{E}_y^2+\mathrm{E}_z^2}$. The components of these fields depend upon the orientation of the co-ordinate system used to describe them, however the magnitude of each field must be independent of the orientation of the co-ordinate system.

(6) The fields E and B are real physical objects. These fields can carry energy, momentum, and angular momentum from one place to another.

(7) The electromagnetic forces on a charged particle, q, can be obtained from a knowledge of the fields E, B generated at the position of q by all other charges. The force in Newtons is given by

$$\overrightarrow{\mathrm{F}}=q[\overrightarrow{\mathrm{E}}+(\overrightarrow{\mathrm{v}}\times\overrightarrow{\mathrm{B}})] \tag{1.1.1}$$

where $\vec{v}$ is the particle velocity in meters/sec. (Notice that $\vec{B}$ has the units of an electric field divided by a velocity). Formula (1.1.1) applies to a spinless particle. In actual fact the situation is more complicated because most particles carry an intrinsic magnetic moment associated with its intrinsic angular momentum (spin). In the rest system of the particle its magnetic moment is subject to a torque due to the presence of the field $\vec{B}$, and to a force due to spatial gradients of $\vec{B}$. These magnetic forces will be discussed later. For the present we shall discuss only spinless charged particles, and we shall ignore the fact that real charged particles are more complicated.

(8) Superposition. Electric and magnetic fields obey the rules of superposition. Given a system of charges which would by themselves produce the fields $\vec{E}_1$, $\vec{B}_1$; given a second system of charges which would by themselves produce the fields $\vec{E}_2$, $\vec{B}_2$; then together the two systems of charges produce the total fields

$$\overrightarrow{\mathrm{E}} = \overrightarrow{\mathrm{E}_1} + \overrightarrow{\mathrm{E}_2}, \quad \overrightarrow{\mathrm{B}} = \overrightarrow{\mathrm{B}_1} + \overrightarrow{\mathrm{B}_2}. \tag{1.1.2}$$

This rule enormously simplifies the calculation of electric and magnetic fields because it can be carried out particle by particle and the total field obtained as the vector sum of all the partial fields due to the individual charges.

(9) A Stationary Charged Particle. In the co-ordinate system in which a charged particle is stationary with respect to the observer the electric and magnetic fields which it generates are very simple:

$$\overrightarrow{\mathrm{E}} = \frac{q}{4\pi\epsilon_0}\left(\frac{\overrightarrow{\mathrm{R}}}{\mathrm{R}^3}\right) \quad \overrightarrow{\mathrm{B}} = 0 \tag{1.1.3}$$

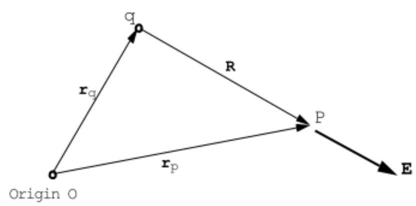


Figure 1.1.2: The fields generated by a point charge of q Coulombs at $\vec{r}_q$ that is stationary with respect to the observer located at $\vec{r}_P$. $\overrightarrow{\mathrm{R}} = \overrightarrow{\mathrm{r}_p} - \overrightarrow{r_q}$.

Equation (1.1.3) is called Coulomb's law. See Figure (1.1.2)

The electric field strength is measured in Volts/meter. The amplitude of the electric field decreases with distance from the charge like the square of the distance ie. $\sim\frac{1}{R^2}$ where the exponent is equal to two within 1 part in 10^{10}. The MKS system of units has been used to write eqn(1.1.3) in which the charge is measured in Coulombs. A current of 1 Amp`ere at some point in a circuit consists of an amount of charge equal to 1 Coulomb passing that point each second. Distances in (1.1.3) are measured in meters. The factor of proportionality is

$$\frac{1}{4\pi\epsilon_0} = 10^{-7} \times c^2 = 8.987 \times 10^9 \quad \text{meters / farad} \tag{1.1.4}$$

where c = 2.9979×10^8 m/sec is the velocity of light in vacuum. The size of

$$\frac{1}{4\pi\epsilon_0}$$

is purely the consequence of the historical definitions of the Volt and the Amp`ere. A second system of units which is very commonly used in the current magnetism literature is the CGS system in which distances are measured in centimeters, mass is measured in grams, and time is measured in seconds. In the CGS system the unit of charge, the statcoulomb, has been chosen to make Coulomb's law very simple. In the CGS system the field due to a stationary point charge is given by

$$\overrightarrow{\mathrm{E}} = q\left(\frac{\overrightarrow{\mathrm{R}}}{\mathrm{R}^3}\right) \quad \text{statvolts /cm} \quad \text{and} \quad \overrightarrow{\mathrm{B}} = 0 \quad \text{Gauss.} \tag{1.1.5}$$

The price that is paid for the simplicity of Equation (1.1.5) is that the conventional engineering units for the current and potential, Amp`eres and Volts, cannot be used. The scaling factors between MKS and CGS electrical units involve the numerical value of the velocity of light, c. For example, in the CGS system the charge on a proton is $e_p = 4.803 \times 10^{-10}$ esu whereas in the MKS system it is $e_p = 1.602 \times 10^{-19}$ Coulombs. The ratio of these two numbers is

$$\frac{e_p|_{\mathrm{esu}}}{e_p|_{MKS}} = 2.9979 \times 10^9. \tag{1.1.6}$$

(10) The Fields generated by a Moving Charged Particle. Consider a co-ordinate system in which a spinless charged particle moves with respect to the observer with a constant velocity v which is much smaller than the speed of light in vacuum, c : ie. (v/c) << 1. The electric and magnetic fields generated by such a slowly moving charge are given by

$$\vec{\mathrm{E}}(\vec{\mathrm{R}}, t) = \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{\mathrm{R}}}{\mathrm{R}^3}\right) \quad V/m \quad \vec{\mathrm{B}}(\vec{\mathrm{R}}, t) = \frac{1}{c^2}(\vec{v} \times \vec{\mathrm{E}}) \quad \text{Webers }/m^2. \tag{1.1.7}$$

These expressions are correct to order $(v/c)^2$. $\vec{R}$ is the vector drawn from the position of the charged particle at the time of observation to the position of the observer. Note that the moving charge generates both an electric and a magnetic field. The above fields can be used to calculate the force on a particle q_2 located at $\vec{R}$:

$$\vec{\mathrm{F}_2}(\vec{\mathrm{R}}, t) = q_2[\vec{\mathrm{E}}(\vec{\mathrm{R}}, t) + (\vec{\mathrm{v}_2}(\vec{\mathrm{R}}, t) \times \vec{\mathrm{B}}(\vec{\mathrm{R}}, t))] \quad \text{Newtons.} \tag{1.1.8}$$

The particle q of Equations (1.1.7) generates the fields that exert forces on the particle q_2. Equations (1.1.7) are simplified versions of the general expressions for the electric and magnetic fields generated by a spinless point charge moving in an arbitrary fashion: see "The Feynman Lectures on Physics", Volume II, page 21-1 (R.P.Feynman, R.B.Leighton, and M.Sands, Addison-Wesley, Reading, Mass., 1964). These general expressions are

$$\vec{\mathrm{E}}(\vec{\mathrm{R}}, t) = \frac{q}{4\pi\epsilon_0}\left[\left(\frac{\vec{\mathrm{R}}}{\mathrm{R}^3}\right) + \left(\frac{\mathrm{R}}{c}\right)\frac{d}{dt}\left(\frac{\vec{\mathrm{R}}}{\mathrm{R}^3}\right) + \frac{1}{\mathrm{c}^2}\frac{d^2}{dt^2}\left(\frac{\vec{\mathrm{R}}}{\mathrm{R}}\right)\right]\Bigg|_{Retarded} \tag{1.1.9}$$

$$c\vec{\mathrm{B}}(\vec{\mathrm{R}}, t) = \frac{\vec{\mathrm{R}}}{\mathrm{R}}\Bigg|_{Retarded} \times \vec{\mathrm{E}}(\vec{\mathrm{R}}, t). \tag{1.1.10}$$

The label "Retarded" refers to the retarded time $t_R = t - \frac{R}{c}$. The distances that appear in Equation (1.1.9) and Equation (1.1.10) are not evaluated at the time of observation, t, but at the earlier time, the retarded time, in order to take into account the finite speed of light. Any change in position requires the minimum time R/c to reach the observer, where c is the speed of light in vacuum. This corresponds to the requirement that changes in the motion of the particle can not be communicated to the observer faster than is permitted by the speed of light in vacuum, see Figure (1.1.3).

For a slowly moving particle, the first two terms of Equation (1.1.9) add together to give Coulomb's law in which the distance R is evaluated at the time of observation rather than at the retarded time; in other words, one can ignore time retardation if v/c is small. The last term in Equation (1.1.9) gives a field that is proportional to the component of acceleration perpendicular to the position vector $\vec{R}$ in the limit (v/c) << 1. This field decreases with distance like 1/R as opposed to the $1/R^2$ decrease of Coulomb's law. It is called the radiation field and is given by the expression

$$\vec{\mathrm{E}}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{[\vec{\mathrm{a}} \times \vec{\mathrm{R}}] \times \vec{\mathrm{R}}}{\mathrm{c}^2 R^3}\Bigg|_{t-\frac{R}{c}} \tag{1.1.11}$$

$$c\vec{\mathrm{B}} = \frac{\vec{\mathrm{R}}}{\mathrm{R}}\Bigg|_{t-\frac{R}{\epsilon}} \times \vec{\mathrm{E}}_{rad}, \tag{1.1.12}$$

where $\vec{a}$ is the acceleration of the charge.

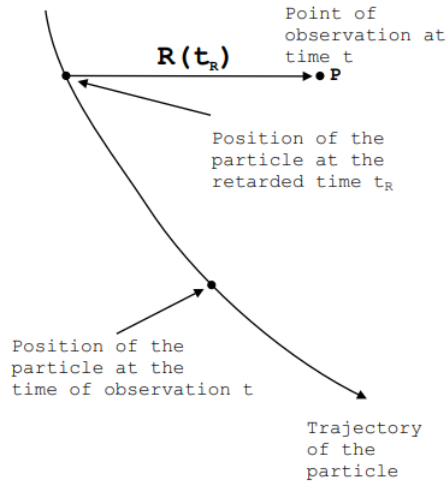


Figure 1.1.3: The electric and magnetic fields generated at the point of observation P at the time t depend upon the position, the velocity, and the acceleration of the charged particle at the retarded time $t_R = \left(t - \frac{R}{c}\right)$.

1.2: Maxwell's Equations

In principle, given the positions of a collection of charged particles at each instant of time one could calculate the electric and magnetic fields at each point in space and at each time from Equations (1.1.9) and (1.1.10). For ordinary matter this is clearly an impossible task. Even a small volume of a solid or a liquid contains enormous numbers of atoms. A cube one micron on a side (10^{-6}m × 10^{-6}m × 10^{-6}m) contains $\sim 10^{11}$ atoms, for example. Each atom consists of a positively charged nucleus surrounded by many negatively charged electrons, all of which are in motion and which will, therefore, generate electric and magnetic fields that fluctuate rapidly both in space and in time. For most purposes one does not wish to know in great detail the space and time variation of the fields. One usually wishes to know about the space and time **averaged** electric and magnetic fields. For example, the magnitude and direction of $\vec{E}$ averaged over a time interval that is determined by the instrument used to measure the field. Typically this might be of order 10^{-6} seconds or more; a time that is very long compared with the time required for an electron to complete an orbit around the atomic nucleus in an atom (10^{-16} to 10^{-21} seconds). Moreover, one is usually interested in the value of these fields averaged over a volume that is small compared with a cube $\sim 10^{-6}$ meters on a side but large compared with atomic dimensions, $\sim 10^{-10}$ meters in diameter. In 1864 J.C.Maxwell proposed a system of differential equations that can be used for calculating electric and magnetic field distributions, and that automatically provide the space and time averaged fields that are of practical interest. These Maxwell's Equations for a macroscopic medium are as follows:

$$\operatorname{curl} \vec{\mathrm{E}} = -\frac{\partial \vec{\mathrm{B}}}{\partial t} \tag{1.2.1}$$

$$\operatorname{div} \vec{\mathrm{B}} = 0 \tag{1.2.2}$$

$$\operatorname{curl} \vec{\mathrm{B}} = \mu_0 \left(\vec{\mathrm{J}}_f + \operatorname{curl} \vec{\mathrm{M}} + \frac{\partial \vec{\mathrm{P}}}{\partial t} \right) + \epsilon_0 \mu_0 \frac{\partial \vec{\mathrm{E}}}{\partial t} \tag{1.2.3}$$

$$\operatorname{div} \vec{\mathrm{E}} = \frac{1}{\epsilon_0} \left(\rho_f - \operatorname{div} \vec{\mathrm{P}} \right) \tag{1.2.4}$$

where $\epsilon_0 \mu_0 = 1/c^2$ and c is the velocity of light in vacuum. These equations underlie all of electrical engineering and much of physics and chemistry. **They should be committed to memory.** In large part, this book is devoted to working out the consequences of Maxwell's equations for special cases that provide the required background and guidance for solving practical problems in electricity and magnetism. In Equations (1.2.13 to 1.2.16) ϵ_0 is the permativity of free space; it has already been introduced in connection with Coulomb's law, Equation (1.1.3). The constant μ_0 is called the permeability of free space. It has the defined value

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{Henries} \,/\, m. \tag{1.2.5}$$

Maxwell's equations as written above contain four new quantities which must be defined: they are

(1) $\vec{J}_f$, the current density due to the charges which are free to move in space, in units of Amp`eres/m^2 ;

(2) ρ_f, the net density of charges in the material, in units of Coulombs/m^3 ;

(3) $\vec{M}$, the density of magnetic dipoles per unit volume in units of Amps/m;

(4) $\vec{P}$, the density of electric dipoles per unit volume in units of Coulombs/m^2 . In Maxwell's scheme these four quantities become the **sources** that generate the electric and magnetic fields. They are related to the space and time averages of the position and velocities of the microscopic charges that make up matter.

1.2.1 Definition of the Free Charge Density, ρ_f.

Construct a small volume element, ΔV , around the particular point in space specified by the position vector $\vec{r}$. Add up all the charges contained in ΔV at a particular instant; let this amount of charge be ΔQ(t). Average ΔQ(t) over a time interval short compared with the measuring time of interest, but long compared with times characteristic of the motion of electrons around the atomic nuclei; let the resulting time averaged charge be < ΔQ(t) >. Then the free charge density is defined to be

$$\rho_f(\vec{r}, t) = \frac{< \Delta Q(t) >}{\Delta V} \quad \text{Coulombs} \,/ m^3. \tag{1.2.6}$$

The dimensions of the volume element ΔV is rather vague; it will depend upon the scale of the spatial variation that is of interest for a particular problem. It should be large compared with atomic dimensions but small compared with the distance over which ρ_f

changes appreciably.

1.2.2 Definition of the Free Current Density, $\vec{J}_f$.

The free charge density, $\rho(\vec{r}, t)$, will in general change with time as charge flows from one place to the other; one need only think of charge flowing along a wire. The rate at which charge flows across an element of area is described by the current density, $\vec{J}_f(\vec{r}, t)$. It is a vector because the charge flow is associated with a direction. The components of the current density vector can be measured by counting the rate of charge flow across a small area ΔA located at the position specified by $\vec{r}$, and whose normal is oriented parallel with one of the co-ordinate axes; parallel with the x-axis, for example (see Figure (1.2.4)).

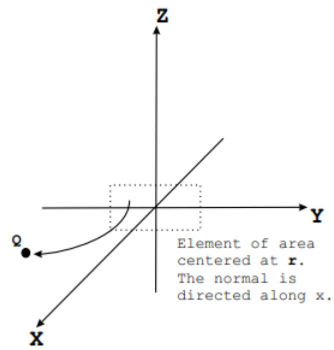


Figure 1.2.4: The x-component of the current density caused by a moving charge distribution. The charge labelled Q is representative of all charges passing through the x-oriented element of area, ΔA per unit time.

Now at time t measure the net amount of charge, < ΔQ >, that has passed through ΔA_x in a small time interval, Δt: positive charge that flows in the direction from +x to -x is counted as a negative contribution; negative charge that flows from -x to +x also makes a negative contribution. The x-component of the current density is given by

$$\left.\overrightarrow{J_f}\right|_x = \frac{< \Delta Q >}{\Delta t \Delta A_x} \quad Amps/m^2. \tag{1.2.7}$$

The other two components of $\vec{J}_f$ are defined in a similar manner. The time interval Δt, and the dimensions of the elements of area, ΔA_x, ΔA_y, and ΔA_z are supposed to be chosen so that they are large compared with atomic times and atomic dimensions, but small compared with the time and length scales appropriate for a particular problem. Free charge density can be visualized as a kind of fluid flowing from place to place with a certain velocity. In terms of this velocity the free current density is given by

$$\vec{J}_f(\vec{r}, t) = \rho_f(\vec{r}, t)\vec{v}(\vec{r}, t). \tag{1.2.8}$$

In the process of charge flow electrical charge can neither be created nor destroyed. Because charge is conserved, it follows that the rate at which charge is carried into a volume must be related to the rate at which the net charge in the volume increases with time. The mathematical expression of this charge conservation law is

$$\frac{\partial \rho_f(\vec{\mathbf{r}}, t)}{\partial t} = -\operatorname{div} \vec{\mathbf{J}}_f(\vec{\mathbf{r}}, t). \tag{1.2.9}$$

1.2.3 Point Dipoles.

In order to discuss the definitions of the two vector functions $\vec{P}(\vec{r}, t)$ and $\vec{M}(\vec{r}, t)$ it is first necessary to discuss the concepts of a point electric dipole and a point magnetic dipole.

The Point Electric Dipole.

Most atoms in a substance are electrically neutral, ie. the charge on the nucleus is compensated by the electrons moving around that nucleus. When examined from a distance that is long compared with atomic dimensions ($\sim 10^{-10}$m) the neutral atom produces no substantial electric or magnetic field. However, if, on average, the centroid of the negative charge distribution is displaced from the position of the nucleus the Coulomb field of the nucleus will no longer cancel the Coulomb field from the electrons. To fix ideas, think of a stationary hydrogen atom consisting of a proton and an electron. The electron moves so fast that on a human time scale its charge appears to be located in a spherical cloud which is tightly distributed around the nucleus (see Figure 1.2.5).

In the absence of an external electric field the centroid of the electronic charge distribution will coincide with the position of the nucleus. Under these circumstances the time-averaged Coulomb fields of the nucleus and the electron cancel each other when observed from distances that are large

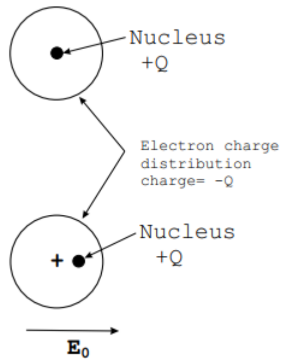


Figure 1.2.5: Upper figure: Sketch of a hydrogen atom in zero applied electric field. The nuclear charge is Q = 1.6 × 10^{-19} Coulombs. The time-averaged electron charge, -Q, is distributed in a spherically symmetric cloud around the nucleus having a radius of approximately 5 × 10^{-11} m. Lower figure: Sketch of a hydrogen atom subjected to a uniform electric field E_0 . The displacement of the centroid of the electron charge density relative to the nucleus has been exaggerated for the sake of clarity.

compared with 10^{-10}m. If the atom is subjected to an external electric field the nucleus is pulled one way and the centroid of the electron cloud is pulled the other way (Equation (1.1.1)): there is an effective charge separation (see Figure 1.2.5). The Coulomb fields due to the nucleus and the electron no longer exactly cancel. Let us use the law of superposition to calculate the field that arises when two point charges no longer coincide; refer to Figure (1.2.6). The electric field at the point of observation, P, due to the positive charge is given by

$$\vec{E}_{+}=\frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r}}{r^3}\right).$$

The electric field at P due to the negative charge is given by

$$\vec{E}_{-}=-\frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r}+\vec{d}}{\left(|\vec{r}+\vec{d}|^3\right.}\right).$$

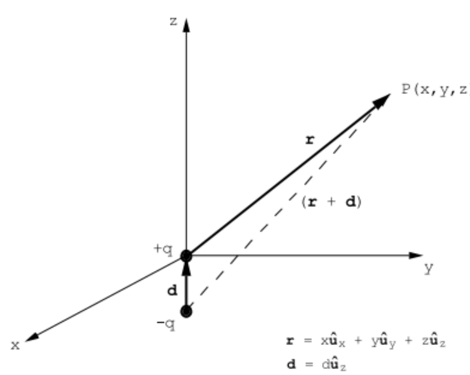


Figure 1.2.6: Two charges equal in magnitude but opposite in sign are separated by the vector distance $\vec{d}$. By definition the dipole moment of this pair of charges is $\vec{p}=q\vec{d}$ where $\vec{d}$ is the vector directed from the negative to the positive charge. $\hat{u}_x, \hat{u}_y$, and $\hat{u}_z$ are unit vectors directed along the x,y, and z axes.

Referring to Figure 1.2.6 one has

$$\vec{r}=x\hat{u}_x+y\hat{u}_y+z\hat{u}_z,$$

and

$$\mathrm{r}=\sqrt{x^2+y^2+z^2}\,.$$

For $\vec{d}$ oriented along the z-axis as shown in Figure 1.2.6,

$$(\vec{\mathrm{r}}+\vec{\mathrm{d}})=x\hat{\mathrm{u}}_x+y\hat{\mathrm{u}}_y+(z+d)\hat{\mathrm{u}}_z$$

so that

$$|\vec{r}+\vec{d}\,|=\left[x^2+y^2+(z+d)^2\right)\Big]^{\;1/2}$$

or

$$|\vec{\mathrm{r}}+\vec{\mathrm{d}}|=\;\left[x^2+y^2+z^2+2zd+d^2\right]^{\;1/2}.$$

Upon dividing out r^2 this gives

$$|\vec{\mathrm{r}}+\vec{\mathrm{d}}|=r\left[1+\left(\frac{2zd+d^2}{r^2}\right)\right]^{1/2}.$$

From this expression one has

$$\frac{1}{|\vec{\mathrm{r}}+\vec{\mathrm{d}}|^3}=\frac{1}{r^3}\left[1+\frac{(2zd+d^2)}{r^2}\right]^{-3/2}.$$

This is so far exact. Now make use of the fact that (d/r) is very small and use the binomial theorem to expand the radical. It is sufficient to keep only terms linear in (d/r). The result is

$$\frac{1}{|\vec{\mathrm{r}}+\vec{\mathrm{d}}|^3}=\frac{1}{r^3}-\frac{3zd}{r^5}.$$

Use this result to calculate the total electric field at the point of observation, P, correct to terms of order (d/r). The terms proportional to

$$\left(\frac{1}{r^2}\right)$$

cancel leaving the field

$$\overrightarrow{\mathrm{E}_d}=\overrightarrow{\mathrm{E}_+}+\overrightarrow{\mathrm{E}_-}=\frac{1}{4\pi\epsilon_0}\left(\frac{(3zqd)\vec{\mathrm{r}}}{r^5}-\frac{q\vec{\mathrm{d}}}{r^3}\right).$$

By definition the dipole moment of the pair of point charges is given by $\vec{\mathrm{p}}=q\vec{\mathrm{d}}$. Moreover, $zqd=\vec{r}\cdot\vec{p}$, ie. it is equal to the scalar product of the dipole moment and the position vector $\vec{r}$. Finally, the expression for the electric field generated by a stationary point dipole can be written

$$\overrightarrow{\mathrm{E}_d}=\frac{1}{4\pi\epsilon_0}\left(\frac{3(\vec{\mathrm{p}}\cdot\vec{\mathrm{r}})\vec{\mathrm{r}}}{r^5}-\frac{\vec{\mathrm{p}}}{r^3}\right).\tag{1.2.10}$$

Although this expression has been obtained for the particular case in which $\vec{p}$ is oriented along the z-axis, the result stated in Equation (1.2.10) is perfectly general and is valid for any orientation of the dipole moment $\vec{p}$.

Formula 1.2.10 is so fundamental that it should be committed to memory along with Coulomb's law. The field distribution around a point dipole is illustrated in Figure 1.2.7. The magnetic field generated by a stationary point dipole is zero; magnetic fields are generated by charges moving with respect to the observer.

It is useful to write the dipole electric field in terms of it's components with respect to a spherical polar co-ordinate system in which the dipole is aligned along the z-axis, see Figure (1.2.8). These components are

$$\mathrm{E_r}=\frac{2p}{4\pi\epsilon_0}\frac{\cos(\theta)}{r^3}\tag{1.2.11}$$

$$E_{\theta} = \frac{p}{4\pi\epsilon_0}\frac{\sin(\theta)}{r^3} \tag{1.2.12}$$

The Point Magnetic Dipole.

Atoms and molecules often carry a magnetic moment. These magnetic moments can arise as a result of the motion of electrons around the nucleus of an atom or around the nuclei in a molecule (see below). In addition,

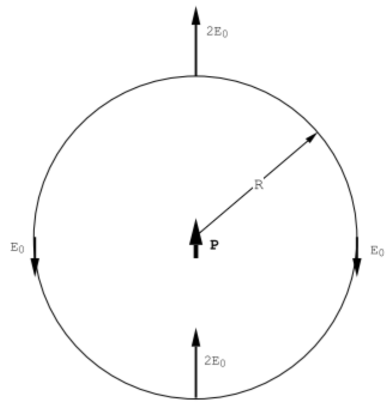


Figure 1.2.7: The electric field intensity at various positions around an electric point dipole, $\vec{P}$. The electric field distribution is cylindrically symmetric around the dipole as axis.

electrons and nuclei carry intrinsic point magnetic moments that are related to their intrinsic angular momentum (spin). Atomic or molecular magnetic moments generate magnetic fields. When these magnetic fields are observed at distances from the atom or molecule that are much larger than atomic or molecular dimensions, and when these fields are averaged over times long compared with atomic or molecular orbital times, the resulting time averaged field can be described by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right). \tag{1.2.13}$$

where μ_0 is the constant of Equation (1.2.5) and $\vec{m}$ is the magnetic moment. In addition to the magnetic field created by a magnetic moment, the atom or molecule, if charged, will generate an electric field given by Coulomb's law, Equation (1.1.3).

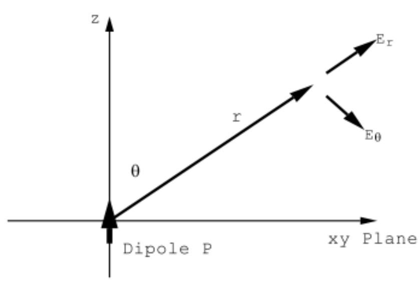


Figure 1.2.8: The electric field generated by a dipole oriented along the z-axis and expressed as spherical polar components.

The generation of a magnetic field due to the orbital motion of a charged particle can be understood using the simple model illustrated in Figure (1.1.9). Let a spinless charged particle, charge= q Coulombs, revolve in a circular orbit of radius a meters with the speed v meters/sec, where $\frac{v}{c} << 1$. One can use Equation (1.1.7) to calculate the electric and magnetic fields that would be measured by an observer whose distance from the center of the current loop is much greater than the orbit radius a. It can be shown that the time averaged electric field is given by coulomb's law

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r}}{r^3}\right) \quad \text{Volts}/\text{m}.$$

This result is obtained by using a binomial expansion in the small quantity a/r and keeping only the lowest order terms; the lowest order correction term upon taking the time average is proportional to $(a/r)^2$, see problem (1.8). The magnetic field can be calculated using Equation (1.1.7). The velocity of the particle is proportional to the orbit radius, and therefore when the time averages are worked out the lowest order non-vanishing term is proportional to $(a/r)^2$; see problem (1.8). The time-averaged magnetic field turns out to be given by Equation (1.2.13). Notice that this expression has exactly the same form as Equation (1.2.10) for the electric field distribution around an electric dipole

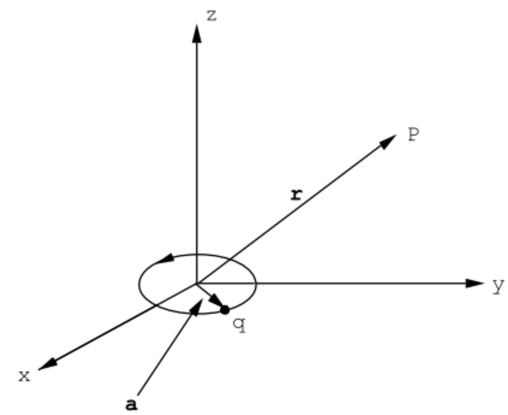


Figure 1.2.9: A particle carrying a charge of q Coulombs and following a circular orbit of radius a meters with the speed v meters/sec generates a magnetic dipole moment $|\vec{\mathrm{m}}|=$ qav $/2\mathrm{Amp}-m^2$.

moment $\vec{p}$. Here the vector m is called the orbital magnetic dipole moment associated with the current loop, and is given by

$$\vec{\mathrm{m}}=\frac{qa^2}{2}\left(\frac{d\phi}{dt}\right)\hat{\mathrm{u}}_z \quad \text{Coulomb } - \text{ meters }^2/\text{ sec.} \tag{1.2.14}$$

Note that $|\vec{\mathrm{m}}|=\mathrm{IA}$ where $\mathrm{I}=\mathrm{qv}/2\pi a$ is the current in the loop, and $A=\pi a^2$ is the area of the loop. Since the speed of the particle is given by $v=a(\mathrm{d}\phi/dt)$, the magnitude of the magnetic moment can also be written in terms of the angular momentum of the circulating charge:

$$|\vec{\mathrm{m}}|=\frac{qa\mathrm{v}}{2}=\left(\frac{q}{2m_p}\right)(m_p av).$$

where the mass of the charged particle is m_p and it carries an angular momentum $\mathrm{L}=\mathrm{m}_p av$. Thus the angular momentum $\vec{m}$ is related to the particle angular momentum $vecL$ by the relation

$$\vec{\mathrm{m}}=\left(\frac{q}{2m_p}\right)\vec{\mathrm{L}}. \tag{1.2.15}$$

For an electron q= - 1.60 × 10^{-19} Coulombs = - |e| so that the magnetic moment and the angular momentum are oppositely directed. The angular momentum is quantized in units of $\hbar$, therefore the magnetic moment of an orbiting particle is also quantized. The quantum of magnetization for an orbiting electron is called the Bohr magneton, μ_B. It has the value

$$\mu_B=\frac{e\hbar}{2m_e}=9.27\times10^{-24} \quad \text{Coulomb } -m^2/\text{sec.}$$

(The units of μ_B can also be expressed as Amp-m^2 or as Joules/Tesla).

In addition to their orbital angular momentum, charged particles possess intrinsic or spin angular momentum, $\vec{S}$. There is also a magnetic moment associated with the spin. The magnetic moment due to spin is usually written

$$\vec{\mathrm{m}}_s=g\left(\frac{q}{2m_p}\right)\vec{\mathrm{s}}. \tag{1.2.16}$$

For an electron q = −|e|, and g = 2.00. The spin of an electron has the magnitude $|\vec{S}|=\hbar/2$; consequently, the intrinsic magnetic moment carried an electron due to its spin is just 1 Bohr magneton, μ_B. The total magnetic moment generated by an orbiting particle that carries a spin moment is given by the vector sum of its orbital and spin magnetic moments. The total magnetic moment associated with an atom is the vector sum of the orbital and spin moments carried by all of its constituent particles, including the nucleus. The magnetic field generated by a stationary atom at distances large compared with the atomic radius is given by Equation (1.2.13) with $\vec{m}$ equal to the total atomic magnetic dipole moment.

1.2.4 The Definitions of the Electric and the Magnetic Dipole Densities.

Let us now turn to the definitions of the electric dipole moment density, $\vec{P}$, and the magnetic dipole density, $\vec{M}$, that occur in Maxwell's equations (1.2.1 to 1.2.4).

The Definition of the Electric Dipole Density, $\vec{P}$.

Think of an idealized model of matter in which all of the atoms are fixed in position. In the presence of an electric field each atom will develop an electric dipole moment; the dipole moment induced on each atom will depend upon the atomic species. Some atomic configurations also carry a permanent electric dipole moment by virtue of their geometric arrangement: the water molecule, for example carries a permanent dipole moment of 6.17×10^{-30} Coulomb-meters (see problem (1.12). Let the dipole moment on atom i be \(\vec p_i\) Coulomb-meters. Select a volume element ΔV located at some position $\vec{r}$ in the matter. At some instant of time, t, measure the dipole moment on each atom contained in ΔV and calculate their vector sum, $\sum_i \vec{\mathrm{p}}_i$. This moment will fluctuate with time, so it is necessary to perform a time average over an interval that is long compared with atomic fluctuations but short compared with times of experimental interest; let this time average be $\langle \sum_i \vec{\mathrm{p}}_i \rangle$. Then the electric dipole density is given by

$$\overrightarrow{\mathrm{P}}(\overrightarrow{\mathrm{r}}, t) = \frac{\left\langle \sum_i \overrightarrow{\mathrm{p}}_i \right\rangle}{\Delta V} \quad \text{Coulombs} / \mathrm{m}^2. \tag{1.2.17}$$

The shape and size of ΔV are unimportant: the volume of ΔV should be large compared with an atom, but small compared with the distance over which $\vec{P}$ varies in space. In a real material the atoms are not generally fixed in position. In a solid they jiggle about more or less fixed sites. In liquids and gasses they may, in addition, take part in mass flow as matter flows from one place to another. This atomic motion considerably complicates the calculation of the electric dipole density because the effective electric dipole on an atom or molecule that is moving with respect to the observer includes a small contribution due to any magnetic dipole moment that might be carried by that atom or molecule. However these correction terms are very small and may be neglected in the limit $\frac{\mathrm{v}}{\mathrm{c}} \ll 1$.

The Definition of the Magnetic Dipole Density, $\vec{M}$.

This vector quantity is defined in a manner that is similar to the definition of the electric dipole moment per unit volume:

$$\vec{\mathrm{M}}(\vec{\mathrm{r}}, t) = \frac{\langle \sum_i \vec{\mathrm{m}}_i \rangle}{\Delta V} \quad \text{Amps / meter.} \tag{1.2.18}$$

$\langle \sum_i \vec{\mathrm{m}}_i \rangle$ is a suitable time average over the atomic magnetic moments contained in a small volume element, ΔV , at time t and centered at the position specified by $\vec{r}$. It is assumed that the atoms are stationary. If they are not, the magnetization density contains contributions which are proportional the velocities of the various atomic electric dipole moments; these velocities are measured with respect to the observer. We shall not be concerned with this correction which is very small if $\frac{\mathrm{v}}{\mathrm{c}} \ll 1$. As above, the volume element ΔV is supposed to be large compared with an atomic dimension but small compared with the length scale over which $\vec{M}$ varies in space.

1.3: Return to Maxwell's Equations

Maxwell's equations (1.2.1, 1.2.2, 1.2.3, 1.2.4) form a system of differential equations that can be solved for the vector fields $\vec{E}$ and $\vec{B}$ given the space and time variation of the four source terms $\rho_f(\vec{r},t)$, $\overrightarrow{J_f}(\vec{r},t)$, $\vec{P}(\vec{r},t)$, and $\vec{M}(\vec{r},t)$. In order to solve Maxwell's equations for a specific problem it is usually convenient to specify each vector field in terms of components in one of the three major co-ordinate systems: (a) cartesian co-ordinates (x,y,z), Figure (1.3.10); (b) cylindrical polar co-ordinates (r,θ,z), Figure (1.3.10); and (c) spherical polar co-ordinates (ρ,θ,φ), Figure (1.3.10).

It is also necessary to be able to calculate the scalar field generated by the divergence of a vector field in each of the above three co-ordinate systems. In addition, one must be able to calculate the three components of the curl in the above three co-ordinate systems. Vector derivatives are reviewed by M.R. Spiegel, Mathematical Handbook of Formulas and Table s, Schaum's Outline Series, McGraw-Hill, New York, 1968, chapter 22. It is also worthwhile reading the discussion contained in The Feynman Lectures on Physics, by R.P. Feynman, R.B. Leighton, and M. Sands, Addison-Wesley, Reading, Mass., 1964, Volume II, chapters 2 and 3.

The following four vector theorems should be read, understood, and committed to memory because they will be used over and over again in the course of solving Maxwell's equations.

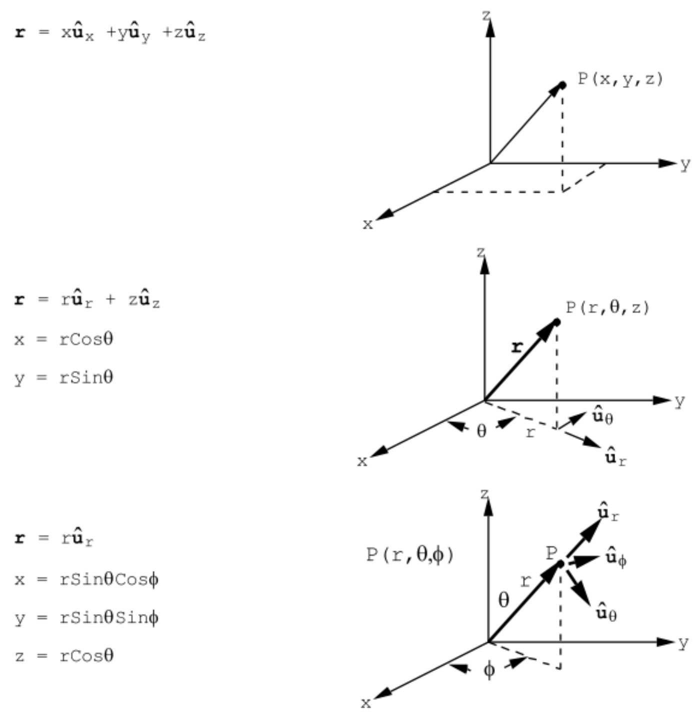


Figure 1.3.10: The three commonly used co-ordinate systems.

1.3.1 The curl of any gradient function is zero.

1.3.2 The divergence of any curl is zero.

1.3.3 Gauss' Theorem.

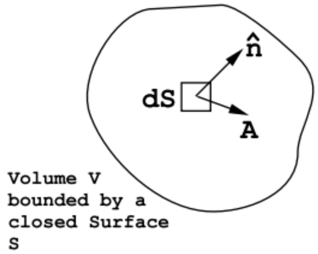


Figure 1.3.11: An application of Gauss' Theorem to a volume V bounded by a closed surface S. $\vec{A}$ is a vector field, and $\hat{\boldsymbol{n}}$ is a unit vector normal to the surface at the element of area dS.

Consider a volume V bounded by a closed surface S, see Figure (1.3.11). An element of area on the surface S can be specified by the vector d$\vec{S}$ = $\hat{\boldsymbol{n}}$dS where dS is the magnitude of the element of area and $\hat{\boldsymbol{n}}$ is a unit vector directed along the outward normal to the surface at the element dS. Let $\vec{A}(\vec{r}$,t) be a vector field that in general may depend upon position and upon time. Then Gauss' Theorem states that

$$\int\int_S (\vec{\mathrm{A}}\cdot\hat{\mathbf{n}})dS = \int\int\int_V \mathrm{div}(\vec{\mathrm{A}})d\tau$$

where dτ is an element of volume. The integrations are to be carried out at a fixed time.

1.3.4 Stokes' Theorem.

Consider a surface S bounded by a closed curve C, see Figure (1.3.12). $\vec{A}(\vec{r}$,t) is any vector field that may in general depend upon position and upon time. At

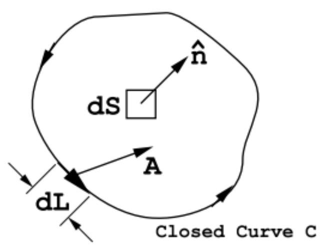


Figure 1.3.12: An application of Stokes' Theorem to a surface bounded by a closed curve C. $d\overrightarrow{\mathrm{L}}$ is an element of length along curve C. $\hat{\boldsymbol{n}}$ is a unit vector normal to the element of surface area, dS.

a fixed time calculate the line integral of $\vec{A}$ around the curve C; the element of length along the line C is $d\overrightarrow{\mathrm{L}}$. Then Stokes' Theorem states that

$$\int_C \overrightarrow{\mathrm{A}}\cdot\overrightarrow{d\mathbf{L}} = \iint_S \mathrm{curl}(\overrightarrow{\mathrm{A}})\cdot\hat{\mathbf{n}}dS$$

where $\hat{\boldsymbol{n}}$ is a unit vector normal to the surface element dS whose direction is related to the direction of traversal around the curve C by the right hand rule.

1.4: The Auxiliary Fields D and H

It is sometimes useful to rewrite Maxwell's equations (1.2.1 to 1.2.4 in terms of $\vec{E}$, $\vec{B}$, and two new vector fields $\vec{D}$and $\vec{H}$. These two new vectors are constructed as follows:

$$\overrightarrow{\mathrm{D}}=\epsilon_0 \overrightarrow{\mathrm{E}}+\overrightarrow{\mathrm{P}}$$

and

$$\overrightarrow{\mathrm{B}}=\mu_0(\overrightarrow{\mathrm{H}}+\overrightarrow{\mathrm{M}})$$

When written using these two new fields Maxwell's equations become

$$\operatorname{curl}(\overrightarrow{\mathrm{E}})=-\frac{\partial \overrightarrow{\mathrm{B}}}{\partial t} \tag{1.4.1}$$

$$\operatorname{div}(\overrightarrow{\mathrm{B}})=0 \tag{1.4.2}$$

$$\operatorname{curl}(\overrightarrow{\mathrm{H}})=\overrightarrow{\mathrm{J}}_f+\frac{\partial \overrightarrow{\mathrm{D}}}{\partial t} \tag{1.4.3}$$

$$\operatorname{div}(\overrightarrow{\mathrm{D}})=\rho_f \tag{1.4.4}$$

Maxwell's equations have a simpler form when written in this way, and may in consequence be easier to remember. Their physical content is, of course, unaltered by the introduction of the two new auxiliary fields $\vec{D}$ and $\vec{H}$.

1.5: The Force Density and Torque Density in Matter

The presence of an electric field, $\vec{E}$, and a magnetic field,$\vec{B}$, in matter results in a force density if the matter is charged and in a torque density if the matter carries electric and magnetic dipole densities. In addition, if the electric field varies in space (the usual case) then a force density is created that is proportional to the electric dipole density and to the electric field gradients. Similarly, if the magnetic field varies in space then a force density is exerted on the matter that is proportional to the magnetic dipole density and to the magnetic field gradients. These force and torque densities are stated below; their proof is left for the problem sets.

1.5.1 The Force Density in Charged and Polarized Matter.

There is a force density that is the direct analogue of Equation (1.1.8), the force acting on a charged particle moving with the velocity $\vec{v}$ in electric and magnetic fields, ie

$$\overrightarrow{\mathrm{f}}=q(\overrightarrow{\mathrm{E}}+[\overrightarrow{\mathrm{v}}\times\overrightarrow{\mathrm{B}}]).$$

If this force acting on each charged particle is averaged in time over periods longer than characteristic atomic or molecular orbital times and summed over the particles contained in a volume, ΔV , where ΔV is large compared with atomic or molecular dimensions, then one can divide this total averaged force by ΔV to obtain the force density

$$\overrightarrow{\mathbf{F}}=\rho_f\overrightarrow{\mathbf{E}}+\left(\overrightarrow{\mathbf{J}}_f\times\overrightarrow{\mathbf{B}}\right)\quad \text{Newtons }/m^3. \tag{1.5.1}$$

If the electric field in matter varies from place to place there is generated a force density proportional to the dipole moment per unit volume, $\vec{P}$, given by

$$\overrightarrow{\mathrm{F}}_E=\left(\overrightarrow{\mathrm{P}}\cdot\nabla E_x\right)\hat{\mathbf{u}}_x+\left(\overrightarrow{\mathrm{P}}\cdot\nabla E_y\right)\hat{\mathbf{u}}_y+\left(\overrightarrow{\mathrm{P}}\cdot\nabla E_z\right)\hat{\mathbf{u}}_z\quad \text{Newtons }/m^3. \tag{1.5.2}$$

In addition, if the magnetic field, $\vec{B}$, varies from place to place there will be generated a force density proportional to the magnetic dipole density, $\vec{M}$, given by

$$\overrightarrow{\mathrm{F}}_B=\left(\overrightarrow{\mathrm{M}}\cdot\nabla B_x\right)\hat{\mathbf{u}}_x+\left(\overrightarrow{\mathrm{M}}\cdot\nabla B_y\right)\hat{\mathbf{u}}_y+\left(\overrightarrow{\mathrm{M}}\cdot\nabla B_z\right)\hat{\mathbf{u}}_z\quad \text{Newtons }/m^3. \tag{1.5.3}$$

The nabla operator denotes the operation of calculating the gradient of a scalar function $\phi(\overrightarrow{\mathrm{r}})$. In cartesian co-ordinates

$$\nabla\phi=\frac{\partial\phi}{\partial x}\hat{\mathbf{u}}_x+\frac{\partial\phi}{\partial y}\hat{\mathbf{u}}_y+\frac{\partial\phi}{\partial z}\hat{\mathbf{u}}_z.$$

1.5.2 The Torque Densities in Polarized Matter.

It can be shown that an electric field exerts a torque on polarized matter. The torque density is given by

$$\overrightarrow{\mathrm{T}}_E=\overrightarrow{\mathrm{P}}\times\overrightarrow{\mathrm{E}}\quad \text{Newtons }/\mathrm{m}^2. \tag{1.5.4}$$

The magnetic field also exerts a torque on magnetized matter. This torque density is given by

$$\overrightarrow{\mathrm{T}}_B=\overrightarrow{\mathrm{M}}\times\overrightarrow{\mathrm{B}}\quad \text{Newtons }/\mathrm{m}^2. \tag{1.5.5}$$

1.6: The CGS System of Units

The CGS system of units is still used by many scientists and they are commonly used in many older articles and books dealing with topics in electricity and magnetism. For that reason it is useful for reference purposes to explicitly display Maxwell's equations written using the CGS system of units.

$$\operatorname{curl}(\vec{\mathrm{E}})=-\frac{1}{c}\frac{\partial\vec{\mathrm{B}}}{\partial t}. \tag{1.6.1}$$

$$\operatorname{div}(\vec{\mathrm{B}})=0. \tag{1.6.2}$$

$$\operatorname{curl}(\vec{\mathrm{B}})=\frac{4\pi}{c}\left(\vec{\mathrm{J}}_f+\operatorname{ccurl}(\vec{\mathrm{M}})+\frac{\partial\vec{\mathrm{P}}}{\partial t}\right)+\frac{1}{c}\frac{\partial\vec{\mathrm{E}}}{\partial t}. \tag{1.6.3}$$

$$\operatorname{div}(\vec{\mathrm{E}})=4\pi\left(\rho_f-\operatorname{div}(\vec{\mathrm{P}})\right). \tag{1.6.4}$$

In this system of units c = 2.998 × 10^{10} cm/sec. and $\vec{E}$ and $\vec{B}$ have the same units (stat-Volts/cm). However, for historical reasons, the units of $\vec{B}$ are
known as Gauss. 10^4 Gauss are equal to 1 Weber/m^2: the unit 1 Weber/m^2 is also called a Tesla. The electric field is measured in stat-Volts/cm where 1 stat-Volt is equal to 299.8 Volts; (yes, these are the same significant figures as occur in the speed of light!). An electric field of 1 stat-Volt/cm (sometimes stated as 1 esu/cm) is approximately equal to 30,000 Volts/m.

If auxillary vector fields $\vec{D}$ and $\vec{H}$ are introduced through the relations

$$\vec{\mathrm{D}}=\vec{\mathrm{E}}+4\pi\vec{\mathrm{P}},$$

and

$$\vec{\mathrm{B}}=\vec{\mathrm{H}}+4\pi\vec{\mathrm{M}},$$

then equations (1.6.3 and 1.6.4) become

$$\operatorname{curl}(\vec{\mathrm{H}})=\frac{4\pi}{c}\vec{\mathrm{J}}_f+\frac{1}{c}\frac{\partial\vec{\mathrm{D}}}{\partial t}, \tag{1.6.5}$$

$$\operatorname{div}(\vec{\mathrm{D}})=4\pi\rho_f \tag{1.6.6}$$

The first two equations, Equations (1.6.1, 1.6.2), remain the same. The vector $\vec{D}$ has the same units as $\vec{E}$, and the vector $\vec{H}$ has the same units as $\vec{B}$, although for historical reasons the units of $\vec{H}$ are called Oersteds.

The relation between charge density and current density in the MKS and the CGS systems can be deduced from the ratio of the proton charge as measured in both sets of units. This ratio is

$$\frac{e_p|_{esu}}{e_p|_{MKS}}=2.9979\times 10^9.$$

It follows from this ratio that 2998 esu/cm^3 is equal to 1 Coulomb/m^3 . Similarly, a current density of 1 Amp`ere/m^2 is equal to 2.998×10^5 esu/cm^2 . The conversion from MKS to CGS magnetic units is easy to remember since the earth's magnetic field is approximately 1 Oersted which is equal to 10^{-4} Tesla (Webers/m^2).

CHAPTER OVERVIEW

2: Electrostatic Field I

The Calculation of the Electrostatic Field Given a Time-independent Source Distribution.

Thumbnail: Field of a positive point charge influenced by a neutral conducting metal sphere. (CC BY-SA 3.0; Geek3 via Wikipedia)

2.1: Introduction

The electrostatic limit is the ideal case in which nothing changes with time. All source distributions are stationary, ie $\frac{\partial}{\partial t}$ is zero. Therefore Maxwell's equations reduce to

$$\operatorname{curl}(\vec{E}) = 0 \tag{2.1.1}$$

$$\operatorname{div}(\vec{B}) = 0 \tag{2.1.2}$$

$$\operatorname{curl}(\vec{B}) = \mu_0\left(\vec{J}_f + \operatorname{curl}(\vec{M})\right) \tag{2.1.3}$$

$$\operatorname{div}(\vec{E}) = \frac{1}{\epsilon_0}\left(\rho_f - \operatorname{div}(\vec{P})\right) \tag{2.1.4}$$

Notice that the magnetic field has become totally uncoupled from the electric field. As far as the static electric field is concerned the magnetic properties of matter are irrelevant. The calculation of the static magnetic field from its sources will be the subject of Chapter(4).

Notice that $-\operatorname{div}(\vec{P})$ is a source of the electrostatic field that is on an equal footing with the free charge density, ρ_f . The electrostatic electric field can be calculated for a given source distribution using the principle of superposition. For example, suppose that

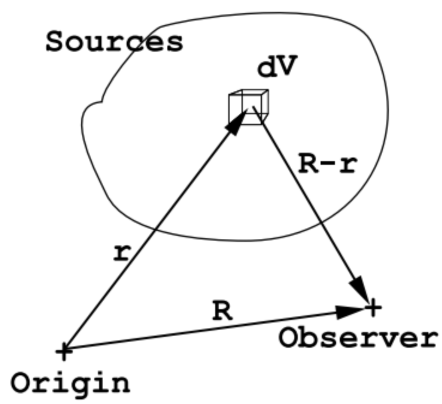


Figure 2.1.1: Given a distribution of sources the electric field at the position of the observer, (X,Y,Z), can be calculated as the sum of the electric fields generated by dividing the source distribution into small volume elements dV=dxdydz and treating the charges or dipole moments in each volume element as a point charge or as a point dipole moment.

one is given a charge density distribution ρ(x,y,z), and let the observer be located at (X,Y,Z). The total charge contained within a volume element dV located at (x,y,z) is given by dQ = ρ(x,y,z)dxdydz. If dV is taken to be sufficiently small the charge dQ can be treated like a point charge. It will generate an electric field contribution at the position of the observer that is given by Coulomb's law:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} dQ \frac{(\vec{R} - \vec{r})}{|\vec{R} - \vec{r}|^3},$$

where $\vec{R}$ is the vector that specifies the position of the observer and $\vec{r}$ is the vector that specifies the location of the volume element, dV (see Figure (2.1.1). The total electric field at the position of the observer can be calculated as the vector sum of the electric field contributions from all volume elements:

$$\vec{E}(X,Y,Z) = \frac{1}{4\pi\epsilon_0}\int\int\int_{AllSpace} dxdydz\rho(x,y,z)\frac{(\vec{R} - \vec{r})}{|\vec{R} - \vec{r}|^3}. \tag{2.1.5}$$

It is very seldom that the above integral can be carried out analytically. In all but a few special cases the integral must be calculated using a computer and small but finite volume elements. Equation (2.1.5) is valid even when the point of observation is located within the charge distribution so that the distance $|\vec{R} - \vec{r}|$ goes to zero for a volume element located at the position of the observer.

Why Equation (2.1.5) still works is not obvious but can be understood using the following argument. Surround the point of observation by a small sphere whose radius, R_0, is finite but is so small that the spatial variation of the charge density within the sphere can be neglected. The electric field at the observer due to charges outside the sphere of radius R_0 can be carried out without problems created by a vanishing denominator in Equation (2.1.5).

To the electric field generated by charges outside the sphere one must add the electric field generated at the center of the sphere by the charges inside the radius R_0. However, the electric field at the center of a uniformly charged sphere vanishes by symmetry, see Figure (2.1.2). For every element of charge at (x,y,z) there is a second equal element of charge at (-x,-y,-z) whose field is equal in magnitude but opposite in direction to the field of the first charge element. Thus the fields generated by these symmetry related charge pairs cancel.

2.1.1 Dipole Moment Density as a Source for the Electric Field.

A point electric dipole generates an electric field according to Equation (1.2.10). This point dipole formula can be used to calculate the electric field at some point in space, (X,Y,Z), generated by a distribution of dipole density $\vec{P}$(x,y,z). The idea is to divide up the source distribution into small volume elements, dV, and then to use the principle of superposition to obtain the electric field as the vector sum of the fields produced by the dipole moments $\vec{P}$(x,y,z)dV treated as point dipoles. The electric field at the position $\vec{R}$(X,Y,Z), the position of the observer- see Figure (2.1.1), can be written

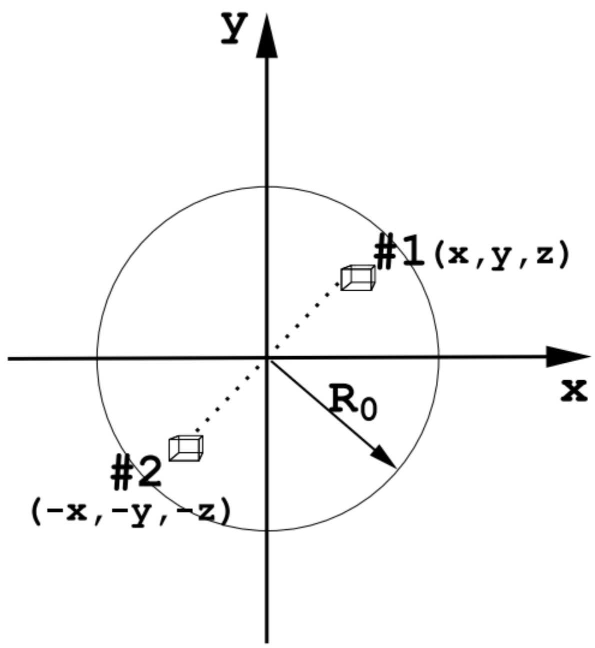


Figure 2.1.2: A sphere of radius R_0 filled with a uniform charge density ρ_0. The electric field at the center of the sphere is zero because the field generated by element number 1 at (x,y,z) is cancelled by the field equal in magnitude but opposite in direction generated by the equal volume element number 2 at (-x,-y,-z). The net field generated by all such symmetry related pairs is zero.

$$\overrightarrow{\mathrm{E}}(X, Y, Z)=\frac{1}{4 \pi \epsilon_0} \int \int \int_{S_{pacc}} d x d y d z\left(\frac{3[\overrightarrow{\mathrm{P}}(x, y, z) \cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})](\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^5}-\frac{\vec{P}(x, y, z)}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}\right) . \tag{2.1.6}$$

This complex formula can be seldom evaluated exactly. Usually it must be evaluated approximately by means of a computer. Eqn. (2.1.6) is valid for points of observation both inside and outside the electric dipole density distribution. If the observation point lies inside the dipole density distribution one must surround it by a small sphere of radius R0 and carry out the summations implied by Equation (2.1.6) for the space outside the sphere. This is required in order to avoid the divergence obtained when $\vec{R}=\vec{r}$. The radius R_0 must be chosen so small that variations of the dipole density, $\vec{P}$, within the sphere can be neglected. After having calculated the contribution to the electric field generated by the dipole density distribution from points outside the sphere, one must add an electric field contribution from the dipoles inside the sphere. It is not clear at this point how to calculate this contribution, but later it will be shown that the electric field at the center of a uniformly polarized sphere, polarization density $\vec{P}_0$, is given by

$$\overrightarrow{\mathrm{E}}_0 = -\frac{\overrightarrow{\mathrm{P}}_0}{3\epsilon_0}.$$

This field $\vec{E}_0$ must be added to the electric field generated by the dipole sources outside the sphere of radius R_0 in order to obtain the total electric field strength at the position of the observer.

The procedure outlined above is very complicated due to the complex form of the electric field generated by a point dipole. A second, simpler approach, is suggested by the Maxwell Equation (2.1.4). Namely, one can use Equation (2.1.5) with the charge density given by

$$\rho(x, y, z) = -\operatorname{div} \overrightarrow{\mathrm{P}}(x, y, z). \tag{2.1.7}$$

It is clear from Equation (2.1.7) that a spatially uniform dipole moment density distribution does not generate an electric field. However, one must be careful: any dipole distribution confined to a finite region of space must vary rapidly at its surfaces. This rapid variation of the dipole density produces an effective charge density distribution that may become very large and is localized near those surfaces. These surface charge density distributions must be taken into account when calculating the electric field.

2.2: The Scalar Potential Function

The direct calculation of the electric field using Coulomb's law as in Equation (2.1.5) is usually inconvenient because of the vector character of the electric field: Equation (2.1.5) is actually three equations, one for each electric field component $\vec{E}_x$, $\vec{E}_y$, and $\vec{E}_z$. It turns out that the electrostatic field can be obtained from a single scalar function, V(x,y,z), called the **potential function**. Usually it is easier to calculate the potential function than it is to calculate the electric field directly. The field $\vec{E}$ can be obtained from the potential function by differentiation:

$$\overrightarrow{\mathrm{E}}(x, y, z)=-\operatorname{grad} V(x, y, z). \tag{2.2.1}$$

That is in cartesian co-ordinates

$$E_x=-\frac{\partial V}{\partial x},$$
$$E_y=-\frac{\partial V}{\partial y},$$
$$E_z=-\frac{\partial V}{\partial z}.$$

According to the Maxwell Equation (2.1.1) the curl($\vec{E}$) must be zero for the electro-static field. This equation is automatically satisfied by Equation (2.2.1) because of the mathematical theorem that states that the curl of any gradient function is zero, see section (1.3.1). The units of the potential function are Volts. Absolute potential has no meaning. One can add or subtract a constant potential from the potential function without changing the electric field; the electric field is the physically meaningful quantity. Since the electric field satisfies the law of superposition it follows that the potential function must also satisfy superposition. This means that the potential function at any point due to a collection of charges must simply be the sum of the potentials generated at that point by each charge acting as if it were alone. One of the virtues of using a potential function is that scalar quantities are easier to add than are vector quantities because one has only to deal with one number at each point in space rather than the three numbers which specify a vector (the three components). Of course, to obtain the electric field from the potential function at some point in space it is necessary to know the potential at that point plus the value of the potential at nearby points in order to be able to calculate the derivatives in grad(V).

The electric field at the point $\vec{R}$, whose co-ordinates are (X,Y,Z), due to a point charge q at $\vec{r}$, whose co-ordinates are (x,y,z), can be calculated from the potential function

$$V(\overrightarrow{\mathrm{R}})=\frac{q}{4 \pi \epsilon_0} \frac{1}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}, \tag{2.2.2}$$

or

$$V(X, Y, Z)=\frac{q}{4 \pi \epsilon_0} \frac{1}{\left[(X-x)^2+(Y-y)^2+(Z-z)^2\right]^{1/2}}.$$

That this is an appropriate potential function can be verified by direct differentiation using

$$E_x=-\frac{\partial V}{\partial X},$$
$$E_y=-\frac{\partial V}{\partial Y},$$

and

$$E_z=-\frac{\partial V}{\partial Z}.$$

These electric field components can be compared with Coulomb's law, Equation (1.1.3).

The potential function Equation (2.2.2) can be used to construct the potential function for **any** charge distribution by using superposition. Consider an arbitrary, but finite, charge distribution, ρ($\vec{r}$), such as that illustrated in Figure (2.1.1). The charge distribution can be divided into a large number of very small volumes. A typical volume element, dV, is shown in the figure. The charge contained in the volume element dV is dq = $\rho(\vec{r})$dV Coulombs. The volume element is supposed to be so small that all the charge contained in it is located at the same distance from the point of observation at $\vec{R}$. The charges contained in dV may be treated like a point charge; they therefore contribute an amount to the total potential at P given by

$$dV_p = \frac{\rho(\overrightarrow{\mathrm{r}})dV}{4\pi\epsilon_0}\frac{1}{|\overrightarrow{\mathrm{R}} - \overrightarrow{\mathrm{r}}|} \quad \text{or}$$

$$dV_p = \frac{\rho(x,y,z)dxdydz}{4\pi\epsilon_0}\frac{1}{[(X-x)^2+(Y-y)^2]+(Z-z)^2]^{\ 1/2}}.$$

(Do not confuse the element of volume, dV, with the element of potential, dV_p.) The total potential as measured by the observer at (X,Y,Z) is obtained by summing the above expression over the entire charge distribution.

$$V_p(X,Y,Z) = \frac{1}{4\pi\epsilon_0}\int\int\int_{All\ Space}\frac{\rho(x,y,z)dxdydz}{[(X-x)^2+(Y-y)^2+(Z-z)^2]^{\ 1/2}}. \tag{2.2.3}$$

Of course, one need not use cartesian co-ordinates. In symbolic notation the above expression, Equation (2.2.3), can be written

$$V_p(\overrightarrow{\mathrm{R}}) = \frac{1}{4\pi\epsilon_0}\int_{Space}\frac{\rho(\overrightarrow{\mathrm{r}})dV}{|\overrightarrow{\mathrm{R}} - \overrightarrow{\mathrm{r}}|}. \tag{2.2.4}$$

This formula, Equation (2.2.4), works even when the point at which the potential is required is located within the charge distribution. It is not obvious that it should work; the proof is based upon Green's theorem (see Electromagnetic Theory by Julius Adams Stratton, McGraw-Hill, NY, 1941, section 3.3). It should also be noted that the total charge density distribution is made up partly of free charges, ρ_f , and partly of the effective charges due to a spatial variation of the dipole density, $\rho_b = -\mathrm{div}(\vec{P})$, where ρ_b is the so-called bound charge density: the total charge density is given by

$$\rho = \rho_f + \rho_b.$$

One can understand why the potential function remains finite even though the integrand in Equation (2.2.4) diverges in the limit as $\vec{r} \rightarrow \vec{R}$. Surround the point of observation at $\vec{R}$ by a small sphere of radius R_0. R_0 is taken to be so small that variations of the charge density inside the sphere may be neglected. The integral in Equation (2.2.4) remains finite at all points outside the sphere and therefore, in principle, the integral can be carried out without problems. Let the resulting contribution to the potential be V_0. Inside the sphere the charge density can be taken to be constant, $\rho(\vec{r}) = \rho_0$, and can therefore be removed from under the integral sign. The remaining integrand in Equation (2.2.4) is spherically symmetric and can be written in spherical polar co-ordinates for which dV = $4\pi r^2$dr. The contribution to the potential at the center of the sphere due to the charge contained within the sphere becomes

$$\Delta V = \frac{\rho_0}{4\pi\epsilon_0}\int_0^{R_0}\frac{4\pi r^2 dr}{r} = \frac{\rho_0 R_0^2}{\epsilon_0 2}.$$

Thus the total potential at the point of observation, $\vec{R}$, is finite and has the value $V(\overrightarrow{\mathrm{R}}) = V_0 + \frac{\rho_0 R_0^2}{2\epsilon_0}$.

Substitute the expression Equation (2.2.1) into the Maxwell Equation (2.1.4) to obtain

$$\mathrm{div}(\mathrm{grad}\, V) = \nabla^2 V = -\frac{1}{\epsilon_0}\left[\rho_f - \mathrm{div}(\vec{P})\right]. \tag{2.2.5}$$

Eqn.(2.2.5) is a differential equation for the potential function, V, given the charge density distribution. This differential equation has been much studied and is called Poisson's equation.

The divergence of a gradient is called the LaPlace operator, div(gradV) = ∇^2V . In cartesian co-ordinates one has

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

The form of the LaPlace operator should be committed to memory for the three major co-ordinate systems: (1) cartesian co-ordinates; (2) plane polar co-ordinates; (3) spherical polar co-ordinates. The LaPlace operator in each of these three systems will keep cropping up over and over again in this book.

2.2.1 The Particular Solution for the Potential Function given the Total Charge Distribution.

We have already written down the potential function which is generated by a given distribution of charge; Equation (2.2.4). From this equation it follows that the particular solution of the differential equation (2.2.5), Poisson's equation, is given by

$$V(\overrightarrow{\mathrm{R}})=\frac{1}{4\pi\epsilon_0}\int_{Space}\frac{\left[\rho_f(\vec{r})-\mathrm{div}(\overrightarrow{\mathrm{P}}(\overrightarrow{\mathrm{r}}))\right]dV}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}. \tag{2.2.6}$$

Eqn.(2.2.6) is called a particular solution of Poisson's equation (Equation (2.2.5)) because it is generated by a particular, local, distribution of charges. Notice that any solution of LaPlace's equation, $\nabla^2 V = 0$, can be added to (2.2.6) and Poisson's equation will still be satisfied: this freedom can be exploited to satisfy boundary conditions for problems that will be treated later.

2.2.2 The Potential Function for a Point Dipole.

As pointed out above, the potential function generated by an electric dipole distribution can be calculated from an effective charge density distribution $\rho_b = -\mathrm{div}(\vec{P})$. However this contribution to the potential function can also be calculated by direct summation of the potential function for a point dipole. The potential generated at a position located $\vec{r}$ from a point dipole, $\vec{p}$, is given by

$$V_{dip}=\frac{1}{4\pi\epsilon_0}\frac{(\overrightarrow{\mathrm{p}}\cdot\overrightarrow{\mathrm{r}})}{\mathrm{r}^3}. \tag{2.2.7}$$

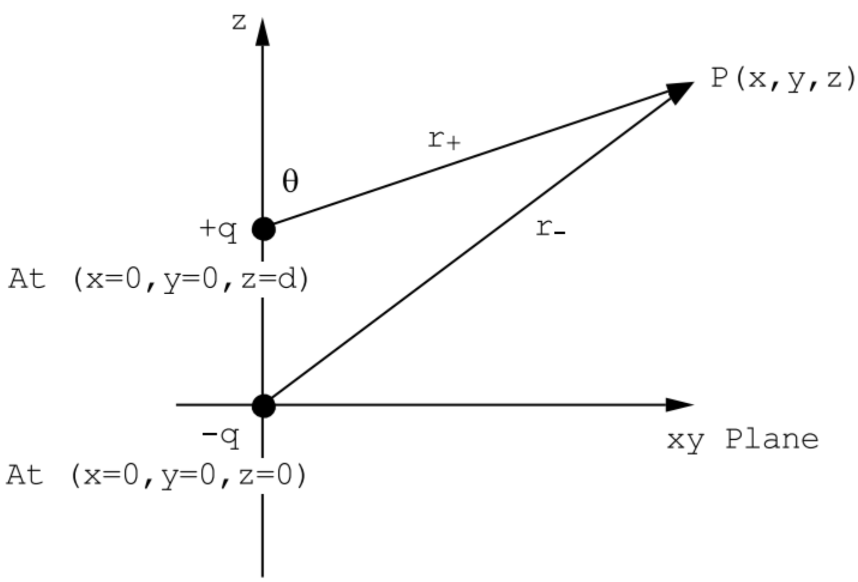


Figure 2.2.3: Model for calculating the potential function for a point dipole. The two charges are separated by the distance d.

This can be shown as follows (see Figure (2.2.3)):

$$\mathrm{r}_+=\left(x^2+y^2+(z-d)^2\right)^{1/2}=\left(x^2+y^2+z^2-2zd+d^2\right)^{1/2}=\mathrm{r}\left[1-\frac{2zd}{\mathrm{r}^2}+\frac{d^2}{\mathrm{r}^2}\right]^{1/2}.$$

Therefore

$$\frac{1}{\mathrm{r}_+}\cong\frac{1}{\mathrm{r}}\left[1+\frac{zd}{\mathrm{r}^2}\right]$$

to first order in the small distance d. Also $1/\mathrm{r}_- = 1/\mathrm{r}$ so that

$$V_{dip} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathrm{r}_+} - \frac{q}{4\pi\epsilon_0}\frac{1}{\mathrm{r}_-} \cong \frac{qzd}{4\pi\epsilon_0}\frac{1}{\mathrm{r}^3}.$$

But $\vec{p}$ = q$\vec{d}$ and $\frac{z}{\mathrm{r}} = \cos(\theta)$ so that V_{dip} is just given by Equation (2.2.7).

The point dipole potential, Equation (2.2.7), can be used to calculate the potential at the point of observation, $\vec{R}$, by superposition of contributions from small volume elements, dV, at $\vec{r}$, each of which acts like a point dipole $\vec{p}$ = $\vec{P}$dV . The result is

$$V_{dip}(\overrightarrow{\mathrm{R}}) = \frac{1}{4\pi\epsilon_0}\int_{Space} dV \frac{\overrightarrow{\mathrm{P}} \cdot (\overrightarrow{\mathrm{R}} - \overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}} - \overrightarrow{\mathrm{r}}|^3}. \tag{2.2.8}$$

Formula (2.2.8) gives the same value for the potential function as does Equation (2.2.6) in which the free charge density, ρ_f , has been set equal to zero. These two ways of calculating the potential due to a distribution of dipoles can be shown to be mathematically equivalent, see Appendix (2A).

2.3: General Theorems

A number of rules, or theorems, can be deduced from Maxwell's equations (2.1.1) and (2.1.4).

2.3.1 Application of Gauss' Theorem

From Maxwell's equations one has

$$\operatorname{div}(\overrightarrow{\mathrm{E}})=\frac{\rho_t}{\epsilon_0}$$

where $\rho_t = \rho_f + \rho_b$, and $\rho_b = -\operatorname{div}(\vec{P})$. Integrate Equation (2.1.4) over any closed volume V:

$$\int\int\int_V dV div(\overrightarrow{\mathrm{E}})=\frac{1}{\epsilon_0}\int\int\int_V dV \rho_t=\frac{Q_t}{\epsilon_0}.$$

But from Gauss' Theorem, section(1.3.3)

$$\int\int\int_V dV div(\vec{E})=\int\int\int_S dS(\vec{E}\cdot\hat{\mathbf{n}}),$$

where S is the surface bounding the volume V, and $\hat{\mathbf{n}}$ is a unit vector normal to the element of surface area, dS, and directed from inside the volume to the outside. Thus the total charge $Q_t=\int\int\int_V dV\rho_t$ contained within the volume V can be calculated from a knowledge of the electric field everywhere on the surface S bounding the volume V:

$$Q_t=\epsilon_0\int\int_S dS(\overrightarrow{\mathrm{E}}\cdot\hat{\mathbf{n}}). \tag{2.3.1}$$

It is often useful to rewrite Equation (2.1.4) in terms of the displacement vector $\overrightarrow{\mathrm{D}}=\epsilon_0\overrightarrow{\mathrm{E}}+\overrightarrow{\mathrm{P}}$. Notice that $\vec{D}$ and $\vec{P}$ have the same units, Coulombs/m^2 , and these units are different from the electric field units of Volts/m. Using the above definition of $\vec{D}$ the fourth Maxwell equation becomes

$$\operatorname{div}(\overrightarrow{\mathrm{D}})=\rho_f, \tag{2.3.2}$$

where ρ_f is the density of free charges. Integrate Equation (2.3.2) over a volume V and apply Gauss' Theorem to obtain

$$\int\int\int_V dV div(\overrightarrow{\mathrm{D}})=\int\int_S dS(\overrightarrow{\mathrm{D}}\cdot\hat{\mathbf{n}})=\int\int\int_V dV\rho_f.$$

It follows from this that the total free charge within a volume V can be calculated from a knowledge of the displacement vector, $\vec{D}$, over the surface S bounding the volume V:

$$Q_f=\iint_S dS(\overrightarrow{\mathrm{D}}\cdot\hat{\mathbf{n}}). \tag{2.3.3}$$

2.3.2 Boundary Condition on $\vec{D}$.

Gauss' theorem in the form of Equation (2.3.3) can be used to show that the normal component of the displacement vector, $\vec{D}$, must be continuous at the boundary between two different materials if that boundary contains no **free surface charges**. Refer to Figure (2.3.4). Let SS be the surface that separates region (1) from region (2). Apply Equation (2.3.3) to a pill-box that straddles the bounding surface SS. The surface area of the pill-box is ΔS and it is dL thick: the thickness dL will be taken to be small compared with the lateral dimensions of the pill-box, $\sim\sqrt{\Delta S}$. The contribution to the surface integral in (2.3.3) from the sides of the pill-box will be negligible because (1) its area will be very small since dL is relatively small, and (2) the components of $\overrightarrow{\mathrm{D}}_1$, $\overrightarrow{\mathrm{D}}_2$ parallel with the surface SS, the tangential components, will be very nearly constant over the dimensions of the pill-box and so as the outward

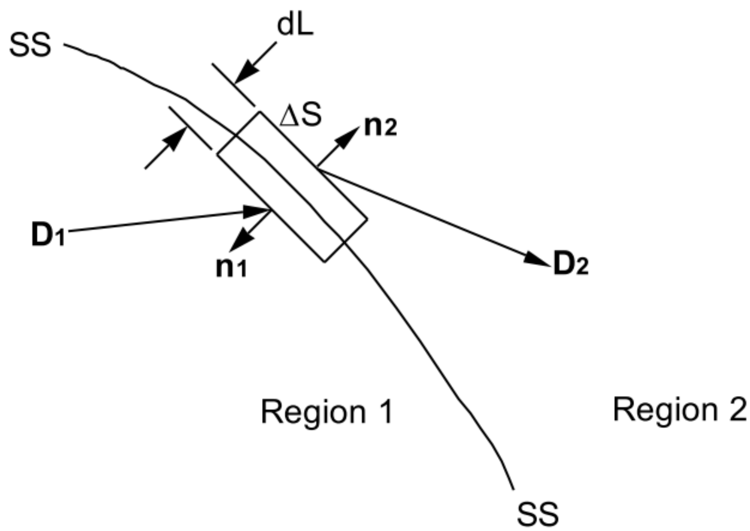


Figure 2.3.4: Application of Gauss' Theorem to a pill-box straddling a surface of discontinuity between two materials.

normal changes direction by 360 degrees around the pill-box perimeter the positive and negative contributions to the surface integral will cancel. Thus in the limit as dL → 0 and ΔA → 0 the surface integral taken over the pill-box surface will be given by

$$\int\int_{\text{Pill-box}} dS(\overrightarrow{\mathrm{D}}\cdot\hat{\mathbf{n}}) = \left(\overrightarrow{\mathrm{D}}_2\cdot\hat{\mathbf{n}}_2\right)\Delta A + \left(\overrightarrow{\mathrm{D}}_1\cdot\hat{\mathbf{n}}_1\right)\Delta A.$$

But $\overrightarrow{\mathrm{D}}_2\cdot\hat{\mathbf{n}}_2 = +D_{2n}$ and $\overrightarrow{\mathrm{D}}_1\cdot\hat{\mathbf{n}}_1 = -D_{n1}$ where D_{2n}, D_{1n} are the components of $\overrightarrow{\mathrm{D}}_2$, $\overrightarrow{\mathrm{D}}_1$ normal to the bounding surface SS. It follows from Gauss' theorem, Equation (2.3.3), that if there are no free charge densities in the two materials then the total charge contained in the pill-box is zero and therefore $D_{2n}-D_{1n} = 0$, so that the normal component of D must be continuous across the bounding surface SS. This result remains valid even if the volume density of free charges is not zero because the total charge contained in the pill-box of Figure (2.3.4) goes to zero as the pill-box volume goes to zero with the pill-box thickness, dL. The normal component of $\vec{D}$ can only be discontinuous if the surface SS carries a surface charge density. If the bounding surface SS carries a surface charge density of σ_f Coulombs/m^2 the total charge contained within the pill-box of Figure (2.3.4) is $\sigma_f\Delta A$ Coulombs, and Equation (2.3.3) gives

$$D_{2n}\Delta A - D_{1n}\Delta A = \sigma_f\Delta A$$

since the charge contained within the pill-box, $\sigma_f\Delta A$, is independent of dL, and does not vanish as dL → 0. It follows that any discontinuity in the normal component of the displacement vector D is an indication and a measure of the presence of a surface charge density:

$$\sigma_f = D_{2n} - D_{1n}. \tag{2.3.4}$$

2.3.3 Discontinuity in the Normal Component of the Polarization Vector.

Gauss' theorem can be used to show that a discontinuity in the normal component of the electrical polarization vector, $\vec{P}$, produces a surface density of bound charges, σ_b. Consider a surface that separates regions having different material properties such as that shown in Figure (2.3.4), and in particular two regions having different polarization densities $\vec{P}_1$ and $\vec{P}_2$. Let there be no free charge distributions, and let there be no free charges on the surface of discontinuity, SS. For this situation Equation (2.1.4) becomes

$$\mathrm{div}(\overrightarrow{\mathrm{E}}) = -\frac{1}{\epsilon_0}\mathrm{div}(\overrightarrow{\mathrm{P}}) = \frac{\rho_b}{\epsilon_0}.$$

Apply Gauss' theorem to this equation for a pill-box straddling the boundary SS such as that illustrated in Figure (2.3.4). In the limit as dL → 0 and ΔA → 0 the surface integral over the pill-box of the electric field gives

$$\int\int_{Pill-box} dS(\overrightarrow{\mathrm{E}}\cdot\hat{\mathbf{n}}) = \left(\overrightarrow{\mathrm{E}}_2\cdot\hat{\mathbf{n}}_2\right)\Delta A + \left(\overrightarrow{\mathrm{E}}_1\cdot\hat{\mathbf{n}}_1\right)\Delta A.$$

But $\overrightarrow{\mathrm{E}}_2 \cdot \hat{\mathbf{n}}_2 = E_{2n}$, and $\overrightarrow{\mathrm{E}}_1 \cdot \hat{\mathbf{n}}_1 = -E_{1n}$ where E_{2n} and E_{1n} are the components of the electric field normal to the bounding surface SS. Thus

$$\int\int\int_{\text{Pill-box}} dV div(\overrightarrow{\mathrm{E}}) = \frac{1}{\epsilon_0}\int\int\int_{\text{Pill-box}} dV \rho_b = \frac{Q_b}{\epsilon_0}$$

and Gauss' Theorem gives

$$\int\int_{Pill-box} dS(\overrightarrow{\mathrm{E}} \cdot \hat{\mathbf{n}}) = (E_{2n} - E_{1n})\,\Delta A = \frac{Q_b}{\epsilon_0}.$$

Q_b is the total bound charge contained in the pill-box. As the thickness of the pill-box shrinks to zero the only bound charges left in the pill-box will be due to surface bound charges, σ_b, and therefore $Q_b = \sigma_b \Delta A$. It follows that

$$\sigma_b = \epsilon_0\left(E_{2n} - E_{1n}\right). \tag{2.3.5}$$

This equation (2.3.5) is the analog of Equation (2.3.4) and the derivations of these two equations are similar. In the present case it has been assumed that there are no free surface charges on the interface surface SS so that from Equation (2.3.4) one has $(D_{2n} - D_{1n}) = 0$ and therefore from the definition of $\vec{D}$

$$D_{2n} - D_{1n} = (\epsilon_0 E_{2n} + P_{2n}) - (\epsilon_0 E_{1n} + P_{1n}) = 0.$$

Using Equation (2.3.5) gives

$$\sigma_b = -\left(P_{2n} - P_{1n}\right). \tag{2.3.6}$$

Any discontinuity in the normal component of the Polarization vector generates a surface density of bound charges. These bound charges generate electric fields and must be explicitly taken into account when the electric field is calculated from its sources using Equation (2.1.5), (the direct application of Coulomb's law), or when Equation (2.2.6) is used to calculate the potential function for a distribution of free and bound charges.

2.4: The Tangential Components of E

It follows from the first Maxwell equation, Equation (2.1.1) curl($\vec{E}$) = 0, that **the tangential components of the electric field vector must be continuous across any surface**. Consider a loop dL long and dw wide that spans a surface SS: the loop has one side in region (1) and the other side in region (2) as shown in Figure (2.4.5); the sides dw are chosen to be perpendicular to the surface SS. E_{t1} and E_{t2} are the electric field components parallel with the surface SS - the tangential electric field components. From Stokes' theorem, Section (1.3.4), one has

$$\iint_{Loop} dS(\hat{\mathbf{n}}\cdot\text{curl}(\overrightarrow{\mathrm{E}})) = \oint_{Loop} \overrightarrow{\mathrm{E}}\cdot d\overrightarrow{\mathrm{L}}.$$

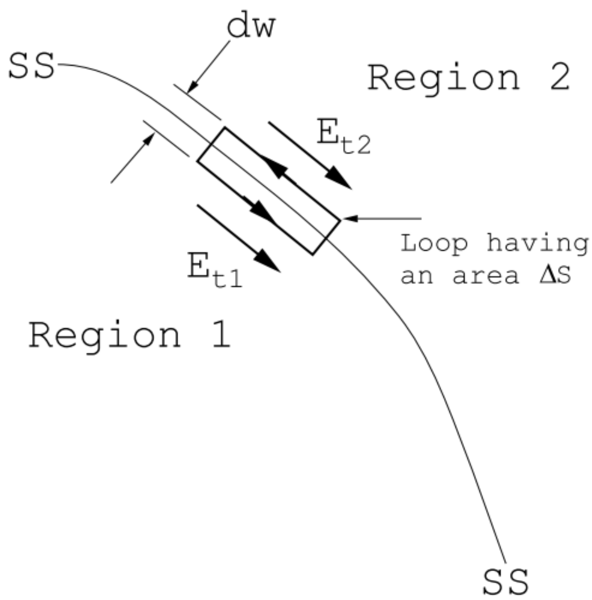


Figure 2.4.5: A rectangular loop having sides dL long and dw wide used for the application of Stokes' Theorem.

But curl($\vec{E}$) = 0, therefore the line integral must vanish:

$$\oint_{\text{Loop}} \overrightarrow{\mathrm{E}}\cdot\overrightarrow{\mathrm{d}} = 0.$$

In calculating the line integral one can take the limit as dw becomes very small so that contributions from the electric field components parallel with dw and therefore normal to the surface can be made negligibly small. In this limit the line integral becomes

$$\oint_{Loop} \overrightarrow{\mathrm{E}}\cdot d\overrightarrow{\mathrm{L}} = E_{t1}dL - E_{t2}dL.$$

The negative sign arises because in Region(2) the loop is traversed in the direction opposite to the direction of E_{t2}. It follows from the fact that the line integral must vanish that

$$E_{t2} = E_{t1}, \tag{2.4.1}$$

or in other words the tangential components of $\vec{E}$ must be continuous across the surface SS. Since SS is an arbitrary surface it follows that the tangential components of the electric field must be continuous across **any surface**.

2.5: A Conducting Body

The electrostatic field must be zero inside a conducting body. A non-zero field would act on mobile charges in the body and so produce currents that would cause the charge distribution to change with time. Any time variation of the field sources must generate time-varying fields in contradiction with the assumption of the electrostatic limit in which nothing changes with time. Since the electrostatic field is zero everywhere inside a conducting body, it follows from Equation (2.4.1) that the electric field just outside a conducting body can have no components parallel with the surface. The electric field just outside a conducting body must be **normal** to the surface of that body. Finally, it follows from an application of Gauss' Theorem to a pill-box spanning the surface of the conducting body that the electric field just outside that conducting body is given by

$$E_n = |\overrightarrow{\mathrm{E}}| = \frac{\sigma_t}{\epsilon_0} = \frac{1}{\epsilon_0}(\sigma_f + \sigma_b)\,, \tag{2.5.1}$$

where σ_f is a **free surface charge density** on the conducting body, and $\sigma_b = -P_n$ is a bound surface charge density due to a discontinuity in the normal component of $\vec{P}$ if the conductor is in contact with a dielectric material.

2.6: Continuity of the Potential Function

The potential function must be continuous at any point in space (with the exception noted below) since a discontinuous jump in its value would correspond to an unphysical infinite electric field strength:

$$E_r = -\frac{dV}{dr},$$

where E_r is the component of the electric field along the direction specified by dr. The exception referred to above occurs at a layer of dipoles; see the example problem discussed below. Let a surface carry a density of dipoles $\vec{P}_d$ per unit area (dimensions of Coulombs/m) oriented such that the dipole density is perpendicular to the plane. Such an electrical dipole layer, or **electrical double layer**, generates no external electric field, but it does generate a jump in potential given by

$$\Delta V = \left|\overrightarrow{\mathrm{P}}_d\right| / \epsilon_0. \tag{2.6.1}$$

Electrical double layers are common in nature. The potential difference that is observed to exist between the fluid inside a living cell and the surrounding fluid is maintained by an electrical double layer on the cell membrane. A double layer is also formed whenever a metal electrode is placed in an electrolytic solution. The potential difference across the double layer is called the **electrode potential**. The potential difference that is observed at the electrodes of a battery is the difference between the electrode potentials of two dissimilar conductors immersed in an electrolyte.

2.7: Example Problems

2.7.1 Plane Symmetry.

(1) A Uniformly Charged Plane.

Consider a plane which is infinite in extent and uniformly charged with a density of σ Coulombs/m^2 ; the normal to the plane lies in the z-direction, Figure (2.7.6). The charge plane is located at z=0. It is clear from symmetry that the electric field can have only a component normal to the plane since for every charge element at x,y there is an exactly similar charge element at -x,-y such that the transverse field components at the point of observation cancel. All of the charges on a ring of radius r produce the same z-component of electric field at the point of observation, P; they may therefore be lumped together to obtain

$$dE_p\big|_z = \frac{dq}{4\pi\epsilon_0}\frac{\cos(\theta)}{R^2} = \frac{\sigma 2\pi r dr}{4\pi\epsilon_0}\frac{z}{R^3},$$

therefore

$$E_z = \left(\frac{\sigma}{2\epsilon_0}\right) z \int_0^\infty \frac{rdr}{\left(z^2+r^2\right)^{3/2}}.$$

Upon carrying out the integration one obtains an electric field that is independent of z and, for positive z, has the value $\sigma/2\epsilon_0$:

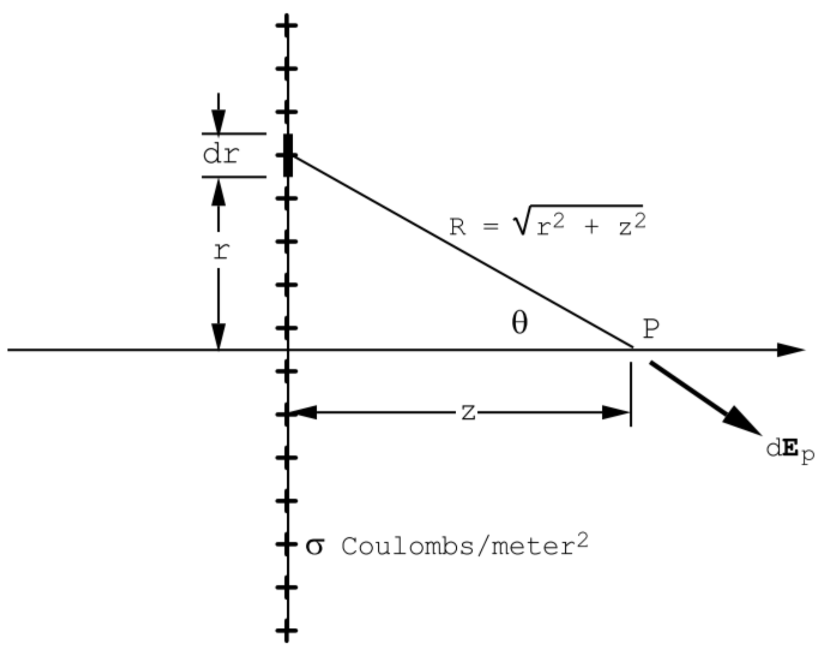


Figure 2.7.6: Calculation of the electric field generated by a uniformly charged plane.

$$E_z = \frac{\sigma}{2\epsilon_0}. \tag{2.7.1}$$

Note that for points to the left of the charge plane the electric field points along -z: ie. $E_z = -\sigma/2\epsilon_0$. There is a discontinuity of magnitude σ/ϵ_0 in the z-component of the electric field at the charge plane.

(2) The Potential Function for a Uniformly Charged Plane.

It is interesting to calculate the potential function from a direct application of Equation (2.2.4) for the potential generated by a charge distribution. Referring to Figure (2.7.6) one can calculate the contribution to the potential at the observation point P:

$$dV_p = \frac{dq}{4\pi\epsilon_0}\frac{1}{R} = \frac{\sigma 2\pi r dr}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+z^2}}$$

and therefore for z>0

$$V_p = \frac{\sigma}{2\epsilon_0}\int_0^\infty \frac{rdr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\epsilon_0}[\sqrt{\infty} - z].$$

This is an example of an integration that does not converge. In order to obtain a proper integral one can use a trick. Suppose that the charged plane was not infinite in extent, but that it was in the form of a large, but finite, disc having a radius D. There would then be no problem; one would have

$$V_p = \frac{D\sigma}{2\epsilon_0} - \frac{z\sigma}{2\epsilon_0} = \text{constant} - \frac{z\sigma}{2\epsilon_0}.$$

One is always free to subtract a constant from the potential without changing the value of the corresponding electric field distribution. One can simply subtract the constant from the above equation to obtain

$$V_p = -\frac{z\sigma}{2\epsilon_0}, \quad \text{for } z > 0.$$

Notice that V_p does not change sign for z less than zero because the integration involves the square root of z^2 . Thus the complete expression for the potential function, valid for all values of z, is given by

$$V_p = -\frac{\sigma}{2\epsilon_0}|z|, \tag{2.7.2}$$

see Figure (2.7.7). In Equation (2.7.2) the zero for the potential function has been chosen so that the potential is zero on the plane. The potential function is continuous as the field point P moves through the plane from left to right in Figure (2.7.7): the electric field component normal to the plane undergoes a discontinuity.

(3) The Field of a Uniformly Charged Plane Using Gauss' Theorem.

We could have deduced that $\vec{E}$ is independent of position, except at the charge plane, directly from the fourth Maxwell equation $\text{div}(\overrightarrow{\text{E}}) = (1/\epsilon_0)\left(\rho_f - \text{div}(\overrightarrow{\text{P}})\right)$. In this application there is no electric dipole density since $\vec{P}$ = 0 everywhere. Consequently, also $\text{div}(\overrightarrow{\text{P}}) = 0$ everywhere.

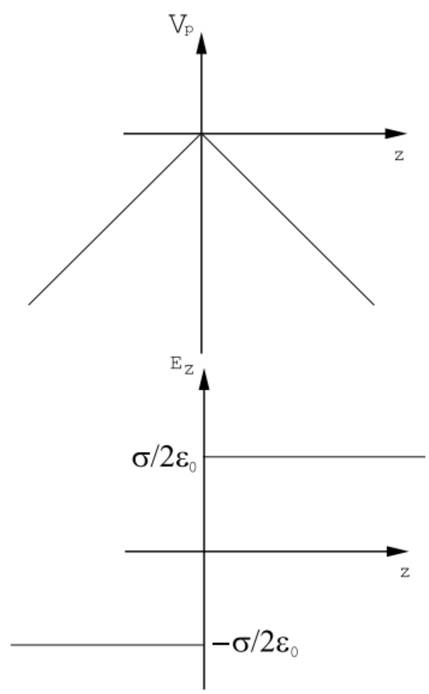


Figure 2.7.7: The potential function (top) and the electric field (bottom) generated by a uniformly charged infinite plane carrying a charge density of σ Coulombs/m^2. $E_z = -\frac{\partial V_p}{\partial z}$.

As we have seen in (1) above using symmetry arguments $\vec{E}$ has only a z-component; moreover, this component, E_z, cannot depend upon x,y if the charge distribution is uniform and infinite in extent: any shift of the infinite charge distribution in the x-y plane can not be detected by a fixed observer. Therefore, in this case,

$$\operatorname{div}(\overrightarrow{\mathrm{E}}) = \frac{\partial E_z}{\partial z} = 0$$

except at z=0 since the charge density is zero everywhere except at z=0. Thus E_z = constant everywhere except at z=0.

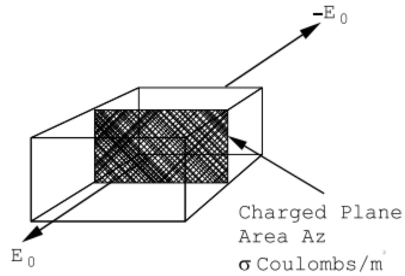


Figure 2.7.8: Geometry for the application of Gauss' Theorem to calculate the electric field strength generated by an infinite, plane, uniformly charged sheet whose density is σ Coulombs/m^2 . The magnitude of the resulting field is $E_0 = \sigma/(2\epsilon_0)$.

Moreover, it is obvious from Coulomb's law and symmetry that the magnitude of the electric field for z < 0 must have the same value as the magnitude of the electric field for z > 0, although the direction of the field switches 180 degrees upon passage through the charge plane. For this very symmetric case one can obtain the magnitude of the electric field directly from Gauss' Theorem.

$$\int\int\int_V dV div(\overrightarrow{\mathrm{E}}) = \int\int_S dS(\overrightarrow{\mathrm{E}} \cdot \hat{\mathbf{n}}) = \frac{Q}{\epsilon_0},$$

where V is the volume bounded by the closed surface S and Q is the total charge contained in the volume. Apply this theorem to a parallelepiped having its edges oriented along the co-ordinate axes as shown in Figure (2.7.8). The box shown in the figure contains an amount of charge σA_z Coulombs since the area of the side of the box whose normal lies along z is just A_z. Now calculate the surface integral of the electric field over the surface of the

parallelepiped. This integral is easy to carry out because the electric field is constant in magnitude and in direction. Over four sides of the box shown in Figure (2.7.8) the direction of the electric field is parallel with the surface, thus perpendicular to the surface normals, and the scalar product of the electric field and the normal to the surface, $\hat{\mathbf{n}}$, vanishes so that these sides contribute nothing to the surface integral. Over the two ends of the box the electric field is parallel with the surface normal so that the scalar product $\vec{E}\cdot \hat{\mathbf{n}}$ just becomes an ordinary product, and the contribution to the surface integral from each end is E_0A_z: notice that at each end the electric field is parallel with the direction of the outward normal because the field reverses direction from one side of the charge sheet to the other. The surface integral in Gauss' theorem is given by

$$\int\int_S dS(\overrightarrow{\mathrm{E}} \cdot \hat{\mathbf{n}}) = 2E_0A_z.$$

But the total charge contained in the box is σA_z, so that the electric field must have the magnitude $E_0 = \sigma/(2\epsilon_0)$ in agreement with Equation (2.7.1).

(4) An Electric Double Layer.

Consider two oppositely charged uniform charge sheets separated by a distance of 2d meters, as illustrated in Figure (2.7.9). The electric field generated by each charge sheet is uniform, independent of z, and directed normal to the planes of charge. In the regions outside the charged planes,ie. z > d or z < d, the electric field is zero because the fields generated by the oppositely charged planes have opposite directions and therefore cancel. Between the two charged planes the fields due to the two planes have the same orientation and therefore add to produce the total field $E_z = -\sigma/\epsilon_0$. A plot of E_z vs z exhibits discontinuities at z= d, Figure (2.7.10). The potential function outside the double layer is constant corresponding to zero electric field. However, the potential on the right hand side of the double layer is different from the potential on the left hand side. The step in the potential is related to the strength of the double layer:

$$\Delta V = \frac{2\sigma d}{\epsilon_0} = P_d/\epsilon_0,$$

where P_d is the dipole moment per unit area in Coulombs/meter.

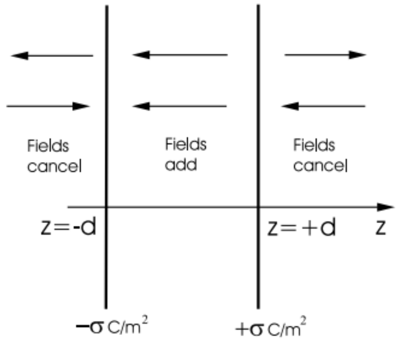


Figure 2.7.9: An electric double layer consists of two infinite plane sheets of charge densities +σ and −σ Coulombs/m^2 separated by a small distance. The electric field is zero everywhere outside the double layer, but is equal to $|E_z| = \sigma/\epsilon_0$ between the two charged sheets.

(5) A Uniformly Polarized Slab.

Consider a slab that is uniformly polarized along the z-axis as shown in Figure (2.7.11). The strength of the polarization density is P_0, and there are no free charges anywhere. One can define a bound charge density from the relation $\rho_b = -\mathrm{div}(\vec{P})$. This bound charge density generates an electric field just as surely as does the free charge density, ρ_f . Inside the slab of Figure (2.7.11) the polarization density is $\overrightarrow{\mathrm{P}} = \mathrm{P}_0\hat{\mathbf{u}}_z$; thus $\partial P_z/\partial z$ is zero inside the slab and therefore div($\vec{P}$), and hence ρ_b, is zero. Outside the slab $\vec{P}$ = 0 so that here also div($\vec{P}$) = 0 and therefore ρ_b = 0. One might be misled by the fact that the bound charge density vanishes both inside and outside the slab into thinking that the bound charge density is zero everywhere. **However, the bound charge density does not vanish on the slab surfaces.** The derivative $\partial P_z/\partial z$ is singular at z=0 and at z=d ie. on the faces of the slab. This singularity is integrable: if one integrates the derivative from z = $-\epsilon$ to

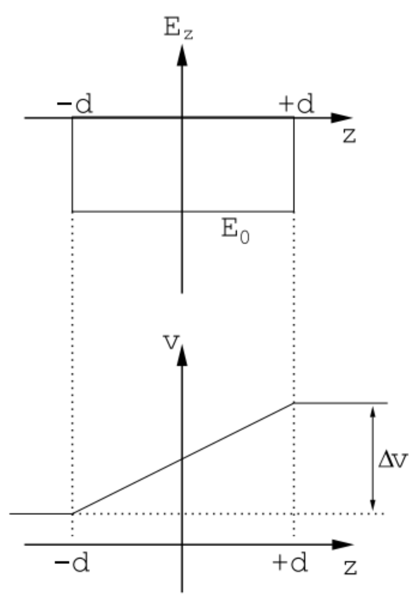


Figure 2.7.10: The electric field distribution and the corresponding potential function generated by an electrical double layer. The electric field intensity inside the double layer is $E_0 = -\sigma/\epsilon_0$. The jump in the potential across the double layer is $\Delta V = 2\sigma d/\epsilon_0$.

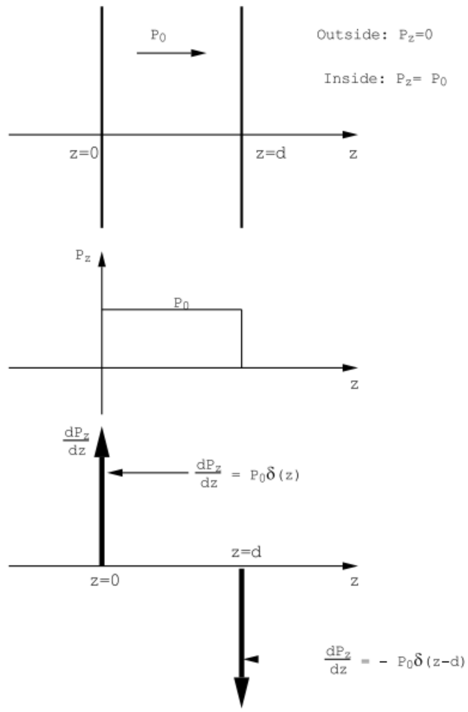


Figure 2.7.11: A uniformly polarized slab. The polarization density, $\vec{P}_0$, is directed along the normal to the slab. The discontinuities in the normal component of the polarization produce effective surface bound charge densities given by $\sigma_b = -\operatorname{div}(\overrightarrow{\mathrm{P}})$.

z = +ϵ, for example, the result is

$$\int_{-\epsilon}^{+\epsilon}\left(\frac{\partial \mathrm{P}_z}{\partial z}\right) dz = \mathrm{P}_z(+\epsilon) - \mathrm{P}_z(-\epsilon) = \mathrm{P}_0,$$

because $\mathrm{P}_z(-\epsilon) = 0$ and $\mathrm{P}_z(+\epsilon) = \mathrm{P}_0$. Notice that the value of the integral is independent of the small interval ϙ, and, in particular, it remains finite even in the limit as $\epsilon \rightarrow 0$. The integrand has the character of a Dirac δ − function: $\partial \mathrm{P}_z/\partial z = \mathrm{P}_0\delta(z)$, where the function δ(z) is a strongly peaked function having zero width but infinite amplitude in such a way that it's integral is just equal to unity. But this means that the charge density on the surface at z=0, $\rho_b = -\partial \mathrm{P}_z/\partial z$, z, is a very sharply peaked integrable function of z: it is in fact a surface charge density of strength $-\mathrm{P}_0$ Coulombs/meter^2 . Similarly, there will be a surface charge density of strength $+\mathrm{P}_0$ Coulombs/meter^2 on the surface at z=d. The electric field distribution produced by a uniformly polarized slab in which the polarization density lies parallel with the slab normal is exactly the same as that which would be produced by two uniformly charged planes carrying charge densities of $\sigma = \pm \mathrm{P}_0$, see Figure (2.7.9). Outside the uniformly polarized slab the electric field is zero. Inside the slab there is a uniform electric field, $\mathrm{E}_z = -\mathrm{P}_0/\epsilon_0$, whose direction is opposite to the direction of the polarization density; it is called a depolarizing field because it tends to act so as to reduce the polarization density. The potential outside the slab will be constant, both on the left and on the right, but the potential on the right will be larger than that on the left by the amount $\Delta V = \mathrm{P}_0 d/\epsilon_0$ Volts. It is interesting to examine the auxiliary field $\overrightarrow{\mathrm{D}} = \epsilon_0 \overrightarrow{\mathrm{E}} + \overrightarrow{\mathrm{P}}$. Outside the slab both $\vec{E}$ and $\vec{P}$ are zero so that $\vec{D}$ must also be zero. Inside the slab $\vec{D}$ has only a z-component, and

$$\mathrm{D}_z = \epsilon_0 \mathrm{E}_z + \mathrm{P}_z = -\mathrm{P}_0 + \mathrm{P}_0 \equiv 0.$$

The normal component of $\vec{D}$ is continuous across the slab surfaces. In general, in the absence of any surface free charge density the normal component of $\vec{D}$ must be continuous across the interface.

Now consider a uniformly polarized slab in which the polarization density vector lies in the plane of the slab as shown in Figure (2.7.12). In this case the polarization has only an x-component, and that x-component, P_x, is a function only of z. This means that $\operatorname{div}(\overrightarrow{\mathrm{P}}) = \partial \mathrm{P}_x/\partial x = 0$ everywhere. There is no bound charge density anywhere, and there is no free charge density, by supposition, so that there are no sources for the electric field. A uniformly

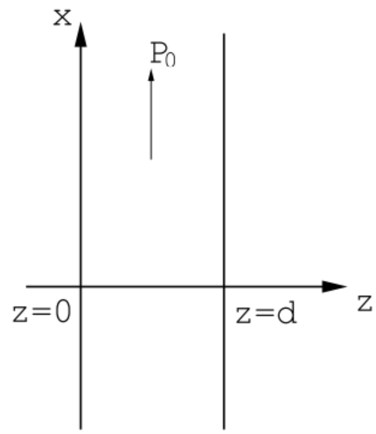


Figure 2.7.12: An infinite slab that is uniformly polarized in plane, $P_x = P_0$.

polarized slab in which the polarization lies in the slab plane generates **no macroscopic electric field**.

It is a general rule that a spatial variation of the polarization density, $\vec{P}$, produces an effective volume charge density $\rho_b = -\operatorname{div}(\overrightarrow{\mathrm{P}})$. In addition to this effective volume charge density, any discontinuity in the normal component of $\vec{P}$ across a surface produces an effective surface charge density given by

$$\sigma_b = -\left(\mathrm{P}_{2n} - \mathrm{P}_{1n}\right).$$

These bound volume and surface charge densities must be used, along with the free charge distributions, to calculate the electric field and potential distributions.

2.7.2 A Spherically Symmetric Charge Distribution.

Consider a charge distribution that is spherically symmetric but one in which the charge distribution ρ(r) may have an arbitrary dependence upon the co-ordinate r, see Figure (2.7.13). In this case the electric field can have only a radial component by symmetry. The magnitude of this radial component, E_r, cannot depend on position on the surface of a spherical surface centered on

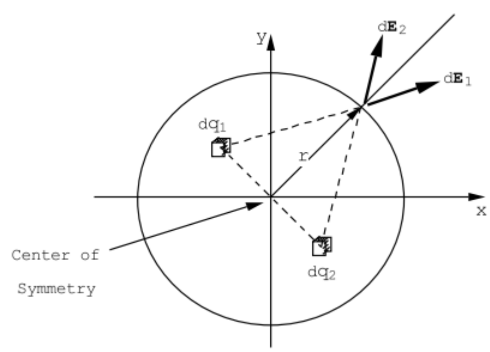


Figure 2.7.13: The electric field intensity generated by a spherically symmetric charge distribution. The electric field has only a radial component because the transverse components generated by two equivalent charges, d_{q1} and d_{q2}, cancel by symmetry.

the center of symmetry of the charges because any rotation of the distribution around the center of symmetry leaves the charge distribution unaltered. It follows that the surface integral of the electric field over the surface of a sphere of radius R is just \(4 \pi R^{2} \mathrm{E}_{r}) so that from Gauss' theorem

$$\mathrm{E}_r = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{R^2}\right)\int_0^R dr\rho(r)4\pi r^2.$$

Thus for the electric field **outside** the charge distribution one has

$$4\pi R^2 \mathrm{E}_r = \frac{1}{\epsilon_0}\int_{Sphere}\rho(r)dVol = \frac{Q}{\epsilon_0}. \tag{2.7.3}$$

Here Q is the total charge contained in the distribution. **The electric field outside a spherically symmetric charge distribution looks like the field of a point charge!**

The potential function generated by any symmetric charge distribution must be independent of the spherical polar co-ordinates θ, φ because E_θ and E_φ are both zero. The potential function can be calculated from ∂V/∂r = −E_r. Note that the potential must be a continuous function of r even if

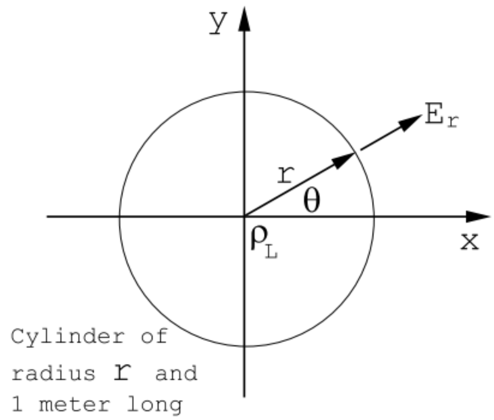


Figure 2.7.14: The electric field generated by a uniformly charged line lying along the z-axis. The line charge density is ρ_L Coulombs/meter. The electric field is radial and has the value $\mathbf{E}_r = (\rho_L/2\pi\epsilon_0)\,(1/r)$ Volts/m.

the charge density distribution, ρ(r), contains discontinuities. The potential function is continuous everywhere except on a surface containing an electrical double layer.

2.7.3 Cylindrical Symmetry.

(1) A uniformly Charged Line.

Let charges be distributed uniformly along the z-axis with a charge density of ρ_L Coulombs/meter as shown in Figure (2.7.14). It is easy to see, using Coulomb's law, that the electric field generated by this distribution can have only a radial component; the transverse components generated by equivalent charge elements symmetrically disposed around the origin at +z and at -z cancel each other out. Furthermore, the radial electric field strength,E_r, cannot depend upon the angle θ because the line charge exhibits rotational symmetry; ie. the charge distribution remains unaltered by a rotation through any angle around the z-axis. E_r also cannot depend upon position along z since the line is taken to be infinitely long. Apply Gauss' theorem to a cylindrical surface centered on the line charge, and 1 meter long and having a radius of r meters. The charge contained within this cylinder is ρ_L Coulombs. The surface integral of the electric field is easy to carry out because the electric field is parallel with the surface normal at every point on the cylinder surface; on the end surfaces the electric field contributes nothing to the surface integral because it lies in the surface, i.e. $\overrightarrow{\mathrm{E}} \cdot \hat{\mathbf{u}}_n = 0$, where $\hat{\mathbf{u}}_n$ is the unit normal to the surface. It follows from Gauss' Theorem that

$$2\pi r \mathrm{E}_r = \frac{\rho_L}{\epsilon_0},$$

and therefore

$$\mathrm{E}_r = \frac{\rho_L}{2\pi\epsilon_0}\left(\frac{1}{r}\right) \quad \text{Volts}/m.$$

Since the electric field has only a radial component it follows that the corresponding potential function, V, cannot depend upon the co-ordinates z,θ. But E_r = −∂V/∂r, so that one can use

$$\mathrm{V}(r) = -\left(\frac{\rho_L}{2\pi\epsilon_0}\right)\ln(r)$$

as the potential function for a uniformly charged line. The logarithmic variation of V(r) with r can also be deduced from a direct application of Poisson's equation, Equation (2.2.5). In this application the free charge density ρ_f = 0 everywhere except at the origin, r=0. There is assumed to be no electric dipole distribution anywhere so that $\vec{P}$ = 0 and $\rho_b = -\text{div}(\vec{P}) = 0$ everywhere. Thus $\nabla^2 V = 0$. But in cylindrical polar co-ordinates, and in the absence of any variation of V with angle or with displacement along z

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0.$$

It follows from this that *r∂V/∂r = constant = a*, and therefore V = *a ln(r) + constant* in agreement with the above expression for V(r): remember that one can add or subtract a constant from the potential function without changing the electric field calculated from it.

(2) A Line of Dipoles.

Consider a line of dipoles uniformly distributed along the z-axis with a density of P_L Coulombs; the dipoles are supposed to be oriented along the x-direction

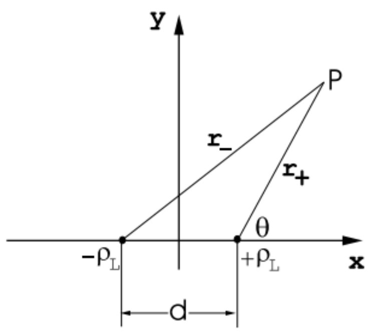


Figure 2.7.15: An infinite line of uniformly distributed point dipoles can be modelled by a uniformly charged positive line separated by a small distance d from a uniformly charged negative line. Let the charge density on the positive line be ρ_L, and let the charge density on the negative line be $-\rho_L$, then the dipole density, a vector, is given by $\left|\overrightarrow{\mathrm{P}}_L\right| = \rho_L d$ Coulombs, and $\vec{P}_L$ is directed from the negative line to the positive line.

as shown in Figure (2.7.15). The line of dipoles can be modelled by two line charges separated by a very small distance d, as shown in Figure (2.7.15). The resulting potential function at P, the point of observation, written in cylindrical polar co-ordinates can be calculated as follows using the potential function for a uniformly charged line:

$$\mathrm{V}_P = -\frac{\rho_L}{2\pi\epsilon_0}\ln(r_+) + \frac{\rho_L}{2\pi\epsilon_0}\ln(r_-) = \frac{\rho_L}{2\pi\epsilon_0}\ln(r_-/r_+).$$

But $r_- \cong r_+ + d\cos(\theta)$ so that

$$\mathrm{V}_P = \frac{\rho_L}{2\pi\epsilon_0}\ln\left(1 + \frac{d\cos(\theta)}{r_+}\right)$$

and

$$\lim_{d\to 0}\mathrm{V}_P = \frac{\rho_L d}{2\pi\epsilon_0}\frac{\cos(\theta)}{r} = \frac{1}{2\pi\epsilon_0}\frac{\overrightarrow{\mathrm{P}}_L\cdot\vec{r}}{r^2},$$

where P_L is the line density of dipoles. The electric field components are

$$\mathrm{E}_r = -\frac{\partial \mathrm{V}_P}{\partial r} = \frac{\mathrm{P}_L}{2\pi\epsilon_0}\frac{\cos(\theta)}{r^2},$$

$$\mathrm{E}_\theta = -\frac{1}{r}\frac{\partial \mathrm{V}_P}{\partial\theta} = \frac{\mathrm{P}_L}{2\pi\epsilon_0}\frac{\sin(\theta)}{r^2},$$

$$\mathrm{E}_z = 0.$$

A cross-section through the electric field distribution in a plane normal to the line of dipoles looks similar to the field distribution around a point dipole. The electric field at θ = 0 is directed along the +x direction and so is the field at θ = π: both fields have the strength $P_L/(2\pi\epsilon_0 r^2)$. The electric fields at θ = π/2, 3π/2 are directed along the -x direction, and have the strength $E_\theta = P_L/(2\pi\epsilon_0 r^2)$.

2.7.4 A Uniformly Polarized Ellipsoidal Body.

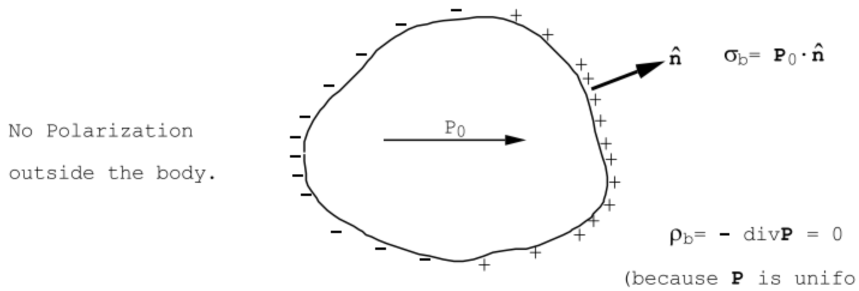


Figure 2.7.16: A uniformly polarized body having an irregular shape. The resulting surface bound charge distribution produces an electric field distribution that is non-uniform both inside and outside the body.

Consider a uniformly polarized body of arbitrary shape that is immersed in vacuum. The bound volume charge density associated with a uniform polarization density is zero since div($\vec{P}$) = 0. The bound surface charge density is not zero because there is a discontinuity in the normal component of $\vec{P}$ on the surface of the body; this surface charge density is given by $\sigma_b = \overrightarrow{P}_0 \cdot \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector normal to the surface of the body. Notice that this surface charge density varies from place to place on the surface and that it

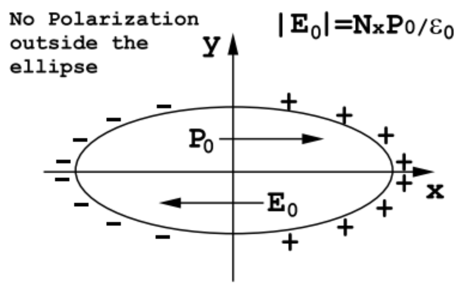


Figure 2.7.17: A uniformly polarized ellipsoidal body; the polarization lies along a principle axis of the ellipse. The resulting surface bound charge distribution produces a uniform electric field **inside** the ellipse. The electric field distribution outside the ellipse is non-uniform.

changes sign when one considers opposite points on the surface, see Figures (2.7.16 and 2.7.17). In fact, it is a consequence of charge conservation that the sum over all surface charges on a body that carries no free charges must be zero. The surface bound charge density distribution can be used to calculate the electric field and potential function everywhere in space. An analytical calculation is usually out of the question, and the problem must usually be solved by means of a numerical summation. One could, in principle, calculate the electric field components by means of Coulomb's law, but it is usually more convenient to work with the integral for the potential function, Equation (2.2.6). For the particular case in which the uniformly polarized body has an ellipsoidal shape the calculation of the potential function and the electric field can be carried out analytically. See, for example, J.A. Stratton, Electromagnetic Theory, McGraw-Hill, N.Y., 1941, section 3.25. The surprising result is that the electric field inside a uniformly polarized ellipsoid is uniform. Usually this internal electric field is not parallel with the direction of the polarization density. The internal electric field and the polarization are parallel only if the polarization vector is directed along a principal axis of the ellipsoid defined by the equation for its surface

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1. \tag{2.7.4}$$

Here a,b,c are the three semi-axes that define the ellipsoid. Outside the ellipsoid the electric field is not uniform: it resembles a distorted dipole field and falls off at large distances like $1/r^3$. In the principle axis co-ordinate system defined by Equation (2.7.4) the electric field components inside the ellipsoid are simply related to the polarization components:

$$E_x = -N_x \frac{P_x}{\epsilon_0}, \tag{2.7.5}$$

$$E_y = -N_y \frac{P_y}{\epsilon_0},$$

$$E_z = -N_z \frac{P_z}{\epsilon_0}.$$

The depolarization coefficients N_x, N_y, and N_z are pure numbers that depend upon the parameters of the ellipsoid (a,b,c of Equation (2.7.4)). They can be calculated using elliptic integrals. The depolarization coefficients obey a **sum rule**:

$$N_x + N_y + N_z = 1. \tag{2.7.6}$$

This sum rule makes it very easy to deduce values for the depolarization coefficients for very symmetrical bodies. Consider the following examples:

(a) **Sphere.** By symmetry Nx = Ny = Nz. Therefore from the sum rule each must be equal to 1/3

(b) **Cylinder.** This is the limiting case in which one dimension of the ellipsoid, the z dimension say, becomes very long. The two transverse depolarizing factors must be equal by symmetry. On the other hand, if the cylinder is polarized along its length any surface bound charge density can only be associated with the ends; but the ends are infinitely far away and consequently any charges on them produce an infinitely small electric field. This means that a cylinder that is uniformly polarized along its length will produce no electric field. It can be concluded that for such a cylinder $N_z = 0$. It follows from the sum rule that if $N_x = N_y$ then each must be equal to 1/2.

(c) **A Thin Flat Disc.** Think of an ellipsoid of revolution which is thin along the z-direction but which has a very large radius R in the xy plane. By symmetry $N_x = N_y$. If the edges of the disc are very far from its center, ie. $R \to \infty$, then the electric field near the center of the disc due to surface charges on the disc edges must become vanishingly small (remember that

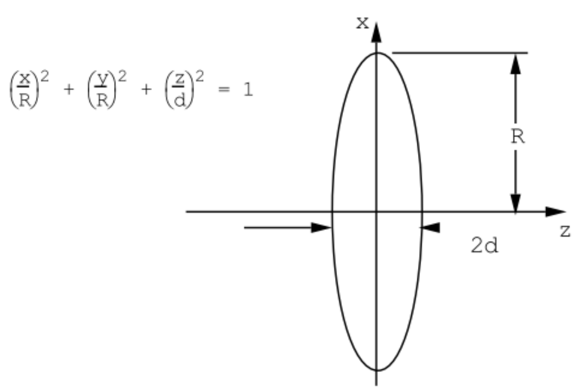


Figure 2.7.18: Oblate ellipsoid of revolution.

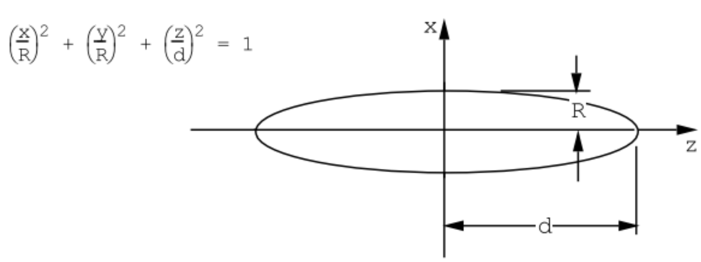


Figure 2.7.19: Cigar shaped ellipsoid of revolution.

the field due to an element of charge decreases like $1/R^2$). It follows that $N_x = N_y \to 0$ in the limit as $R \to \infty$. From the sum rule Equation (2.7.6) one can conclude that $N_z \to 1$. The field inside an infinite disc that is polarized parallel with its axis of symmetry has the value $E_z = -P_z/\epsilon_0$ in agreement with the result previously deduced for a transversely polarized slab, section (5) above, and Figure (2.7.11).

Two commonly encountered special cases of ellipsoids of revolution are shown in Figures (2.7.18 and 2.7.19). In each case one need only specify the depolarizing coefficient for the axis of revolution, the z-axis. The other two depolarizing factors are equal and can be calculated from the sum rule Equation (2.7.6). Case(a) is a pancake shaped ellipsoid, Figure (2.7.18). For this case

$$N_z = \frac{R^2 d}{\left(R^2 - d^2\right)^{3/2}}\left(\frac{\sqrt{R^2-d^2}}{d} - \arctan\left(\frac{\sqrt{R^2-d^2}}{d}\right)\right), \quad \text{where } (d/R) < 1. \tag{2.7.7}$$

In the limit $d/R \to 0$ the depolarizing factor is given approximately by

$$N_z \cong 1 - \frac{\pi}{2}\left(\frac{d}{R}\right). \tag{2.7.8}$$

Case(b) is a cigar shaped ellipsoid, Figure (2.7.19). For this case

$$N_z = \frac{(1-\psi^2)}{\psi^3}\left(\frac{1}{2}\ln\left(\frac{1+\psi}{1-\psi}\right) - \psi\right), \text{ where } \psi = \sqrt{1-(R/d)^2}. \tag{2.7.9}$$

In the limit as the cigar becomes very long, $(d/R) \to \infty$ the demagnetizing coefficient can be expressed as

$$N_z \cong \left(\frac{R}{d}\right)^2 \left(\ln\left[\frac{2d}{R}\right] - 1\right). \tag{2.7.10}$$

For a general ellipsoid it can be shown that

$$\begin{aligned} N_x &= \left(\tfrac{abc}{2}\right)\int_0^\infty \tfrac{ds}{(s+a^2)R_s} \\ N_y &= \left(\tfrac{abc}{2}\right)\int_0^\infty \tfrac{ds}{(s+b^2)R_s} \\ N_z &= \left(\tfrac{abc}{2}\right)\int_0^\infty \tfrac{ds}{(s+c^2)R_s} \end{aligned} \tag{2.7.11}$$

where $R_s = \left(\left[s+a^2\right]\left[s+b^2\right]\left[s+c^2\right]\right)^{1/2}$.

(See J.A. Stratton, Electromagnetic Theory, McGraw-Hill, N.Y., 1941. Section 3.27.)

2.8: Appendix 2A

2.8 Appendix 2A.

It was pointed out in sections 2.2.1 and 2.2.2 above that the potential function, V($\vec{R}$) generated by a distribution of electric dipoles, P($\vec{r}$), can be calculated in two ways:

$$(1)\quad \mathrm{V}(\overrightarrow{\mathrm{R}}) = \frac{1}{4\pi\epsilon_0}\int\int\int_{S_{\text{pace}}\,dV_{\text{vol}}} \frac{(-\operatorname{div}(\overrightarrow{\mathrm{P}}))}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}. \tag{2.8.1}$$

This equation for the potential function is calculated from the distribution of bound charges, $\rho_b = -\operatorname{div}(\vec{P})$. The second equation for the potential function can be written as the potential due to point dipoles $\vec{P}dV_{vol}$ summed over the entire distribution of dipoles:

$$(2)\quad \mathrm{V}(\overrightarrow{\mathrm{R}}) = \frac{1}{4\pi\epsilon_0}\int\int\int_{S_{pace}} dV_{\text{vol}} \frac{\overrightarrow{\mathrm{P}}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}. \tag{2.8.2}$$

These two formulae, Equations (2.8.1 and 2.8.2), give the same potential function apart from a possible constant that has no effect on the resulting electric field. This statement can be proved by applying Gauss' Theorem to the function

$$\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right) = \operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{\sqrt{[X-x]^2+[Y-y]^2+[Z-z]^2}}\right).$$

The divergence is calculated with respect to the co-ordinates of the source point, (x,y,z):

$$\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right) = \frac{\partial}{\partial x}\left(\frac{\mathrm{P}_x}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right) + \frac{\partial}{\partial y}\left(\frac{\mathrm{P}_y}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right) + \frac{\partial}{\partial z}\left(\frac{\mathrm{P}_z}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right).$$

By direct differentiation one can readily show that

$$\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right) = \frac{\operatorname{div}(\overrightarrow{\mathrm{P}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|} + \frac{\overrightarrow{\mathrm{P}}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}.$$

Remember that the differentiations are with respect to the co-ordinates of $\vec{r}$, (x,y,z), and not with respect to the observer co-ordinates $\vec{R}$, (X,Y,Z). Integrate the above equation over a volume, V_{vol}, bounded by a surface S and apply Gauss' Theorem, section 1.3.3, to the term on the left. The result is

$$\int\int_S \frac{dS(\overrightarrow{\mathrm{P}}\cdot\hat{\mathbf{n}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|} = \int\int\int_{V_{\text{rot}}} \frac{dV_{\text{vol}}\, div(\overrightarrow{\mathrm{P}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|} + \int\int\int_{V_{\text{rot}}} \frac{dV_{\text{vol}}\,\overrightarrow{\mathrm{P}}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}.$$

Now let the volume V_{vol} become very large so that the surface S recedes to infinity. If the polarization distribution is limited to a finite region of space, as we shall assume, the surface integral must vanish because the polarization density on the surface, S, is zero. We are left with the identity

$$-\int\int\int_{V_{\text{vol}}} \frac{dV_{\text{vol}}\, div(\overrightarrow{\mathrm{P}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|} = \int\int\int_{V_{\text{red}}} \frac{dV_{\text{vol}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}) \tag{2.8.3}$$

Upon multiplying both sides of Equation (2.8.3) by $1/(4\pi\epsilon_0)$ one obtains the integral of Equation (2.8.1) on the left and the integral of Equation (2.8.2) on the right. It follows that the same value for the potential will be obtained, aside from a possible unimportant constant, whether one uses the formulation based upon the potential for a point charge or the formulation based upon the potential function for a point dipole.

CHAPTER OVERVIEW

3: Electrostatic Field II

Electrostatic Boundary Value Problems for an Isotropic, Linear, Dielectric Material.

Thumbnail: The field of a positive charge above a flat conducting surface, found by the method of images. (CC BY 3.0; Geek3 via Wikipedia)

3.1: Introduction

Chapter(2) demonstrated that the electrostatic field could be calculated everywhere in space from a knowledge of the spatial distribution of free charges and a knowledge of the spatial distribution of the electric dipole moment density. However, in most cases of interest one does not initially know the free charge and dipole moment distributions. In a typical problem one is given two or more metal electrodes embedded in a material medium, in which the electrode potentials are specified. In this kind of problem the free charge and dipole moment distributions must be determined as part of the problem solution. In order to solve such problems it is necessary to know the relation between the electric field in a material and the dipole density that is induced in that material by the electric field. In general such problems are extremely difficult unless the dipole density, $\vec{P}(\vec{r})$, at a point $\vec{r}$ in the material is linearly related to the electric field at that same point, $\vec{E}(\vec{r})$. In this chapter we shall assume that we have to do with **linear, isotropic, media** such that

$$\vec{\mathrm{P}}(\vec{\mathrm{r}})=\chi\epsilon_0\vec{E}(\vec{r}), \tag{3.1.1}$$

where χ is a pure number called the static dielectric susceptibility.χ is supposed to be independent of position within a given material; it will exhibit discontinuous jumps at the boundary between two different materials. Fora dielectric material characterized by χ Maxwell's equations for the electrostatic field become (no variation with time)

$$\operatorname{curl}(\vec{\mathrm{E}})=0, \tag{3.1.2}$$

$$\operatorname{div}(\vec{\mathrm{E}})=\frac{1}{\epsilon_0}\left[\rho_f-\operatorname{div}(\vec{\mathrm{P}})\right]=\frac{1}{\epsilon_0}\left[\rho_f-\chi\epsilon_0\operatorname{div}(\vec{\mathrm{E}})\right].$$

This last equation can be written $\epsilon_0(1+\chi)\operatorname{div}(\vec{\mathrm{E}})=\rho_f$. But $\epsilon_0(1+\chi)$ is independent of position within a given dielectric material so that

$$\operatorname{div}\left[(1+\chi)\epsilon_0\vec{\mathrm{E}}\right]=\rho_f=\operatorname{div}(\vec{\mathrm{D}}), \tag{3.1.3}$$

since

$$\vec{D}=\epsilon_0\vec{\mathrm{E}}+\vec{\mathrm{P}}=(1+\chi)\epsilon_0\vec{E}. \tag{3.1.4}$$

The number $(1+\chi)$ is called the **relative dielectric constant**, ϵ_r. Thus for a linear isotropic material

$$\vec{\mathrm{D}}=\epsilon_r\epsilon_0\vec{\mathrm{E}}=\epsilon\vec{\mathrm{E}}, \tag{3.1.5}$$

and

$$\operatorname{div}(\vec{\mathrm{D}})=\rho_f \quad \text{or} \quad \operatorname{div}(\vec{E})=\rho_f/\epsilon. \tag{3.1.6}$$

Equation(3.5) defines the dielectric constant ϵ which has the same units as ϵ_0, namely Farads/meter or Coulombs/Volt meter.

As in Chapter(2) one can introduce a potential function, V($\vec{r}$), such that

$$\vec{\mathrm{E}}(\vec{\mathrm{r}})=-\operatorname{grad}V(\vec{\mathrm{r}}). \tag{3.1.7}$$

This definition guarantees that the Maxwell Equation (3.1.2) will be satisfied because the curl of any gradient is zero. Using Equation (3.1.7) in Equation (3.1.6) gives

$$-\operatorname{divgrad}V(\vec{\mathrm{r}})=\rho_f(\vec{\mathrm{r}})/\epsilon$$

or

$$\nabla^2V(\vec{r})=-\frac{1}{\epsilon}\rho_f(\vec{r}) \tag{3.1.8}$$

The differential equation, Equation (3.1.8), is called Poisson's equation. It is similar in form to Equation (2.2.5) of Chapter(2) for $\vec{\mathrm{P}}=0$ except that ϵ_0 is replaced by ϵ. It follows by analogy with Equation (2.2.6) of Chapter(2) that the particular solution of Equation (3.1.8) is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon}\int\int\int_{Space} dV_{vol}\frac{\rho_f(\vec{r})}{|\vec{\mathrm{R}}-\vec{r}|} \tag{3.1.9}$$

Unfortunately, Equation (3.1.9) is seldom helpful because one does not usually know a priori the free charge distribution, $\rho_f(\vec{\mathrm{r}})$. The usual problem involves a number of conducting electrodes embedded in a dielectric medium: either the potential or the total charge on each electrode is specified as a boundary condition. Generally the dielectric medium is taken to be either charge free, $\rho_f = 0$, or else slightly conducting where the current density at any point in the medium is proportional to the electric field strength at that point

$$\vec{J}_f = \sigma\vec{\mathrm{E}}. \tag{3.1.10}$$

For the charge-free case Poisson's equation, Equation (3.1.8), becomes

$$\nabla^2 V = 0. \tag{3.1.11}$$

This is called LaPlace's equation. For a conducting medium the charge density will not generally be zero. However, any distribution of charges must be time independent because, by hypothesis, we are dealing with static field distributions which are independent of the time. It is a consequence of charge conservation that the current density, $\vec{J}_f$, and the charge density, ρ_f, must satisfy the equation

$$\operatorname{div}\left(\vec{\mathrm{J}}_f(\vec{\mathrm{r}})\right) + \frac{\partial\rho_f(\vec{\mathrm{r}})}{\partial t} = 0, \tag{3.1.12}$$

so that if the charge distribution does not depend upon time the current density must be divergence free;

$$\operatorname{div}\left(\vec{\mathrm{J}}_f\right) = 0. \tag{3.1.13}$$

But if the current density is proportional to the electric field, Equation (3.1.10), it follows that a divergence free current density must be paired with a divergence free electric field,

$$\operatorname{div}(\vec{E}) = 0, \tag{3.1.14}$$

and therefore that the potential function must satisfy LaPlace's equation, Equation (3.1.11), $\nabla^2 V = 0$. The time-independent problem associated with conductors having a specified potential embedded in a linear, isotropic, chargefree dielectric medium has exactly the same potential distribution as the

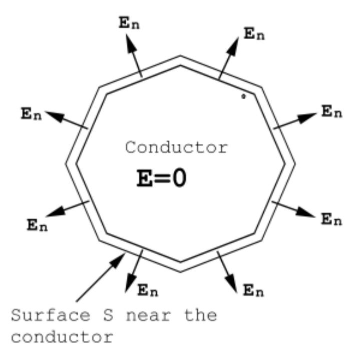


Figure 3.1.1: A charged conductor. The free charge on the conductor is given by $Q = \int\int_S \vec{\mathrm{D}}\cdot \mathrm{d}\vec{S}$.

problem of the same conductors embedded in an isotropic, conducting, linear dielectric medium. In either case one requires a potential distribution that satisfies LaPlace's equation, Equation (3.1.11), and for which the potential on each conducting electrode either reduces to a specified value, or else the total charge on each conductor calculated from the potential distribution gives a specified value. A conductor must have the same potential throughout by definition because the electric field within a conductor is zero; a zero electric field means that the potential function has no dependence upon position. The electric field within a conductor must be zero because otherwise mobile charges within the conductor would flow to its surface and build up time dependent electric fields in contradiction with the original assumption that the electric fields were time independent. It is a general rule that the tangential components of the electrostatic field must be continuous across any surface (see sections 2.4 and 2.5). Therefore, if $\vec{E} =$

0 inside the conducting body it follows that the tangential components of the electrostatic field must be zero just outside the conductor surface. In other words, **the electrostatic field must be normal to the surface of a conductor**.

The total charge on a conductor, which must be located entirely on the surface of the conductor, can be calculated by means of an application of Gauss' theorem to Equation (3.1.6): see Figure (3.1.1).

$$Q = \int\int_{\text{Surface}} \overrightarrow{d\vec{S}} \cdot \vec{D} = \epsilon \int\int_{\text{Surface}} \overrightarrow{dS} \cdot \vec{E},$$

where the integral is evaluated on a surface that lies just outside the actual conductor surface. The electric field just outside the conductor is normal to the surface and has the magnitude $E_n = -\partial V/\partial n$, where $\partial V/\partial n$ is the gradient of the potential at the surface of the conductor. The total charge contained on the conductor is therefore related to the normal gradient of the potential function at the conductor

$$Q = -\epsilon \int\int_{\text{Surface}} d\vec{S} \cdot \left(\frac{\partial V}{\partial n}\right). \tag{3.1.15}$$

It can be proved mathematically, and it makes sense physically, that there is only one solution of LaPlace's equation, $\nabla^2 V = 0$, which satisfies the given boundary conditions. In other words, a solution of LaPlace's equation that satisfies the boundary conditions is unique; it is **the solution**. The boundary conditions may be of three different types: (1) the potential on each conductor is specified; (2) the total charge on each conductor is specified; (3) the potential on some of the conductors is specified and the total charge on the remaining conductors is specified. In addition, the fields very far from any charges must fall to zero at least as fast as $1/R^2$. This requirement follows from the fact that any collection of charges when viewed from very far away must look very nearly like a point charge, and hence the potential function must fall off like 1/R where R is the mean distance to the group of charges.

In general it is very difficult to find a solution of LaPlace's equation that satisfies the boundary conditions imposed by a particular problem. Usually it is necessary to resort to numerical or approximate techniques in order to find a suitable solution. In the following sections a number of standard problems are posed and their solutions are discussed. The solutions of these standard problems enable one to build up, by analogy, a picture of how the electrostatic field should behave given a problem whose parameters lie outside those corresponding to one of the categories discussed below.

3.2: Soluble Problems

3.2.1 (1) Orthogonal Systems.

The only problems that can be solved analytically are those for which the conducting electrodes can be described by u1=const., or u2=const., or u3=const. where u1, u2, u3 form a system of orthogonal co-ordinates. Stratton discusses eight such orthogonal systems (Electromagnetic Theory by Julius Adams Stratton, McGraw-Hill, New York, 1941). We shall be interested only in the three most commonly used systems (1) cartesian co-ordinates, (2) cylindrical polar co-ordinates, and (3) spherical polar co-ordinates.

(a) Cartesian Co-ordinates.

In the Cartesian co-ordinate system LaPlace's equation becomes

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{V}}{\partial \mathrm{y}^2}+\frac{\partial^2 \mathrm{V}}{\partial \mathrm{z}^2}=0.$$

Let the surfaces of two semi-infinite electrodes lie at z=0 and at z=D, Figure (3.2.2). In this case the potential function must be independent of x and y by symmetry: the potential on any plane z=constant must be featureless because there are no edges with which to locate oneself in the plane. In other words, any shift of the electrodes in the x-y plane does not change the geometry of the problem. Thus LaPlace's equation is reduced to

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{z}^2}=0.$$

This simple equation has the general solution

$$\mathrm{V}(\mathrm{z})=A+Bz, \tag{3.2.1}$$

where A,B are constants that must be determined from the boundary conditions. For z=0 the potential is required to be V_1 and therefore A = V_1. For z=D the potential must equal V_2 and therefore B = $(V_2 - V_1)/D$. Thus the required potential function for this problem is given by

$$\mathrm{V}(\mathrm{z})=\mathrm{V}_1+\frac{(\mathrm{V}_2-\mathrm{V}_1)\,z}{\mathrm{D}},$$

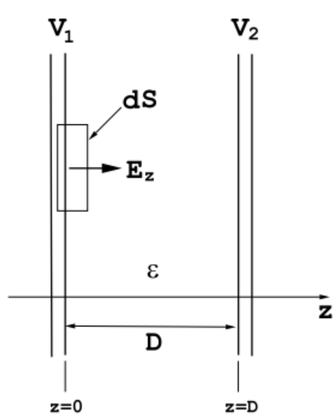


Figure 3.2.2: Two plane parallel, semi-infinite metal electrodes separated by a distance D. The electrode potentials are V_1 and V_2. The space between the electrodes is filled with a material having a dielectric constant ϵ.

and this solution is unique. It corresponds to an electric field whose components are:

$$\mathrm{E}_{\mathrm{x}}=0$$
$$\mathrm{E}_{y}=0$$
$$\text{and}\quad \mathrm{E}_{\mathrm{z}}=\frac{(\mathrm{V}_1-\mathrm{V}_2)}{\mathrm{D}}\quad \text{Volts}/\mathrm{m}.$$

The electric field is forced to be uniform simply because the potential function has no spatial variation along x or y.

The surface charge density on each electrode must also be independent of x and y, and the charge density on the two electrodes are equal in magnitude but opposite in sign. They may be calculated by means of Gauss' theorem. Consider a small pillbox that spans an electrode surface such as that shown in Figure (3.2.2). According to Gauss' Theorem the surface integral of the normal component of D over the pillbox is equal to the total free charge contained within the pillbox:

$$\mathrm{Q}=\rho_s dS=\int\int_{Pillbox}\overrightarrow{\mathrm{dS}}\cdot\vec{\mathrm{D}}.$$

But $\vec{\mathrm{D}}=\epsilon\vec{\mathrm{E}}$ so that the surface integral of $\vec{D}$ can be written as a surface integral of $\vec{E}$. Since $\vec{E}$ is zero inside the metal electrode it follows that the only contribution to the surface integral comes from the surface of the pillbox that lies in the dielectric; the surface integral of $\vec{E}$ gives dSE_z. Thus one finds

$$\mathrm{Q}=\mathrm{dS}\rho_s=\epsilon\mathrm{dSE}_z$$

and therefore

$$\rho_s=\epsilon\frac{(\mathrm{V}_1-\mathrm{V}_2)}{\mathrm{D}}.$$

This expression can be used to estimate the relation between total charge and voltage difference on a parallel plate capacitor. Consider two parallel plate electrodes each having an area of A $meters^2$, and let the charge on one plate be Q Coulombs and on the other plate be -Q Coulombs. If edge effects are neglected, and if it is assumed that the charge density is uniform, one can write ρ_s = Q/A. It follows that

$$\mathrm{Q}=\left(\frac{\epsilon\mathrm{A}}{\mathrm{D}}\right)(\mathrm{V}_1-\mathrm{V}_2),$$

or writing $(V_1 - V_2) = \Delta V$,

$$\mathrm{C}=\frac{\mathrm{Q}}{\Delta\mathrm{V}}=\left(\frac{\epsilon\mathrm{A}}{\mathrm{D}}\right)\quad\text{Farads. .}$$

C is the capacitance of the parallel plate system.

Variations of this problem involve two regions having different dielectric constants, see Figure (3.2.3).

The potential function in each region must satisfy Laplace's equation (3.1.11), and therefore in the infinite plate approximation

$$\frac{\partial^2\mathrm{V}}{\partial\mathrm{z}^2}=0,$$

since the potential function can not depend upon the co-ordinates x and y. It follows from $E_z = -dV/dz$ that the electric field in each region must be

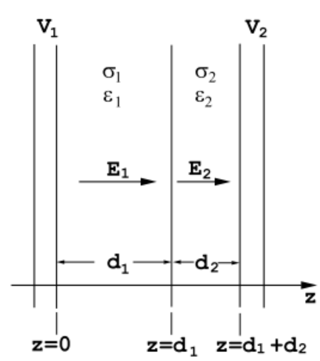


Figure 3.2.3: The parallel plate capacitor problem with two different dielectric materials. The electric field in each region is independent of position.

independent of position. The magnitude of the electric field in each region depends upon whether or not the dielectric material is, or is not, conducting. There are two main limiting cases: (1) the conductivity, σ in each region is zero; and (2) each region is conducting with a current density given by $\vec{J} = \sigma \vec{E}$.

Case(1). If the conductivities are zero there can be no free charges anywhere in the dielectric materials. As a consequence it follows from Maxwell's equation(2.3.2) that $\vec{D}$ must be divergence free, ie div($\vec{D}$) = 0. This in turn means that the normal component of $\vec{D}$ must be continuous across the interface between the two dielectrics, or $D_1 = D_2$. This implies that

$$\epsilon_1 E_1 = \epsilon_2 E_2. \tag{3.2.2}$$

But also

$$d_1 E_1 + d_2 E_2 = (V_1 - V_2) = \Delta V. \tag{3.2.3}$$

These two equations, Equations (3.2.2 and 3.2.3) can be solved to obtain the electric field strengths in each region of the dielectric insulators. The charge density on each electrode has the magnitude $\rho_s = D_1 = D_2$.

Case(2). If the dielectric materials are conducting the current density in each region must be the same in the steady state in order to prevent a time dependent build up of charge at the interface between the two dielectric slabs. But $J_1 = \sigma_1 E_1$ and $J_2 = \sigma_2 E_2$ so that

$$\sigma_1 E_1 = \sigma_2 E_2. \tag{3.2.4}$$

Equation(3.2.4) replaces Equation (3.2.2) which is only valid providing that there is no charge flow through the dielectric slabs. Eqns.(3.2.3 and 3.2.4) form a system of two equations that may be solved for the two unknowns E_1 and E_2. Notice that for this case of conducting materials the displacement vector will have a different value in each of the two regions:

$$D_1 = \epsilon_1 E_1 = \epsilon_1 \left(\frac{\sigma_2}{\sigma_1}\right) E_2,$$

and

$$D_2 = \epsilon_2 E_2.$$

Notice that the free surface charge density on each electrode will be different in magnitude because $\rho_s = D_1$ for the positive electrode and $\rho_s = -D_2$ for the negative electrode. The surface free charge density at the interface between the two dielectric slabs is given by $\rho_s = (D_2 - D_1)$.

(b) A Leaky Capacitor.

The potential function for a leaky capacitor is the same as the potential function for a non-leaky capacitor because in both cases the potential must satisfy Equation (3.1.11), $\nabla^2 V = 0$, and in both cases the potential must satisfy the same boundary conditions. In the infinite electrode approximation in which edge effects are neglected the plane symmetry requires that the potential function have the form $V = a + bz$, Equation (3.2.1), where a and b are constants. This means that $E_z = -(\partial V/\partial z)$ must be independent of position. If the potential difference between the electrodes is ΔV, see Figure (3.2.4), the electric field strength is $E = \Delta V/d$, and the corresponding strength of the displacement vector is $D = \epsilon E = \epsilon \Delta V/d$. But from Gauss' Theorem

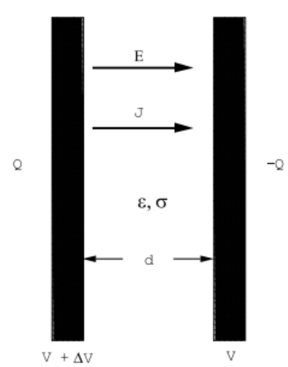


Figure 3.2.4: Charge decay through a leaky capacitor. $\epsilon = \epsilon_r \epsilon_0$ is the dielectric constant for the spacer material. σ is the conductivity of the spacer material.

applied to div($\vec{D}$) = ρ_f it follows that the surface charge density on the positive electrode is given by $\sigma_f = D = \epsilon \Delta V/d = Q/A$, where Q is the total charge on the electrode and A is the electrode area. (Do not confuse the free charge density, σ_f, with the conductivity, σ). The capacitance is defined by $C = Q/\Delta V$ so

$$C = \frac{\epsilon A}{d} \quad \text{Farads},$$

exactly the same formula as for a non-leaky capacitor! However, there is a flow of charge between the two electrodes of a leaky capacitor. The current density is given by

$$J = \sigma E = \frac{\sigma \Delta V}{d} = \frac{\sigma Q}{\epsilon A}.$$

The total current is

$$I = JA = \frac{\sigma Q}{\epsilon} \quad Amps.$$

Unless the current is maintained by some external source such as a battery this current flow must deplete the electrodes of charge. For an isolated capacitor the charge on the positive electrode must change with time according to the equation of charge conservation:

$$\frac{dQ}{dt} = -I = -\frac{\sigma Q}{\epsilon}.$$

The solution of this differential equation is

$$Q(t) = Q_0 \exp -\sigma t/\epsilon. \tag{3.2.5}$$

Thus the charge on a leaky capacitor dies away exponentially with a time constant, τ, given by

$$\tau = \frac{\epsilon}{\sigma} = \rho\epsilon \quad \text{seconds}, \tag{3.2.6}$$

where ρ is the resistivity of the material between the conducting electrodes (not the charge density!), see Figure (3.2.4). Whether or not a capacitor should be treated as leaky depends entirely upon the time scale associated with the problem. For most materials the relative dielectric constant, ϵ_r, lies between 1 and 10, so that differences in the intrinsic time constant, τ, from one material to another are determined primarily by the resistivity. Resistivities for some selected materials are listed in Table (3.2.1). The most striking feature of this Table is the wide range of resistivities exhibited by these solid materials. It is clear that the best candidates for an insulating dielectric material listed in the Table are yellow sulphur and paraffin wax.

(c) Cylindrical Co-ordinates.

Consider a problem that exhibits cylindrical symmetry so that the potential function does not depend upon the z co-ordinate. LaPlace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0. \tag{3.2.7}$$

The general solution of this equation can be written

$$V(r,\theta) = a + b\ln r + \sum_{n=1}^{\infty}\left(a_n r^n + \frac{b_n}{r^n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n r^n + \frac{d_n}{r^n}\right)\sin n\theta, \tag{3.2.8}$$

where a_n, b_n, c_n, and d_n are arbitrary constants. The series (3.2.8) satisfies the equation $\nabla^2 V = 0$ term by term as can be verified by direct differentiation.

Table 3.2.1: Resistivities and time constants for some selected materials at a temperature of 20C. (Handbook of Chemistry and Physics, 53rd Ed., CRC Press (1972). The dielectric constant has been taken to be ε_0 for convenience.

Material ρ (Ohm m)	**Material ρ (Ohm m)**	$\tau_0 = \epsilon_0\rho(\text{ seconds })$
Copper	1.67×10^{-8}	1.48×10^{-19}
Intrinsic Ge	0.46	4.1×10^{-12}
Boron	1.8×10^{4}	1.6×10^{-7}
Yellow Sulpher	2×10^{15}	1.8×10^{4}= 4.9 hours
Pyrex 7060	1.3×10^{5}	1.2×10^{-6}

Material ρ (Ohm m)	Material ρ (Ohm m)	$\tau_0 = \epsilon_0 \rho($ seconds $)$
Pyrex 1710	2.5×10^7	2×10^{-4}
Fused Silica	$\sim 10^8$	$\sim 9 \times 10^{-4}$
Beeswax	$\sim 10^{13}$	~ 89
Paraffin	$10^{13} - 10^{17}$	$89 - 8.9 \times 10^5$ (up to 247 hours)
Wet Ground	$10^2 - 10^3$	$10^{-9} - 10^{-8}$

The constants in (3.2.8) have to be chosen so as to satisfy the boundary conditions for a particular problem. The term *b ln r* corresponds to the potential generated by an infinite line charge, see Section (2.7.3) of Chpt.(2). A line charge of strength ρ_L Coulombs/meter in free space generates the potential

$$\mathrm{V(r)} = -\left(\frac{\rho_L}{2\pi\epsilon_0}\right)\ln \mathrm{r}. \tag{3.2.9}$$

If the line charge is immersed in a medium of dielectric constant ϵ then ϵ_0 must be replaced by ϵ in Equation (3.2.9).

The term $V_1 = (b_1/r) \cos\theta$ corresponds to a line of dipoles in which the dipole moment is oriented along the x-axis, see Chpt.(2), Section (2.7.3). The potential generated by a line of dipoles in free space and having a strength of P_x Coulombs is given by

$$\mathrm{V_x(r,\theta)} = \frac{\mathrm{P_x}\cos\theta}{2\pi\epsilon_0 \mathrm{r}}.$$

If the dipole moments are oriented along the y-axis the potential is given by

$$V_y(r,\theta) = \frac{P_y \sin\theta}{2\pi\epsilon_0 r}:$$

this is one of the terms proportional to sin θ in Equation (3.2.8).

The terms $a_1 r\cos(\theta)$ and $c_1 r\sin(\theta)$ in (3.2.8) correspond to uniform fields along x and y. This can be seen by using the substitutions

$$x = r\cos\theta,$$

and

$$y = r\sin\theta,$$

to obtain $V_x = a_1 x$, corresponding to an electric field $E_x = -a_1$, and $V_y = c_1 y$, corresponding to the electric field $E_y = -c_1$. Such terms are appropriate for discussing the problem of a uniform dielectric cylinder immersed in a uniform applied electric field, Figure (3.2.5). If the applied electric field, E_0, is taken to lie along the x-direction it is clear from the symmetry of the problem that the potential function must be symmetric in y: a reflection of the system through the xz plane gives one exactly the same problem. This implies that the potential for θ and for −θ must be the same. This being the case, the amplitudes of all the sin nθ terms in the expansion (3.2.8) must be zero for

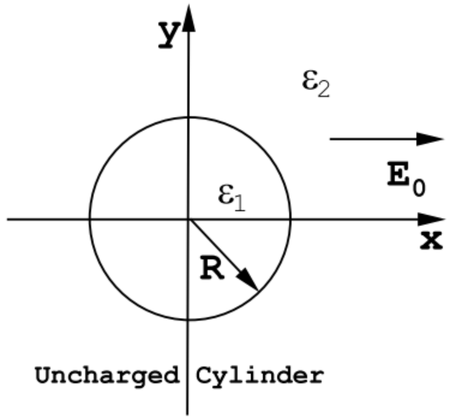


Figure 3.2.5: A uniform cylinder, infinitely long in the z-direction, and immersed in a uniform electric field $E_x = E_0$. The cylinder is characterized by a dielectric constant ϵ_1. It is situated in a medium whose dielectric constant is $\epsilon 2$.

each n because $\sin(n\theta)$ is an odd function of its argument. Furthermore, the cylinder is uncharged and therefore the amplitude, b, of the ln(r) term in the expansion must be zero because b is proportional to the strength of the line charge, ρ_L, which generates this term in the potential. A second argument requires this term to be absent in the potential function for this problem: the potential function inside the cylinder is required to remain finite at r=0, and ln(r) diverges at r=0. In addition, the potential function in the region outside the cylinder is required to approach the value corresponding to a uniform electric field, $V(r,\theta) = -E_0 r \cos(\theta)$, for very large distances r, whereas ln(r) diverges at large r much more slowly. On the basis of these arguments one may conclude that the potential function required for the problem of a cylinder immersed in a uniform electric field must have the form

$$V(r,\theta) = a + a_1 r\cos\theta + \frac{b_1\cos\theta}{r} + a_2 r^2\cos 2\theta + \frac{b_2\cos 2\theta}{r^2} + \cdots \tag{3.2.10}$$

In the limit as $r \to \infty$ the potential function outside the cylinder, V_0, must converge to the potential corresponding to a uniform electric field, E_0, along the x direction. That is

$$\lim_{r\to\infty} V_0 \to -E_0 r\cos\theta.$$

This condition requires all the terms an to vanish for n > 1. It also requires $a_1 = -E_0$.

Inside the cylinder the potential function, V_i , must remain finite in the limit as $r \to 0$: there are no charges inside the cylinder to produce any singularity in the potential. Thus inside the cylinder all the terms proportional to $1/r^n$ must vanish for all n. These considerations now leave the following possibilities for the potential functions inside and outside the cylinder:

Inside(r ≤ R)

$$V_i(r,\theta) = a + a_1 r\cos\theta + a_2 r^2\cos 2\theta + a_3 r^3\cos 3\theta\cdots$$

Outside(r ≥ R)

$$V_0(r,\theta) = A - E_0 r\cos\theta + \frac{b_1\cos\theta}{r} + \frac{b_2\cos 2\theta}{r^2} + \frac{b_3\cos 3\theta}{r^3} + \cdots$$

These two series for the potential functions inside and outside the cylinder must be matched on the surface of the cylinder in order to satisfy two conditions: (1) the tangential component of $\vec{E}$ must be continuous across the interface (from curl$\vec{E}$ = 0); and (2) the normal component of $\vec{D}$ must be continuous across the interface at r=R because there is no free charge density (from div($\vec{D}$) = ρ_f). Condition (1) will obviously be satisfied if the potential function is forced to be continuous at r=R,

$$V_i(R,\theta) = V_0(R,\theta).$$

Condition (2) requires that

$$\epsilon_1\left(\frac{\partial V_i}{\partial r}\right)_{r=R} = \epsilon_2\left(\frac{\partial V_0}{\partial r}\right)_{r=R}.$$

These two conditions must be satisfied for every angle θ, and this means that they must be separately satisfied for each term cos (nθ) in the above two series. For example:

n=0

$$a = A$$

n=1

$$a_1 \mathrm{R} \cos\theta = \left(-\mathrm{E}_0\mathrm{R} + \frac{b_1}{\mathrm{R}}\right)\cos\theta,$$

and

$$\epsilon_1 a_1 \cos\theta = -\epsilon_2\left(\mathrm{E}_0 + \frac{b_1}{\mathrm{R}^2}\right)\cos\theta.$$

These last two equations can be solved to give

$$a_1 = \frac{-2\epsilon_2\mathrm{E}_0}{(\epsilon_1+\epsilon_2)}, \tag{3.2.11}$$

$$\frac{b_1}{\mathrm{R}^2} = \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)\mathrm{E}_0. \tag{3.2.12}$$

n=2

$$a_2\mathrm{R}^2\cos 2\theta = \frac{b_2}{\mathrm{R}^2}\cos 2\theta,$$

and

$$2\epsilon_1 a_2 R\cos 2\theta = \frac{-2\epsilon_2 b_2}{R^3}\cos 2\theta.$$

The latter two equations have only the solution $a_2 = b_2 = 0$. These procedures can be continued for all n with the result that all coefficients for n ≥ 2 are zero. The potential function for the problem of an infinite cylinder subjected to a uniform applied field turns out to be rather simple:

Inside the cylinder (r ≤ R)

$$V_i(r,\theta) = a - \frac{2\epsilon_2 E_0 r\cos\theta}{(\epsilon_1+\epsilon_2)}. \tag{3.2.13}$$

Outside the cylinder (r ≥ R)

$$\mathrm{V}_0(\mathrm{r},\theta) = a - \mathrm{E}_0\mathrm{r}\cos\theta + \mathrm{R}^2\left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)\frac{\mathrm{E}_0\cos\theta}{\mathrm{r}}. \tag{3.2.14}$$

The constant a has no physical significance and could just as well have been set equal to zero. The potential function inside the cylinder corresponds to a uniform electric field along the x-direction:

$$\mathrm{E_x} = \frac{2\epsilon_2}{(\epsilon_1+\epsilon_2)}\mathrm{E}_0. \tag{3.2.15}$$

This means that the material inside the cylinder is uniformly polarized along the x-direction. This is an example of the depolarizing coefficients discussed in Section (2.7.4) of Chpt.(2). In order to make contact with the treatment of Chpt.(2), consider the problem of a cylinder characterized by a dielectric constant ϵ, surrounded by free space $(\epsilon_2 = \epsilon_0)$, and located in a uniform external field $E_x = E_0$. A uniform polarization density transverse to the cylinder axis, P_x, produces an internal field given by

$$\mathrm{E_x} = -\frac{\mathrm{P_x}}{2\epsilon_0},$$

because the depolarization coefficient for this geometry is 1/2. When this field is added to the applied field the total electric field along the x-direction inside the cylinder is given by

$$E_x = E_0 - \frac{P_x}{2\epsilon_0}. \tag{3.2.16}$$

It is this total field that polarizes the material of the cylinder. By definition

$$D_x = \epsilon E_x = \epsilon_0 E_x + P_x;$$

therefore

$$P_x = (\epsilon - \epsilon_0) E_x.$$

Now substitute Equation (3.2.16) for the electric field to obtain

$$P_x = (\epsilon - \epsilon_0)\left(E_0 - \frac{P_x}{2\epsilon_0}\right).$$

Solving for P_x this gives

$$P_x = 2\epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right). \tag{3.2.17}$$

From Equation (3.2.16) the total electric field inside the cylinder is

$$E_x = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0,$$

in agreement with Equation (3.2.15) deduced from the potential function for the case $\epsilon_2 = \epsilon_0$ and $\epsilon_1 = \epsilon$.

The potential function outside the cylinder corresponds to the uniform applied electric field, E_0, plus the potential due to a line of dipoles whose dipole moment per unit length is given by

$$P_{Lx} = \left(\pi R^2\right) 2\epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right)$$

(by comparison of Equation (3.2.14) with the expression for the potential function for a line of dipoles given in Section(2.7.3)). This is equivalent to a dipole moment per unit volume

$$P_x = \frac{P_{Lx}}{\pi R^2} = 2\epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right),$$

in agreement with Equation (3.2.17).

This problem has been treated in detail because it is the prototype for all problems in cylindrical polar co-ordinates that involve boundaries describable by the form r= constant. At each surface of discontinuity one must require the potential function to be continuous through the surface. In addition the normal component of the displacement vector, $vecD$, is required to be continuous through the surface if that surface contains no surface free charge density. These conditions, together with the requirement that the potential function behave properly in the limits as r approaches zero and as r approaches infinity, serve to determine the coefficients in the expansions Equation (3.2.8). The solution so found is guaranteed to be **the solution** apart from an additive constant.

(d) Spherical Polar Co-ordinates.

LaPlace's equation written in spherical polar co-ordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right) = 0. \tag{3.2.18}$$

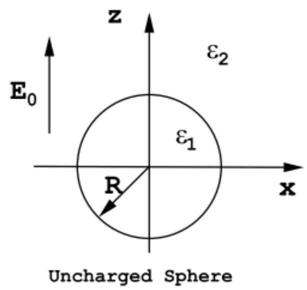


Figure 3.2.6: An uncharged dielectric sphere, dielectric constant ϵ_1, situated in a medium characterized by a dielectric constant, $\epsilon 2$, in the presence of a uniform electric field $E_z = E_0$.

For simplicity consider problems that are symmetric around the z-axis so that the potential function does not depend upon the angle variable ϕ. The general solution of LaPlace's equation for that case is

$$V(r,\theta) = \sum_{n=0}^{n=\infty}\left(a_n r^n + \frac{b_n}{r^{n+1}}\right)P_n(\cos\theta). \tag{3.2.19}$$

The angular functions $P_n(\cos\theta)$ are called Legendre polynomials: the first few of them are listed in Table (3.2.2). The coefficients a_n, b_n must be chosen to satisfy the boundary conditions for a particular problem. As an example, consider a dielectric sphere having a dielectric constant ϵ_1 surrounded by a medium characterized by a dielectric constant $\epsilon 2$ and immersed in a uniform applied field, $E_z = E_0$, see Figure (3.2.6). The electric field is directed along the z-axis, and is supposed to be produced by sources that are very far removed from the position of the sphere. Far from the sphere the potential function must have the form

$$V(r,\theta) \to -E_0 r\cos\theta,$$

corresponding to a uniform field E_0. This suggests that the potential both inside and outside the sphere should be proportional to the Legendre polynomial $P_1 = \cos(\theta)$. One is therefore led to try

Table 3.2.2: The first five Legendre polynomials $P_n(x)$: see Schaum's Outline Series "Mathematical Handbook" by Murray R. Spiegel, McGraw-Hill, N.Y.,1968. The multiplicative constant in front of each polynomial has been chosen so that the polynomials satisfy the condition $\int_{-1}^{1} dx P_m P_n = \left(\frac{2}{2n+1}\right)\delta_{mn}$, where $\delta_{mn} = 1$ 1 if m=n, and zero otherwise.

$P_0 = 1$
$P_1 = x$
$P_2 = \frac{1}{2}(3x^2 - 1)$
$P_3 = \frac{1}{2}(5x^3 - 3x)$
$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$
$P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

Inside.

$$V_i = Ar\cos\theta.$$

There is no term b_1/r^2 term because there is no charge at r=0 that would cause the potential function to be singular at the origin.

Outside.

$$V_o = \left(ar + \frac{b}{r^2}\right)\cos\theta,$$

where $a = -E_0$ in order that the potential reduce to that corresponding to a uniform field of strength E_0 at distances far from the sphere.

On the surface of the sphere the potential must be continuous on passing from the inside to the outside the sphere; this continuity of the potential function guarantees that the tangential component of E will be continuous across the surface of the sphere as is required by the Maxwell equation $\mathrm{curl}(\vec{E}) = 0$. One finds, for r=R,

$$AR = -E_0 R + \frac{b}{R^2},$$

or

$$A = -E_0 + \frac{b}{R^3}. \tag{3.2.20}$$

On the surface of the sphere the normal component of $\vec{D}$ must be continuous. This condition gives

$$\epsilon_1 A = -\epsilon_2 \left(E_0 + \frac{2b}{R^3} \right). \tag{3.2.21}$$

Equations (3.2.20) and (3.2.21) can be solved for A and b. The result of the calculation is

$$A = -\left(\frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} \right) E_0,$$

and

$$b = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \right) R^3 E_0.$$

The function

$$V_i(r, \theta) = -\frac{3\epsilon_2 E_0 r \cos\theta}{(\epsilon_1 + 2\epsilon_2)}.$$

satisfies $\nabla^2 V = 0$ everywhere in the region inside the sphere. The function

$$V_o(r, \theta) = -E_0 r \cos\theta + \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} \right) \frac{R^3 E_0 \cos\theta}{r^2}$$

satisfies $\nabla^2 V = 0$ everywhere in the region outside the sphere. Moreover, these two functions satisfy all of the boundary conditions for this problem. The uniqueness theorem guarantees that this is **the solution** of the problem of an uncharged dielectric sphere subject to a uniform applied electrostatic field.

For the particular case in which an uncharged dielectric sphere characterized by a dielectric constant ϵ is located in free space, dielectric constant ϵ_0, the above result reduces to

$$V_i(r, \theta) = -\left(\frac{3E_0}{2 + \epsilon_r} \right) r \cos\theta, \tag{3.2.22}$$

and

$$V_o(r, \theta) = -E_0 r \cos\theta + \left(\frac{\epsilon_r - 1}{2 + \epsilon_r} \right) \frac{R^3 E_0 \cos\theta}{r^2}, \tag{3.2.23}$$

where the relative dielectric constant is $\epsilon_r = (\epsilon_2/\epsilon_0)$. These expressions are consistent with the results of Chpt.(2), Section(2.7.4) in which it was stated that the depolarization factor for a sphere is 1/3. The second term in Equation (3.2.23) corresponds to the potential generated by a point dipole at the center of the sphere having the strength

$$p_z = 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{2 + \epsilon_r} \right) R^3 E_0.$$

This moment corresponds to a polarization per unit volume directed along z and having the value

$$P = p_z / \frac{4\pi R^3}{3} = 3\epsilon_0 \left(\frac{\epsilon_r - 1}{2 + \epsilon_r} \right) E_0 \quad \text{Coulombs}/m^2. \tag{3.2.24}$$

This uniform polarization would produce a depolarizing field within the sphere given by

$$E_z = -\left(\frac{\epsilon_r - 1}{2 + \epsilon_r} \right) E_0 \quad \text{Volts/m}.$$

When this is combined with the applied field, E_0, the total field within the dielectric sphere becomes

$$E_i = \frac{3E_0}{2+\epsilon_r} \quad \text{Volts}/\text{m}$$

in agreement with the inner field calculated from the potential function Equation (3.2.22).

3.2.2 The Method of Images.

(a) A Charge Located Near a Plane Interface.

This is a very specialized technique for solving electrostatic problems that involves setting up a distribution of non-existent charges in such a way that the boundary conditions on the real problem are satisfied. For example, consider a point charge, q, located in vacuum and at a distance d in front of an infinite conducting plane, Figure (3.2.7). In the region to the left of the interface the potential function must satisfy $\nabla^2 V = 0$. The boundary conditions are:

(1) Very near the position of the charge the potential function must have the form required for a point charge q, i.e.

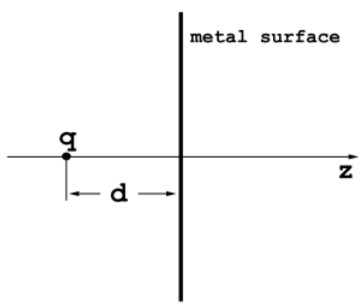
$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}\right).$$

(2) The conducting surface must be an equipotential surface, i.e. V=const.

These two boundary conditions are satisfied by the system of two charges shown in the bottom diagram of Figure (3.2.7). The real problem involving a conducting surface has been replaced by an image problem which just involves two charges in free space. The potential at any point in space for the image problem is

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right).$$

On the symmetry plane $r_1 = r_2$ and therefore V=0, and is constant, everywhere on the symmetry plane. Moreover, this potential function satisfies $\nabla^2 V = 0$ everywhere, except right at the two charges, because it is the sum of two point charge potentials each of which separately satisfies LaPlace's



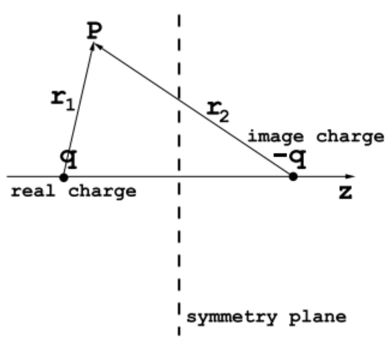


Figure 3.2.7: Top figure: a point charge located a distance d in front of an infinite conducting metal plane. Bottom figure: The system of charges whose electrostatic potential satisfies $\nabla^2 V = 0$ as well as the boundary conditions for the problem posed in the top figure. This solution is only valid in the vacuum region: in the metal V = 0.

equation. This potential function obviously approaches the limit $\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}\right)$ as $r_1 \rightarrow 0$. Therefore the potential function for the image problem of Figure (3.2.7) is also the potential function which satisfies all the requirements for the problem shown in the top diagram of Figure (3.2.7) in the region outside the conductor, ($z \leq 0$). According to the uniqueness theorem, this is therefore the required solution. Of course this solution is only valid for the region outside the conductor: inside the conductor the potential is constant and equal to zero. This image problem can be easily generalized to the problem in which the space outside the conducting plane is filled with a dielectric material ϵ simply by replacing ϵ_0 with ϵ.

(b) A Charged Particle Located Near an Interface between Two Dielectric Materials.

The problem of a point charge outside a plane interface of discontinuity in the dielectric constant can also be solved by the method of images, although in this case the required image charge distribution is not so obvious. Refer to Figure (3.2.8). Let the potential function in Region(1) be that due to the real charge q plus an image charge q_1 symmetrically placed with respect to the interface, see Figure (3.2.8). If space were homogeneously filled with material characterized by a dielectric constant ϵ_1 the resulting potential would be given by

$$\mathrm{V_L} = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\mathrm{r_1}} + \frac{q_1}{\mathrm{r_2}}\right). \tag{3.2.25}$$

Let the potential to the right of the interface be the same as that due to an image charge q_2 located at the position of the real charge, but a charge that is immersed in a homogeneous dielectric material characterized by a dielectric constant ϵ_2:

$$\mathrm{V_R} = \frac{1}{4\pi\epsilon_2}\left(\frac{q_2}{\mathrm{r_1}}\right). \tag{3.2.26}$$

Clearly both V_L and V_R satisfy LaPlace's equation. The trick now is to choose the image charges q_1,q_2 so as to satisfy the boundary conditions

(1) $V_L = V_R$ on the interface between the two dielectrics; and

(2) the normal component of $\vec{D}$ must be continuous across the interface

between the two dielectrics, ie.

$$\epsilon_1\left(\frac{\partial \mathrm{V_L}}{\partial \mathrm{n}}\right) = \epsilon_2\left(\frac{\partial \mathrm{V_R}}{\partial \mathrm{n}}\right).$$

Boundary condition (1) gives

$$\frac{1}{4\pi\epsilon_1}\left(\frac{q}{\mathrm{r_0}} + \frac{q_1}{\mathrm{r_0}}\right) = \frac{1}{4\pi\epsilon_2}\left(\frac{q_2}{\mathrm{r_0}}\right), \tag{3.2.27}$$

where $r_1 = r_2 = r_0$ on the boundary between the two dielectrics. Boundary condition (2) gives

$$-\frac{q\mathrm{d}}{\mathrm{r_0^3}} + \frac{q_1\mathrm{d}}{\mathrm{r_0^3}} = -\frac{q_2\mathrm{d}}{\mathrm{r_0^3}}. \tag{3.2.28}$$

Equation(3.2.28) is the consequence of the relation

$$\frac{\partial}{\partial \mathrm{n}}\left(\frac{1}{\mathrm{r_1}}\right) = \frac{\partial}{\partial \mathrm{z}}\left(\frac{1}{\sqrt{\mathrm{x^2+y^2+(z+d)^2}}}\right) = -\frac{\mathrm{(z+d)}}{\mathrm{(x^2+y^2+(z+d)^2)}^{\,3/2}}.$$

When this expression is evaluated on the interface, ie. at $z = 0$, the result is

$$\frac{\partial}{\partial \mathrm{z}}\left(\frac{1}{\mathrm{r_1}}\right) = \frac{-d}{\mathrm{r_0^3}}.$$

Similarly

$$\frac{\partial}{\partial \mathrm{z}}\left(\frac{1}{\mathrm{r_2}}\right) = \frac{+d}{\mathrm{r_0^3}}.$$

The boundary condition Equation (3.2.28) requires

$$q-q_1=q_2.$$

When this is combined with the first boundary condition, Equation (3.2.27), one obtains

$$q_1=\frac{\left(\frac{\epsilon_1}{\epsilon_2}-1\right)}{\left(\frac{\epsilon_1}{\epsilon_2}+1\right)}q \tag{3.2.29}$$

and

$$q_2=\frac{2q}{\left(\frac{\epsilon_1}{\epsilon_2}+1\right)}. \tag{3.2.30}$$

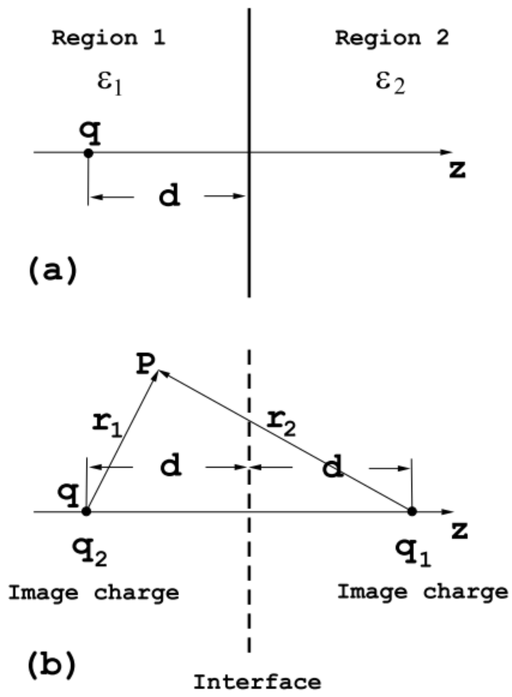


Figure 3.2.8: (a) The real problem: a charge q located a distance d from the interface between two uncharged dielectric media. (b) The configuration of image charges that produce a potential that satisfies $\nabla^2 V = 0$ and that can be used to satisfy the required boundary conditions.

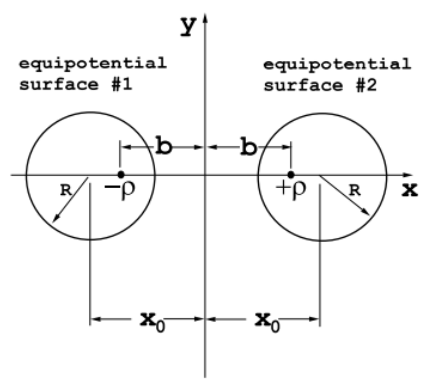


Figure 3.2.9: Equipotential surfaces for two line charges of equal strength but of opposite sign.

The solution of the original problem illustrated in part (a) of Figure (3.2.8) is given by Equation (3.2.25), valid for region(1) characterized by ϵ_1, and by Equation (3.2.26) in region(2) characterized by ϵ_2, where q_1 and q_2 are given by Equations (3.2.29 and

3.2.30). The force acting on the real charge q is just q multiplied by the electric field generated at the position of q by the image charge q1: ie. the electric field $\vec{E}$ = −grad(V_L). It follows that if $\epsilon_2 > \epsilon_1$ the charge q is attracted to the interface, but if $\epsilon_2 < \epsilon_1$ then the charge q is repelled by the interface.

(c) Parallel Conducting Cylinders

Let two line charges of strengths −ρ and +ρ Coulombs/m be separated by the distance 2b along the x-axis as shown in Figure (3.2.9). In the first instance let these line charges be immersed in vacuum. The potential generated by a line charge of strength ρ is given by

$$\mathrm{V}(\mathrm{r})=-\left(\frac{\rho}{2\pi\epsilon_0}\right)\ln(\mathrm{r}),$$

(see Section(2.7.3)). If r_1 is the distance from −ρ to an observer at P , and if r_2 is the distance from the line charge +ρ to the observer at P, then the potential at P is given by

$$\mathrm{V}_\mathrm{P}=\frac{\rho}{2\pi\epsilon_0}\ln(\mathrm{r}_1/\mathrm{r}_2).$$

Let (r_1/r_2) = k, a constant, so that

$$\mathrm{V}=\frac{\rho}{2\pi\epsilon_0}\ln(k)\quad \text{Volts}\,.$$

This is the potential on all points that satisfy the condition $r_1 = kr_2$, or$r_1^2=k^2r_2^2$. This last condition can be written out explicitly in cartesian co-ordinates:

$$(x+b)^2+y^2=k^2\left((x-b)^2+y^2\right).$$

With the application of some tedious algebra this last expression may be put in the form:

$$\left(x+b\left(\frac{1+k^2}{1-k^2}\right)\right)^2+y^2=\frac{4b^2k^2}{\left(1-k^2\right)^2}. \tag{3.2.31}$$

Equation (3.2.31) describes a circle centered at

$$\mathrm{x}_0=-\mathrm{b}\left(\frac{1+\mathrm{k}^2}{1-\mathrm{k}^2}\right), \tag{3.2.32}$$

with a radius

$$\mathrm{R}=\frac{2\mathrm{bk}}{\left|1-\mathrm{k}^2\right|}. \tag{3.2.33}$$

Notice that k′ = 1/k corresponds to an equipotential surface centered at

$$\mathrm{x}_0'=\mathrm{b}\left(\frac{1+\mathrm{k}^2}{1-\mathrm{k}^2}\right)=-\mathrm{x}_0,$$

and having the same radius R as the equipotential surface corresponding to k and centered at x_0. Equipotential surfaces for k and 1/k are illustrated in Figure (3.2.9). The two equipotential surfaces shown correspond to different potentials. The cylinder on the left corresponds to r_1/r_2 = k; the cylinder on the right corresponds to r_1/r_2 = 1/k. It follows that the potential of the cylinder on the right is equal in magnitude but opposite in sign to the potential of the cylinder on the left.

These families of displaced equipotential cylindrical surfaces can be used to solve a number of parallel conducting cylinder problems. The same treatment works if the cylinders are immersed in a dielectric medium; one has only to replace $epsilon_0$ by the dielectric constant for the medium, ϵ.

(d) A Point Charge Outside a Conducting Sphere.

The problem of the potential function generated by a point charge located outside a conducting sphere can also be solved using the method of images. Consider the geometry shown in Figure (3.2.10). The potential everywhere outside the conducting sphere is the same as the potential generated by two point charges: (1) the original point charge q; and (2) an image charge $-q_1$. (The minus sign has been introduced for convenience; it makes sense that the charges induced on the sphere should have a sign that is opposite to that of the charge q.) The assertion is that if the position of the charge q_1 as well as its magnitude are properly chosen then these two charges will create a potential that is constant on the surface of the sphere. This assertion is by no means obvious, but let us see how this comes about. The potential generated at the point of observation, P, is given by

$$V_p = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} - \frac{q_1}{r_1}\right),$$

where

$$r = \sqrt{x^2 + y^2 + (z-d)^2}$$

and

$$r_1 = \sqrt{x^2 + y^2 + (z-e)^2}\ .$$

It is now obvious from the form of the above potential that the potential will be zero on all points such that

$$\frac{q}{r} = \frac{q_1}{r_1}.$$

So let $qr_1 = q_1 r$, or more conveniently let $q^2 r_1^2 = q_1^2 r^2$. Write out this last equation explicitly in cartesian co-ordinates:

$$x^2 + y^2 + (z-e)^2 = \left(\frac{q_1}{q}\right)^2 \left[x^2 + y^2 + (z-d)^2\right],$$

or

$$x^2 + y^2 + z^2 - 2ez + e^2 = \left(\frac{q_1}{q}\right)^2 \left[x^2 + y^2 + z^2 - 2zd + d^2\right].$$

Gather terms to obtain

$$\left[1 - \left(\frac{q_1}{q}\right)^2\right](x^2 + y^2 + z^2) - 2z\left[e - \left(\frac{q_1}{q}\right)^2 d\right] = \left(\frac{q_1}{q}\right)^2 (d^2) - e^2.$$

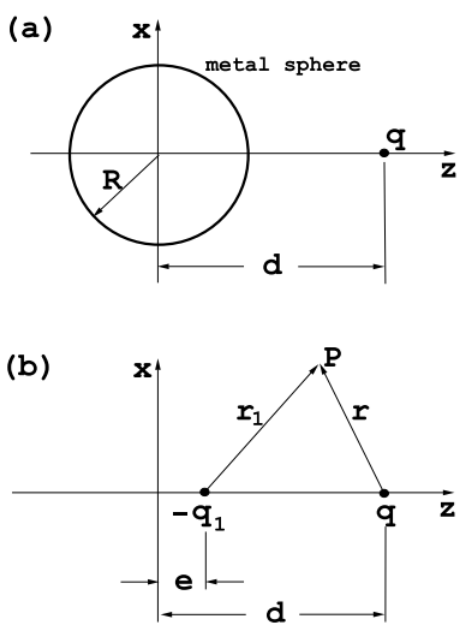


Figure 3.2.10: The real problem of a point charge located a distance d from the center of a metal sphere of radius R is shown in (a). In part (b) the real problem has been replaced by two point charges as shown in the figure.

Note that if

$$e=\left(\frac{q_1}{q}\right)^2$$

then the potential V = 0 at all points such that

$$x^2+y^2+z^2=\frac{\left(\frac{a_1}{q}\right)^2 d^2-e^2}{\left[1-\left(\frac{q_1}{q}\right)^2\right]},$$

or using

$$e=\left(\frac{q_1}{q}\right)^2$$

$$x^2+y^2+z^2=\left(\frac{q_1 d}{q}\right)^2. \tag{3.2.34}$$

This means that the spherical equipotential surface corresponding to V = 0 will coincide with the surface of the metal sphere if

$$q_1=q\left(\frac{R}{d}\right), \tag{3.2.35}$$

and

$$e=\left(\frac{R^2}{d}\right). \tag{3.2.36}$$

So if Equations (3.2.35 and 3.2.36) are satisfied then the potential everywhere outside the metal sphere will satisfy $\nabla^2 V = 0$ because the potential is the sum of two point charge potentials each of which satisfies the LaPlace equation. Moreover, this potential function satisfies the boundary condition that the surface of the sphere be an equipotential surface. This solution corresponds to the special case in which charge is allowed to flow onto the sphere as the driving charge q is brought up from infinity. It can be shown using Gauss' theorem that the charge induced on the sphere is just $-q_1$. The problem of an isolated sphere such that the net charge on it is zero can be solved by adding a third charge of strength $+q_1$ to the position of the center of the sphere in the image problem of Figure (3.2.10(b)).The potential in the region outside the sphere is now given by

$$V(x,y,z)=\frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_2}+\frac{q}{r}-\frac{q_1}{r_1}\right], \tag{3.2.37}$$

where

$$r_2=\sqrt{x^2+y^2+z^2}.$$

The potential function Equation (3.2.37) satisfies the LaPlace equation, $\nabla^2 V = 0$, the surface of the sphere is an equipotential surface, and it corresponds to a net charge of zero on the sphere. The potential of the sphere is just

$$V_s=\frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{R}\right)=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{d}\right),$$

because the last two terms in Equation (3.2.37) cancel each other, by construction, on the surface of the sphere.

3.2.3 Two-dimensional Problems.

(a) The Theory of Complex Variables.

The theory of complex variables may be useful for solving problems in which the potential function does not depend upon one co-ordinate- the z coordinate, say. Let z=x+iy represent a complex number, and let F(z)= U(x,y)+iV(x,y) be an analytic function of the complex variable z. In order for F(z) to exhibit a well-defined derivative it can be shown that the Cauchy-Riemann equations must be satisfied:

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y},$$

and

$$\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}.$$

See, for example, Schaum's Outline Series: Complex Variables by Murray R. Spiegel, McGraw-Hill, N.Y.,1964. It follows from the Cauchy-Riemann relations by direct differentiation that

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,$$

and

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0.$$

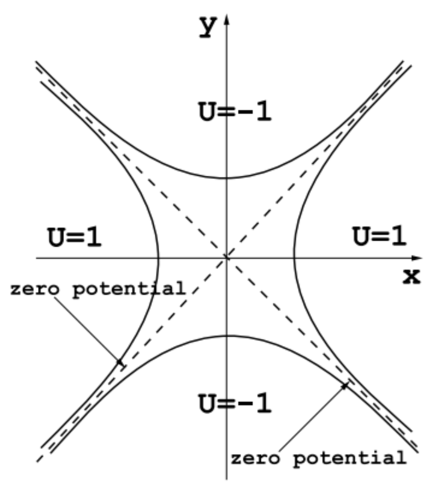


Figure 3.2.11: Electrodes in the form of cylindrical hyperboloids. One pair of electrodes is held at a potential U=-1 Volt; the other pair of electrodes is held at a potential U=+1 Volt. The potential function in the space between the electrodes is given by $U = x^2 - y^2$. The electric field components in the space between the electrodes are given by $E_x = -2x$ and $E_y = +2y$.

That is, both of the functions U(x,y) and V(x,y) satisfy LaPlace's equation. Both U and V are therefore candidates for the solution of some problem in electrostatics. Consider, for example, the analytic function

$$F(z)=z^2=\left(x^2-y^2\right)+2ixy. \tag{3.2.38}$$

In this case

$$U(x,y)=x^2-y^2$$

and

$$V(x,y)=2xy.$$

The families of curves U= const. and V= const. are orthogonal to each other. If the equipotential surfaces are represented by U(x,y)= const. (see Figure (3.2.11)) then the curves V(x,y)= const represent the electric field lines: electric field lines are constructed so that their tangent at each point is parallel with the direction of the electric field. Conversely, if the equipotential surfaces are described by the curves V(x,y)= const. then the curves U(x,y)= const. represent the field lines. Other examples are described in the Feynman Lectures on Physics,Vol.(II), section 7-2. In principle, the technique of conformal mapping (described in Schaum's Outline Series: Complex Variables, loc.cit.) can be used to determine the potential distribution around electrodes whose shape can be represented by a polygon.

(b) Analogue Solution Using Conducting Paper.

Two dimensional problems can also be solved by means of a kind of analogue computer. The desired electrode configuration is painted on conducting paper using a metallic conducting paint, eg. silver dag, see Figure (3.2.12). The silver paint electrodes portrayed in this figure would represent an infinitely long metal, circular, cylinder placed between two infinitely long parallel metal plates. The electrodes are held at fixed potentials V_1, V_2, and V_3. The currents which flow in the conducting paper must be such that there is no charge build-up anywhere; they must, therefore, satisfy the equation

$$\mathrm{div}(\mathrm{J}) = 0.$$

But in a conducting medium one has

$$\mathrm{J} = \sigma \mathrm{E}$$

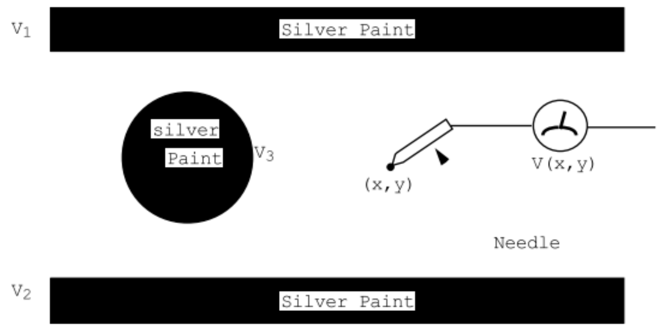


Figure 3.2.12: Electrodes of silver paint drawn on a sheet of conducting paper. The resistivity of the paper is much larger than that of the silver paint electrodes.The equipotential lines can be mapped out by means of a voltmeter connected to a pointed probe.

so that

$$\mathrm{div}(\mathrm{E}) = 0.$$

From this last equation it follows that the potential distribution in the conducting paper must satisfy $\nabla^2 V = 0$ because $E = -\mathrm{grad}V$. The equipotential lines corresponding to a desired potential value can be traced out on the conducting paper by means of a high input impedance voltmeter connected to a sharply pointed probe. The electric field can, of course, be obtained from the known potential distribution via a numerical differentiation.

3.3: Electrostatic Field Energy

It will be shown in Chapter(8) that it costs energy to set up an electric field. As the electric field increases from zero the energy density stored in the electrostatic field, W_E, increases according to

$$\frac{\partial W_E}{\partial t} = \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}.$$

For the particular case in which the electric field is set up in a dielectric medium that can be described by a dielectric constant so that $\vec{D} = \epsilon \vec{E}$, this expression can be written

$$\frac{\partial W_E}{\partial t} = \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\epsilon}{2} \frac{\partial E^2}{\partial t}. \tag{3.3.1}$$

Eqn.(3.3.1) can be integrated immediately to obtain

$$W_E = \frac{\epsilon E^2}{2} = \frac{1}{2} \vec{E} \cdot \vec{D} \quad \text{Joules } / m^3. \tag{3.3.2}$$

In the above expressions the zero of energy has been chosen to be zero when the electrostatic field is everywhere zero. The total energy stored in the electrostatic field is obtained as an integral of W_E over all space. This total energy, U_E, can be expressed in terms of the potentials and charges on the electrodes that created the electric field. This can be shown by starting from the vector identity

$$\operatorname{div}(V\vec{D}) = V\operatorname{div}(\vec{D}) + \vec{D} \cdot \operatorname{grad}(V), \tag{3.3.3}$$

where $\vec{D}$ is any vector field and V is a scalar function. This identity can be proved by writing out the divergence in cartesian co-ordinates and by carrying out the differentiations. But from Maxwell's equations $\operatorname{div}(\vec{D}) = \rho_f$, and by definition $\vec{E} = -\operatorname{grad}(V)$, so that

$$\iiint_{Volume} \operatorname{div}(VD) d(\mathrm{Vol}) = \iiint_{Volume} \left(\rho_f V - \vec{E} \cdot \vec{D}\right) d(\mathrm{Vol}). \tag{3.3.4}$$

The volume integral on the left can be replaced by a surface integral by using Gauss' theorem:

$$\iiint_{Volume} \operatorname{div}(V\vec{D}) d(\mathrm{Vol}) = \iint_{Surface} VD \cdot \overrightarrow{dS}.$$

As the volume becomes very large and the surface S recedes to infinity, the surface integral becomes very small. Very far from all charges the potential V must decrease at least as fast as 1/R (the potential due to a point charge) and $|\vec{D}|$ must decrease at least as fast as $1/R^2$ (again a point charge) whereas the surface area increases like R^2. It follows that the surface integral must decrease at least as fast as 1/R in the limit as the dimensions of the surface become infinitely large. It follows from Equation (3.3.4) that

$$\iiint_{Volume} \rho_f V d(\mathrm{Vol}\,) = \iiint_{Volume} (\vec{E} \cdot \vec{D}) d(\mathrm{Vol}\,),$$

and therefore

$$U_E = \iiint_{Space} W_E d(\mathrm{Vol}) = \frac{1}{2} \iiint_{Space} \rho_f V d(\mathrm{Vol}). \tag{3.3.5}$$

For a collection of conductors embedded in a non-conducting dielectric medium all of the charges are on the conductor surfaces and the charges on a given conductor are all at the same potential. In that case the integrals in Equation (3.3.5) simply give the product of electrode potential and the total charge on the electrode:

$$U_E = \frac{1}{2} \sum_n Q_n V_n. \tag{3.3.6}$$

3.3.1 Generalized Capacitance Coefficients

Maxwell's equations are linear, therefore the potentials associated with electrodes embedded in a material that obeys linear response must obey the principle of superposition. The potential distribution that is generated by a particular charge is proportional to the quantity of that charge. It follows from superposition that for any collection of charges the potential at any point must be a linear function of the charge strengths. The converse must also be true. Given a collection of conducting electrodes embedded in a linear dielectric medium the charge on each of the electrodes must be a linear function of the electrode potentials: if the potentials are doubled then so must the charge on each electrode be doubled and vice versa.

This linear dependence of the charge on potentials can be expressed as follows (see Figure (3.3.13):

$$\begin{aligned} Q_1 &= C_{11}V_1 + C_{12}V_2 + \ldots + C_{1n}V_n \\ Q_2 &= C_{21}V_1 + C_{22}V_2 + \ldots + C_{2n}V_n \\ &\vdots \quad \vdots \\ Q_n &= C_{n1}V_1 + C_{n2}V_2 + \ldots + C_{nn}V_n \end{aligned} \tag{3.3.7}$$

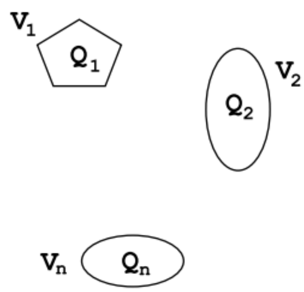


Figure 3.3.13: Charged conductors embedded in a linear dielectric medium. The charges are a linear function of the potentials, see Equation (3.3.8) in the text.

The factors of proportionality, C_{mn}, are called **capacitance coefficients**; they have the units of Farads. These equations express the observation that a change in the potential of one electrode causes a change in the amount of charge stored on every electrode, not just on the electrode whose potential was altered. The energy stored in the electric field, which can be calculated from Equation (3.3.6), must be independent of how the charging process was carried out. It must not matter, for example, whether electrode (1) is first charged, then electrode (2), then electrode (3), and so on, or whether (3) is charged first, then (2), then (1), then (4), and so on. The energy contained in the final state of the system must be independent of the way in which that final state was reached: in order that this be so, it can be shown that

$$C_{mn} = C_{nm}.$$

Instead of N^2 independent capacitance coefficients there are only N(N+1)/2 of them. Notice that these capacitance coefficients are geometry dependent. Any change in the shape of any electrode, or a change in the position of any electrode, will result in a change in all of the capacitance coefficients. It follows also that the energy stored in the electric field must change. This change in field energy can, in principle, be used to calculate the electrostatic forces on the conductors or on the dielectric medium.

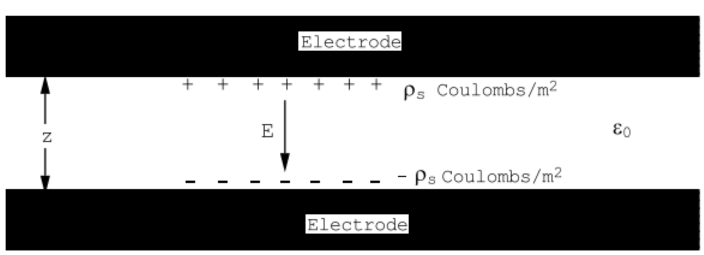


Figure 3.3.14: A parallel plate capacitor. The two plates have an area A and are separated by a distance z. The charge density on each plate is ρ_s Coulombs/m^2 .

3.3.2 Electrostatic Forces.

Case(1) The Charges are Fixed.

The charges on each conductor are held fixed, and one of the conductors is allowed to undergo a slight displacement $\overrightarrow{\delta \mathrm{r}}$. During this displacement the electric forces will do an amount of work

$$\delta \mathrm{w}=\overrightarrow{\mathrm{F}} \cdot \overrightarrow{\delta \mathrm{r}} .$$

This work can only be done at the expense of the energy stored in the electric field since there are no other energy sources. Consequently

$$\overrightarrow{\mathrm{F}_{\mathrm{E}}} \cdot \overrightarrow{\delta \mathrm{r}}=-\delta \mathrm{U}_{\mathrm{E}} .$$

The energy stored in the electric field acts like a potential function for the electrical forces. As an example, consider the parallel plate capacitor of Figure (3.3.14). It is convenient in this case to work with a unit area of electrode surface, and to take metal plates that are so large that edge effects can be neglected. For a fixed surface charge density on each electrode the electric field strength between the plates is independent of the electrode spacing, z. The energy stored in the electric field per unit area of electrode can be calculated from the energy density Equation (3.3.2); the result of the calculation is

$$\mathrm{U}_{\mathrm{E}}=\left(\frac{\rho_{s}^{2}}{2 \epsilon_{0}}\right) \mathrm{z}$$

since the electric field strength is given by $\mathrm{E}=\rho_{s} / \epsilon_{0}$. Let the plates be moved apart by a small increment dz. The work done on the displaced plate by the electrical force per unit area is given by Fdz. This work must be done at the cost of the stored electrical energy, therefore

$$\mathrm{F} \mathrm{dz}=-\left(\frac{\rho_{s}^{2}}{2 \epsilon_{0}}\right) \mathrm{dz},$$

or

$$\mathrm{F}=-\left(\frac{\rho_{s}^{2}}{2 \epsilon_{0}}\right)=-\frac{1}{2} \rho_{s} \mathrm{E} \quad \text { newtons } / \mathrm{m}^{2} . \tag{3.3.8}$$

The electric forces act in such a way as to pull the electrodes together. This is the expected result because one plate carries a positive charge and the other plate carries a negative charge. As a guess, one might have thought that the force per unit area on a given electrode would just be given by the charge density multiplied by the electric field at the surface of the electrode, i.e. ρ_sE. The result Equation (3.3.8) shows that the average field acting on the charges must be used to calculate the force (remember that E=0 inside the conductor).

Although the above result for the force on a conductor has been derived for a plane parallel plate, it turns out to be valid for the electric force per unit area acting on the surface of any conductor facing vacuum. There is a negative pressure acting on the conductor surface that depends only upon the local values of the field strength and the surface charge density. This negative pressure, or tension t_E, is given by

$$\mathrm{t}_{\mathrm{E}}=\frac{\rho_{s} \mathrm{E}}{2}=\frac{\epsilon_{0}}{2} \mathrm{E}^{2} \quad \text { Newtons } / \mathrm{m}^{2} .$$

For a conducting surface immersed in a fluid characterized by a dielectric constant ϵ it is easy to show that this tension becomes

$$\mathrm{t}_{\mathrm{E}}=\frac{1}{2} \vec{\mathrm{E}} \cdot \vec{\mathrm{D}}=\frac{\epsilon}{2} \mathrm{E}^{2} \quad \text { Newtons } / \mathrm{m}^{2} . \tag{3.3.9}$$

Case(2) The Potentials are Fixed.

In many instances it is convenient to investigate the electrical force distribution under circumstances in which the electrode potentials are held fixed. Any change in the electrode configuration at fixed potentials that results in a change in the capacitance coefficients will also lead to a change in the amount of charge carried by each conductor. If the change in the charge carried by a particular electrode is $\delta \mathrm{Q}_{\mathrm{M}}$, the work required to add this charge to the conductor is $\delta \mathrm{W}_{\mathrm{B}}=\mathrm{V}_{\mathrm{M}} \delta \mathrm{Q}_{\mathrm{M}}$ and this energy is provided by the source of emf that is attached to conductor M, i.e. by the battery which is used to maintain the constant potential. The change in energy stored in the electric field can be calculated from Equation (3.3.6); the result for constant potentials is

$$\delta \mathrm{U_E} = \frac{1}{2}\sum_N \mathrm{V_N}\delta \mathrm{Q_N}. \tag{3.3.10}$$

The energy provided by the batteries that hold the potentials $\mathrm{V_N}$ constant is given by

$$\delta \mathrm{W_B} = \sum_N \mathrm{V_N}\delta \mathrm{Q_N}. \tag{3.3.11}$$

The energy supplied from the batteries is exactly twice the increase in the energy stored in the electric field. The work done by the electrical forces in moving an electrode is $\vec{\mathrm{F}}_\mathrm{E}\cdot\vec{\mathrm{dr}}$. Conservation of energy now gives

$$\vec{\mathrm{F}}_\mathrm{E}\cdot\vec{\mathrm{dr}} + \delta \mathrm{U_E} = \delta \mathrm{W_B},$$

or

$$\vec{\mathrm{F}}_\mathrm{E}\cdot\vec{\mathrm{dr}} = \delta \mathrm{U_E} = \frac{1}{2}\sum_N \mathrm{V_N}\delta \mathrm{Q_N}, \tag{3.3.12}$$

since $\delta \mathrm{W_B} = 2\delta \mathrm{U_E}$. For this case the increase in electrical energy stored in the field is exactly equal to the external work done by the electrical forces in changing the electrode geometry.

As an example, consider the configuration shown in Figure (3.3.15). A slab of dielectric material characterized by a dielectric constant ϵ, lies with one end near the center of a plane parallel capacitor and the other end lies well outside the capacitor. The slab has a thickness d meters and a width w meters. The specimen is so long that the electric field at the end that lies

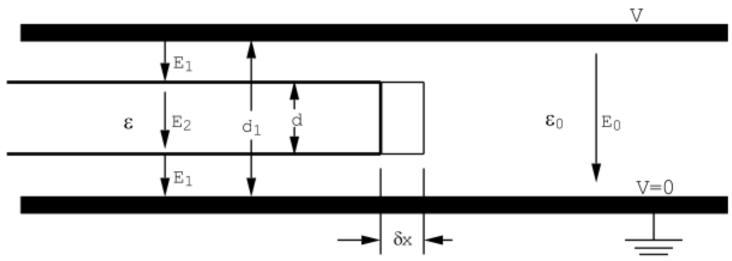


Figure 3.3.15: A plane parallel capacitor partially containing a slab of dielectric material d thick and w wide. The dielectric constant of the slab is ϵ. The objective is to calculate the electric forces acting on the slab.

outside the capacitor is nearly zero and may be neglected. An uncomplicated but tedious calculation gives (refer to Figure (3.3.15)):

$$\mathrm{E_1} = \frac{\mathrm{V}}{\left[\mathrm{d_1} + \left(\frac{\epsilon_0}{\epsilon} - 1\right)\mathrm{d}\right]} \quad \mathrm{Volts/m};$$

$$\mathrm{E_2} = \frac{\left(\frac{\epsilon_0}{\epsilon}\right)\mathrm{V}}{\left[\mathrm{d_1} + \left(\frac{\epsilon_0}{\epsilon} - 1\right)\mathrm{d}\right]} \quad \mathrm{Volts\ /m};$$

and

$$\mathrm{E_0} = \frac{\mathrm{V}}{d_1} \quad \mathrm{Volts/m}.$$

$\mathrm{E_1}$ is the field in the vacuum in a region occupied by the dielectric slab, but far enough from the end of the slab so that inhomogeneities in the field can be neglected: in practice, this means that one is considering a position several slab thicknesses, d, from the end. The quantity $\mathrm{E_2}$ is the electric field strength in the dielectric slab, but at a position several d removed from its end. $\mathrm{E_0}$ is the electric field strength in the region of the capacitor where there is no slab, and far enough from the end of the slab so that fringing fields can be neglected. Now let the slab be inserted δx farther between the capacitor plates. The change in energy stored in the electric field will just be that corresponding to removing a volume $(d_1 w)\,\delta x$ of dielectric-free space where the field is $\mathrm{E_0}$ Volts/m and replacing it with the volume (wd) δx of dielectric material subject to the field $\mathrm{E_2}$ plus the vacuum volume

$w\left(d_1-d\right)\delta x$ subject to the field E_1. This change in energy will be independent of the exact shape of the end of the slab providing that the extent of the non-uniform field region around the end of the slab is very small compared with the lateral dimensions, D, of the capacitor plates, i.e. providing that d/D ≪ 1. The change in stored electrostatic energy for a small displacement δx is given by

$$\delta \mathrm{U_E}=\mathrm{wd}\,\delta\mathrm{x}\left(\frac{\epsilon \mathrm{E}_2^2}{2}+\frac{\epsilon_0\mathrm{E}_1^2}{2}\left[\frac{\mathrm{d}_1}{\mathrm{d}}-1\right]\right)-\left(\mathrm{wd}_1\delta\mathrm{x}\right)\frac{\epsilon_0\mathrm{E}_0^2}{2}.$$

After some algebra this may be written

$$\delta \mathrm{U_E}=(\mathrm{w}\delta\mathrm{x})\left(\frac{\epsilon_0\mathrm{V}^2}{2\mathrm{d}_1}\right)\left(\frac{1-\frac{\epsilon_0}{\epsilon}}{\frac{\epsilon_0}{\epsilon}-1+\frac{\mathrm{d}_1}{\mathrm{d}}}\right).$$

In general the dielectric constant ϵ is greater than ϵ_0 so that the electrostatic energy stored in the field increases if the dielectric slab moves farther into the capacitor. For constant applied voltage this means that the electric forces are such as to pull the slab further between the capacitor plates: at constant applied potential the geometry tends to change so as to maximize the energy stored in the field. The force on the slab is given by

$$\mathrm{F_x}=\mathrm{w}\left(\frac{\epsilon_0\mathrm{V}^2}{2\mathrm{d}_1}\right)\left(\frac{1-\frac{\epsilon_0}{\epsilon}}{\frac{\epsilon_0}{\epsilon}-1+\frac{\mathrm{d}_1}{\mathrm{d}}}\right)\quad \text{Newtons.}\tag{3.3.13}$$

The force on a dielectric slab may be measured and used to obtain the dielectric constant for the slab material, ϵ. A variant of this method is often used to measure the dielectric constant of a fluid, see Figure (3.3.16). Eqn.(3.3.13) becomes much simpler if the thickness of the dielectric slab is the same, or nearly the same, as the spacing between the capacitor plates. When d = d_1 one finds

$$\mathrm{F_x}=\mathrm{w}\left(\frac{\epsilon_0\mathrm{V}^2}{2\mathrm{d}}\right)\left(\frac{\epsilon}{\epsilon_0}-1\right)=\mathrm{w}\left(\frac{\epsilon_0\mathrm{V}^2}{2\mathrm{d}}\right)\chi_e,\tag{3.3.14}$$

where χ_e is the electrical susceptibility defined by $\epsilon=\epsilon_0\left(1+\chi_e\right)$. In the opposite limit, d/d_1 ≪ 1, the force is given by

$$\mathrm{F_x}=\mathrm{wd}\left(\frac{\epsilon_0\mathrm{V}^2}{2\mathrm{d}_1^2}\right)\left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)=\mathrm{wd}\left(\frac{\epsilon_0\mathrm{V}^2}{2\mathrm{d}_1^2}\right)\left(\frac{\chi_e}{1+\chi_e}\right)\quad \text{Newtons.}\tag{3.3.15}$$

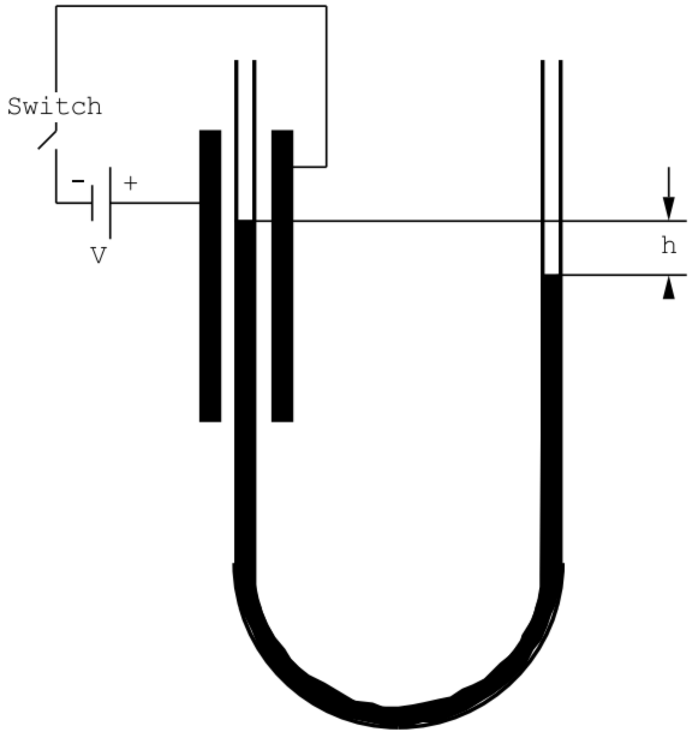


Figure 3.3.16: Quincke's method for measuring the dielectric constant of a fluid. Upon the application of the voltage V to the capacitor plates the dielectric fluid is sucked up between the plates. In equilibrium the electrical force on the fluid just balances the gravitational force. The gravitational force is proportional to the level difference h. The electrical force per unit area, t, is given by t = ρgh where ρ is the fluid density.

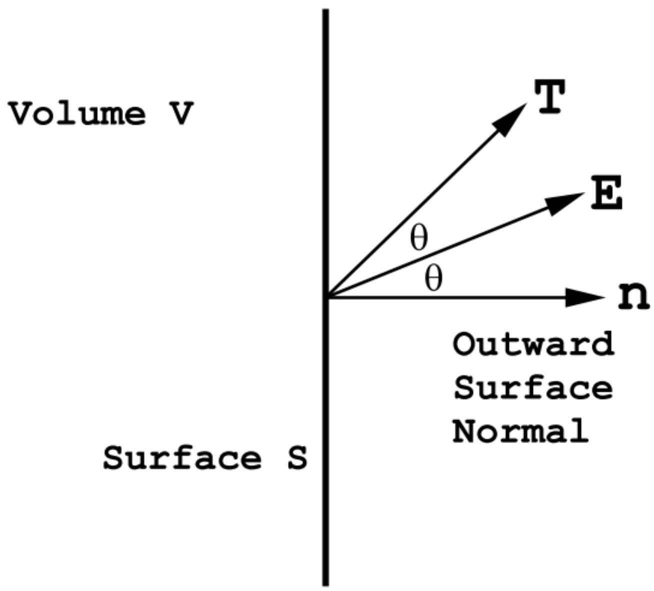


Figure 3.3.17: The force acting on the matter contained within a volume V can be obtained as the surface integral of a vector $\vec{T}$ over a surface S that encloses V. It is assumed that $\vec{D}$ is everywhere inside S proportional to the electric field, $\vec{E}$. It is further assumed that the surface S is immersed in a fluid that can support no shear stresses, and that $\vec{D}$ and $\vec{E}$ are parallel on S. The force per unit area is given by $|\mathrm{T}| = \vec{\mathrm{E}} \cdot \vec{\mathrm{D}}/2$ and the direction of the force per unit area is such that the angle between $\vec{T}$ and the surface normal is bisected by the direction of the electric field.

3.3.3 The Maxwell Stress Tensor

The forces acting on a static charge distribution located in a linear isotropic dielectric medium can be obtained as the divergence of an object called the **Maxwell stress tensor**. It can be shown that there exists a vector $\vec{T}$ associated with the elements of the stress tensor such that the surface integral of $\vec{T}$ over a closed surface S enclosing a volume V gives the net force acting on the charges within V: see, for example, Electromagnetic Theory by J.A.Stratton, section 2.5, (McGraw-Hill, N.Y., 1941). One can write

$$\overrightarrow{\mathrm{F}_{\mathrm{E}}}=\int\int_{\mathrm{S}} \vec{\mathrm{T}} \mathrm{dS}. \tag{3.3.16}$$

In this integral $\vec{T}$ is a vector whose magnitude is given by $|\vec{\mathrm{T}}|=(\vec{\mathrm{E}} \cdot \vec{\mathrm{D}})/2$ and whose direction is given by the construction shown in Figure (3.3.17). Note that the element of area, dS, in Equation (3.3.16) is not represented by a vector; it is simply a scalar quantity. When the electric field, E, is directed parallel with the outward normal to the surface element the force contribution is a tension, but when E lies in the surface the contribution to the force is a pressure. It is an interesting exercise to show that the force between two charges in vacuum is given by $\left(\mathrm{q}_1 \mathrm{q}_2 / 4 \pi \epsilon_0 \mathrm{R}^2\right)$ if one integrates the vector $\vec{T}$ over a suitably chosen surface that completely surrounds one of the charges.

3.4: The Field Energy as Minimum

Consider a group of conductors embedded in a dielectric medium as shown in Figure (3.3.13). Let the actual potential distribution be V corresponding to the electric field $\overrightarrow{\mathrm{E}}=-\operatorname{grad}(\mathrm{V})$; the potential function V satisfies LaPlace's equation, $\nabla^2 V = 0$, and it satisfies the boundary conditions. Now consider a second potential distribution, V_e, that is not the correct one: $\mathrm{V}_{\mathrm{e}}=\mathrm{V}+\delta \mathrm{V}$, where $\delta \mathrm{V}=0$ on the conductors. This situation might arise, for example, if one tried to guess the potential distribution given the potential on each of the conductors. The field energy calculated using the wrong potential function V_e would be

$$\mathrm{U}_{\mathrm{E}}^{1}=\iiint_{S p a c e}(\mathrm{dVol}) \frac{\epsilon}{2}(\overrightarrow{\mathrm{E}}+\delta \overrightarrow{\mathrm{E}}) \cdot(\overrightarrow{\mathrm{E}}+\delta \overrightarrow{\mathrm{E}}),$$

where $\overrightarrow{\mathrm{E}}=-\operatorname{grad}(\mathrm{V})$ and $\delta \mathrm{E}=-\operatorname{grad}(\delta \mathrm{V})$. Upon multiplying out the factors under the integral sign one obtains

$$\mathrm{U}_{\mathrm{E}}^{1}=\iiint_{S p a c e}(\mathrm{dVol})\left(\frac{\epsilon \mathrm{E}^{2}}{2}+\epsilon \overrightarrow{\mathrm{E}} \cdot \delta \overrightarrow{\mathrm{E}}+\frac{\epsilon}{2}(\delta \overrightarrow{\mathrm{E}})^{2}\right). \tag{3.4.1}$$

But

$$\operatorname{div}(\overrightarrow{\mathrm{D}} \delta \mathrm{V})=\delta \mathrm{V} \operatorname{div}(\overrightarrow{\mathrm{D}})+\overrightarrow{\mathrm{D}} \cdot \operatorname{grad}(\delta \mathrm{V})=\rho_{f}(\delta \mathrm{V})-\overrightarrow{\mathrm{D}} \cdot \delta \overrightarrow{\mathrm{E}}.$$

If $\operatorname{div}(\overrightarrow{\mathrm{D}} \delta \mathrm{V})$ is integrated over all space the result is

$$\iiint_{S p a c e}(\mathrm{dVol}) \operatorname{div}(\overrightarrow{\mathrm{D}} \delta \mathrm{V})=\iint_{S u r f a c e} \overrightarrow{\mathrm{dS}} \cdot \overrightarrow{\mathrm{D}} \delta \mathrm{V}=\iiint_{S p a c e}(\mathrm{dVol}) \rho_{f} \delta \mathrm{V}-\iiint_{S p a c e}(\mathrm{dVol}) \overrightarrow{\mathrm{D}} \cdot \delta \overrightarrow{\mathrm{V}}. \tag{3.4.2}$$

The surface integral in Equation (3.4.2) goes to zero for the usual reasons as the surface S becomes infinitely large. Namely, very far from any sources $|\vec{D}|$ goes to zero at least as fast as $1/R^2$ and the surface area increases like R^2 so that the surface integral contribution must vanish providing that the product $|\overrightarrow{\mathrm{D}}| \delta \mathrm{V}$ goes to zero at least as fast as $1/R^3$, ie. δV must go to zero at least as fast as 1/R. Also by hypothesis the charge density, ρ_f , vanishes everywhere except on the surfaces of the conductors where $\delta V = 0$, by hypothesis. Consequently, Equation (3.4.2) gives the result

$$\iiint_{S p a c e}(\mathrm{dVol}) \overrightarrow{\mathrm{D}} \cdot \delta \overrightarrow{\mathrm{E}}=\iiint_{S p a c e}(\mathrm{dVol}) \epsilon \overrightarrow{\mathrm{E}} \cdot \delta \overrightarrow{\mathrm{E}}=0.$$

It follows from this and from Equation (3.4.1) that the incorrect energy U_E^1 exceeds the correct energy U_E, where

$$\mathrm{U}_{\mathrm{E}}=\iiint_{S_{\mathrm{P} n} \alpha_{\mathrm{e}}}(\mathrm{dVol}) \frac{\epsilon}{2} \mathrm{E}^{2},$$

by a positive definite amount:

$$\delta \mathrm{U}_{\mathrm{E}}=\mathrm{U}_{\mathrm{E}}^{1}-\mathrm{U}_{\mathrm{E}}=\iiint_{S_{\text {pace }}}(\mathrm{dVol}) \frac{\epsilon}{2}(\delta \overrightarrow{\mathrm{E}})^{2}. \tag{3.4.3}$$

This demonstrates that the electric field energy is a minimum for the correct field distribution. This fact can be made the basis for an approximate method for solving electrostatic field problems: one guesses at the form of the potential using a reasonable function that contains a number of adjustable constants a,b,c etc. These constants are adjusted so as to obtain the minimum electrostatic energy. The solution so obtained represents the given functional form that most closely approximates the exact solution. This method has been illustrated in the last part of Chpt.(19) of the Feynman Lectures on Physics, Vol.(II), using a cylindrical capacitor as an example.

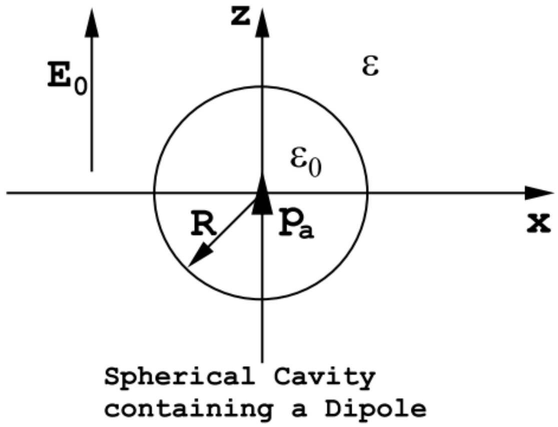


Figure 3.4.18: A point dipole p_a is located at the center of an empty spherical cavity of radius R cut out of an otherwise homogeneous dielectric material characterized by a dielectric constant ϵ. The electric field far from the cavity is E_0 Volts/m, and is uniform and directed along the z-axis.

3.5: Appendix(A) - The Onsager Problem

An interesting variant of the problem of a sphere in a uniform field has been discussed by Onsager in connection with the calculation of the dielectric constant of a material from its atomic polarizability; L.Onsager, J.Amer.Chem.Soc.58, 1486-1493 (1936). When an isolated atom is placed in a uniform external electric field it develops a dipole moment, p_a, that is proportional to the applied field E_0;

$$\mathrm{p_a} = \alpha\epsilon_0 \mathrm{E_0},$$

where the polarizability α has the dimensions of a volume, and can in principle be calculated using quantum mechanics. In a solid or a liquid the atom is not isolated, but its electric moment is influenced by the electric fields due to its neighbours. As a crude approximation one may imagine that the atom plus its associated electric moment is located at the center of a spherical cavity of radius R cut out of an otherwise homogeneous dielectric material characterized by a dielectric constant ϵ, see Figure (3.4.18). Far from the cavity the electric field is E_0 and directed along the z-axis corresponding to the potential function

$$\mathrm{V} = -\mathrm{E_0 z} = -\mathrm{E_0 r}\cos\theta,$$

where r and θ are spherical polar co-ordinates. The problem is to determine the field inside the cavity that acts to polarize the atom. The externally applied electric field is derived from a potential function whose angular dependence is proportional to cos (θ); one is therefore motivated to seek a solution of this problem that corresponds to the use of the terms proportional to cos (θ) in the expansion for the potential, Equation (3.2.19). Inside the cavity the potential near r=0 must be dominated by the dipole potential

$$\frac{\mathrm{p_a}}{4\pi\epsilon_0}\frac{\cos\theta}{\mathrm{r}^2}.$$

One is therefore led to try

Inside: r < R

$$\mathrm{V_i}(\mathrm{r},\theta) = \left(\frac{\mathrm{p_a}\cos\theta}{4\pi\epsilon_0}\right)\frac{1}{\mathrm{r}^2} - A\mathrm{r}\cos\theta. \tag{3.5.1}$$

and

Outside: r > R

$$\mathrm{V_o}(\mathrm{r},\theta) = -\mathrm{E_0 r}\cos\theta + \frac{b\cos\theta}{\mathrm{r}^2}. \tag{3.5.2}$$

The requirements that the potential function and the normal components of $\vec{D}$ be continuous across the surface of the sphere, r=R, lead to the two equations

$$\begin{aligned} A + \frac{b}{\mathrm{R}^3} &= \left(\frac{\mathrm{p_a}}{4\pi\epsilon_0}\right)\frac{1}{\mathrm{R}^3} + \mathrm{E_0} \\ -A + \frac{2\epsilon_r b}{\mathrm{R}^3} &= \left(\frac{2\mathrm{p_a}}{4\pi\epsilon_0}\right)\frac{1}{\mathrm{R}^3} - \epsilon_r \mathrm{E_0}, \end{aligned}$$

where $\epsilon_r = \epsilon/\epsilon_0$. From these two equations one finds

$$A = \left(\frac{3\epsilon_r}{2\epsilon_r + 1}\right)\mathrm{E_0} + \left(\frac{\epsilon_r - 1}{2\epsilon_r + 1}\right)\frac{2\mathrm{p_a}}{4\pi\epsilon_0 \mathrm{R}^3}, \tag{3.5.3}$$

and

$$\frac{b}{\mathrm{R}^3} = \left(\frac{1-\epsilon_r}{2\epsilon_r + 1}\right)\mathrm{E_0} + \left(\frac{3}{2\epsilon_r + 1}\right)\frac{\mathrm{p_a}}{4\pi\epsilon_0 \mathrm{R}^3}. \tag{3.5.4}$$

But A is just the value of the uniform field inside the cavity that is responsible for the induced dipole moment on the atom, therefore from the definition of the polarizability one has

$$\mathrm{p_a} = \alpha\epsilon_0 A. \tag{3.5.5}$$

This value can be substituted into Equation (3.5.3) for the constant A to obtain

$$A=\left(\frac{3\epsilon_r}{2\epsilon_r+1}\right)\mathrm{E}_0+\left(\frac{\epsilon_r-1}{2\epsilon_r+1}\right)\frac{2\alpha A}{4\pi\mathrm{R}^3}. \tag{3.5.6}$$

Eqn.(3.5.6) can be solved for A in terms of the applied electric field E_0, and this result can be used in Equation (3.5.5) to calculate the atomic dipole moment p_a:

$$\mathrm{p}_\mathrm{a}=\left(\frac{3\epsilon_r\epsilon_0}{2\epsilon_r+1-\left(\frac{2\alpha}{4\pi\mathrm{R}^3}\right)(\epsilon_r-1)}\right)\alpha\mathrm{E}_0. \tag{3.5.7}$$

But the dipole moment per atom can be used to calculate the dipole moment per unit volume, $\vec{P}$:

$$|\overrightarrow{\mathrm{P}}|=\mathrm{P}=\mathrm{N}\mathrm{p}_\mathrm{a}, \tag{3.5.8}$$

where N is the number of atoms per unit volume. From the definition

$$\mathrm{D}=\epsilon_0\mathrm{E}_0+\mathrm{P}$$

one has

$$\mathrm{P}=(\epsilon_r-1)\,\epsilon_0\mathrm{E}_0. \tag{3.5.9}$$

(Notice that one can drop the vector signs on D, E_0, and P because all of these vectors are parallel with the z-axis). Using Equations (3.5.9,3.5.8, and 3.5.7) one can obtain a relation between the relative dielectric constant, ϵ_r and the polarizability α:

$$\epsilon_r-1=\left(\frac{3\epsilon_r}{2\epsilon_r+1-(\epsilon_r-1)\left(\frac{\alpha}{2\pi R^3}\right)}\right)\mathrm{N}\alpha.$$

The latter expression can be solved to obtain the polarizability in terms of the relative dielectric constant, ϵ_r:

$$\alpha=\frac{(2\epsilon_r+1)}{\left(\epsilon_r-1+\left(\frac{3\epsilon_r}{\epsilon_r-1}\right)2\pi\mathrm{N}\mathrm{R}^3\right)}2\pi\mathrm{R}^3. \tag{3.5.10}$$

Eqn.(3.5.10) can be used to calculate the atomic polarizability from measured values of the relative dielectric constant, ϵ_r. These values of α can then be compared with values calculated from atomic theory.

CHAPTER OVERVIEW

4: The Magnetostatic Field I

The Calculation of Magnetic Fields Given a Time-independent Distribution of Sources.

Thumbnail: Magnetic B-field inside and outside of a cylindrical bar magnet. (CC BY-SA 4.0 International; Geek3 via Wikipedia)

4.1: Introduction

If nothing changes with time Maxwell's equations become:

$$\mathrm{curl}\,\vec{\mathrm{E}} = 0 \tag{4.1.1}$$

$$\mathrm{div}\,\vec{\mathrm{B}} = 0 \tag{4.1.2}$$

$$\mathrm{curl}\,\vec{\mathrm{B}} = \mu_0 \left(\vec{\mathrm{J}}_{\mathrm{f}} + \mathrm{curl}\,\vec{\mathrm{M}}\right) \tag{4.1.3}$$

$$\mathrm{div}\,\vec{\mathrm{E}} = \frac{1}{\epsilon_0} \left(\rho_f - \mathrm{div}\,\vec{\mathrm{P}}\right) \tag{4.1.4}$$

The magnetic field has become completely uncoupled from the electric field. The magnetostatic field, B, is generated by current flow and by a spatial variation of the magnetization density, M. It is customary to introduce a vector potential function $\vec{A}$ through the relation

$$\vec{\mathrm{B}}(\mathbf{r}) = \mathrm{curl}\,\vec{\mathrm{A}}(\mathbf{r}), \tag{4.1.5}$$

where **r** is the position vector corresponding to some point in space. The divergence of any curl of a vector field is zero, therefore Equation (4.1.5) automatically guarantees that the equation div$\vec{B}$ = 0 will be satisfied. Notice that the equation div$\vec{B}$ = 0 requires the normal component of $\vec{B}$ to be continuous across any surface. This conclusion is based upon an application of Gauss' theorem similar to that used in section(2.3.2) in chapter(2). Upon substituting for $\vec{B}$ in Equation (4.1.3) one obtains

$$\mathrm{curl}\,\mathrm{curl}(\vec{\mathrm{A}}) = \mu_0 \left(\vec{\mathrm{J}}_{\mathrm{f}} + \mathrm{curl}(\vec{\mathrm{M}})\right). \tag{4.1.6}$$

The free current density, $\vec{J}_{\mathrm{f}}$, and the function curl($\vec{M}$) both act in exactly the same way to generate a magnetic field. It is useful, therefore, to define an effective current density by the relation

$$\overrightarrow{\mathrm{J}_{\mathrm{M}}} = \mathrm{curl}(\vec{\mathrm{M}}),$$

and a total current density by

$$\vec{\mathrm{J}}_{\mathrm{T}} = \overrightarrow{\mathrm{J}_{\mathrm{f}}} + \overrightarrow{\mathrm{J}_{\mathrm{M}}}, \tag{4.1.7}$$

The total current density is just the sum of the current density due to the motion of charges and the effective current density due to a spatial variation of the magnetization density, $\vec{M}$. With this notation, Equation (4.1.6) becomes

$$\mathrm{curl}\;\mathrm{curl}(\vec{\mathrm{A}}) = \mu_0 \vec{\mathrm{J}}_{\mathrm{T}}. \tag{4.1.8}$$

The vector operator *curl curl* has a particularly simple form when written out in cartesian co-ordinates:

$$\mathrm{curl}\,\mathrm{curl}(\vec{\mathrm{A}}) = -\left(\nabla^2 \mathrm{A}_{\mathrm{x}} \hat{\mathbf{u}}_{\mathrm{x}} + \nabla^2 \mathrm{A}_{\mathrm{y}} \hat{\mathbf{u}}_{\mathbf{y}} + \nabla^2 \mathrm{A}_{\mathrm{z}} \hat{\mathbf{u}}_{\mathbf{z}}\right) + \mathrm{grad}(\mathrm{div}\,\vec{\mathrm{A}}), \tag{4.1.9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \mathrm{x}^2} + \frac{\partial^2}{\partial \mathrm{y}^2} + \frac{\partial^2}{\partial \mathrm{z}^2}$$

is the LaPlacian operator, and $\hat{\mathbf{u}}_{\mathbf{x}}$, $\hat{\mathbf{u}}_{\mathbf{y}}$ and $\hat{\mathbf{u}}_{\mathbf{z}}$ are unit vectors. Eqn.(4.1.8) is actually three equations when written in cartesian co-ordinates: one equation for each of the three components.

$$-\nabla^2 \mathrm{A}_{\mathrm{x}} + \frac{\partial}{\partial \mathrm{x}}(\mathrm{div}\,\vec{\mathrm{A}}) = \mu_0 \mathrm{J}_{\mathrm{T}}\Big)_{\mathrm{x}}$$

$$-\nabla^2 \mathrm{A}_y + \frac{\partial}{\partial \mathrm{y}}(\mathrm{div}\,\vec{\mathrm{A}}) = \mu_0 \mathrm{J_T}\Big)_{\mathrm{y}} \tag{4.1.10}$$

$$-\nabla^2 \mathrm{A}_z + \frac{\partial}{\partial \mathrm{z}}(\mathrm{div}\,\vec{\mathrm{A}}) = \mu_0 \mathrm{J_T}\Big)_{z}$$

At this point the vector field $\vec{A}$ has not been uniquely defined because so far all that has been specified is its curl through the requirement that curl($\vec{A}$) = $\vec{B}$. In order to uniquely specify a vector field, apart from a constant vector, it is necessary to specify both its curl and its divergence. There are many fields $\vec{A}$ whose curl give the same field $\vec{B}$. For example, let us define a new field from the old vector potential, $\vec{A}$, by means of the relation

$$\overrightarrow{\mathrm{A}'} = \vec{\mathrm{A}} + \mathrm{grad(F)} \tag{4.1.11}$$

where F is any scalar function of position. Both $\vec{A}'$ and $\vec{A}$ give exactly the same field, $\vec{B}$, because the curl of any gradient is zero. This property of the curl was used in Chapter(2) in order to introduce the electrical potential function. The arbitrariness in the vector potential A illustrated by Equation (4.1.11) means that one can choose the vector potential so that its divergence has a convenient value. It turns out that div($\vec{A}$) = 0 is a convenient choice because it causes the differential equations (4.1.11) to assume a familiar form:

$$\begin{aligned}\nabla^2 \mathrm{A_x} &= -\mu_0 \mathrm{J_T}\big)_x,\\ \nabla^2 \mathrm{A_y} &= -\mu_0 \mathrm{J_T}\big)_y,\\ \nabla^2 \mathrm{A}_z &= -\mu_0 \mathrm{J_T}\big)_z.\end{aligned} \tag{4.1.12}$$

Each of these equations has exactly the form as Equation (2.2.5) encountered in Chapter(2) for the electrostatic potential. The particular solutions for Equations (4.1.12) can therefore be written down immediately by analogy with Equation (2.2.6) of Chapter(2):

$$\begin{aligned}\mathrm{A_x}(\vec{\mathrm{R}}) &= \frac{\mu_0}{4\pi}\iiint_{Space} d\tau \frac{\mathrm{J_T})_x(\vec{\mathrm{r}})}{|\vec{\mathrm{R}}-\vec{\mathrm{r}}|}\\ \mathrm{A_y}(\vec{\mathrm{R}}) &= \frac{\mu_0}{4\pi}\iiint_{Space} d\tau \frac{\mathrm{J_T})_y(\vec{\mathrm{r}})}{|\vec{\mathrm{R}}-\vec{\mathrm{r}}|}\\ \mathrm{A_z}(\vec{\mathrm{R}}) &= \frac{\mu_0}{4\pi}\iiint_{Space} d\tau \frac{\mathrm{J_T})_z(\vec{\mathrm{r}})}{|\vec{\mathrm{R}}-\vec{\mathrm{r}}|}\end{aligned}$$

where dτ is the element of volume.

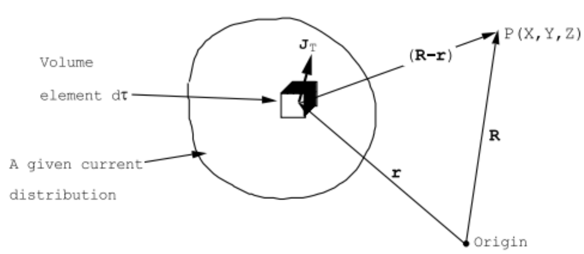


Figure 4.1.1: The geometry used to calculate the vector potential at the point P($\vec{R}$) generated by a given current density distribution $\vec{J}_{\mathrm{T}}(vecr)$.

But these equations are just the three cartesian components of a single vector equation

$$\vec{A}(\vec{R}) = \frac{\mu_0}{4\pi} \iiint_{Space} d\tau \frac{\left(\vec{J}_f + \mathrm{curl}(\vec{M})\right)}{|\vec{R} - \vec{r}|}. \tag{4.1.13}$$

It is instructive to rewrite this equation explicitly in terms of the co-ordinates that specify the point of observation, $\vec{R} = X\hat{u}_x + Y\hat{u}_y + Z\hat{u}_z$ and the coordinates that specify the position of the source element of volume $\vec{r} = x\hat{u}_x + y\hat{u}_y + z\hat{u}_z$, see Figure (4.1.1):

$$\vec{A}(X, Y, Z) = \frac{\mu_0}{4\pi} \int\int\int_{Space} dxdydz \frac{\vec{J}_T(x, y, z)}{\sqrt{(X - x)^2 + (Y - y)^2 + (Z - z)^2}}. \tag{4.1.14}$$

The particular solution, Equation (4.1.14), corresponds to the choice div($\vec{A}$) = 0.

Derivatives of the components of $\vec{A}$ with respect to the field co-ordinates (X,Y,Z) can be calculated using Equation (4.1.14) by differentiating under the integral sign. For example,

$$\frac{\partial A_x}{\partial Y} = -\frac{\mu_0}{4\pi} \int\int\int_{Space} dxdydz \frac{J_T)_x(x, y, z)(Y - y)}{\left[(X - x)^2 + (Y - y)^2 + (Z - z)^2\right]^{3/2}}.$$

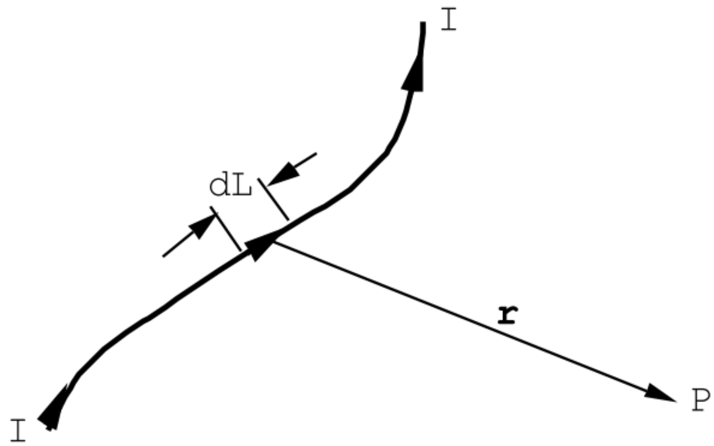


Figure 4.1.2: A thin wire carrying a current of I Amps. The field is to be calculated at P, the point of observation.

Carrying out the differentiations term by term one can show that

$$\vec{B}(\vec{R}) = \mathrm{curl}(\vec{A}) = \frac{\mu_0}{4\pi} \int\int\int_{Space} d\tau \frac{(\vec{J} \times (\vec{R} - \vec{r}))}{|\vec{R} - \vec{r}|^3}, \tag{4.1.15}$$

where $d\tau$ is the element of volume.

4.2: The Law of Biot-Savart

It often happens that the current density is confined to a relatively small cross-section. Consider, for example, the case of a thin wire carrying a current, see Figure (4.1.2). For this case the current density is I/S inside the wire, where S is the cross-sectional area of the wire, and the current density is zero outside the wire. If the thickness of the wire is very small compared with the distance to the point of observation, one can neglect the very small variations of $|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|=\sqrt{(\mathrm{X}-\mathrm{x})^{2}+(\mathrm{Y}-\mathrm{y})^{2}+(\mathrm{Z}-\mathrm{z})^{2}}$ for the various elements across the wire section, so that when integrated over the wire cross-section Equations (4.1.14) and (4.1.15) become line integrals:

$$\vec{A}_{P}=\frac{\mu_{0} I}{4 \pi} \int_{Wire} \frac{d \vec{L}}{|\vec{r}|}, \tag{4.2.1}$$

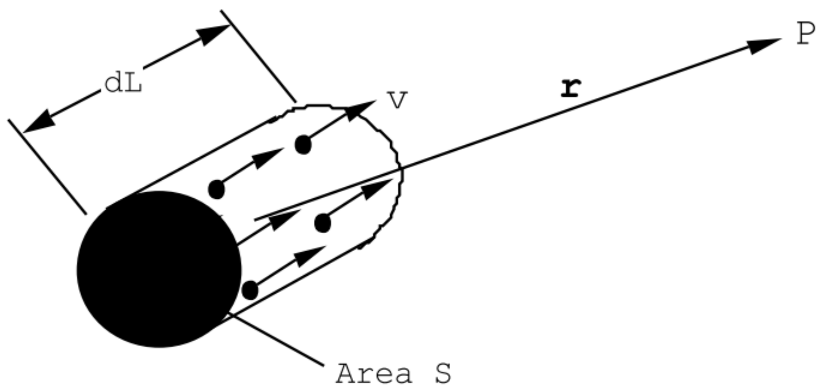


Figure 4.2.3: Derivation of the law of Biot-Savart from the fields generated by a slowly moving point charge.

and

$$\overrightarrow{\mathrm{B}}_{P}=\frac{\mu_{0} \mathrm{I}}{4 \pi} \int_{\text{Wire}} \frac{(\overrightarrow{\mathrm{d}} \overrightarrow{\mathrm{L}} \times \overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{r}}|^{3}}. \tag{4.2.2}$$

Equation (4.2.2) is called the law of Biot-Savart. Notice that the magnetic fieldstrength falls off like $1/r^2$ with distance from a small element of current. The law of Biot-Savart can also be deduced directly from the expression for the magnetic field generated by a slowly moving point charge, Chapter(1), Equation (1.1.7). Consider a small element of a wire containing N charges q per unit volume that are moving along the wire with a velocity v, see Figure (4.2.3). The charge density contributed by the mobile charge carriers are supposed to be exactly compensated by an equal number density of fixed charges of opposite sign. The current flowing through the wire is numerically equal to the charge contained in a cylinder of area S and equal in length to the velocity, v: all of the mobile charges in such a cylinder will pass through a given cross-section in 1 second,

$$\mathrm{I}=\mathrm{NSq}_{\mathrm{V}} \quad \text{Ampères}\,. \tag{4.2.3}$$

The electric field due to the mobile charge carriers contained in a piece of wire dL long is just given by

$$\overrightarrow{\mathrm{dE}}_{P}=\frac{1}{4 \pi \epsilon_{0}} \mathrm{NSqdL} \frac{\overrightarrow{\mathrm{r}}}{|\overrightarrow{\mathrm{r}}|^{3}}.$$

These charges produce a magnetic field because of their motion

$$\mathrm{d} \overrightarrow{\mathrm{B}}_{P}=\frac{1}{c^{2}}\left[\overrightarrow{\mathrm{v}} \times \overrightarrow{\mathrm{dE}}_{P}\right]=\frac{1}{4 \pi \epsilon_{0} c^{2}} \mathrm{NSqdL} \frac{[\overrightarrow{\mathrm{v}} \times \overrightarrow{\mathrm{r}}]}{|\overrightarrow{\mathrm{r}}|^{3}}.$$

The compensating stationary charges produce no magnetic field because their velocity relative to the observer is zero. They do, however, produce an electric field that cancels the electric field due to the mobile charges. Now use the fact that $\vec{v}$ and $\mathrm{d}\vec{L}$ are parallel, along with Equation (4.2.3), to obtain

$$\overrightarrow{\mathrm{dB}}_P = \frac{I}{4\pi\epsilon_0 c^2}\frac{[\mathrm{d}\overrightarrow{\mathrm{L}} \times \overrightarrow{\mathrm{r}}]}{|\overrightarrow{\mathrm{r}}|^3}.$$

This leads directly to the integral expression Equation (4.2.2) for the law of Biot and Savart since $c^2 = 1/\epsilon_0\mu_0$.

4.3: Standard Problems

4.3.1 A Long Straight Wire.

Each element of the wire, $d\vec{L}$, is directed along z, and therefore A has only a z-component, see Figure (4.3.4) and Equation (4.1.16):

$$dA_z = \frac{\mu_0}{4\pi} \frac{I_0 dz}{\sqrt{z^2 + x^2}}. \tag{4.3.1}$$

Unfortunately, the integral of Equation (4.3.1) diverges if it is evaluated over the interval $-\infty \le z \le \infty$. This indicates that an infinitely long wire is unphysical; eventually the two ends of the wire must be connected in order to complete the steady state current loop. In order to proceed, one can calculate the contribution to the vector potential from the large but finite wire segment $-L \le z \le +L$. The result is

$$A_z(x) = \frac{\mu_0 I_0}{4\pi} \ln\left(\frac{\sqrt{L^2 + x^2} + L}{\sqrt{L^2 + x^2} - L} \right).$$

Clearly A_z must have the same value everywhere on a circle of radius x centered on the origin and lying in the x-y plane. The expression for the

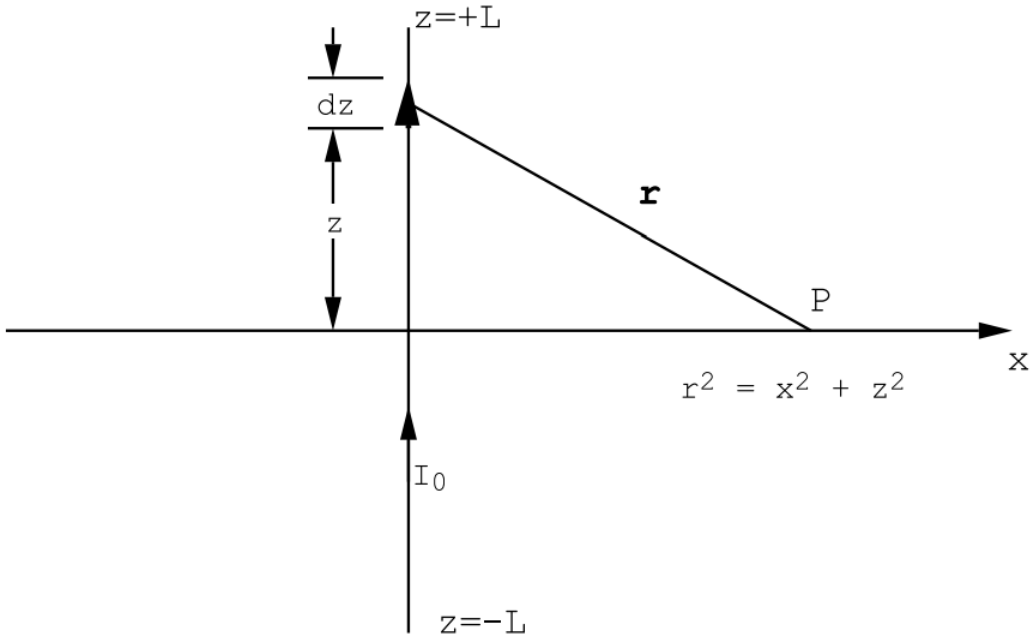


Figure 4.3.4: A straight wire 2L meters long, and carrying a current of I_0 Amp`eres, used to calculate the vector potential and the magnetic field generated at a point P in the central plane.

vector potential may therefore be written in cylindrical polar co-ordinates as

$$A_z(r) = \frac{\mu_0 I_0}{4\pi} \ln\left(\frac{\sqrt{L^2 + r^2} + L}{\sqrt{L^2 + r^2} - L} \right).$$

Although this expression is strictly valid only for points in the x-y plane, it is clear from symmetry arguments that for large L and small z the vector potential must be essentially independent of z. The corresponding magnetic field is given by $\vec{B}$ = curl($\vec{A}$), and since $\vec{A}$ has only a z-component, and since this z-component is independent of the angle θ, the magnetic field has only a θ-component:

$$B_\theta = -\left(\frac{\partial A_z}{\partial r}\right) = \frac{\mu_0 I_0}{2\pi}\frac{L}{r\sqrt{L^2+r^2}}.$$

If $L \gg r$ this expression reduces to

$$B_\theta = \frac{\mu_0 I_0}{2\pi}\frac{1}{r}. \tag{4.3.2}$$

4.3.2 A Long Straight Wire Revisited.

The result Equation (4.3.2) for the magnetic field generated by a long straight wire is so simple that it suggests that there must be an easy method for obtaining it: a method based upon the symmetry of the problem. Magnetic problems in which the current distribution is very symmetric may often be solved by means of an application of Stokes' theorem (Chpt.(1), Section(1.3.4)). Stokes' theorem states that the surface integral of the curl of any vector field over a surface bounded by a closed curve C can be replaced by the line integral of that vector over the curve C. Apply this theorem to the Maxwell equation

$$\operatorname{curl}(\overrightarrow{B}) = \mu_0\left(\overrightarrow{J}_f + \operatorname{curl}(\overrightarrow{M})\right) = \mu_0 \overrightarrow{J}_T.$$

For the present problem there is no magnetization density; $\overrightarrow{M} = 0$ everywhere and therefore $\operatorname{curl}(\overrightarrow{M}) = 0$ everywhere and $\overrightarrow{J}_T = \overrightarrow{J_f}$. The current flow is confined to the cross-section of the wire so that if one applies Stokes' theorem to the surface bounded by the circle of radius R shown in Figure (4.3.5) one obtains

$$\iint_{Surface} dS\operatorname{curl}(\overrightarrow{B})\cdot\hat{\mathbf{n}} = \mu_0\iint_{\text{Surface}} dS\,\overrightarrow{J}_f\cdot\hat{\mathbf{n}} = \mu_0 I_0,$$

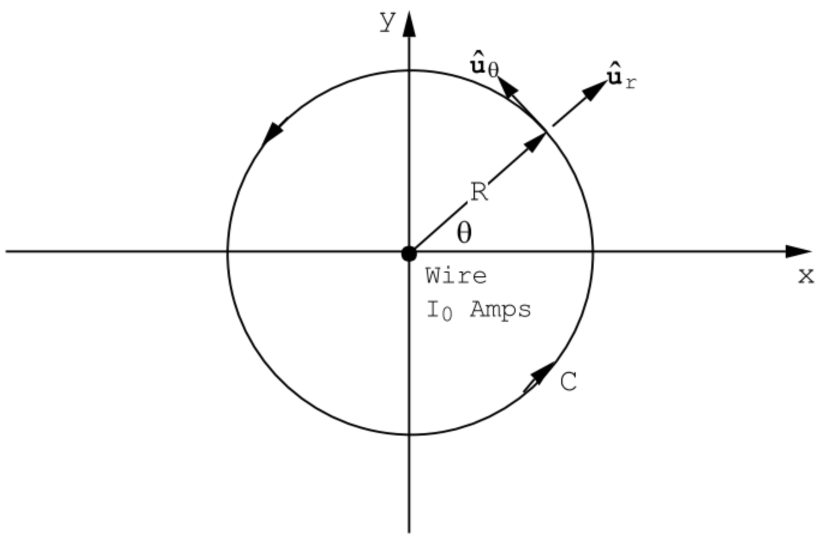


Figure 4.3.5: Geometry used to calculate the magnetic field generated by a long straight wire carrying a current of I_0 Amp`eres.

where dS is the element of area, and $\hat{\mathbf{n}}$ is a unit vector normal to the element of surface area. But from Stokes' Theorem

$$\iint_{Surface} dS\operatorname{curl}(\overrightarrow{B})\cdot\hat{\mathbf{n}} = \oint_C \overrightarrow{B}\cdot d\overrightarrow{L},$$

$$\oint_C \overrightarrow{B}\cdot\overrightarrow{d} = \mu_0 I_0.$$

The law of Biot-Savart, Equation (4.1.17), can be used to convince oneself that $\vec{B}$ has only a component in the direction tangent to the circle C of Figure (4.3.5). By symmetry this component must be independent of position along the circumference of the circle, and the line integral in is very easy to carry out.

$$\oint_C \overrightarrow{\mathrm{B}} \cdot \mathrm{d}\overrightarrow{\mathrm{L}} = 2\pi \mathrm{R} \mathrm{B}_\theta = \mu_0 \mathrm{I}_0,$$

or

$$\mathrm{B}_\theta = \frac{\mu_0 \mathrm{I}_0}{2\pi \mathrm{R}}, \tag{4.3.3}$$

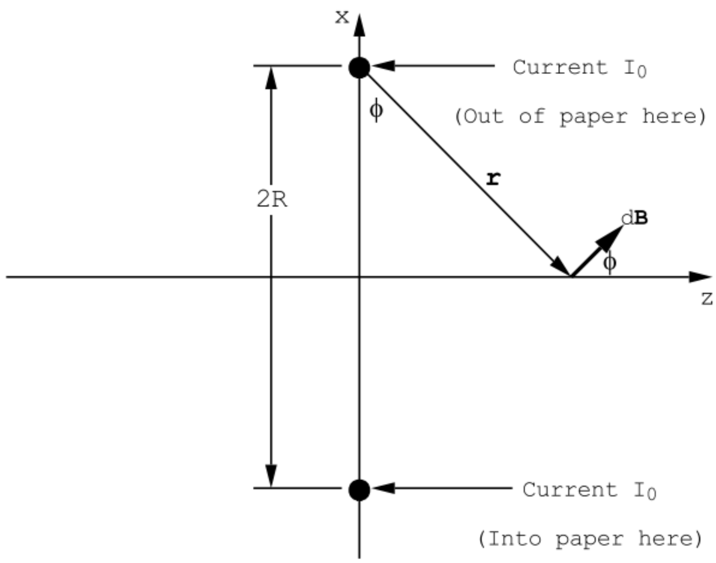


Figure 4.3.6: The magnetic field generated along the axis of a circular current loop.

in agreement with Equation (4.3.2) deduced from the vector potential. Unfortunately, most problems do not exhibit sufficient symmetry to be so simply solved.

4.3.3 A Circular Loop.

Refer to Figure (4.3.6). In this case the field is most simply calculated by direct application of the law of Biot-Savart, Equation (4.1.17). The element of magnetic field, $\mathrm{d}\vec{B}$, generated by any small element of length, $\mathrm{d}\vec{L}$, along the wire is perpendicular both to $\mathrm{d}\vec{L}$ and to $\vec{r}$ as shown in Figure (4.3.6). The transverse component of $\mathrm{d}\vec{B}$ is cancelled by symmetry by the contribution from the element of length that is diametrically opposite to $\mathrm{d}\vec{L}$. Thus along the axis of the loop there is only a z-component of magnetic field:

$$\mathrm{dB_z} = \frac{\mu_0 \mathrm{I}_0}{4\pi}\left(\frac{\mathrm{dL}}{\mathrm{r}^2}\right)\cos\phi = \frac{\mu_0 \mathrm{I}_0}{4\pi}\frac{\mathrm{RdL}}{\mathrm{r}^3}.$$

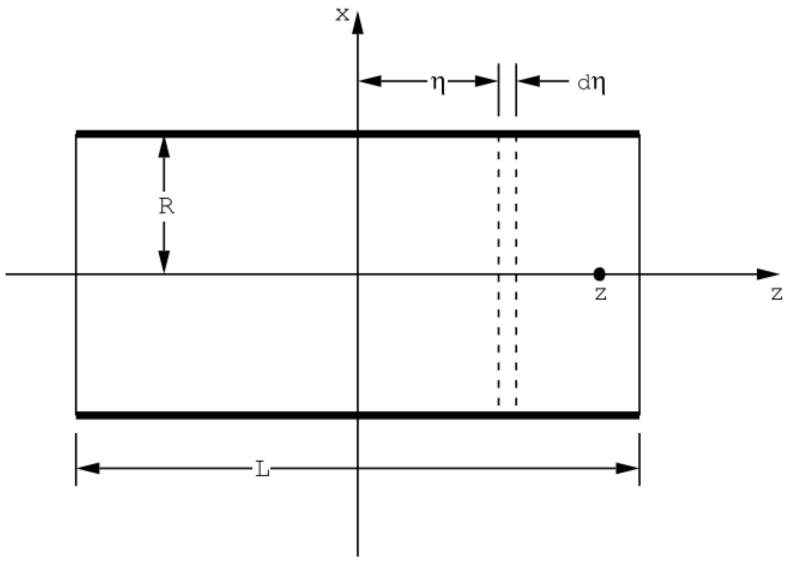


Figure 4.3.7: The magnetic field along the axis of a solenoid L meters long, R meters in radius, and having N turns per meter.

This expression can be readily integrated because the distance r does not depend upon position around the circumference of the wire. Thus

$$B_z = \frac{\mu_0 I_0}{4\pi}\left(\frac{R}{r^3}\right)\oint_C d\overrightarrow{L} = \frac{\mu_0 I_0}{2}\frac{R^2}{\left[z^2+R^2\right]^{3/2}}. \tag{4.3.4}$$

This expression, together with the principle of superposition, can be used to calculate the magnetic field along the axis of a solenoid.

4.3.4 The Magnetic Field along the Axis of a Solenoid.

Consider a coil L meters long that is uniformly wound with N turns/meter. The magnetic field at a point on the axis of the coil can be calculated as the sum of the fields generated by each turn separately using the principle of superposition. The field generated by a single turn located at η is given by

$$B_z = \left(\frac{\mu_0 I_0 R^2}{2}\right)\frac{1}{\left((z-\eta)^2+R^2\right)^{3/2}},$$

where I_0 is the current; this follows from Equation (4.3.4). The field generated at z by the $Nd\eta$ turns contained in the element of length $d\eta$ is given by

$$dB_z = \left(\frac{\mu_0 I_0 R^2}{2}\right) N\frac{d\eta}{\left((z-\eta)^2+R^2\right)^{3/2}}.$$

Upon integration over η the total field becomes

$$B_z = \left(\frac{\mu_0 I_0 R^2 N}{2}\right)\int_{-L/2}^{L/2}\frac{d\eta}{\left((z-\eta)^2+R^2\right)^{3/2}}.$$

This is a standard integral:

$$B_z = \left(\frac{\mu_0 I_0 N}{2}\right)\left[\frac{([L/2]+z)}{\sqrt{([L/2]+z)^2+R^2}}+\frac{([L/2]-z)}{\sqrt{([L/2]-z)^2+R^2}}\right]. \tag{4.3.5}$$

The calculation of the field strength at an off-axis position is more difficult, and must be carried out numerically. In the limit as L becomes very large, ie. (z/L)≪ 1, the z dependence drops out to give

$$B_z \to \mu_0 NI_0 \left(\frac{L/2}{\sqrt{(L/2)^2 + R^2}} \right) \to \mu_0 NI_0 \quad \text{if } (R/L) \ll 1. \tag{4.3.6}$$

(Note that N is not the total number of turns on the solenoid but is the total number of turns divided by the length L.)

4.3.5 The Magnetic Field of an Infinite Solenoid.

The field in an infinite solenoid cannot depend upon position along z because the coil appears the same to a fixed observer even if it is shifted along its axis through any finite interval, Δz. The current flow in the solenoid turns is transverse to the solenoid axis, therefore according to the expression (4.2.1) for the vector potential, $\vec{A}$ must be purely transverse; ie. the vector potential $\vec{A}$ can have only the components A_r and A_θ when written in cylindrical polar co-ordinates. These components cannot depend upon the angle θ because any rotation of the solenoid around its axis leaves the current distribution unchanged. The curl of a vector that has only the components A_r and A_θ,

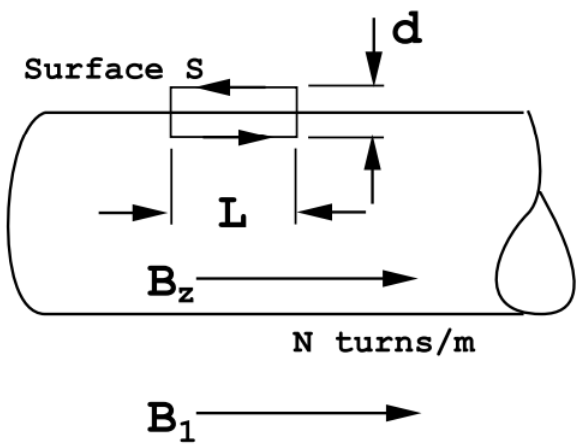


Figure 4.3.8: Diagram to illustrate the use of Stokes' Theorem to show that the field outside an infinite solenoid is zero.

and for which these components depend only upon the radial co-ordinate,r, has only a z-component,

$$B_z = \text{curl}(\overrightarrow{A})_z = \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r}.$$

We conclude, therefore, that the magnetic field can have only one component, B_z, and that component can depend only upon the distance r from the solenoid axis. Further note that everywhere inside the solenoid curl($\vec{B}$) = 0 from Maxwell's equations since there is no free current density and no magnetization density by hypothesis. But since $\vec{B}$ has only a z-component that is independent of θ and z, its curl has only the component

$$\text{curl}(\overrightarrow{B})_\theta = -\frac{\partial B_z}{\partial r} = 0$$

and therefore B_z is independent of the distance from the solenoid axis. A similar line of argument applies equally to the region outside the solenoid. It follows from Equation (4.3.6), the expression for the field at the center of a long solenoid, that the field **everywhere** inside an infinite solenoid must be given by

$$B_z = \mu_0 NI_0 \quad \text{Teslas}. \tag{4.3.7}$$

As was shown above, outside the infinite solenoid the field must be a constant, $B_z = B_1$ say. The value of B_1 may be calculated by means of Stokes' theorem, Figure (4.3.8). Apply Stokes' theorem to an area bounded by the rectangle L long and d wide that is oriented perpendicular to the current flow in the windings. From Maxwell's equations

$$\text{curl}(\overrightarrow{B}) = \mu_0 \overrightarrow{J}_f,$$

since there is no magnetization density and the fields are static. Therefore

$$\int\int_S \mathrm{dS\,curl}(\vec{\mathrm{B}})\cdot\hat{\mathbf{n}} = \mu_0 \iint_S \mathrm{dS}\,\vec{\mathrm{J}}_{\mathrm{f}}\cdot\hat{\mathbf{n}} = \mu_0 \mathrm{NI_0 L}.$$

This last result follows because $\vec{J}_{\mathrm{f}} = 0$ except on the cross-section of each wire. But, referring to Figure (4.3.8)

$$\int\int_S \mathrm{dS\,curl}(\vec{\mathrm{B}})\cdot\hat{\mathbf{n}} = \oint_C \vec{\mathrm{B}}\cdot\mathrm{d}\vec{\mathrm{L}} = \mathrm{B_z L} - \mathrm{B_1 L}.$$

The sides of the loop d meters long contribute nothing because they are perpendicular to the magnetic field. It can therefore be concluded that

$$(\mathrm{B_z} - \mathrm{B_1}) = \mu_0 \mathrm{NI_0}. \tag{4.3.8}$$

But inside the solenoid the field is $B_z = \mu_0 NI_0$, and Equation (4.3.8) then requires that the field outside the solenoid be zero. **The fields generated by an infinitely long solenoid are zero everywhere outside the solenoid, and a uniform field parallel with the axis, $B_z = \mu_0 NI_0$, everywhere inside the solenoid.**

4.3.6 The Field generated by a Point Magnetic Dipole.

Consider a current loop of radius a meters centered on the origin and lying in the x-y plane as shown in Figure (4.3.9). For simplicity, let the point of observation, P, lie in the y-z plane; this assumption involves no loss of generality because the vector potential and the field must be independent of angle around the z-axis. The contribution of the line element $\overrightarrow{\mathrm{dL}} = ad\phi\,[-\sin\phi\hat{\mathbf{u}}_x + \cos\phi\hat{\mathbf{u}}_y]$ to the vector potential at P(0,Y,Z) is

$$\mathrm{d}\vec{\mathrm{A}}_{\mathrm{P}} = \frac{\mu_0 \mathrm{I_0}}{4\pi}\frac{\mathrm{d}\vec{\mathrm{L}}}{|\vec{\mathrm{R}} - \vec{\mathrm{a}}|} = \frac{\mu_0 \mathrm{I_0}}{4\pi}\,\mathrm{a\,d}\,\frac{(-\sin\phi\hat{\mathbf{u}}_x + \cos\phi\hat{\mathbf{u}}_y)}{|\vec{\mathrm{R}} - \vec{\mathrm{a}}|}.$$

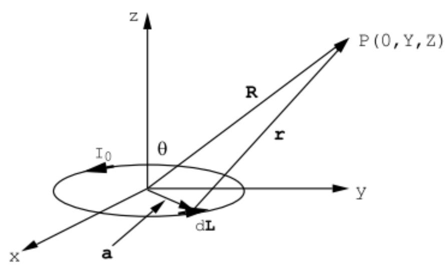


Figure 4.3.9: Calculation of the vector potential generated by a current loop of radius a carrying a current of I_0 Amps.

As usual $\hat{\mathbf{u}}_x$ and $\hat{\mathbf{u}}_y$ are unit vectors directed along x and y.

$$|\vec{\mathrm{R}} - \vec{\mathrm{a}}| = \sqrt{\mathrm{a}^2\cos\phi^2 + [\mathrm{Y} - \mathrm{a}\sin\phi]^2 + \mathrm{Z}^2}\,,$$

or

$$|\vec{\mathrm{R}} - \vec{\mathrm{a}}| = \mathrm{R}\sqrt{1 - \frac{2\mathrm{aY}\sin\phi}{\mathrm{R}^2} + \frac{\mathrm{a}^2}{\mathrm{R}^2}}\,,$$

where $R^2 = Y^2 + Z^2$. Using the binomial expansion theorem along with the condition $(a/R) \ll 1$ one finds, to first order in (a/R),

$$\overrightarrow{\mathrm{dA}}_P = \frac{\mu_0 \mathrm{I_0}}{4\pi}\frac{\mathrm{a}d\phi}{\mathrm{R}}(-\sin\phi\hat{\mathbf{u}}_x + \cos\phi\hat{\mathbf{u}}_y)\left(1 + \frac{\mathrm{aY}\sin\phi}{\mathrm{R}^2}\right).$$

Integrate over the angle ϕ from $\phi = 0$ to $\phi = 2\pi$. The integrals over $\sin\phi$, $\cos(\phi)$, and $\sin(\phi)\cos(\phi)$ all vanish. However, the integral over $\sin(\phi^{2)}$ gives π. Thus $\vec{A}_{\mathrm{P}}$ will have only a component parallel with the x-axis for the above choice of P lying in the Y-Z plane at (0,Y,Z):

$$A_x = -\frac{\mu_0}{4\pi}(\pi a^2 I_0)\frac{Y}{R^3}.$$

This result indicates, because of the symmetry around the z-axis, that in spherical polar co-ordinates the vector potential has only one component, A_ϕ. Let $m = \pi a^2 I_0$; then

$$A_\phi = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{R^2}.$$

In vector notation this result can be written

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{(\vec{m}\times\vec{R})}{R^3} \quad \text{Tesla} - \text{meters}\,, \tag{4.3.9}$$

where $|\vec{m}| = I_0 A$, and $A = \pi a^2$, the area of the current loop. It can be shown that this same result is obtained for any small current loop, whatever its shape may be, in the limit as the dimensions of the loop become very small compared with the distance to the point of observation, R.

It is simple, but tedious, to show that the magnetic field corresponding to Equation (4.3.9) is given by

$$\vec{B}_{\text{dip}} = \text{curl}\left(\vec{A}_{\text{dip}}\right) = \frac{\mu_0}{4\pi}\left(\frac{3[\vec{m}\cdot\vec{R}]\vec{R}}{R^5} - \frac{\vec{m}}{R^3}\right). \tag{4.3.10}$$

This result can best be obtained by calculating the curl using cartesian coordinates. Eqn.(4.3.10) for the magnetic field generated by a magnetic point dipole has exactly the same form as Equation (1.2.10), the electric field produced by an electric point dipole.

4.3.7 A Long Uniformly Magnetized Rod.

Let a cylindrical rod be magnetized uniformly along its axis. Inside the rod the magnetization density, $M_z = M_0$, is independent of position, ie. $\partial M_z/\partial r = 0$, $\partial M_z/\partial\phi = 0$ and $\partial M_z/\partial z = 0$. Therefore, *curl* ($\vec{M}$) = 0 everywhere inside the rod. Similarly, *curl* ($\vec{M}$) = 0 everywhere outside the rod. However, *curl* ($\vec{M}$) does not vanish on the surface of the rod, see Figure (4.3.10). In cylindrical polar co-ordinates one finds only one non-zero component, $\text{curl}(\vec{M})_\theta = -\partial M_2/\partial r$ the radial component of *curl* ($\vec{M}$) is zero because the magnetization density does not depend upon the azimuthal angle, ϕ. Notice that $\partial M_z/\partial r$ is zero everywhere **except** on the surface where M_z varies rapidly from M_0 on the inside to $M_z = 0$ on the outside of the rod. This rapid radial variation of M_z introduces an integrable singularity into the angular component of *curl* ($\vec{M}$):

$$\text{curl}(\vec{M})_\theta = -\frac{\partial M_z}{\partial r} = M_0\delta(r-R).$$

where $\delta(r-R)$ is the Dirac δ-function that vanishes except at the radius r=R. The quantity *curl* ($\vec{M}$) is equivalent to a real current density as far as

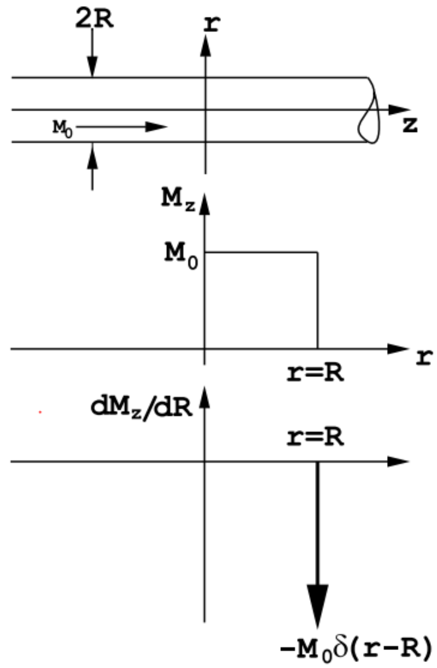


Figure 4.3.10: A long cylindrical rod magnetized along the axis.

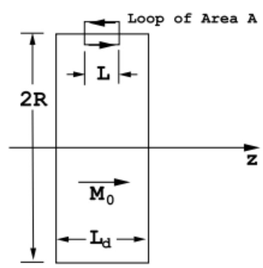


Figure 4.3.11: A uniformly magnetized disc.

producing a magnetic field is concerned. The above surface current density produces exactly the same magnetic fields as a surface current sheet having a strength of M_0 Amps/meter; in terms of the windings on a solenoid, it is equivalent to N turns/meter carrying a current of I_0 Amps where $NI_0 = M_0$ Amps/meter. The field inside a uniformly magnetized rod is given by the infinite solenoid formula, Equation (4.3.7),

$$B_z = \mu_0 M_0 \quad \text{Teslas. .} \tag{4.3.11}$$

Unfortunately this field is not accessible. The field outside an infinitely long magnetized rod is zero.

4.3.8 A Uniformly Magnetized Disc.

The discontinuity in the tangential component of the magnetization density at the surfaces of a uniformly magnetized disc produces an effective surface current density that sets up a magnetic field whose distribution is exactly equivalent to the field set up by a solenoid of the same length. The strength of the effective current sheet is M_0 Amps/meter, and is equivalent to N turns/meter carrying I_0 Amps, where $NI_0 = M_0$. This can be shown using Stokes' Theorem applied to a small loop of area A that spans the surface of the disc as shown in Figure (4.3.11). The field along the axis of a disc of thickness Ld is given by Equation (4.3.5) applied to this case :

$$B_z = \frac{\mu_0 M_0}{2}\left(\frac{[(L_d/2)+z]}{\sqrt{[(L_d/2)+z]^2+R^2}} + \frac{[(L_d/2)-z]}{\sqrt{[(L_d/2)-z]^2+R^2}}\right). \tag{4.3.12}$$

The field generated by a uniformly magnetized disc having a finite thickness is accessible at points outside the disc. The strength of the field at the center of the disc surface at r=0 is given by

$$B_z = \frac{\mu_0 M_0 L_d}{2} \frac{1}{\sqrt{L_d^2 + R^2}} \quad \text{Teslas.}$$

Permanent magnets are available for which $\mu_0 M_0 \approx 1$ Tesla. The external fields produced by such magnets can be quite large- the order of 0.2 Teslas or greater.

4.4: A Second Approach to Magnetostatics

When time variations of the source terms can be neglected we have seen that Maxwell's equations for the magnetostatic field become

$$\operatorname{div}(\vec{B})=0 \tag{4.4.1}$$

$$\operatorname{curl}(\vec{B})=\mu_0\left(\vec{J}_f+\operatorname{curl}(\vec{M})\right). \tag{4.4.2}$$

The auxillary vector $\vec{H}$ was introduced in chapter(1), section(1.4), through the relation

$$\vec{B}=\mu_0(\vec{H}+\vec{M}). \tag{4.4.3}$$

When (4.4.3) is used in (4.4.1) and (4.4.2) to replace $\vec{B}$ by $\vec{H}$ the result is

$$\operatorname{curl}(\vec{H})=\vec{J}_f \tag{4.4.4}$$

$$\operatorname{div}(\vec{H})=-\operatorname{div}(\vec{M}). \tag{4.4.5}$$

For problems in which there is no free current density, $\vec{J}_f$, these equations reduce to

$$\operatorname{curl}(\vec{H})=0, \tag{4.4.6}$$

$$\operatorname{div}(\vec{H})=-\operatorname{div}(\vec{M})=\rho_M. \tag{4.4.7}$$

The form of these equations for the field $\vec{H}$ is exactly the same as the form of Maxwell's equations for the electrostatic field in the absence of a free charge density, ie. (see section(2.1))

$$\operatorname{curl}(\vec{E})=0,$$

$$\operatorname{div}(\vec{E})=-\frac{1}{\epsilon_0}\operatorname{div}(\vec{P})=\frac{\rho_b}{\epsilon_0}.$$

The analogy between these equations for the electrostatic field and the above equations for the magnetic field, $\vec{H}$, in a current free region suggests that $\vec{H}$ can be obtained from a magnetic potential function, V_M; $\vec{H}=-\operatorname{grad}(V_M)$. Notice that if there are no free currents, curlH=0, and **therefore in the absence of a current density the tangential components of H must be continuous everywhere**. The truth of this statement can be demonstrated by means of an application of Stokes' theorem, section(1.3.4). The argument is the same as that used to derive Equation (2.4.1) which states that the tangential component of the electrostatic field must be continuous across a boundary. In the electrostatic case continuity of the tangential component of E can be guaranteed by the requirement that the electrostatic potential function be continuous. In the equivalent magnetostatic case the continuity of the tangential component of H is guaranteed by the requirement that the magnetostatic potential function, V_M, be continuous across a boundary.

The machinery that was set up in Chapter(2) to calculate the electrostatic field from a given charge distribution can be taken over intact to calculate the magnetostatic field from a given "magnetic charge density" distribution, ρ_M, where

$$\rho_M=-\operatorname{div}(\vec{M}). \tag{4.4.8}$$

From now on Equation (4.4.8) will be used to define what is meant by the term magnetic charge density. There is no real magnetic charge density; to this date (2004) no one has been able to discover a magnetic monopole, the magnetic analogue of an electric charge. If a magnetic monopole were to be discovered it would have the units of Amp-meters, and it would produce a field

$$\vec{H}=\frac{1}{4\pi}\frac{q_m}{r^2}\left(\frac{\vec{r}}{r}\right) \quad \text{Amps / meter},$$

by analogy with the electrostatic case, where q_m is the strength of the magnetic charge.

If curl($\vec{H}$) = 0, ie. no free current density, the magnetic field can be written as the gradient of a magnetic scalar potential, V_M:

$$\overrightarrow{\mathrm{H}}=-\operatorname{grad}(\mathrm{V_M}). \tag{4.4.9}$$

Eqn.(4.4.9) guarantees that curl($\vec{H}$) = 0 since the curl of a gradient is always zero. Notice that an arbitrary constant can be added to the potential without changing the magnetic field, $\vec{H}$. This constant is usually chosen to make the expression for the potential function as simple as possible. Upon substituting Equation (4.4.9) into Equation (4.4.7) for the divergence of $\vec{H}$ one obtains

$$\operatorname{div}\operatorname{grad}(\mathrm{V_M})=-\rho_{\mathrm{M}},$$

or

$$\nabla^2\mathrm{V_M}=-\rho_M. \tag{4.4.10}$$

By analogy with the electrostatic case, Equation (2.2.4), the particular solution for the magnetic potential can be written

$$\mathrm{V_M}(\overrightarrow{\mathrm{R}})=\frac{1}{4\pi}\iiint_{Spa\infty}\mathrm{dVol}\,\frac{\rho_M(\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}. \tag{4.4.11}$$

In the application of Equation (4.4.11) it must be remembered that a discontinuity in the normal component of the magnetization, $\vec{M}$, will produce a surface density of magnetic charges just as a discontinuity in the normal component of the electric dipole moment, $\vec{P}$, produces a surface density of bound electric charges, Chapter(2), section(2.3.3). The magnetic surface charge density contributes to the magnetic potential, $V_M(vecR)$, and must be included in Equation (4.4.11) as a surface integral. It is often easier to calculate the fields generated by a given configuration of magnetization density by means of the magnetic scalar potential than it is to use the equivalent current density, $\vec{J}_f$ = curl($\vec{M}$), and the generalized law of Biot-Savart, Equation (4.1.15). Examples follow of magnetic field distributions calculated from given magnetization distributions using the magnetic scalar potential.

4.4.1 An Infinitely Long Uniformly Magnetized Rod.

See Figure (4.3.11). For this case div($\vec{M}$) = 0 everywhere, so that ρ_M = 0 everywhere. There are no surface charge densities because there are no discontinuities in the normal component of $\vec{M}$. This means that the magnetic potential must be independent of position, see Equation (4.4.11), and thus

$$\overrightarrow{\mathrm{H}}=-\operatorname{grad}(\mathrm{V_{M1}})=0.$$

But by definition

$$\overrightarrow{\mathrm{H}}=\left(\frac{\overrightarrow{\mathrm{B}}}{\mu_0}-\overrightarrow{\mathrm{M}}\right),$$

therefore if $\vec{H}$ = 0 it follows that $\vec{B}=\mu_0\vec{M}$ in agreement with Equation (4.3.11) which was earlier obtained using the law of Biot and Savart;(see section(4.3.7) above).

4.4.2 A Thin Disc Uniformly Magnetized along its Axis.

Consider a disc of radius R meters and having a thickness of L meters, that is uniformly magnetized parallel with its axis as shown in Figure (4.4.12); $M_z = M_0$. The discontinuity in the normal component of the magnetization at the front and rear surfaces produces a surface magnetic charge density given by $\sigma_M = +M_0$ per m_2 on the front surface and $\sigma_M = -M_0$ per m_2 on the rear surface. These magnetic charge densities produce a magnetic field along the axis of the disc that can be obtained from the scalar potential, V_M, calculated using Equation (4.4.11), where, for this example, the volume integral reduces to a surface integral. The field $\vec{H}$ so calculated can be used to calculate $\vec{B}$ along the axis: the result is given by Equation (4.3.12) of section(4.3.8).

If (R/L) ≫ 1 the configuration of charges illustrated in Figure (4.4.12) is the magnetic analogue of the electrostatic double layer problem, section(2.7.1) example(4), Figures (2.7.9) and (2.7.10). By analogy with the electrostatic double layer, one can immediately deduce that outside the disc the magnetic field $\vec{H}$ is zero, but inside the disc $H_z = -M_0$. From the definition $\overrightarrow{\mathrm{B}}=\mu_0(\overrightarrow{\mathrm{H}}+\overrightarrow{\mathrm{M}})$ this means that, for a disc having an infinite radius, the field $\vec{B}$ is zero both inside and outside the disc. This conclusion is in agreement with Equation (4.3.12) in which the field $\vec{B}$ was calculated along the axis from the equivalent surface current density on the edge of the disc. Notice that the normal component of $\vec{B}$ is continuous across the interface between the outside and inside of the magnet. It is a general consequence of the Maxwell equation div($\vec{B}$) = 0 that **the normal component of B must be continuous across any interface**.

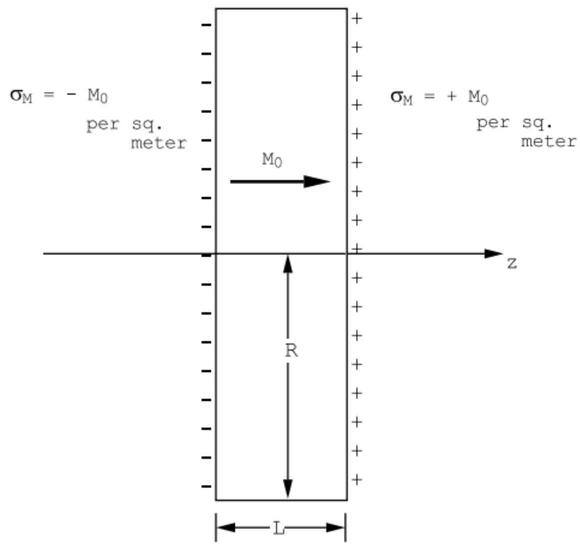


Figure 4.4.12: A uniformly magnetized disc. The discontinuities in the normal component of the magnetization generate an effective magnetic surface charge density on the front and back surfaces of the disc.

4.4.3 A Uniformly Magnetized Ellipsoid.

The results of section(2.7.4) for a uniformly polarized ellipsoid can be taken over for the magnetic case because of the similarity between the equations for the electrostatic field, $\vec{E}$, and those for the magnetic field, $\vec{H}$, in a current free region. Consider the ellipsoid whose surface is described by

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

Let the components of the magnetization in the principle axis system be M_x, M_y, M_z. There exist demagnetizing coefficients, N_α, such that the field $\vec{H}$ inside the ellipsoid is uniform with

$$\begin{aligned} H_x &= -N_x M_x, \\ H_y &= -N_y M_y, \\ H_z &= -N_z M_z. \end{aligned} \tag{4.4.12}$$

Moreover, the demagnetizing coefficients satisfy the sum rule

$$N_x + N_y + N_z = 1. \tag{4.4.13}$$

Equations (4.4.12) and (4.4.13) are the magnetic analogues of eqns,(2.7.5) and (2.7.6) for a uniformly polarized ellipsoid in the electrostatic case. Demagnetizing factors for simple degenerate limits of the ellipsoid of revolution can be deduced immediately from the sum rule and symmetry arguments, just as for the electrostatic case:

(1) A uniformly magnetized sphere: $N_x = N_y = N_z = 1/3$.

(2) A long cylinder magnetized transverse to its axis. In this case the demagnetizing factor for the long axis, the z-axis say, is zero, ie. $N_z = 0$. Therefore since the other two demagnetizing factors are equal, one must have $N_x = N_y = 1/2$.

(3) A flat disc having a very large radius and magnetized along its axis. In the limit of infinite radius the in-plane demagnetizing factors go to zero, and therefore from the sum rule $N_z = 1$.

For the general ellipsoid the demagnetizing factors are given by Equations (2.7.11), and for ellipsoids of revolution by Equations (2.7.7 and 2.7.9).

The magnetic field $\vec{H}$ outside a uniformly magnetized ellipsoid is generally not uniform even though the field $\vec{H}$ inside the ellipsoid is uniform. Analytical expressions for the field H, and therefore also for the field B, are available but they are complicated and are written using generalized elliptic co-ordinate systems. See Electromagnetic Theory by J.A. Stratton, McGraw-Hill, N.Y., 1941, sections 3.25 to 3.27.

4.4.4 A Magnetic Point Dipole.

By analogy with the electrostatic case, the magnetic field around a point magnetic dipole can be obtained from a magnetic potential function of the form

$$\mathrm{V_M} = \frac{1}{4\pi}\left(\frac{\vec{\mathrm{m}}\cdot\vec{\mathrm{r}}}{\mathrm{r}^3}\right). \tag{4.4.14}$$

This potential function gives the magnetic field

$$\vec{\mathrm{H}}(\vec{\mathrm{r}}) = \frac{1}{4\pi}\left(\frac{3[\vec{\mathrm{m}}\cdot\vec{\mathrm{r}}]\vec{\mathrm{r}}}{\mathrm{r}^5} - \frac{\vec{\mathrm{m}}}{\mathrm{r}^3}\right). \tag{4.4.15}$$

The components of this field when written in the spherical polar co-ordinate system are (see Figure (4.4.13))

$$\begin{gathered}\mathrm{H_r} = \frac{2\mathrm{m}}{4\pi}\frac{\cos\theta}{\mathrm{r}^3}, \\ \mathrm{H}_\theta = \frac{\mathrm{m}}{4\pi}\frac{\sin\theta}{\mathrm{r}^3}, \\ \mathrm{H}_\phi = 0.\end{gathered} \tag{4.4.16}$$

The components of $\vec{B}$ are obtained from the components of $\vec{H}$ by multiplying by the permeability of free space, μ_0. The resulting expressions are exactly the same as the ones obtained earlier, Equation (4.3.10), from the vector potential for a point dipole, Equation (4.3.9). Thus, the field due to a magnetic point dipole can be calculated either from a magnetic vector potential or from a magnetic scalar potential.

The magnetic scalar potential corresponding to a given distribution of magnetization density can be calculated by superposition using the magnetic

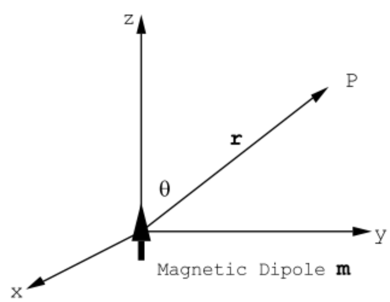


Figure 4.4.13: A magnetic point dipole oriented along the z-axis.

potential due to a point dipole, Equation (4.4.14), see Figure (4.4.14). The element of volume, $d\tau$, has associated with it a magnetic dipole moment $\vec{\mathrm{m}} = \vec{\mathrm{M}}(\vec{\mathrm{r}})\mathrm{d}\tau$. This contributes to the magnetic scalar potential at point P(X,Y,Z) an amount given by

$$\mathrm{dV_P}(\vec{\mathrm{R}}) = \frac{1}{4\pi}\frac{\vec{\mathrm{M}}(\vec{\mathrm{r}})\cdot[\vec{\mathrm{R}}-\vec{\mathrm{r}}]}{|\vec{\mathrm{R}}-\vec{\mathrm{r}}|^3}\mathrm{d}\tau. \tag{4.4.17}$$

Sum Equation (4.4.17) over the entire magnetization distribution to obtain

$$V_P(\vec{R}) = \frac{1}{4\pi} \int\int\int_{Space} d\tau \frac{\vec{M}(\vec{r}) \cdot [\vec{R} - \vec{r}]}{|\vec{R} - \vec{r}|^3}. \tag{4.4.18}$$

The potential calculated using Equation (4.4.18) will give the same fields as that calculated from the equivalent magnetic charge distribution and Equation (4.4.11) which is based upon the superposition of the magnetic potentials generated by fictitious point magnetic charges. The proof that the potential calculated in these two different ways is the same, except, possibly for a constant, is based on the identity

$$\operatorname{div}\left(\frac{\pi}{\left(\frac{\pi}{1} - \bar{r}\right)}\right) = \frac{\operatorname{div}(\vec{x})}{|\vec{R} - \vec{r}|} + \overrightarrow{11} \cdot \operatorname{grad}\left(\frac{1}{|\vec{R} - \vec{r}|}\right).$$

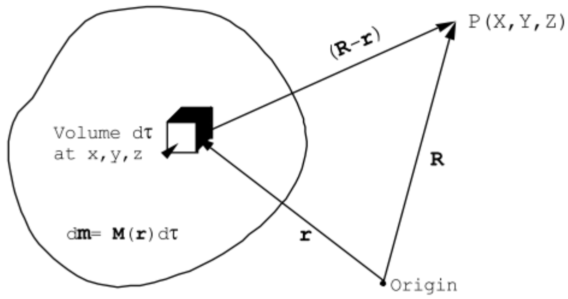


Figure 4.4.14: The calculation of the magnetic scalar potential for a given distribution of magnetization density, $\vec{M}(\vec{r})$, using superposition and the potential function for a point magnetic dipole.

The argument proceeds in exactly the same fashion as for the analogous electrostatic case; see Chpt.(2), section(2.8).

CHAPTER OVERVIEW

5: The Magnetostatic Field II

Magnetostatic Boundary Value Problems for a Linear, Isotropic, Magnetic Material.

Thumbnail: Magnetic H-field inside and outside of a cylindrical bar magnet. (CC BY-SA 4.0; Geek3 via Wikipedia)

5.1: Introduction- Sources in a Uniform Permeable Material

The equations of magnetostatics are given by Equation (4.1.2)

$$\operatorname{div}(\overrightarrow{\mathrm{B}})=0,$$

and Equation (4.1.3)

$$\operatorname{curl}(\overrightarrow{\mathrm{B}})=\mu_0\left(\overrightarrow{\mathrm{J}}_f+\operatorname{curl}(\overrightarrow{\mathrm{M}})\right).$$

(refer to section(4.1)). For a linear, isotropic, magnetic medium $\vec{B}$ is proportional to $\vec{H}$ where the factor of proportionality is called the permeability.

$$\overrightarrow{\mathrm{B}}=\mu\overrightarrow{\mathrm{H}}=\mu_0(\overrightarrow{\mathrm{H}}+\overrightarrow{\mathrm{M}}),$$

so that

$$\overrightarrow{\mathrm{M}}=\left(\frac{\mu}{\mu_0}-1\right)\overrightarrow{\mathrm{H}},$$

or

$$\overrightarrow{\mathrm{M}}=(\mu_r-1)\overrightarrow{\mathrm{H}}. \tag{5.1.1}$$

In Equation (5.1.1) $\mu r = \mu/\mu 0$ is the relative permeability. The second of the above Maxwell's equations can be re-written in the form

$$\operatorname{curl}(\overrightarrow{\mathrm{H}})=\overrightarrow{\mathrm{J}}_f,$$

or

$$\operatorname{curl}(\overrightarrow{\mathrm{B}})=\mu\overrightarrow{\mathrm{J}}_f. \tag{5.1.2}$$

The substitution $\vec{B}$ = curl($\vec{A}$) ensures that Equation (4.1.2) will be satisfied since the divergence of any curl is zero. Using this substitution in Equation (5.1.2) gives

$$\operatorname{curlcurl}(\overrightarrow{\mathrm{A}})=\mu\overrightarrow{\mathrm{J}}_f. \tag{5.1.3}$$

If in addition one chooses

$$\operatorname{div}(\overrightarrow{\mathrm{A}})=0, \tag{5.1.4}$$

then

$$\nabla^2\overrightarrow{\mathrm{A}}=-\mu\overrightarrow{\mathrm{J}}_f, \tag{5.1.5}$$

and this equation has the particular solution

$$\overrightarrow{\mathrm{A}}(\overrightarrow{\mathrm{R}})=\frac{\mu}{4\pi}iiint_{Space}\mathrm{d}\tau\frac{\overrightarrow{\mathrm{J}}_f(\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}, \tag{5.1.6}$$

where $\mathrm{d}\tau$ is an element of volume. This development exactly follows the procedure described in Chpt.(4); the only difference is that the integration in Equation (5.1.6) is carried out over the free current density distribution, and the fields due to the effective current density curl($\vec{M}$) are taken into account through the permeability μ that multiplies the integral. It should be noted that this procedure only works if μ does not depend upon position in space. If there are regions characterized by different values of μ the problem of calculating the magnetic field distribution becomes much more difficult. This is because at the boundaries between regions having different permeabilities there are discontinuities in the normal and tangential components of $\vec{M}$ that act as field sources.

In the usual situation the current density is zero except within a finite number of thin wires. For a current of I Amps carried in a wire of negligible cross-section Equation (5.1.6) becomes

$$\overrightarrow{\mathrm{A}}_{P}=\frac{\mu \mathrm{I}}{4 \pi} \int_{Wire} \frac{|\mathrm{d} \overrightarrow{\mathrm{L}}}{|\overrightarrow{\mathrm{r}}|}, \tag{5.1.7}$$

where $\vec{r}$ is the vector from the element of length d$\vec{L}$ to the point P where the vector potential $\vec{A}$ is to be calculated. From $\vec{B}$ = curl($\vec{A}$) one obtains

$$\overrightarrow{\mathrm{B}}(\overrightarrow{\mathrm{r}})=\frac{\mu \mathrm{I}}{4 \pi} \int_{Wire} \frac{\mathrm{d} \overrightarrow{\mathrm{L}} \times \overrightarrow{\mathrm{r}}}{|\overrightarrow{\mathrm{r}}|^{3}}. \tag{5.1.8}$$

These formulae are very similar to Equations (4.2.1) and (4.17) of Chpt.(4). The fields corresponding to the standard problems of a long straight wire, the field along the axis of a circular loop, and along the axis of a finite solenoid are given by Equations (4.3.3), (4.3.4), and (4.3.5) where the permeability of free space, μ_0, is replaced by the permeability μ. In particular, the field of an infinite solenoid that is filled with a magnetic material is given by

$$\overrightarrow{\mathrm{B}}=\mu \mathrm{NI}, \tag{5.1.9}$$

where N is the number of turns per meter.

5.2: Calculation of off-axis Fields

It is relatively easy to calculate the magnetic field along the symmetry axis of an axially symmetric coil system using the law of Biot-Savart, Equation (5.1.8). The calculation can be easily carried out because the magnetic field has only one component, an axial component, and the cylindrical symmetry makes the integration over the current distribution relatively simple. M.W.Garrett has pointed out that off-axis fields can be readily calculated from the magnetic scalar potential (M.W.Garrett, J.Appl.Phys.22,1091-1107(1951); ”Axially Symmetric Systems for Generating and Measuring Magnetic Fields. Part I”). In any current-free region

$$\operatorname{curl}(\overrightarrow{\mathrm{H}})=0 \tag{5.2.1}$$

and therefore one can write

$$\overrightarrow{\mathrm{H}}=-\operatorname{grad}(\mathrm{V_m}), \tag{5.2.2}$$

where $\mathrm{V_m}$ is a scalar function of position. In a uniform medium for which $\overrightarrow{\mathrm{B}}=\mu\overrightarrow{\mathrm{H}}$ the equation $\operatorname{div}(\overrightarrow{\mathrm{B}})=0$ can be re-written as

$$\operatorname{div}(\overrightarrow{\mathrm{H}})=0. \tag{5.2.3}$$

It follows from Equation (5.2.2) that the magnetic scalar potential must satisfy LaPlace’s equation in any region free of currents. In spherical polar coordinates LaPlace’s equation is written

$$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2\frac{\partial \mathrm{V_m}}{\partial \mathrm{r}}\right)+\frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathrm{V_m}}{\partial\theta}\right)+\frac{1}{\mathrm{r}^2\sin^2\theta}\frac{\partial^2 \mathrm{V_m}}{\partial\phi^2}=0. \tag{5.2.4}$$

The magnetic scalar potential cannot depend upon the azimuthal angle ϕ for an axially symmetric coil system, so that $\nabla^2\mathrm{V_m}=0$ reduces to

$$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2\frac{\partial \mathrm{V_m}}{\partial \mathrm{r}}\right)+\frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathrm{V_m}}{\partial\theta}\right)=0. \tag{5.2.5}$$

The general solution of this equation can be written as a series expansion in Legendre polynomials:

$$\mathrm{V_m}=\sum_{n=0}^{\infty}\left(\mathrm{a_n r^n}+\frac{\mathrm{b_n}}{\mathrm{r^{n+1}}}\right)\mathrm{P_n}(\cos\theta). \tag{5.2.6}$$

This is the same expansion as was used for the electrostatic potential in Chpt.(3),section(3.2.1(d)) to treat the problem of a dielectric sphere in a uniform applied electric field. The functions $\mathrm{P_n(x)}$ are Legendre polynomials, the first five of which are listed in Table (3.2.2). The terms proportional to $1/\mathrm{r^{n+1}}$ in the expansion (5.2.6) are not acceptable for describing the magnetic potential function for a system of axially symmetric coils because they blow up at r=0; there are no singularities in the magnetic field along the axis of the coil system. This means that the magnetic potential must be describable by the series

$$\mathrm{V_m}=\sum_{n=1}^{\infty}\mathrm{a_n r^n P_n}(\cos\theta). \tag{5.2.7}$$

(The n=0 term corresponds to a constant; it is not important and may be set equal to zero because any constant may be added to $\mathrm{V_m}$ without changing the magnetic field). Along the axis of the coil system, the z-axis of the spherical polar co-ordinate system, the angle θ is fixed; cos θ=+1 for the region z > 0 and cos θ=-1 for the region z < 0. Moreover,along the axis of the coil system r =| z |, so that along the axis Equation (5.2.7) becomes a power series in z. The magnetic field calculated from this power series may be compared term by term with the power series for the magnetic field calculated

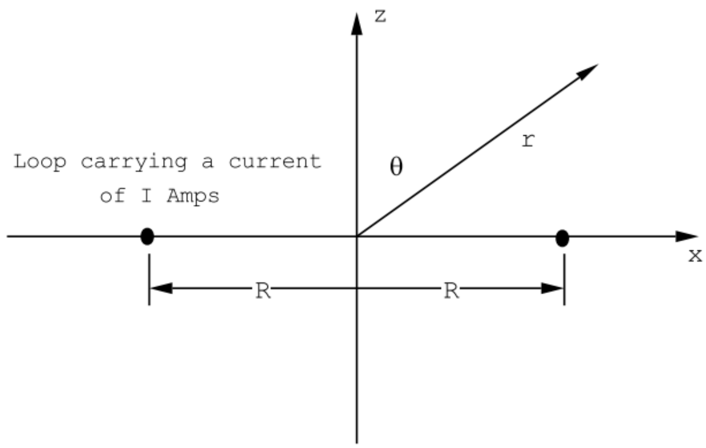


Figure 5.2.1: A circular loop of radius R carrying a current of I Amps and lying in the x-y plane.

directly from the law of Biot-Savart. The comparison of the two series yields values for the coefficients an that appear in the expansion for the magnetic potential, Equation (5.2.7). Once the coefficients an have been determined the magnetic field at any point within the coil system can be readily calculated from $\vec{H}$ = −grad(V_m).

This procedure can be illustrated for a single loop of wire lying in the xy plane, Figure (5.1.1). The magnetic field along the axis of such a loop is given by

$$B_z = \frac{\mu IR^2}{2} \frac{1}{\left(z^2 + R^2\right)^{3/2}}, \tag{5.2.8}$$

see Equation (4.3.4) of Chpt.(4). This expression can be expanded in a Taylor seies in the variable z:

$$B_z(z) = B_z(0) + \left(\frac{dB_z}{dz}\right)_{z=0} z + \left(\frac{d^2B_z}{dz^2}\right)_{z=0} \left(\frac{z^2}{2}\right) + + \left(\frac{d^3B_z}{dz^3}\right)_{z=0} \left(\frac{z^3}{6}\right) + \cdots + \left(\frac{d^nB_z}{dz^n}\right)_{z=0} \left(\frac{z^n}{n!}\right) + \cdots \tag{5.2.9}$$

For the single current loop of Figure (5.1.1) this Taylor's series becomes

$$B_z(z) = \left(\frac{\mu I}{2R}\right)\left(1 - \frac{3}{2}\left(\frac{z}{R}\right)^2 + \frac{45}{24}\left(\frac{z}{R}\right)^4 + \cdots\right). \tag{5.2.10}$$

Notice that this series contains only even powers of (z/R) because the magnetic field is symmetric with respect to the plane of the coil, i.e. $B_z(-z)$ = Bz(z). Now $B_z(z)$ is derived from the magnetic potential function through a differentiation with respect to z:

$$B_z(z) = -\frac{\partial V_m}{\partial z}, \tag{5.2.11}$$

The series (5.2.11) must be compared with the general series Equation (5.2.7) using r=z and cos θ = 1, i.e. with

$$V_m(z, 0) = a_1 zP_1(1) + a_2 z^2 P_2(1) + a_3 z^3 P_3(1) + a_4 z^4 P_4(1) + a_5 z^5 P_5(1) + \cdots \tag{5.2.12}$$

It is clear from this comparison that the coefficients of all the even terms must be zero. The Legendre polynomials are normalized so that $P_n(1) = 1$ (see Table (3.2.2), section(3.2.1(d))). It follows from a comparison of (5.2.12) with (5.2.11) that

$$\begin{aligned} a_1 &= -\frac{\mu I}{2R} \\ a_3 &= \frac{\mu I}{2}\left(\frac{1}{2R^3}\right) \\ a_5 &= -\frac{\mu I}{2}\left(\frac{9}{24R^5}\right), \quad \text{etc.} \end{aligned}$$

The first three terms in the expansion for the potential function, valid for any point in space such that (r/R) < 1, are given by

$$V_m(r, \theta) = -\frac{\mu I}{2}\left[\left(\frac{r}{R}\right) P_1(\cos\theta) - \frac{1}{2}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{9}{24}\left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \cdots\right], \tag{5.2.13}$$

where

$$P_1(x) = x$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

and

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right);$$

(see Schaum's Outline Series: Mathematical Handbook, by Murray R. Spiegel, McGraw-Hill,N.Y., 1968)). The components of the magnetic field can be calculated from

$$B_r = -\frac{\partial V_m}{\partial r},$$

and

$$B_\theta = -\frac{1}{r}\frac{\partial V_m}{\partial \theta}.$$

These fields can be calculated very readily for particular values of r, θ by means of a modern digital computer; programs for calculating Legendre polynomials and their derivatives are readily available.

5.3: A Discontinuity in the Permeability

Problems that involve currents embedded in materials for which the permeability varies from place to place are very difficult even if all of those materials exhibit linear response. Such problems can usually be solved by successive approximations: (1) the fields are calculated as if the source currents are immersed in a uniform permeable medium; (2) the resulting magnetization distribution is calculated from Equation (5.1.1) using the actual permeabilities; (3) the fields due to the effective current distribution $\vec{J}_{\text{eff}}$ = curl(M) are added to the fields previously calculated; (4) the resulting total field is used to calculate a new magnetization distribution; (5) the cycle is repeated until adequate convergence has been secured, i.e. until the input field and the output field are essentially the same.

There is a class of magnetic problems that are very similar to electrostatic problems. In a region of space that is current free it is appropriate to use a magnetic scalar potential because curl(H) = 0. In any region in which the permeability does not depend upon position B = μH and therefore div(H) = 0 because div(B) = 0; in such a region the magnetic scalar potential, V_m, must satisfy LaPlace's equation. The magnetic potential must also satisfy boundary conditions at a surface of discontinuity between regions which are characterized by different permeabilities. These boundary conditions are:

(1) V_m must be continuous across the interface between two materials. This condition is a consequence of curl($\vec{H}$) = 0; the tangential components of $\vec{H}$ must be continuous across the interface, as can be shown using Stokes' theorem.

(2) The normal component of B must be continuous across the interface between two different materials. This condition is a consequence of div($\vec{B}$) = 0; the continuity of the normal component of B can be deduced using Gauss' theorem. In terms of the permeabilities one has

$$\mu_1\left(\frac{\partial \mathrm{V}_{\mathrm{m}}^{1}}{\partial \mathrm{n}}\right)_{\text{Boundary}}=\mu_2\left(\frac{\partial \mathrm{V}_{\mathrm{m}}^{2}}{\partial \mathrm{n}}\right)_{\text{Boundary}}, \tag{5.3.1}$$

where $\frac{\partial \mathrm{V}_{\mathrm{m}}}{\partial \mathrm{n}}$ is the derivative of the magnetic potential along the normal to the interface. These two boundary conditions are essentially the same as the boundary conditions imposed upon the electrostatic potential function. In addition, the magnetic potential must exhibit the proper behaviour at infinity and at the origin just like the electrostatic potential. The electrostatic potential and the magnetic potential both satisfy LaPlace's equation, and both satisfy similar boundary conditions; it follows that similar problems must have correspondingly similar solutions. In particular, the magnetic potential functions for a sphere in a uniform applied field, and for a cylinder in a uniform field applied transverse to the cylinder axis must have the same form as those for the corresponding electrostatic problems discussed in Chpt.(3), sections(3.2.1 (c) and (d)) and Figures (3.2.4) and (3.2.5). For the magnetic case see Figures (5.3.2 and 5.3.3).

5.3.1 A Permeable Sphere in a Uniform Magnetic Field.

The potential function **inside** the sphere is given by

$$\mathrm{V}_{\mathrm{m}}^{\mathrm{i}}=-\frac{3\left(\frac{\mu_2}{\mu_1}\right)\mathrm{H}_0}{\left(1+2\left(\frac{\mu_2}{\mu_1}\right)\right)}\mathrm{r}\cos\theta. \tag{5.3.2}$$

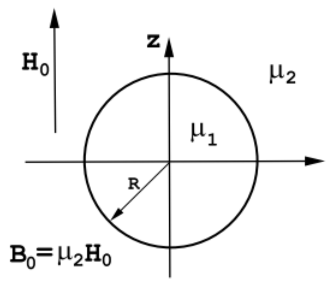


Figure 5.3.2: A permeable sphere placed in a uniform magnetic H-field of strength H_0 Amps/meter. The permeability of the sphere is μ_1, and it is placed in a medium having a permeability μ_2.

This corresponds to a uniform magnetic field along the z-axis. The potential function **outside** the sphere is given by

$$V_m^o = -H_0 r\cos\theta + \frac{(1-(\mu_2/\mu_1))}{(1+2(\mu_2/\mu_1))} R^3 H_0 \frac{\cos\theta}{r^2}. \tag{5.3.3}$$

This corresponds to a uniform magnetic field, H_0, plus the field of a point dipole located at the center of the sphere and having a strength

$$m = \frac{(1-[\mu_2/\mu_1])}{(1+2[\mu_2/\mu_1])} 4\pi R^3 H_0 \quad Amp - m^2.$$

5.3.2 An Infinitely Long permeable Cylinder in a Uniform Magnetic Field.

The potential function **inside** the cylinder is given by

$$V_m^i = -\frac{2(\mu_2/\mu_1)H_0}{(1+(\mu_2/\mu_1))} r\cos\theta. \tag{5.3.4}$$

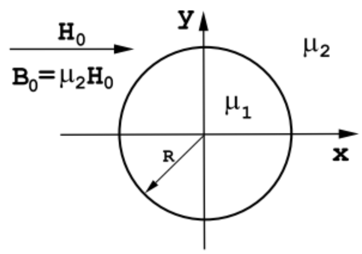


Figure 5.3.3: An infinitely long permeable cylinder, radius R, placed in a uniform transverse magnetic field, H_0 Amps/meter. The axis of the cylinder is oriented along the z-axis. The permeability of the cylinder is μ_1 and the cylinder is immersed in a medium whose permeability is μ_2.

This corresponds to a uniform field along the x-axis, parallel with the applied magnetic field. The potential function **outside** the cylinder is given by

$$V_m^o = -H_0 r\cos\theta + \frac{(1-[\mu_2/\mu_1])}{(1+[\mu_2/\mu_1])} H_0 R^2 \frac{\cos\theta}{r}. \tag{5.3.5}$$

This corresponds to a uniform magnetic field, H_0, plus the field due to a line of dipoles located on the cylinder axis and having a strength

$$m = 2\pi R^2 H_0; \frac{(1-[\mu_2/\mu_1])}{(1+[\mu_2/\mu_1])} H_0 R^2 \frac{\cos\theta}{r}.$$

5.3.3 A Point Magnetic Dipole Near a Permeable Plane.

There are no magnetic point charges, nevertheless the field of a point dipole can be considered to have its origin in two magnetic poles that are very close together. This suggests that the problem of a magnetic dipole near the plane interface between two magnetically dissimilar materials may be treated by the method of images by analogy with the corresponding electrostatic problem: see section(3.2.2(a) and (b)). The potential function for a magnetic point charge of strength q_m is given by

$$V_m = \frac{q_m}{4\pi}\left(\frac{1}{r}\right)$$

and the corresponding magnetic field, $\vec{H}$, generated by a hypothetical point magnetic charge of strength q_m is given by the Coulomb law

$$\vec{H} = \frac{q_m}{4\pi}\left(\frac{\vec{r}}{r^3}\right). \tag{5.3.6}$$

The corresponding field B is given by

$$\vec{B} = \frac{\mu q_m}{4\pi}\left(\frac{\vec{r}}{r^3}\right). \tag{5.3.7}$$

Eqn.(5.3.6) can be used to calculate the magnetic field generated by a magnetic dipole. Let a magnetic charge q_m be located at x=0, y=0, and z=d/2. Let a second magnetic charge $-q_m$ be located at x=0, y=0, and z=- d/2. This pair of charges forms a magnetic dipole of strength m=qd oriented along the z-axis and positioned at z=0. The magnetic field of the dipole is given by (using Equation (5.3.6))

$$\overrightarrow{H}=\frac{q_m}{4\pi}\frac{(x\hat{u}_x+y\hat{u}_y+(z-d/2)\hat{u}_z)}{(x^2+y^2+(z-d/2)^2)^{3/2}}-\frac{q_m}{4\pi}\frac{(x\hat{u}_x+y\hat{u}_y+(z+d/2)\hat{u}_z)}{(x^2+y^2+(z+d/2)^2)^{3/2}},$$

or in the limit as d → 0 the expression for the magnetic field can be written

$$\vec{H}=\frac{1}{4\pi}\left(\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5}-\frac{\vec{m}}{r^3}\right). \tag{5.3.8}$$

Using $\vec{B}=\mu\vec{H}$ this is the same result as stated in Equation (1.2.13) of Chpt.(1) for the case $\mu = \mu_0$.

The plane interface image problem for magnetic monopoles is shown in Figure (5.3.4). The H-field in the region on the left of the interface where the permeability is μ_1 is that due to the original magnetic charge q_m plus that due to an image charge of strength $-q_m(\mu_2-\mu_1)/(\mu_2+\mu_1)$ located the

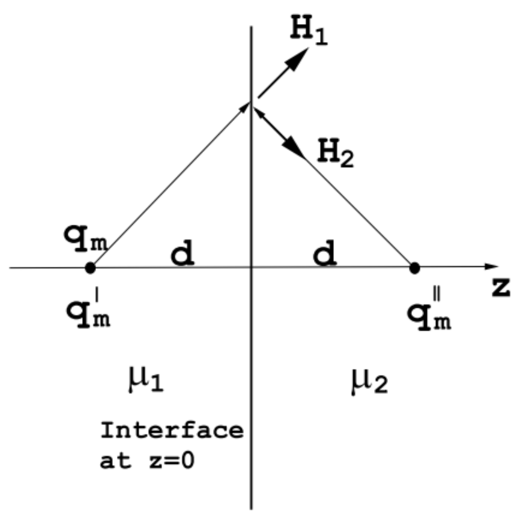


Figure 5.3.4: A fictitious magnetic point charge, qm, located outside the plane boundary between two permeable materials. The permeability to the left of the boundary is μ_1. The permeability on the right hand side of the boundary is μ_2. The magnetic charge is located a distance d from the interface. The other two magnetic charges, q_m' and q_m'' , are image charges.

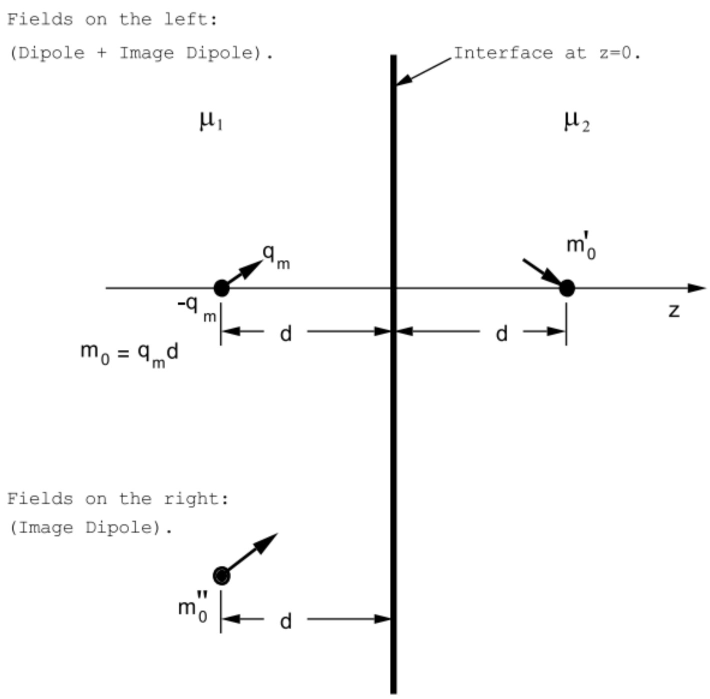


Figure 5.3.5: Image problem for a magnetic dipole m0 located near the plane boundary between two permeable materials. The image dipole used for calculating fields on the left of the boundary is $\mathrm{m}_0^{\prime}=\mathrm{m}_0\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)$. The image dipole used for calculating fields on the right of the plane boundary is $\mathrm{m}^{\prime\prime}=2\mathrm{m}_0\left(\frac{\mu_1}{\mu_2+\mu_1}\right)$. (N.B. the dipole strength is given by $m_0 = q_m d$ where d is the small separation between the plus and minus magnetic charges, **NOT** the distance d between the dipole and the interface at z=0.)

same distance behind the interface as q_m is located in front of the interface. The field in the right hand space, permeability μ_2, is the field due to a magnetic charge that is located at the same place as qm but whose strength is $2\mathrm{q}_{\mathrm{m}}\mu_1/\left(\mu_2+\mu_1\right)$. It is easy to show that the fields in the two regions are such that the components of H parallel with the interface are continuous across the interface. It is also easy to show that the normal components of H in the two regions is such that at the interface $\mu_1(\vec{\mathrm{H}})_n=\mu_2(\vec{\mathrm{H}})_n$ so that the normal component of $\vec{B}$ is continuous across the interface. Notice that the image charge is the negative of q_m for the case $\mu_2 > \mu_1$, but that the image charge has the same sign as qm when $\mu_2 < \mu_1$. The field on the left of the interface is given by

$$\vec{\mathrm{H}}_L=\frac{\mathrm{q}_{\mathrm{m}}}{4\pi}\left(\frac{\vec{\mathrm{r}}_1}{\mathrm{r}_1^3}\right)-\frac{\mathrm{q}_{\mathrm{m}}}{4\pi}\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\left(\frac{\vec{\mathrm{r}}_2}{\mathrm{r}_2^3}\right).$$

The field on the right side of the interface is given by

$$\vec{\mathrm{H}}_R=\frac{1}{4\pi}\left(\frac{2\mathrm{q}_{\mathrm{m}}\mu_1}{\mu_2+\mu_1}\right)\left(\frac{\vec{\mathrm{r}}_1}{\mathrm{r}_1^3}\right).$$

This point charge solution can be extended by means of superposition to treat the problem of a magnetic dipole near a plane interface, Figure (5.3.5). The field on the left, in the region of permeability μ_1, is due to the dipole plus its image as shown in the figure. The fields in the region on the left due to the image dipole can be used to calculate the force and torque on the real dipole. The force on a magnetic monopole is given by

$$\vec{\mathrm{F}}=\mathrm{q}_{\mathrm{m}}\vec{\mathrm{B}},\tag{5.3.9}$$

The force on the magnetic dipole is just the sum of the forces acting on the two monopoles that make up the dipole: the dipole force is proportional to the field gradientr at the position of the dipole. The torque exerted on a magnetic dipole is given by

$$\vec{\mathrm{T}}=\vec{\mathrm{m}}\times\vec{\mathrm{B}}.\tag{5.3.10}$$

If $\mu_2 > \mu_1$ the magnetic dipole is attracted to the interface. Clearly this treatment can be generalized to discuss any permanently magnetized body located near a plane interface between two linear magnetic materials: the solution can be obtained as the superposition of the fields generated by the elementary dipoles of which the body is composed.

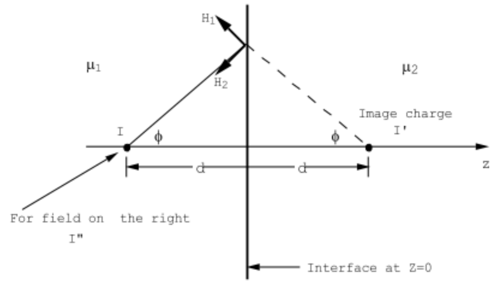


Figure 5.3.6: The image problem for the case of a line current of I Amps running parallel with the plane interface between two permeable materials and located a distance d from the interface.

5.3.4 A Wire Parallel with an Interface and carrying a Current of I Amps.

Another problem that can be solved using the method of images is that of a thin wire carrying a current, and oriented parallel with the plane interface between two different permeable regions as shown in Figure (5.3.6). The field in the region on the left is ascribed to the real current I plus an image current I' located the same distance, d, to the right of the interface as the real current is located to the left of the interface. Any point P on the interface is equidistant from both the current I and its image, I'; let that distance be R. Let the line joining the position of the current to the point P on the interface make an angle ϕ with the normal to the interface, as shown in Figure (5.3.6). The field lines generated by a long straight current carrying wire form concentric cylinders around the wire, and the strength of the field in free space is given by $B_\theta = \mu_0 I/2\pi R$, where R is the distance from the wire; see Chpt.(4), Equation (4.3.3). This corresponds to an H-field $H_\theta = I/2\pi R$ Amps/m. The component of the magnetic field, $\vec{H}$, parallel to the interface that is generated by the current I and its image I' in Figure (5.3.6) is given by

$$\left.\mathrm{H}_1\right|_{\text{parallel}}=\frac{1}{2\pi\mathrm{R}}\left(\mathrm{I}\cos\phi-\mathrm{I}'\cos\phi\right). \tag{5.3.11}$$

The field component normal to the interface is given by

$$\left.\mathrm{H}_1\right|_{\text{normal}}=\frac{1}{2\pi\mathrm{R}}\left(\mathrm{I}\sin\phi+\mathrm{I}'\sin\phi\right). \tag{5.3.12}$$

Let the field in the region of space to the right of the interface be generated by an image current I" located at the position of the real current I. The magnetic field component parallel with the interface generated by I" is given by

$$\left.\mathrm{H}_2\right|_{\text{parallel}}=\frac{1}{2\pi\mathrm{R}}\mathrm{I}''\cos\phi, \tag{5.3.13}$$

and its normal component is given by

$$\left.\mathrm{H}_2\right|_{\text{normal}}=\frac{1}{2\pi\mathrm{R}}\mathrm{I}''\sin\phi, \tag{5.3.14}$$

The tangential component of H must be continuous across the interface, and therefore from Equations (5.3.11 and 5.3.13)

$$\mathrm{I}-\mathrm{I}'=\mathrm{I}''. \tag{5.3.15}$$

The normal component of $\vec{B}$ must be continuous across the interface so that from Equations (5.3.12 and 5.3.14) one finds

$$\mu_1\left(\mathrm{I}+\mathrm{I}'\right)=\mu_2\mathrm{I}''. \tag{5.3.16}$$

The linear equations (5.3.15 and 5.3.16) can be solved to obtain the required image current strengths.The results are

$$\mathrm{I}'=\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\mathrm{I}, \tag{5.3.17}$$

and

$$\mathrm{I}''=\frac{2\mathrm{I}\mu_1}{\left(\mu_2+\mu_1\right)}. \tag{5.3.18}$$

The $\vec{H}$ field in the space to the left of the interface is that generated by the current I plus the image current I’. The $\vec{H}$ field to the right of the interface is that generated by the image current I”.

5.4: The Magnetostatic Field Energy

Energy is required to establish a magnetic field. The energy density stored in a magnetostatic field established in a linear isotropic material is given by

$$\mathrm{W_B}=\frac{\mu}{2}\mathrm{H}^2=\frac{\vec{\mathrm{H}}\cdot\vec{\mathrm{B}}}{2}\quad \text{Joules}/\mathrm{m}^3. \tag{5.4.1}$$

The total energy stored in the magnetostatic field is obtained by integrating the energy density, $\mathrm{W_B}$, over all space (the element of volume is $d\tau$):

$$\mathrm{U_B}=\iiint_{Space}\mathrm{d}\tau\left(\frac{\vec{\mathrm{H}}\cdot\vec{\mathrm{B}}}{2}\right). \tag{5.4.2}$$

This expression for the total energy, $\mathrm{U_B}$, can be transformed into an integral over the sources of the magnetostatic field. The transformation can be carried out by means of the vector identity

$$\operatorname{div}(\vec{\mathrm{A}}\times\vec{\mathrm{H}})=\vec{\mathrm{H}}\cdot(\vec{\nabla}\times\vec{\mathrm{A}})-\vec{\mathrm{A}}\cdot(\vec{\nabla}\times\vec{\mathrm{H}}). \tag{5.4.3}$$

(There is a nice discussion of this identity in The Feynman Lectures on Physics, Vol.II, section 27.3, by R.P.Feynman, R.B.Leighton, and M.Sands, Addison-Wesley, Reading, Mass.,1964). Proceed by integrating Equation (5.4.3) over all space, then use Gauss' theorem to transform the left hand side into a surface integral. The result is

$$\iint_{Surface}(\vec{A}\times\vec{H})\cdot d\vec{S}=\iiint_{Volume}d\tau\left(\vec{H}\cdot\vec{B}-\vec{J}_f\cdot\vec{A}\right), \tag{5.4.4}$$

where $d\vec{S}$ is the element of surface area, $\vec{\mathrm{B}}=\vec{\nabla}\times\vec{\mathrm{A}}=\operatorname{curl}(\vec{\mathrm{A}})$, and $\vec{\nabla}\times\vec{\mathrm{H}}=\operatorname{curl}(\vec{\mathrm{H}})=\vec{\mathrm{J}}_f$. Here $\vec{A}$ is the vector potential and $\vec{J}_f$ is the current density. When the integrals in Equation (5.4.4) are extended over all space the surface integral goes to zero: the surface area of a sphere of large radius R is proportional to R^2 but for currents confined to a finite region of space $|\vec{A}|$ must decrease at least as fast as a dipole source, i.e. $\propto 1/\mathrm{R}^2$, and $|\vec{H}|$ must decrease at least as fast as $1/\mathrm{R}^3$. It follows that in the large R limit the surface integral must go to zero like $1/\mathrm{R}^3$. This requires the two terms on the right hand side of (5.4.4) to be equal, and this result can be used to rewrite the expression (5.4.2) in terms of the vector potential and the source current density:

$$\mathrm{U_B}=\frac{1}{2}\iiint_{Space}\mathrm{d}\tau(\vec{\mathrm{H}}\cdot\vec{\mathrm{B}})=\frac{1}{2}\iiint_{Space}\mathrm{d}\tau\left(\vec{\mathrm{J}}_f\cdot\vec{\mathrm{A}}\right). \tag{5.4.5}$$

In many problems the current density is confined to a wire whose dimensions are small compared with other lengths in the problem. For such a circuit the contribution to the second volume integral in (5.4.5) vanishes except for points within the wire, and therefore the volume integral can be replaced by a line integral along the wire providing that the variation of the vector potential, $vecA$, over the cross-section of the wire can be neglected. For a wire of negligible thickness

$$\iiint_{Space}\mathrm{d}\tau\left(\vec{\mathrm{J}}_f\cdot\vec{\mathrm{A}}\right)\rightarrow\mathrm{I}\oint_C\vec{\mathrm{A}}\cdot\mathrm{d}\vec{\mathrm{L}}, \tag{5.4.6}$$

where I is the current through the wire; the current must be the same, of course, at all points along the circuit. The line integral of the vector potential around a closed circuit is equal to the magnetic flux, Φ, through the circuit. This equivalence can be seen by using the definition $\vec{B}=\operatorname{curl}(\vec{A})$ along with Stokes' theorem to transform the integral for the flux:

$$\Phi=\iint_S\vec{\mathrm{B}}\cdot\mathrm{d}\vec{\mathrm{S}}=\iint_S\operatorname{curl}(\vec{\mathrm{A}})\cdot\mathrm{d}\vec{\mathrm{S}}=\oint_C\vec{\mathrm{A}}\cdot\mathrm{d}\vec{\mathrm{L}}, \tag{5.4.7}$$

where the curve C bounds the surface S. Combining Equations (5.4.7) and (5.4.5), the magnetic energy associated with a single circuit can be written

$$\mathrm{U_B}=\frac{1}{2}\iiint_{Space}\mathrm{d}\tau\left(\vec{\mathrm{J}}_f\cdot\vec{\mathrm{A}}\right)=\frac{1}{2}\mathrm{I}\Phi, \tag{5.4.8}$$

and for a number of circuits, N,

$$U_B = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k. \tag{5.4.9}$$

The latter expression is similar to Equation (3.3.6) for the electrostatic energy associated with a collection of charged conductors: currents in the magnetostatic case play a role similar to that of charges in the electrostatic case, and flux plays a role that is similar to the role played by the potentials.

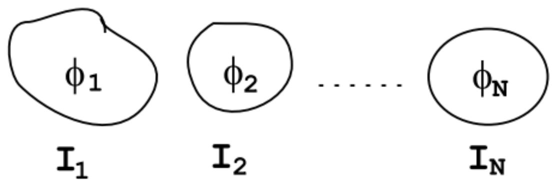


Figure 5.4.7: N circuits embedded in a linear, isotropic medium. The magnetic fluxes, Φ_k, are linear functions of the currents I_j. The inductance coefficients satisfy the symmetry relations $L_{MN} = L_{NM}$.

5.5: Inductance Coefficients

Consider N circuits embedded in a linear, isotropic medium characterized by a permeability μ. The magnetic flux through a given circuit will depend upon the currents in all of the circuits. However, the magnetic field generated by the current in a particular circuit will be a linear function of the current in that circuit; if the current is doubled then the magnetic field due to that current will also be doubled because Maxwell's equations are linear in the current density. Since the magnetic field at any point is a linear function of the currents it follows that the flux through each circuit must be a linear function of the currents: i.e.

$$\begin{aligned}\Phi_1 &= L_{11}I_1 + L_{12}I_2 + \cdots + L_{1N}I_N \\ \Phi_2 &= L_{21}I_1 + L_{22}I_2 + \cdots + L_{2N}I_N \\ \vdots \\ \Phi_N &= L_{N1}I_1 + L_{N2}I_2 + \cdots + L_{NN}I_N\end{aligned} .$$

The coefficients L_{MN} are called **coefficients of induction**. They have units of Henries.

The magnetostatic energy, Equation (5.4.9), can be written in terms of the current in each circuit and the induction coefficients. The magnetic energy must be independent of the order in which the circuit currents attains their final values. This condition requires that

$$L_{MN} = L_{NM}. \tag{5.5.1}$$

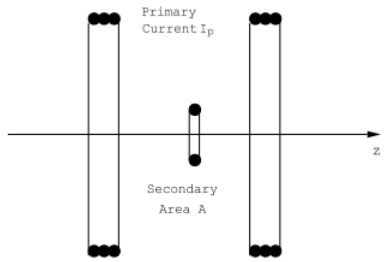


Figure 5.5.8: A system consisting of a primary coil carrying a current I_p and a secondary coil consisting of a single turn of wire enclosing an area A m^2.

There are, therefore, only N(N+1)/2 independent induction coefficients rather than N^2 of them. This symmetry property of the induction coefficients can be used to determine the flux produced in a coil system by a magnetized body. Consider a primary coil system carrying a current I_p. Let there be a small secondary coil as shown in Figure (5.5.8). The magnetic field produced by the primary coil at the position of the secondary coil is

$$B_z = KI_p \quad \text{Teslas} ,$$

K is just a constant that depends upon the geometry of the primary coil. If the area of the secondary coil is A, supposed to be very small, the flux through the secondary coil due to the primary current is given by

$$\Phi_s = B_z A = KAI_p, \tag{5.5.2}$$

and therefore the relevant inductance coefficient, L_{sp}, is

$$L_{sp} = KA. \tag{5.5.3}$$

But this means that the flux through the primary coil system due to a current I_s in the secondary coil is given by

$$\Phi_p = L_{pr}I_s = L_{xp}I_z = KAI_s.$$

The small secondary coil carrying a current I_s constitutes a magnetic moment $m_z = I_s A \; Amp - m^2$. It follows that a magnetic dipole m_z produces a flux through a primary coil system given by

$$\Phi_p = Km_z, \tag{5.5.4}$$

where the coil system produces the field $B_z = KI_p$ Teslas at the position of the magnetic moment. If the field produced by the coil system is uniform over a magnetic body it follows from the principle of superposition that the flux produced in the coil system by the body is proportional to its total magnetic moment. In particular, the flux through an infinite solenoid produced by a magnetic dipole, $\vec{m}$, oriented along the solenoid axis is given by

$$\Phi=\mu \mathrm{Nm}, \tag{5.5.5}$$

if the system is immersed in a medium whose permeability is μ. In most applications μ can be taken to be the permeability of free space, μ_o.

5.6: Forces on Magnetic Circuits

The induction coefficients, Equation (5.5.2), are functions of coil position and coil geometry. If a circuit is moved, or if its shape is altered, the fluxes through all the circuits will be altered. A changing magnetic flux through a circuit induces an emf in that circuit:

$$\mathrm{curl}(\mathrm{E})=-\frac{\partial\vec{\mathrm{B}}}{\partial\mathrm{t}},$$

so that

$$\int\int_{Surface}\mathrm{curl}(\vec{\mathrm{E}})\cdot\mathrm{d}\vec{\mathrm{S}}=-\int\int_{Surface}\frac{\partial\vec{\mathrm{B}}}{\partial\mathrm{t}}\cdot\mathrm{d}\vec{\mathrm{S}}=-\frac{\partial}{\partial\mathrm{t}}\int\int_{Surface}\vec{\mathrm{B}}\cdot\mathrm{d}\vec{\mathrm{S}}.$$

But the last term above is just the time rate of change of the magnetic flux, therefore

$$\int\int_{Surface}\mathrm{curl}(\vec{\mathrm{E}})\cdot\mathrm{d}\vec{\mathrm{S}}=-\frac{\mathrm{d}\Phi}{\mathrm{dt}}.$$

Stokes' theorem can be used to transform the surface integral of curl($\vec{E}$) over the area of a circuit into a line integral of the electric field around the circuit contour C:

$$e=\oint_C\vec{\mathrm{E}}\cdot\mathrm{d}\vec{\mathrm{L}}=-\frac{\mathrm{d}\Phi}{\mathrm{dt}}.\tag{5.6.1}$$

In order to be able to use conservation of energy to derive formulae that relate the forces exerted by the magnetic field on circuits to the change in magnetic energy that accompanies any shift in position or distortion in shape of those circuits, it is useful to consider wires that are made of perfectly superconducting metals so that resistive losses can be neglected. Magnetic forces can not depend upon the circuit resistances because these forces depend only upon the interaction between a current element and the magnetic field acting upon that current element. The emf around a closed superconducting loop must be zero because the electric field in a perfect conductor must always be zero; any electric field strength would produce an infinite current. It follows that the fluxes through closed superconducting circuits cannot change. Any relative motion of the circuits, or any distortion of the circuits, must be accompanied by changes in their currents such that the fluxes remain constant. In addition, energy must be conserved, and therefore any external work done by the magnetic forces during a displacement must be at the expense of the magnetic energy stored in the system

$$\vec{\mathrm{F}}_{\mathrm{M}}\cdot\delta\vec{\mathrm{r}}=-\delta\mathrm{U}_{\mathrm{B}}.\tag{5.6.2}$$

The magnetic energy acts like a potential energy function for the magnetic forces.

Usually it is convenient to arrange to keep the current in each circuit constant during the circuit motion or deformation. The current can be kept constant, in principle, by including a superconducting generator in each circuit as is shown in Figure (5.6.9(a)). This is a device by means of which external electrical work can be done on the circuit. Now the flux through the circuit can be allowed to change because the electromotive force (emf) due to the changing flux can be cancelled out by the external emf of the generator so that the net emf in the system can be maintained at zero. In order to cancel out the emf due to the changing flux, work must be done on the generator at the rate

$$\frac{\mathrm{dW}}{\mathrm{dt}}=\mathrm{e}_{\mathrm{gen}}\mathrm{I}=-\mathrm{eI}=\mathrm{I}\frac{\mathrm{d}\Phi}{\mathrm{dt}},$$

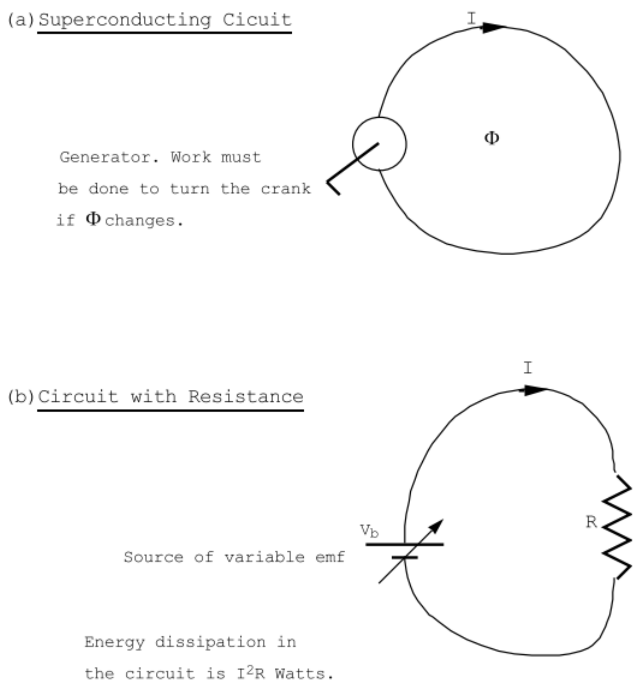


Figure 5.6.9: A change in flux through a circuit must be accompanied by an applied emf $e = d\Phi/dt$ in order to maintain the current constant.

or the change in flux, dΦ, must be accompanied by an input of external work (to turn the crank on the generator) such that

$$\mathrm{dW} = \mathrm{Id}\Phi.$$

A similar argument can be applied to each circuit of N coupled circuits so that taken all together the total external energy input through the generators is

$$\mathrm{dW} = \sum_{n=1}^{\mathrm{N}} \mathrm{I_N d\Phi_N}. \tag{5.6.3}$$

As mentioned above, the result (5.6.3) cannot depend upon whether or not the circuits contain resistance elements. A circuit that contains a resistance must be provided with a source of emf in order to maintain a constant current (see Figure (5.6.9(b)). This emf, that can be imagined to be a power supply whose terminal voltage is variable, must supply energy at the rate of I^2R Watts, where R is the circuit resistance. If the flux through the circuit increases the power supply voltage must be increased by e = dΦ/dt in order to maintain the current constant. This means that the power extracted from the power supply must also increase by the amount

$$\frac{\mathrm{dW}}{\mathrm{dt}} = \mathrm{eI} = \mathrm{I}\frac{\mathrm{d}\Phi}{\mathrm{dt}}.$$

In order to change the flux through the circuit by dΦ at fixed current the power supply must add extra energy in the amount

$$\mathrm{dW} = \mathrm{Id}\Phi,$$

and this clearly does not depend upon whether there is, or is not, resistance in the circuit. A generalization of this result to a collection of N circuits leads directly to Equation (5.6.3). Any displacement of a circuit, or a deformation of a circuit, that results in a change of the inductance coefficients must result in a change in the magnetic energy stored in the field if the current in each of the circuits is held fixed. This energy change is given by

$$\mathrm{dU_B} = \frac{1}{2}\sum_{N} \mathrm{I_N d}\Phi_{N},$$

see Equation (5.4.9). But in view of Equation (5.6.3), any flux changes at constant current that cause the energy stored in the magnetic field to increase must extract twice that energy increase from the external energy sources which do work on the generators in order to keep the currents constant:

$$dW = 2dU_B.$$

The excess of the work done on the generators, dW, over the energy increase in the magnetostatic field, dU_B, must represent the external work done by the magnetic forces during the circuit displacement or deformation, therefore

$$\vec{F}_M \cdot \delta\vec{r} = \delta U_B. \tag{5.6.4}$$

At constant currents work expended to keep the currents fixed goes one half into increasing the stored magnetic energy and one half into the work done by the magnetic forces.

5.6.1 Forces on a Magnetic Dipole.

Let us apply the above ideas to calculate the force on a magnetic dipole. Consider a magnetic moment located in an inhomogeneous field that is directed along the z-axis . The magnetic moment can be represented as a small loop carrying a fixed current I, see Figure (5.10). The magnetic moment of the loop, IA, is also directed along z. If the loop moves a distance dz along z, the magnetic field will change from B to B + $(dB_z/dz)dz$. The magnetic energy of the system is given by

$$U_B = \frac{L_{11}}{2} I_p^2 + L_{12} I_p I + \frac{L_{22}}{2} I^2.$$

During the displacement dz only the mutual inductance coefficient L_{12} changes: L_{11} and L_{22} are not altered. The change in magnetic energy due to the displacement of the loop at constant currents is given by

$$dU_B = I_p dL_{12} I = I d\Phi.$$

But

$$d\Phi = A dB_z = A\left(\frac{dB_z}{dz}\right) dz,$$

therefore

$$dU_B = IA\left(\frac{dB_z}{dz}\right) dz.$$

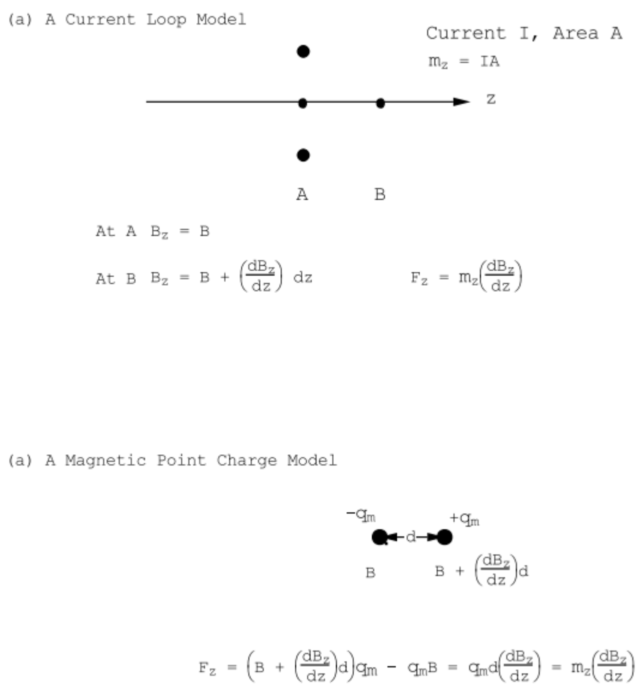


Figure 5.6.10: A magnetic dipole located in an inhomogeneous magnetic field. The magnetic moment and the applied field, $\vec{B}$, are both directed along the z-axis.

From an application of Equation (5.6.4) one can conclude that the magnetic force on the loop must be

$$F_x = IA\left(\frac{dB_z}{dz}\right) = m_x\left(\frac{dB_z}{dz}\right) \quad \text{Newtons.} \tag{5.6.5}$$

This equation for the force on a dipole was derived using arguments based upon circuits in which the currents were held fixed. However, the validity of the expression for the force does not depend upon the manner of its derivation, and Equation (5.6.5) is correct for any point dipole.

The result (5.6.5) can be generalized by considering displacements along x and y. For a magnetic moment that has only a z-component the result is

$$\vec{F} = m_z \overrightarrow{grad}(B_z).$$

The total force acting on a magnetic moment that has components along all three co-ordinate axes is obtained from a further generalization of the above arguments. The result is

$$\vec{F} = m_x gr\vec{a}d(B_x) + m_y \text{ grad }(B_y) + m_z gr\vec{a}d(B_z). \tag{5.6.6}$$

The expression (5.6.6) can be further transformed by using the fact that *curl* $(\vec{B})$ = 0 in a homogeneous medium free from currents; because for a current free region *curl* $(\vec{H})$ = 0, and in a homogeneous permeable region $\vec{B} = \mu\vec{H}$. From *curl* $(\vec{B})$ = 0

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x},$$
$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x},$$
$$\frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y}.$$

Using these relations, Equation (5.6.6) can be written in the form

$$\begin{aligned} F_x &= m_x\frac{\partial B_x}{\partial x} + m_y\frac{\partial B_x}{\partial y} + m_z\frac{\partial B_x}{\partial z}, \\ F_y &= m_x\frac{\partial B_y}{\partial x} + m_y\frac{\partial B_y}{\partial y} + m_z\frac{\partial B_y}{\partial z}, \\ F_z &= m_x\frac{\partial B_z}{\partial x} + m_y\frac{\partial B_z}{\partial y} + m_z\frac{\partial B_z}{\partial z}. \end{aligned} \tag{5.6.7}$$

or

$$\vec{F} = (\vec{m}\cdot\overrightarrow{grad})\vec{B}. \tag{5.6.8}$$

Equations (5.6.7) are the expressions which would be deuced from a magnetic point charge model in which the force on a magnetic pole is given by $q_m\vec{B}$ (see Figure (5.6.10)), and the dipole is represented by a magnetic charge $+q_m$ separated an infinitesimal distance d from a magnetic point charge $-q_m$: the magnetic moment is given by

$$\vec{m} = q_m d.$$

The generalization of Equation (5.6.8) to the dipoles contained in a volume element dVol gives the magnetic force per unit volume quoted in Chpt.(1), Section(1.5.1), Equation (1.5.3):

$$\overrightarrow{F_M} = \left(\vec{M}\cdot\vec{\nabla}B_x\right)\hat{u}_x + \left(\vec{M}\cdot\vec{\nabla}B_y\right)\hat{u}_y + \left(\vec{M}\cdot\vec{\nabla}B_z\right)\hat{u}_z \quad \text{Newtons/m}^3.$$

Here $\vec{M}$ is the magnetization per unit volume.

5.6.2 Torque on a Magnetic Dipole.

Consider a small current loop placed in a uniform magnetic field generated by a primary coil system carrying a current I_1. If the loop is allowed to rotate the flux linkage between it and the primary coil system changes; however, neither its own self-inductance coefficient, L_{22}, nor the self-inductance coefficient of the primary system, L_{22}, changes with the angle of rotation, θ, Figure (5.6.11). The mutual inductance coefficient, $L_{12} = L_{21}$, does change with angle, and therefore a small rotation through dθ causes a change in the magnetic energy for fixed currents I_1,I_2. The change in magnetostatic energy of the system for a small rotation is given by

$$\mathrm{dU_B = I_1 I_2 dL_{12}}.$$

The mutual inductance coefficient can be calculated from the flux through the current loop:

$$\Phi_2 = \left(\pi \mathrm{R}^2\right) \mathrm{B_z} \cos\theta = \left(\pi \mathrm{R}^2 \mathrm{K}\right) \mathrm{I_1} \cos\theta.$$

Therefore

$$\mathrm{L_{12}} = \left(\pi \mathrm{R}^2 \mathrm{K}\right) \cos\theta \quad \text{Henries},$$

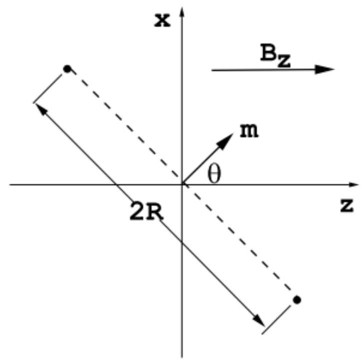


Figure 5.6.11: A loop of wire carrying a current I_2 Amps, and having an area πR^2, is located in a uniform magnetic field B_z. The magnetic field is generated by a coil system carrying a current I_1 Amps: $B_z = KI_1$. The field exerts a torque on the loop given by $T_y = - B_z m \sin\theta$ Newton-meters.

so that

$$\mathrm{dL_{12}} = -\left(\pi \mathrm{R}^2 \mathrm{K} \sin\theta\right) \mathrm{d}\theta.$$

It follows that the change in the magnetostatic energy of the system as a result of the change in angle is given by

$$\mathrm{dU_B} = -\left(\pi \mathrm{R}^2 \mathrm{I_2}\right)\left(\mathrm{KI_1}\right) \sin\theta \mathrm{d}\theta.$$

The change in magnetic energy can be expressed in terms of the magnetic moment of the loop, m, and the value of the magnetic field at the position of the loop (it is assumed that the loop is small enough that the variation of the magnetic field across its diameter can be ignored):

$$\mathrm{dU_B} = -\mathrm{B_z m} \sin\theta \mathrm{d}\theta.$$

This change in the energy must be equal to the external work done by the magnetic torque on the loop (Equation (5.6.4)); $T_y d\theta = dU_B$, where the torque is directed along the y-axis if the rotation takes place in the xz plane, Figure (5.6.11). Therefore

$$\mathrm{T_y} = -\mathrm{mB_z} \sin\theta. \tag{5.6.9}$$

The direction of the torque is such as to orient the normal to the plane of the loop along the direction of the applied magnetic field: in other words, the torque is such as to cause both the flux through the loop and the magnetostatic energy to become as large as possible. The magnetic moment associated with the loop is perpendicular to the plane of the loop, consequently Equation (5.6.9) can be written in the form

$$\vec{\mathrm{T}} = (\vec{\mathrm{m}} \times \vec{\mathrm{B}}). \tag{5.6.10}$$

The torque (5.6.10) can be derived from an effective potential energy

$$\mathrm{W_B} = -\vec{\mathrm{m}} \cdot \vec{\mathrm{B}} \quad \text{Joules}, \tag{5.6.11}$$

using

$$T_\theta = -\frac{\partial W_B}{\partial \theta}. \tag{5.6.12}$$

Equations (5.6.11 and 5.6.12) are particularly useful for treating the problem of a collection of permanent dipoles (a permanently magnetized body) that interacts with an applied magnetic field.

5.6.3 Forces on a Solenoid.

Consider a coil in the form of a very long cylindrical solenoid that is wound with N turns per meter. The length of the solenoid is L meters, and the mean radius of the windings is R meters, Figure (5.6.12). The field at the center of a very long solenoid is given by Equation (4.3.7) of Chpt.(4):

$$B_z = \mu_0 NI \quad \text{Teslas}, \tag{5.6.13}$$

where the coil carries a current of I Amps. The flux through one turn of the solenoid is $AB_z = \pi R^2 B_z$ near the center of the solenoid; neglecting end effects which play a relatively small role in the limit $L/R \to \infty$, the total flux through all of the windings is the flux per turn multiplied by the total number of turns

$$\Phi = \pi R^2 B_z NL,$$

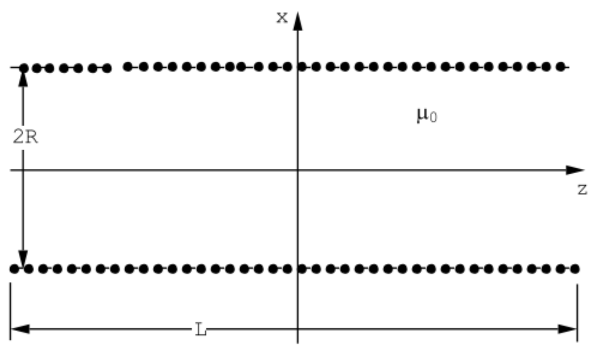


Figure 5.6.12: A long cylindrical solenoid. The solenoid is L meters long and has a radius of R meters. It is wound with N turns per meter.

or

$$\Phi = \mu_0 N^2 \left(\pi R^2 L\right) I. \tag{5.6.14}$$

The magnetic energy stored in the field is $U_B = I\Phi/2$ Joules, and therefore

$$U_B = \frac{\mu_0}{2} N^2 I^2 \left(\pi R^2 L\right). \tag{5.6.15}$$

In order to discuss magnetic forces it is necessary to consider distortions of the solenoid; for example, a change in the length or in its diameter. During those distortions the current will be held fixed. The total number of turns on the solenoid, $N_{tot} = NL$, must also remain fixed, and therefore the magnetic energy will be re-written in terms of the total number of turns on the winding:

$$U_B = \mu_0 N_{tot}^2 I^2 \frac{\pi R^2}{2L}. \tag{5.6.16}$$

It is apparent from this expression that an increase in solenoid length will reduce the magnetostatic energy, U_B. We can conclude that the magnetic forces on the solenoid windings will act in such a way as to shorten the solenoid because any system of coils carrying fixed currents will attempt to arrange themselves so as to maximize the magnetostatic energy of the system. Similarly, any increase in the solenoid radius results in an increase in the magnetostatic energy; it can therefore be concluded that the magnetic forces will place the windings under tension. The magnitude of the magnetic forces can be obtained from an application of Equation (5.6.4) which equates the work done by the magnetic forces during a small change in geometry to the increase in magnetic energy. For example, let the length of the solenoid increase by δL; the work done by the magnetic forces is $F_M \delta L$. The change in magnetic energy for a displacement δL is

$$\delta U_B = -\frac{\mu_0}{2}\left(\frac{N_{tot}^2}{L^2}\right) I^2 \pi R^2 \delta L;$$

consequently, from $\delta U_B = F_M \delta L$ one finds

$$F_M = -\left(\frac{\mu_0 N^2 I^2}{2}\right)\pi R^2 \quad \text{Newtons}\,. \tag{5.6.17}$$

The force that tends to squeeze the turns together is the force that would be generated by a pressure $p_B = BH/2$ Newtons/m^2 acting over the crosssectional area of the solenoid.

A similar argument can be used to calculate the tension in the solenoid windings due to magnetic forces. Let the radius of the solenoid of Figure (5.6.12) expand slightly. The work done by a uniform pressure, p_M, during this expansion would be

$$\delta W = p_M (2\pi R)\delta R$$

per meter of length because the work done per unit area of wall per unit length is given by $p_M \delta R$. The total work done by the magnetic pressure, p_M, during the change δR for a cylinder L meters long is given by

$$\delta W = p_M (2\pi RL)\delta R.$$

The corresponding increase in magnetostatic energy, from Equation (5.6.15), is

$$\delta U_B = \mu_0 N^2 I^2 (\pi RL)\delta R.$$

But the work done by the magnetic forces must be equal to the increase in stored magnetic energy if the current is held fixed, and therefore

$$p_M = \frac{\mu_0}{2} N^2 I^2 = \frac{BH}{2} \quad \text{Newtons}\,/\text{m}^2. \tag{5.6.18}$$

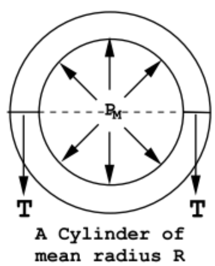


Figure 5.6.13: The hoop stresses, T, acting on a cylindrical shell due to an internal pressure of p_M Newtons per m^2 . The hoop stresses T exerted by the bottom half-cylinder on the top half-cylinder are required to overcome the internal pressure forces. The thickness of the shell has been exaggerated.

The magnetic field exerts a pressure on the solenoid windings: this pressure results in a tension given by

$$T = p_M R = R\left(\frac{BH}{2}\right) \quad \text{Newtons/meter}\,, \tag{5.6.19}$$

see Figure (5.6.13). There are N wires per meter of length, and therefore the tension on each wire will be given by

$$t = T/N = \frac{R}{N}\left(\frac{BH}{2}\right). \tag{5.6.20}$$

The forces exerted on the windings of a solenoid can be quite large. For example, superconducting solenoids are available that can be used to generate fields in excess of 10 Teslas. The magnetic pressure in a 10 Tesla field is $p_M = B^2/2\mu_0 = 3.98 \times 10^7$ Newtons/m2 . The pressure of one atmosphere is 1.01×10^5 Newtons/m^2 , therefore the magnetic pressure associated with a field of 10 Teslas is equivalent to 394 atmospheres. The windings on such a superconducting solenoid must be firmly anchored in order to prevent them from moving

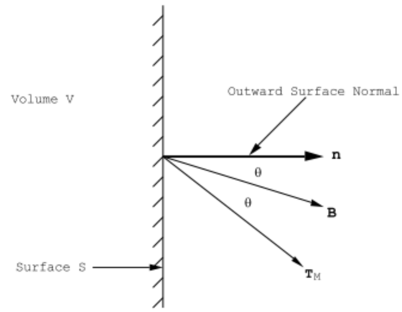


Figure 5.6.14: The magnetic force acting on the matter contained within a volume V can be obtained as the integral of a vector $\vec{T}_{\mathrm{M}}$ over the closed surface S which encloses V. It is assumed that $\vec{B}$ is proportional to $\vec{H}$ everywhere inside V, that $\vec{B}$ and $\vec{H}$ are parallel on S, and that S is immersed in a fluid that can support no shear stresses. $\left|\vec{\mathrm{T}}_M\right| = \frac{\vec{\mathrm{B}} \cdot \vec{\mathrm{H}}}{2}$. The direction of the force per unit area, $\vec{T}_{\mathrm{M}}$, is such that the angle between $\vec{T}_{\mathrm{M}}$ and the surface normal is bisected by the direction of the magnetic field $\vec{B}$.

5.7: The Maxwell Stress Tensor

In analogy with the electrostatic case, the forces due to the magnetic field acting on the current distribution in a body can be obtained from a magnetic Maxwell stress tensor, see J.A.Stratton, Electromagnetic Theory, section 2.5, (McGraw-Hill, N.Y., 1941). If the magnetic materials in the system are linear so that $\vec{B}$ is proportional to $\vec{H}$, it can be shown that there exists a vector $\vec{T}_{\mathrm{M}}$ associated with the elements of the stress tensor such that the surface integral of $\vec{T}_{\mathrm{M}}$ over a closed surface S gives the net force acting on the material in the volume V enclosed by the surface S: it is assumed that the surface S is contained entirely within a fluid that can support no shearing stresses. The magnetic force acting on the material within the volume V can be calculated from

$$\overrightarrow{\mathrm{F}}_{M}=\int\int_{S}\overrightarrow{\mathrm{T}}_{M}\cdot\overrightarrow{\mathrm{dS}},\tag{5.7.1}$$

where the magnitude of the Maxwell stress vector for a linear, isotropic material, is

$$\left|\overrightarrow{\mathrm{T}}_{M}\right|=\frac{\overrightarrow{\mathrm{B}}\cdot\overrightarrow{\mathrm{H}}}{2},\tag{5.7.2}$$

and its direction is given by the construction shown in Figure (5.6.14). The stress vector $\vec{T}_{\mathrm{M}}$ is turned away from the surface normal through an angle that is twice the angle that the magnetic field $\vec{B}$ (or $\vec{H}$) makes with the surface normal. When $\vec{B}$ lies along the surface normal the magnetic force is a tension, but when the field $\vec{B}$ lies in the surface the magnetic force is a pressure.

CHAPTER OVERVIEW

6: Ferromagnetism

A description of the ferromagnetic state with an application to magnetic recording.

6.1: Introduction to Ferromagnetism

The magnetization density, $\vec{M}$, in most materials at room temperatures is proportional to the magnetic field, $\vec{H}$:

$$\vec{\mathrm{M}} = \chi\vec{\mathrm{H}}, \tag{6.1.1}$$

The factor of proportionality, χ, is called the **magnetic susceptibility**. Since $\vec{M}$, and $\vec{H}$ have the same units (Amps/meter) the magnetic susceptibility has no dimensions. Typical values of the susceptibility at room temperatures for some common substances are listed in Table (6.1.1). It is clear from this Table that for such non-magnetic substances the magnetization per unit volume is negligible compared with values of the impressed magnetic field, $\vec{H}$. The situation is quite different for ferromagnetic substances such as iron, nickel, or cobalt. In a ferromagnet the magnetic moments are held parallel by very strong forces called **exchange forces**. The magnetization per unit volume, $|\vec{M}_s|$, is very large and essentially independent of applied magnetic field at temperatures low compared with a critical temperature called the **Curie Temperature**. The Curie temperature, T_c, is that temperature at which the magnetization in zero applied magnetic field goes to zero. The Curie temperature is material dependent. The Curie temperatures for iron,nickel and cobalt are 1044, 631, and 1393 K respectively. The magnetization varies slowly with temperature at low temperatures. The temperature dependence of the magnetization in iron is depicted in Figure (6.1.1). The temperature dependence of M_s for other ferromagnets is very similar. If a cylindrical rod of iron were to be uniformly magnetized along its length the magnetic field strength near its end surfaces would be very large, see Equation (4.3.12) and Figure (4.3.11) of Chapter(4).

Substance	Susceptibility in units of 10^{-6}
Al	+20.7
Cu	-9.68
Au	-34.6
Si	-4.0
SiO_2	-16.3
H_2O	-9.05

Table 6.1.1: The magnetic susceptibility at room temperature for some common non-magnetic substances.

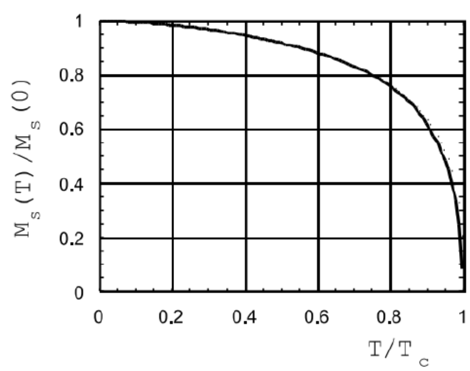


Figure 6.1.1: The variation with temperature of the reduced magnetization for pure iron. The Curie temperature is T_c = 1044 K, and the magnetization at T=0 K is $M_s(0)$ = 22.1 kOe= 1.76 × 10^7 Amps/m.

The field just outside an end face and near the cylinder axis is given by $B_z = \mu_0 M_0/2$ for a cylinder whose length, L_d, is much greater than its radius, R. For iron at room temperature this field is approximately 1 Tesla. It is, however, common experience that the fields around a length of iron rod are very weak, of the order of 0.01 Teslas or less. The field outside the rod is weak because the magnetization is broken up into a very large number of small domains. Each domain carries a large magnetization, M_s , but the direction of the magnetization changes from domain to domain in such a way that the average magnetization density is very nearly zero. It can be shown that the energy due to a magnetization distribution can be calculated from

$$\mathrm{U_m} = \frac{\mu_0}{2}\int\int\int_{Space} \mathrm{d}\tau \mathrm{H_m^2}, \tag{6.1.2}$$

where H_m is the magnetic field generated by the magnetic charge density $\rho_m = -\operatorname{div}(\vec{\mathrm{M}})$ and $d\tau$ is the element of volume. It follows that in the absence of an applied external field the domain magnetization vectors will attempt to orient their magnetization vectors so as to make (\ div (\vec M\)) as nearly zero as possible. This tendency is called ”the magnetic pole avoidance principle”. The size of the magnetic domains in the absence of an applied magnetic field depends very strongly on the structure of the material (whether the specimen is a polycrystal or a single crystal), upon the concentration of impurities, and upon the presence of internal stresses. The domain dimensions in an annealed polycrystalline iron bar are of the order of 1/10 mm on a side. The domains are therefore very large compared with atomic dimensions, but are small on a macroscopic scale. In very perfect single crystalline prisms of iron in which the cubic iron axes are accurately parallel with the edges of the specimen the domains may be as long as a cm or more: see Figure (6.1.2).

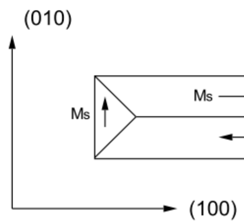


Figure $6.1.2$: The domain structure at room temperature in a perfect iron single crystal in which the edges of the crystal are accurately parallel with the crystalline axes. The magnetization is uniform along the z-direction (out of the paper). Simple domain structures are observed only in single crystalline specimens.

6.2: B-H Curves

The magnetic properties of ferromagnets at a fixed temperature are often described by curves of magnetic induction B vs. H, see Figure (6.2.3). H is the internal magnetic field: its sources include $\rho_m = -\operatorname{div}(\vec{\mathrm{M}})$ in the magnetic body as well as any externally applied magnetic field generated by a system of coils outside the magnetic body. $\vec{\mathrm{B}} = \mu_0(\vec{\mathrm{H}} + \vec{\mathrm{M}})$, and in the simplest case the three vectors $\vec{B}$, $\vec{H}$, $\vec{M}$ are all parallel. Starting from the fully demagnetized state (H=0, M=0) the magnetization increases with H, and B follows the curve labeled ”virgin curve”. In the virgin state with H=0, an equal number of domains have positive magnetization as have negative magnetization so that the net magnetization is zero. As H increases those domains having a magnetization oriented along the applied field direction grow in volume at the expense of domains having a magnetization oriented opposite to the applied field direction. Eventually those domains having a component of magnetization opposed to the direction of H have been eliminated (at the point marked A in Figure (6.2.3)). However, iron has a cubic crystal structure and exhibits the property that the magnetization strongly prefers to orient itself along a direction corresponding to one of the three equivalent

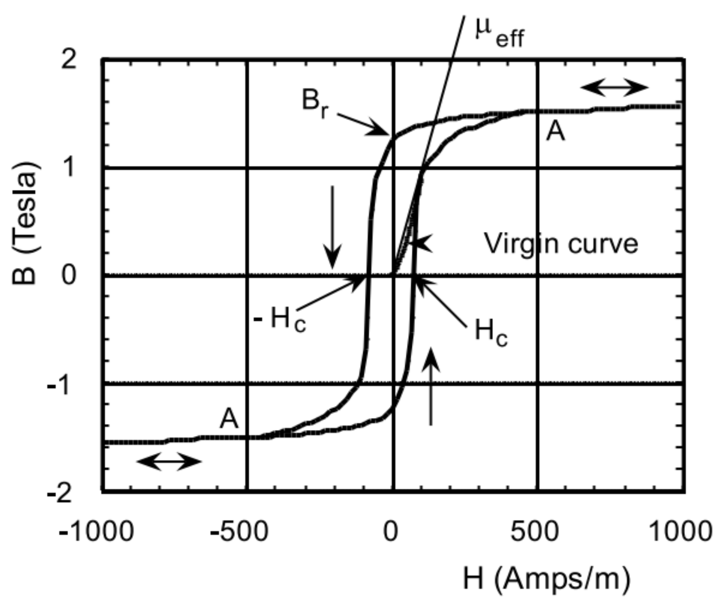


Figure 6.2.3: A hysteresis loop for a polycrystalline specimen of pure iron. The details of the B-H loop are specimen sensitive. The saturation magnetization at room temperature is 2.14 Teslas. The remanent field is B_r= 1.22 T, and the coercive field is H_c= 79 Amps/m.

cubic axes. In a polycrystalline material at a field corresponding to point A in Figure (6.2.3) the domain magnetizations, each having a strength M_s per unit volume, are oriented at angles with respect to the applied field ranging from 0 to $\pm 90^\circ$. As H increases these domain magnetizations gradually rotate into the applied field direction: during this portion of the B-H loop the curve is reversible. Ultimately, the magnetization reaches the saturation value, M_s , and the magnetization density becomes uniform throughout the specimen. The field necessary to achieve the saturated state in iron, $\sim 2\times 10^5$ Amps/m, is very large because of the large magnetocrystalline anisotropy energy that resists the rotation of the magnetization away from a cubic axis. Very soft magnetic materials such as Supermalloy (79% Ni, 16% Fe, and 5% Mo, see Table (6.2.2)) have compositions corresponding to a relatively small magnetocrystalline anisotropy. The approach to saturation in such materials occurs at much lower applied fields than for iron, see Figure (6.2.4). It is also worth mentioning that a pure single crystal of iron for which the domain magnetizations are oriented along the cubic axes, Figure (6.1.2), exhibits a very large maximum effective permeability, see Table (6.2.2), and can be saturated in fields less than H= 100 Amps/m. However, polycrystalline iron is cheap and is therefore used extensively in the construction of electromagnets and large generators.

At saturation the magnetic domains have been eliminated so that the magnetization density is uniform throughout the body and has the value M_s . But $\vec{\mathrm{B}} = \mu_0(\vec{\mathrm{H}} + \vec{\mathrm{M}})$ so after saturation the B-field continues to increase with H, although the rate of increase of B with H becomes negligibly slow compared with the rate of increase leading up to saturation; the variation of B with H becomes imperceptible on the scale of Figure (6.2.3). Upon reducing the field H after having increased it to values larger than that corresponding to point A in Figure (6.2.3), B follows the upper curve in Figure (6.2.3) and when H=0 the magnetic induction reaches the remanent field value $B = B_r$. B gradually falls as domains with a reversed magnetization orientation gradually reform in the body. As H is further reduced B continues to follow the upper curve and eventually B reaches zero at a negative value of H called the coercive field, H_c. Continued reduction of H ultimately leads to magnetic saturation in the negative direction with uniform magnetization having the value $-M_s$. As H is increased from applied field values more negative than the field corresponding to point A in the third quadrant the B-field increases along the lower curve, the magnetization becomes less negative as the number of domains having

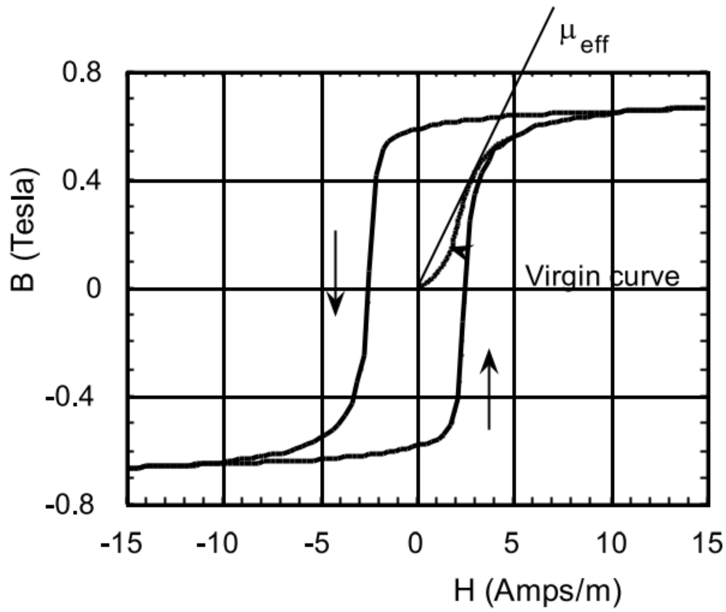


Figure 6.2.4: The hysteresis loop for the soft ferromagnet 4-79 Permalloy (79% Ni, 17% Fe, 4% Mo). The maximum effective permeability is 0.14 (the relative permeability is $\mu_R = 1.1\times10^5$), and H_c = 2.45 Amps/m. The saturation field is B_s= 0.87 Teslas.

a positive magnetization increases, and ultimately B becomes positive. At the coercive field H_c the magnetic induction is zero; B=0. At sufficiently large values of the magnetic field the specimen once again becomes saturated with a uniform magnetization having the value Ms. For fields larger than 500 Amps/m, or for fields less than -500 Amps/m, the curve of B vs. H is reversible. The loop defined by the upper and lower curves in Figure (6.2.3) is called a major hysteresis loop. Two questions arise immediately: (1) What happens if H is decreased before point A is reached? and (2) How can the virgin state with H=0, M=0, and B=0 be attained? If H is reduced before the reversible part of the B-H loop has been attained the B-field decreases along a minor hysteresis loop such as those shown in Figure (6.2.5). If a sinusoidal driving field is applied having an amplitude sufficient to drive the specimen into the reversible part the hysteresis loop the magnetic state is carried around the hysteresis loop from one extreme in the plus direction to an extreme in the negative direction many times per second. If now the amplitude of the driving field is slowly reduced to zero the hysteresis curve collapses to zero symmetrically around the origin. The specimen will be left in the virgin state in which B=M=H=0.

Important parameters associated with the hysteresis loop are (1) the remanent field, B_r , (2) the coercive field, H_c, and (3) the maximum effective permeability defined by the maximum slope of the straight line joining the origin to a point on the virgin magnetization curve as shown in Figures (6.2.3, 6.2.4). There are two major classes of ferromagnetic materials: soft ferromagnets and hard ferromagnets. Soft magnetic materials are characterized by very small values of the coercive field, see Table (6.2.2). For such materials the dependence of B on H is almost linear for $H < H_c$, and as a reasonable approximation one can write $\vec{\mathrm{B}} = \mu_{eff}\vec{\mathrm{H}}$. It is useful to express the effective permeability as a dimensionless number

$$\mu_{eff} = \mu_R\mu_0.$$

Pure polycrystalline iron is a soft ferromagnet characterized by H_c = 60 Amps/m and μ_{eff}= 0.013, or $\mu_R \sim$ 10,000. There are a number of alloys that behave very nearly like perfectly soft ferromagnets, see Table (6.2.2).

Hard magnetic materials are characterized by large values of the coercive field and remanent field B_r , see Table (6.2.3). Very hard ferromagnets such as SmCo alloys, NdFeB alloys, and Strontium ferrites can be better described in terms of M vs. H. The variation of magnetization with internal magnetic field, H, is shown for a commercial Barium ferrite in Figure (6.2.6); only negative

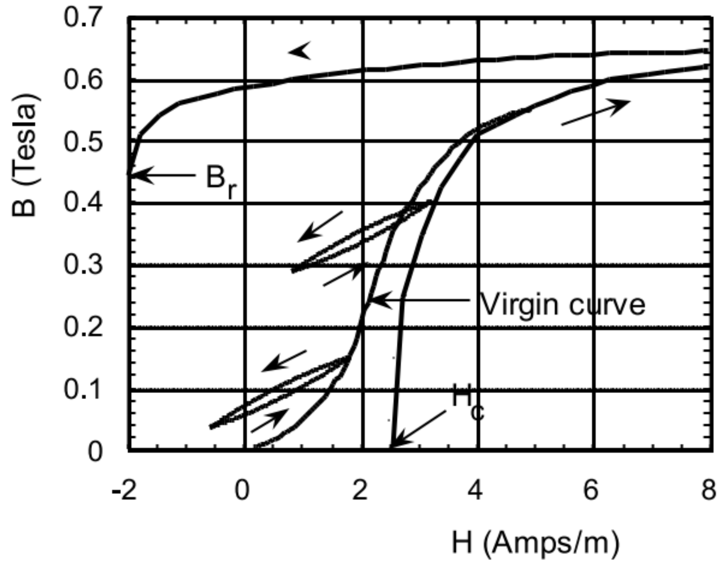


Figure (6.2.4). For the lowest minor loop the field was reduced after having partially traversed the virgin magnetization curve. For the upper minor loop the field was reduced after having traversed the major hysteresis curve cycle at least once.

Material	B_s Saturation Field, Teslas	Curie Temp. (C)	Coercive Field Amps/m	Maximum Relative Perm. μ_R
Iron (Fig. (6.3))	2.14	770	79	6600
Single Cryst. Fe	2.14	770	0.8	1.2×10^6
Permalloy	1.08	600	4	100,000
Supermalloy	0.75	400	0.16	10^6

Table 6.2.2: Magnetic properties of some soft magnetic materials. B = μ_{eff} (H + M), where $\mu_{eff} = \mu_R\mu_0$. One Tesla equals 10 kGauss, and 79 Amps/m equals 1 Oersted. Permalloy is an alloy of 78.5 atomic % Ni and 21.5 atomic % Fe. Supermalloy is composed of 79 atomic % Ni, 16 atomic % Fe, and 5 atomic % Mo.

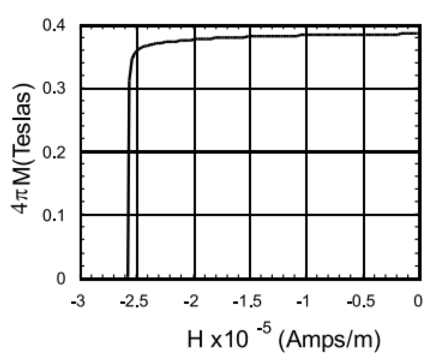


Figure 6.2.6: The variation with internal magnetic field, H, of the magnetization at room temperature for a commercial Strontium Ferrite. The coercive field is 2.59×10^5 Amps/m.

internal fields, H, are shown because this portion of the magnetization curve is the one required for practical applications. The hysteresis loop is very nearly rectangular, meaning that to a good approximation the magnetization is independent of the H-field until it flips 180° at, or near, the coercive field. For very hard materials such as those listed in Table (6.2.3) the concept of a permeability is not very useful.

Warning! Commercial hysteresis loops are usually displayed using CGS units. In the CGS system the fields B,M,H all have the same units, although for historical reasons the units of B,M are called Gauss whereas the units of H are called Oersteds. The conversion from CGS to MKS units is relatively simple: 1 Tesla= 10,000 Gauss. 79.6 Amps/m = 1 Oersted. M in Amps/m = [4 πM(in Gauss)] × 79.6.

In the CGS system $\vec{B} = \vec{H} + 4\pi\vec{M}$. The relative permeability is the same ~ for both systems.

Material	Residual Magnetic Field at 300C Teslas	Curie Temp. (C)	Coercive Field at 300 C Amps/m
Sr Ferrite	0.39	450	2.59×10^5
Samarium Cobalt	1.07	820	1.43×10^6
Sintered NdFeB	1.29	310	1.03×10^6

Table 6.2.3: Magnetic properties of some commercial hard magnet materials. (See, for example, www.dextermag.com).

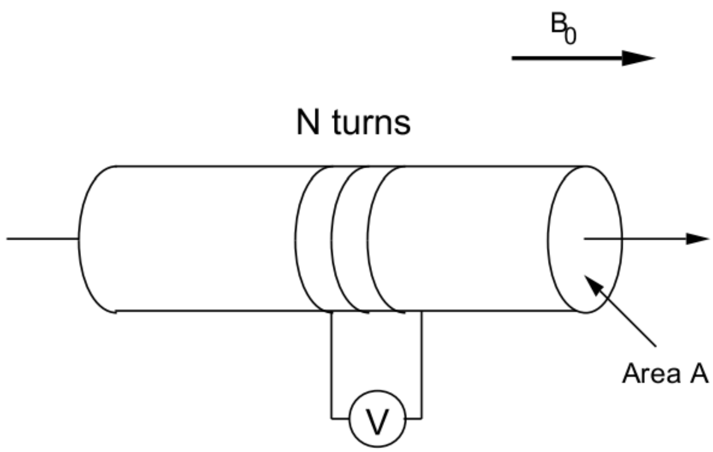


Figure 6.2.7: Device for measuring the axial magnetic field, B, inside a cylinder having a cross-sectional area A. A coil of N turns is connected to a voltmeter V.

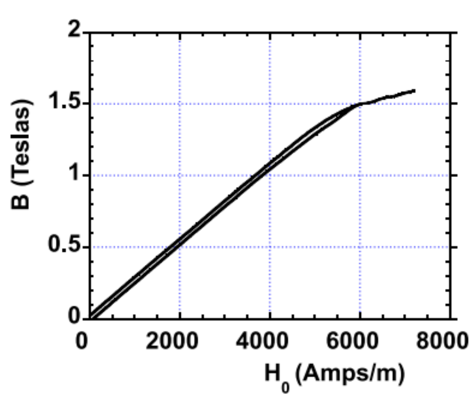


Figure 6.2.8: A plot of B inside a long iron rod vs the externally applied magnetic field $H_0 = B_0/\mu_0$. The length to diameter ratio is 25 corresponding to a demagnetizing coefficient N_z= 0.00467.

6.2.1 Measuring the B-H Loop.

It is relatively easy to measure the axial magnetic flux density, B, in a specimen. It is only necessary to wind a few turns of wire closely around a specimen and to measure the emf developed across the coil terminals as an external field B_0 is changed with time, see Figure (6.2.7). The emf across the coil terminals is given by Faraday's law:

$$V(t) = NA\frac{dB}{dt},$$

where N is the number of turns on the coil and A is the cross-sectional area of the specimen in m^2 . Upon integration of the voltage signal starting from a known initial condition (B=0 at t=0 say) one obtains B inside the specimen corresponding to a particular value of the applied field $H_0 = B_0/\mu_0$. In this way one can trace out the hysteresis loop of B vs H_0 as the specimen is saturated first in one direction and then in the other direction. Unfortunately the hysteresis loop so obtained is not what is wanted: it depends more on the geometry of the specimen than on its intrinsic magnetic properties.

What is wanted is the variation of B inside the cylinder with the value of H inside the cylinder. But H inside the cylinder is the sum of the applied field $H_0 = B_0/\mu_0$ plus the contribution generated by the magnetic pole density distribution $\rho_M = -\operatorname{div}(\vec{M})$. This pole field very nearly cancels out the applied field H_0 in a material having a large permeability. The net result is that the curve of B vs H_0 measures an effective demagnetizing coefficient for the body under test, and does not provide a satisfactory measure of an intrinsic magnetic property of the material of the test body. In order to see how this comes about consider a particular example: consider a cylindrical bar whose length is 25 times its diameter (25 cm long by 1 cm in diameter, for example). Let this bar be characterized by the hysteresis loop shown in Figure (6.2.3). The pole field inside this bar can be approximated by $H_D = -N_zM$, where N_z is the demagnetizing factor for an ellipsoid of revolution having the same length to diameter ratio as the cylinder; M is the magnetization density in the bar. The demagnetization factor for an ellipsoid of revolution having a length to diameter ratio of 25 is N_z=0.00467; see Chpt.(2), Figure (2.7.19) and eqn.(2.5.1). Inside the rod one has $B/\mu_0 = H+M$. Given the co-ordinates of a point on the B-H loop one can calculate the magnetization, M. Consider the point in Figure (6.2.3) B=1.0 Tesla and H=110 Amps/m. For this point $B/\mu_0 = 1.0/(4\pi \times 10^{-7}) = 0.796 \times 10^6$ Amps/m. Thus H is negligible and M= 0.796×10^6 Amps/m. The resulting pole field is $H_D = -N_zM = -3.72 \times 10^3$ Amps/m. In order to obtain a net value H= +110 Amps/m it is necessary to apply a field $H_0 = (3.72\times10^3)+110$ Amps/m for a total field $H_0 = 3.83\times10^3$ Amps/m. In this same way one can calculate B vs H_0 for all of the points on the hysteresis loop. The results of such a calculation are shown in Figure (6.2.8). The most obvious result is that B is nearly a linear function of the applied field H_0 for H_0 less than 5800 Amps/m; the slope of the line corresponds to a relative permeability μ_R= 212 (NB. $1/N_z$= 214). Moreover, the hysteretic behaviour has been reduced to a very small value: approximately 0.05 Tesla in B. It is easy to show that if the material properties are such that B= $\mu_R\mu_0H$ then for large μR one has B= μ_0H_0/N_z; ie. a straight line having a slope corresponding to $\mu_R = 1/N_z$. The argument runs as follows:

$$\begin{aligned} B &= \mu H = \mu_0(H+M), \\ \frac{\mu}{\mu_0}H &= (H+M), \end{aligned}$$

so

$$M = (\mu_R - 1)H.$$

Since the pole field is $H_D = -N_zM$, the applied field must overcome this pole field and in addition supply the field H. Therefore the applied field is given

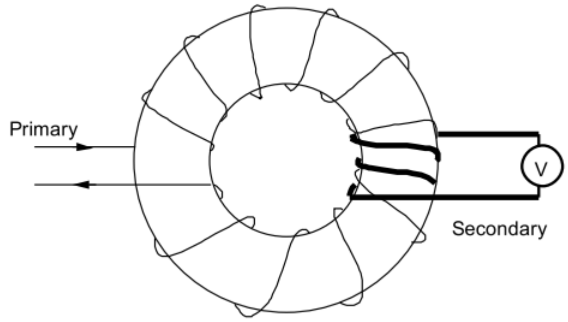


Figure 6.2.9: A ring shaped specimen used to measure the intrinsic magnetic properties of a soft magnetic material. The primary winding of N_p turns is used to generate the field H. The secondary coil of N_s turns is used to measure the field B in the ring.

by

$$H_0 = N_z(\mu_R - 1)H + H,$$

or

$$H = \frac{H_0}{[N_z(\mu_R - 1) + 1]},$$

and

$$B = \frac{\mu_0 \mu_R H_0}{[N_z(\mu_R - 1) + 1]}.$$

Upon dividing through by μ_R and taking the limit such that $\mu_R \gg 1$, one obtains

$$B \cong \mu_0 H_0 / N_z. \tag{6.2.1}$$

The point is that in order to measure the intrinsic response of a soft magnetic material it is necessary to avoid spatial variations in the magnetization that give rise to magnetic pole fields. This can be done by using a specimen having the topology of a ring, Figure (6.2.9). This ring can be supplied with a uniformly wound primary coil of N_p turns used to generate the applied field, H_0, plus a secondary coil of N_s turns used to measure the flux density in the specimen. There are no magnetic poles if M is uniform around the ring, therefore the field in the material is just $H = N_p I/L$, where I is the primary coil current in Amps and L is the length in meters measured along the centerline of the ring. The B-field can be calculated from the emf developed across the secondary windings as the primary current is changed; according to Faraday's law

$$V = N_s A(dB/dt),$$

where A is the cross-sectional area of the ring.

6.3: Digital Magnetic Recording

A magnetic hard disc for use as a magnetic memory storage device for a computer consists of a very smooth circular substrate upon which has been deposited a very thin coating of a magnetic cobalt alloy 50 nm or less thick. This disc is rotated at a very high rate. The remanent magnetization of this magnetic thin film is such that $B_R \sim 1/2$ Tesla, and the coercive field is approximately 105 Amps/m. The magnetization lies in the plane of the disc and contains many small, oblong regions in which the magnetization is oriented either parallel or antiparallel to the disc velocity. These magnetization regions are written into the disc magnetization by means of a write head: an extremely simplified drawing of a write head is shown in Figure (6.3.10).

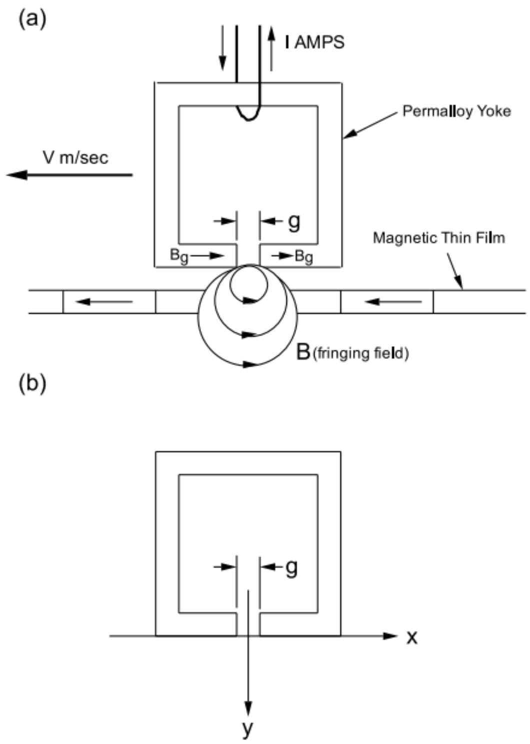


Figure 6.3.10: (a) A schematic representation of a hard disc write head. (b) The co-ordinate system used to write the spatial dependence of the write head field in the Karlqvist approximation.

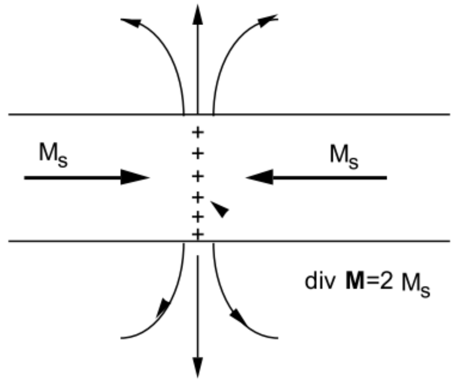


Figure 6.3.11: At the juncture between two regions of oppositely directed magnetizations there is a large surface charge density, and this gives rise to large fringing fields.

The write head is basically an electromagnet constructed of a soft magnetic permalloy yoke (the saturation field is 1 Tesla and the coercive field is $H_c \sim 4$ Amps/m). This electromagnet is driven by the current through a single turn. The yoke contains a narrow gap, g, approximately 50 nm wide. The write head ”flies” over the surface of the disc at an altitude of approximately 25 nm, and the magnetic film on the disc is magnetized by the fringing field produced at the magnet gap. A field of approximately 3 times the coercive field is used to write magnetization regions into the disc magnetic film that are either parallel or antiparallel to the disc velocity. The spatial dependencies of the fringing field components near the gap are given in the **Karlqvist approximation** by:

$$H_x = \frac{B_g}{\mu_0 \pi} \arctan\left[\frac{yg}{x^2+y^2-(g^2/4)}\right], \tag{6.3.1}$$

$$H_y = \frac{B_g}{2\pi} \ln\left[\frac{(x-g/2)^2+y^2}{(x+g/2)^2+y^2}\right], \tag{6.3.2}$$

where B_g is the B-field in the middle of the gap region, and g is the gap width: the co-ordinate axes are shown in Figure (6.3.10b). Bits of information are stored as magnetization reversals (also called flux reversals). It is only at those places where the magnetizations are directed opposite to one another that the fringing field is large enough to be detected by the read head. This is illustrated in Figure (6.3.11). The absence of a flux reversal is taken to be a zero; the presence of a flux reversal is taken to be a 1. The magnetization profile for a typical run of data might look like that shown in Figure (6.3.12). In practice, each data byte of input is stored using a complex code that uses more than the nominal 8 bits per byte in order to build in the capability to detect and correct errors.

Modern read heads use a complicated structure of thin films. The magnetic field due to a magnetization change on the hard disc is detected by means of a change in resistance of a magnetoresistance element. Write and read heads are combined in a single write/read unit.

As of November 1999 IBM demonstrated a hard disc drive having the

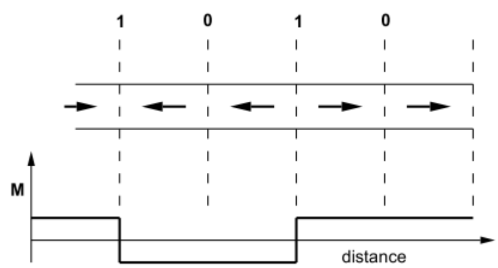


Figure 6.3.12: Example of encoded data bits on a hard disc drive.

capability to store 3.5×10^{10} bits per square inch using 522,000 bits per inch and 67,300 tracks per inch. This means that each magnetization cell was only 49 nm long by 377 nm wide. The disc spun at 10,000 revolutions per minute, the seek time was 4.9 msecs, and information was read in and out at the rate of 18×10^6 bytes per second. The uncorrected error rate was $1{:}10^8$; after correction this error rate decreased to less than $1{:}10^{12}$.

Further Reading

- S. Chikazumi, ”Physics of Magnetism”. John Wiley and Sons,New York, 1964.
- John C. Mallinson, ”The Foundations of Magnetic Recording”, Second Edition. Academic Press, San Diego, 1993.
- John C. Mallinson, ”Magneto-Resistive Heads”. Academic Press, San Diego, 1996.

6.4: Electromagnets

Consider the electromagnet shown in Figure 6.4.13. As a first approximation let the B-field in the ferromagnetic yoke be uniform with the same value B Teslas everywhere. The field in the gap measured along the centerline will also be B to a good approximation. This follows from the Maxwell equation $\div B = 0$ which requires the normal component of B to be continuous across a material discontinuity. If the gap field is B then the H-field in the gap is $H_g = B/\mu_0$. The permeability of free space, $\mu_0 = 4\pi \times 10^{-7}$, is a small number therefore H_g will be quite large: if B=1.0 Tesla then $H_g = 7.96\times10^5$ Amps/m. This H-field is much larger than H within the soft ferromagnetic yoke material. For example, in iron the field H cannot exceed 100 Amps/m if B= 1.0 Tesla, see Figure (6.3). According to another Maxwell equation for the static magnetic field

$$\mathrm{curl}(\vec{\mathrm{H}}) = \vec{\mathrm{J}},$$

or

$$\oint_C \vec{\mathrm{H}} \cdot \mathrm{d}\vec{\mathrm{L}} = \iint_{Area} \vec{\mathrm{J}} \cdot \mathrm{d}\vec{\mathrm{A}}, \tag{6.4.1}$$

from Stokes' theorem where the Area of the surface integration is bounded by the curve C.

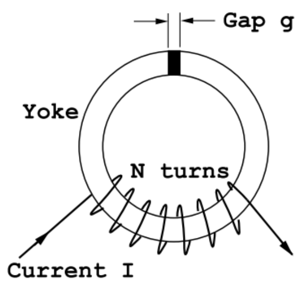


Figure 6.4.13: An electromagnet consisting of a soft ferromagnetic yoke wound with N turns of wire and containing a gap of width g meters. The length of the ferromagnetic yoke along its center line is L meters.

Apply Equation (6.4.1) to the closed line running along the centerline of the magnet. The integral of the current density over the area bounded by the magnet center line is just NI so that the line integral of H becomes

$$\mathrm{LH} + \mathrm{gH_g} = \mathrm{NI},$$

where L is the length of the path in the ferromagnetic yoke and g is the width of the gap. Given a value for the B-field one can look up the corresponding value of the H-field from the ferromagnetic hysteresis loop. Then the current required to produce that B-field is

$$\mathrm{I} = \frac{1}{\mathrm{N}}(\mathrm{LH} + [\mathrm{gB}/\mu_0])\,. \tag{6.4.2}$$

In this way one can construct a graph of B vs. I corresponding to various points on the B-H loop. It should be noted that this simple construction fails when the ferromagnet approaches magnetic saturation.

CHAPTER OVERVIEW

7: Time Dependent Electromagnetic Fields.

Chapters 2-5 treated the problem of how to calculate electric and magnetic field distributions given time independent charge and current distributions. This chapter discusses the more general problem of how to calculate electric and magnetic fields given time varying charge and current distributions. It turns out that the solution to this general problem is most easily developed using the scalar and vector potentials discussed in chapters 2 and 4. By way of example, the formalism is applied to the generation of radio waves by currents flowing in an antenna, and to the generation of light waves by oscillating atomic dipole moments.

General Reference: The Feynman Lectures in Physics, Vol.(II), by R.P.Feynman, R.B.Leighton, and M.Sands, Addison Wesley, Reading, Mass., 1964.

7.2: Time Dependent Maxwell's Equations

Start from Maxwell's equations in the form

$$\begin{aligned} \operatorname{curl}(\vec{E}) &= -\frac{\partial \vec{B}}{\partial t}, \\ \operatorname{div}(\vec{B}) &= 0, \\ \operatorname{curl}(\vec{B}) &= \mu_0 \vec{J}_T + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}, \\ \operatorname{div}(\vec{E}) &= \frac{1}{\epsilon_0} \rho_T, \end{aligned} \tag{7.2.1}$$

where

$$\rho_T = \rho_f - \operatorname{div}(\vec{P}), \tag{7.2.2}$$

and

$$\vec{J}_T = \vec{J}_f + \operatorname{curl}(\vec{M}) + \frac{\partial \vec{P}}{\partial t}. \tag{7.2.3}$$

Recall that ρ_f is the density of free charges, $\vec{J}_f$ is the free current density due to the motion of the free charges, $\vec{P}$ is the electric dipole moment density, and $\vec{M}$ is the magnetic dipole density. It is presumed that the total charge density, ρ_T , and the total current density, $\vec{J}_T$, are prescribed functions of position and of time. The equation $\operatorname{div}(\vec{B}) = 0$ can be satisfied by setting

$$\vec{B} = \operatorname{curl}(\vec{A}). \tag{7.2.4}$$

because the divergence of any curl is equal to zero. The first of Equations (7.2.1) becomes, with the help of Equation (7.2.4),

$$\operatorname{curl}(\vec{E}) = -\frac{\partial}{\partial t} \operatorname{curl}(\vec{A}) = -\operatorname{curl}\left(\frac{\partial \vec{A}}{\partial t}\right),$$

where it has been assumed that the order of the space and time derivatives can be interchanged. It follows that the curl of the sum of the electric field and the time derivative of the vector potential is zero,

$$\operatorname{curl}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{7.2.5}$$

The curl of **any gradient** is zero so that the requirement Equation (7.2.5) can be satisfied by putting

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\operatorname{grad} V,$$

or

$$\vec{E} = -\overrightarrow{\operatorname{grad}} V - \frac{\partial \vec{A}}{\partial t}. \tag{7.2.6}$$

The introduction of the vector potential, $\vec{A}$, and the scalar potential, $\vec{V}$, enables one to satisfy the first two of Maxwell's equations (7.2.1). Write $\vec{E}$ and $\vec{B}$ in terms of the potentials in the second pair of Maxwell's equations to obtain

$$\operatorname{curl} \operatorname{curl}(\vec{A}) = \mu_0 \vec{J}_T + \epsilon_0 \mu_0 \left(-\overrightarrow{\operatorname{grad}} \frac{\partial V}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2}\right),$$

and

$$-\operatorname{div} \overrightarrow{\operatorname{grad}} V - \frac{\partial}{\partial t} \operatorname{div}(\vec{A}) = \frac{\rho_T}{\epsilon_0}.$$

In cartesian co-ordinates, **but only in cartesian co-ordinates**, the vector operator curl curl can be written

$$curlcurl = -\nabla^2 + \overrightarrow{\text{grad}}\, div. \tag{7.2.7}$$

Using Equation (7.2.7) one obtains

$$-\nabla^2\vec{\text{A}} + \epsilon_0\mu_0\frac{\partial^2\vec{\text{A}}}{\partial\text{t}^2} + \overrightarrow{\text{grad}}\left(\text{div}(\vec{\text{A}}) + \epsilon_0\mu_0\frac{\partial\text{V}}{\partial\text{t}}\right) = \mu_0\vec{\text{J}}_T. \tag{7.2.8}$$

In order to completely specify a vector field one must give both its curl and its divergence. But at this point only the curl of $\vec{A}$ has been fixed by the requirement that $\vec{B}$ = curl($\vec{A}$); one is still free to impose some constraint on the divergence of $\vec{A}$. It is convenient to choose the vector potential so that it satisfies the condition

$$\text{div}(\vec{\text{A}}) + \epsilon_0\mu_0\left(\frac{\partial\text{V}}{\partial\text{t}}\right) = 0. \tag{7.2.9}$$

This choice of div($\vec{A}$) is called the Lorentz gauge. In the **Lorentz gauge** Equation (7.2.8) simplifies to become

$$\nabla^2\vec{\text{A}} - \epsilon_0\mu_0\frac{\partial^2\vec{\text{A}}}{\partial\text{t}^2} = -\mu_0\vec{\text{J}}_T, \tag{7.2.10}$$

or in component form

$$\begin{aligned} \nabla^2\text{A}_\text{x} - \epsilon_0\mu_0\frac{\partial^2\text{A}_\text{x}}{\partial\text{t}^2} &= -\mu_0\text{J}_\text{T}\big|_x, \\ \nabla^2\text{A}_\text{y} - \epsilon_0\mu_0\frac{\partial^2\text{A}_\text{y}}{\partial\text{t}^2} &= -\mu_0\text{J}_\text{T}\big|_y, \\ \nabla^2\text{A}_\text{z} - \epsilon_0\mu_0\frac{\partial^2\text{A}_\text{z}}{\partial\text{t}^2} &= -\mu_0\text{J}_\text{T}\big|_z, \end{aligned} \tag{7.2.11}$$

Similarly, if the last of Maxwell's Equations (7.2.1) is combined with Equation (7.2.6) and with the Lorentz condition (7.2.9) one finds

$$\nabla^2\text{V} - \epsilon_0\mu_0\frac{\partial^2\text{V}}{\partial\text{t}^2} = -\frac{\rho_T}{\epsilon_0}. \tag{7.2.12}$$

Obviously, the four equations (7.2.11 plus 7.2.12) are very similar and the form of a solution that satisfies one of them must also satisfy the other three. (The fact that A_x, A_y, A_z, V all satisfy equations of the same form is no accident: according to the special theory of relativity these four quantities are related to the four components of a single vector in four-dimensional space-time). Consider the homogeneous equation

$$\nabla^2\text{V} - \epsilon_0\mu_0\frac{\partial^2\text{V}}{\partial\text{t}^2} = 0;$$

or, since $c^2 = 1/(\epsilon_0\mu_0)$,

$$\nabla^2\text{V} - \frac{1}{\text{c}^2}\frac{\partial^2\text{V}}{\partial\text{t}^2} = 0. \tag{7.2.13}$$

This equation is called the wave equation. A spherically symmetric solution that satisfies the wave equation is

$$\text{V} = \frac{\text{f}(\text{t} - [\text{r}/\text{c}])}{\text{r}}. \tag{7.2.14}$$

where f(x) is any function whatsoever. It is instructive to substitute the function (7.2.14) into the wave equation. Since the function does not depend upon either of the angular co-ordinates, θ or ϕ, the Laplacian operator becomes

$$\nabla^2\text{V} = \frac{1}{\text{r}^2}\frac{\partial}{\partial\text{r}}\left(\text{r}^2\frac{\partial\text{V}}{\partial\text{r}}\right).$$

Inserting the function (7.14) one obtains

$$\frac{\partial \mathrm{V}}{\partial \mathrm{r}}=-\frac{f}{\mathrm{r}^{2}}-\frac{\dot{\mathrm{f}}}{\mathrm{cr}},$$

since

$$\frac{\partial \mathrm{f}}{\partial \mathrm{r}}=\left(\frac{\partial \mathrm{f}}{\partial \mathrm{t}}\right)\left(\frac{\partial}{\partial \mathrm{r}}\left[\mathrm{t}-\frac{\mathrm{r}}{\mathrm{c}}\right]\right)=-\frac{\dot{\mathrm{f}}}{\mathrm{c}}.$$

Therefore

$$r^{2} \frac{\partial V}{\partial r}=-f-\frac{r \dot{f}}{c},$$

and

$$\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{2} \frac{\partial \mathrm{V}}{\partial \mathrm{r}}\right)=-\frac{\partial \mathrm{f}}{\partial \mathrm{r}}-\frac{\dot{\mathrm{f}}}{\mathrm{c}}+\frac{\mathrm{r} \ddot{\mathrm{f}}}{\mathrm{c}^{2}}=\frac{\mathrm{r} \ddot{\mathrm{f}}}{\mathrm{c}^{2}},$$

thus

$$\nabla^{2} \mathrm{V}=\frac{\ddot{\mathrm{f}}}{\mathrm{rc}^{2}}.$$

But

$$\frac{\partial^{2} V}{\partial t^{2}}=\frac{\ddot{f}}{r c^{2}},$$

and therefore the wave equation (7.2.13) is satisfied by a potential function of the form Equation (7.2.14) where f(x) is an arbitrary function of its argument, x. Apart from the appearance of the retarded time, $t_R = t - r/c$, the form of Equation (7.2.14) is very similar to the potential function for a point charge. It is therefore natural to suppose that the potential function that is generated by a time-varying point charge q(t) located at the origin is given by

$$\mathrm{V}(\mathrm{r}, \mathrm{t})=\frac{1}{4 \pi \epsilon_{0}} \frac{\mathrm{q}(\mathrm{t}-\mathrm{r} / \mathrm{c})}{\mathrm{r}}, \tag{7.2.15}$$

where the value of the charge at the retarded time must be used to calculate the potential at the time of observation, t: the retarded time must be used in order to allow for the finite time required to propagate a signal from the charge to the observer at the speed of light. The notion of a time-dependent charge is an unusual one: think of a tiny volume at the origin into which charge can flow with time. Then the potential function (7.2.15) describes the contribution to the potential at the position of the observer due to the charge in that tiny volume element at the origin. The potential function (7.2.15) goes

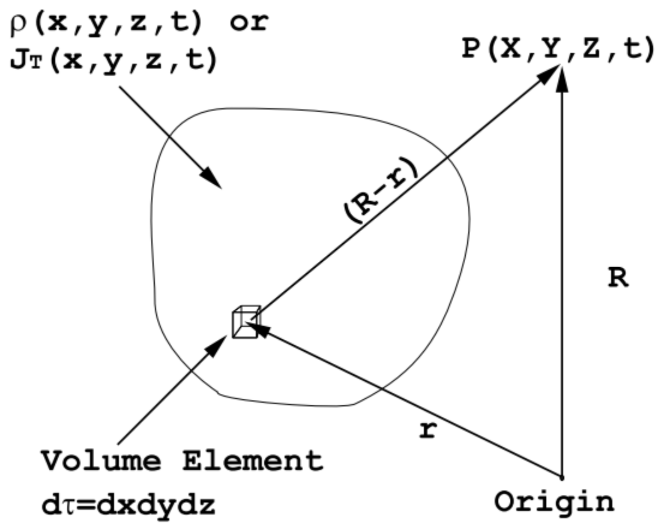


Figure 7.2.1: A space and time dependent source charge density $\rho(\vec{r},t)$ and a space and time dependent source current density $\vec{J}_T(\vec{r},t)$ generate time dependent electric and magnetic fields at the position, P, of an observer.

over into the electrostatic potential for a point charge if the observer is so close to the origin that (r/c) can be neglected, or if the charge q is independent of the time.

The elementary solution (7.2.15) of the wave equation can be used, together with the principle of superposition, to construct a particular solution of the wave equation given a space and time varying distribution of charge density (see Figure (7.2.1)):

$$V(\vec{R},t)=\frac{1}{4\pi\epsilon_0}\int\int\int_{Space} d\tau \frac{\rho_T(\vec{r},t_R)}{|\vec{R}-\vec{r}|}, \tag{7.2.16}$$

where $d\tau$ is the element of volume and the retarded time is given by

$$t_R=t-\frac{|\vec{R}-\vec{r}|}{c}, \tag{7.2.17}$$

It may be helpful to write out Equation (7.2.16) explicitly in cartesian co-ordinates (see Figure (7.2.1):

$$V_P(X,Y,Z,t)=\frac{1}{4\pi\epsilon_0}\int\int\int_{Space} dxdydz \frac{\rho_T(x,y,z,t_R)}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}},$$

where

$$t_R=t-\frac{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}{c},$$

If Equation (7.2.16) is the required solution of the inhomogeneous wave equation (7.2.12) for the potential function $V(\vec{R},t)$, then by analogy the solution of each of the three Equations (7.2.11) must have the same form. The particular solution for the vector potential that is generated by the current density $\vec{J}_T(\vec{r},t)$ is given by

$$\vec{A}(\vec{R},t)=\frac{\mu_0}{4\pi}\int\int\int_{Space} d\tau \frac{\vec{J}_T(\vec{r},t_R)}{|\vec{R}-\vec{r}|}. \tag{7.2.18}$$

Here again t_R is the retarded time. These solutions, which satisfy Maxwell's equations for the case in which the charge and current distributions depend upon time, have exactly the same form as the solution for the electrostatic potential, Equation (2.2.4), and the solution for the magnetostatic vector potential, Equation (4.1.13), except that the retarded time must be used in the source terms. The presence of the retarded time in the integrals makes the calculation of the scalar and vector potentials much more complicated than the equivalent calculations for the static limit. It can be shown, after much work, that the potential functions (7.2.16) and (7.2.18) satisfy the Lorentz condition, Equation (7.2.9).

7.3: A Simple Radio Antenna

See (Electromagnetic Theory by J.A.Stratton, McGraw-Hill, N.Y., 1941. Section 8.7 and the following).

Consider a center fed linear antenna such as that depicted in Figure (7.3.2). In order to apply Equation (7.2.18) to an antenna of finite length it is necessary to know the current distribution along the wire. An exact solution of this problem is very difficult. A useful approximation assumes that the current distribution along the antenna is sinusoidal if the time variation of the current is sinusoidal. For a thin wire the current must be zero at the ends of the wire since there is no place to store charge. At other places along the wire charge may be stored on the wire surfaces and so the current need not be the same at every cross-section. The antenna is supposed to be driven by a sinusoidal generator at the circular frequency ω. A wave of current propagates along the wire, which can be regarded as a transmission line, and is reflected from the open ends of the wire. The resulting current distribution is a sinusoidal

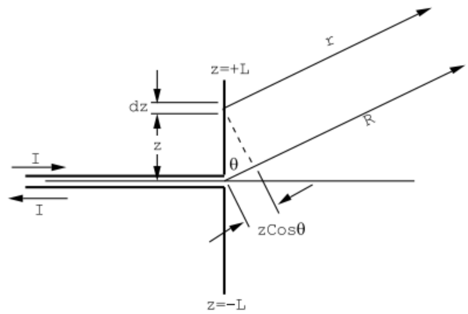


Figure 7.3.2: A center fed linear radio antenna. The current forms a standing wave with nodes at the wire ends.

standing wave along the wire having zero current at the ends of the wire at z = ±L. Such a current distribution can be described by

$$\begin{aligned} \mathrm{I}(\mathrm{z}, \mathrm{t}) &= \mathrm{I}_0 \exp(-i\omega \mathrm{t}) \sin \tfrac{\omega}{\mathrm{c}}(\mathrm{L}-\mathrm{z}), \quad \mathrm{z} \geq 0 \\ \mathrm{I}(\mathrm{z}, \mathrm{t}) &= \mathrm{I}_0 \exp(-i\omega \mathrm{t}) \sin \tfrac{\omega}{\mathrm{c}}(\mathrm{L}+\mathrm{z}), \quad \mathrm{z} \leq 0 \end{aligned} \tag{7.3.1}$$

see Figure (7.3.2). Use this current distribution in Equation (7.2.18) to calculate the vector potential. The currents flow only in the z-direction so the vector potential will only have a z-component. Further, it is assumed that we will be interested only in those fields that are far removed from the antenna: this means that for an observer at distance R from the antenna we shall assume that R ≫ L,λ where the antenna length is 2L and λ is the wavelength of the electric and magnetic fields produced by the antenna where λ = c/f, and ω = 2πf where f is the frequency. The antenna wire is assumed to be very thin so that every point on a cross-section at z can be considered to be at the same distance from the observer. This means that in the integral of (7.2.18) the integral of current density over x,y simply gives the total current at that place along the wire. Eqn.(7.2.18) can be written

$$\mathrm{A}_{\mathrm{z}}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{t}) = \frac{\mu_0}{4\pi} \int_{-\mathrm{L}}^{\mathrm{L}} \mathrm{dz} \frac{\mathrm{I}_{\mathrm{z}}(\mathrm{z}, \mathrm{t}_{\mathrm{R}})}{\mathrm{r}},$$

where I_z is given by (7.3.1), t_R= t - r/c, and

$$r^2 = X^2 + Y^2 + (Z-z)^2.$$

Since | z |< L and R ≫ L one can use the binomial theorem to expand r in powers of (z/R):

$$\mathrm{r} = \mathrm{R}\left[1 - \frac{2\mathrm{Zz}}{\mathrm{R}^2} + \left(\frac{\mathrm{z}}{\mathrm{R}}\right)^2\right]^{1/2},$$

or

$$\mathrm{r} \cong \mathrm{R} - \frac{\mathrm{Zz}}{\mathrm{R}},$$

neglecting terms of order $(z/R)^2$ or higher. Thus the distance to the observer from a point z on the antenna is given by

$$\mathrm{r} = \mathrm{R} - \mathrm{z}\cos\theta,$$

where θ is the angle between the direction of $\vec{R}$ and the antenna, see Figure (7.3.2). The vector potential can therefore be written

$$A_z(R,\theta,t)=\frac{\mu_0}{4\pi}\int_{-L}^{0}dz\frac{I_0\sin\left[\frac{\omega}{c}(L+z)\right]}{[R-z\cos\theta]}\exp(-i\omega t)\exp\left(\frac{i\omega}{c}[R-z\cos\theta]\right)+\frac{\mu_0}{4\pi}\int_0^L dz\frac{I_0\sin\left[\frac{\omega}{c}(L-z)\right]}{[R-z\cos\theta]}\exp(-i\omega t)\exp\left(\frac{i\omega}{c}[R-z\cos\theta]\right). \tag{7.3.2}$$

This rather formidable appearing expression can be simplified if one notices that z cos θ in the denominator can be neglected compared with the distance R since R≫L. On the other hand, the term (ωz cos θ/c) in the exponentials cannot be neglected because L is usually comparable with λ and (ωz/c) = 2πz/λ. With the above simplification we have

$$A_z(R,\theta,t)=\frac{\mu_0}{4\pi}\frac{I_0\exp(-i\omega[t-R/c])}{R}[I_1+I_2], \tag{7.3.3}$$

where

$$I_1=\int_{-L}^{0}dz\sin\left[\frac{\omega}{c}(L+z)\right]\exp\left(-i\frac{\omega}{c}z\cos\theta\right)$$

and

$$I_2=\int_0^L dz\sin\left[\frac{\omega}{c}(L-z)\right]\exp\left(-i\frac{\omega}{c}z\cos\theta\right).$$

The integrals are messy but can be easily carried out. The result is

$$F(\theta)=I_1+I_2=\frac{2}{\frac{\omega}{c}\sin^2\theta}\left[\cos\left(\frac{\omega L\cos\theta}{c}\right)-\cos\left(\frac{\omega L}{c}\right)\right]. \tag{7.3.4}$$

The next step uses (7.3.3) to calculate the B-field from $\vec{B}=\text{curl}(\vec{A})$. For this purpose it is convenient to work in spherical polar co-ordinates:

$$A_R=A_z\cos\theta$$
$$A_\theta=-A_z\sin\theta$$
$$A_\phi=0$$

But since A_ϕ = 0 and there is no angular dependence on the angle ϕ, it follows that

$$B_R=B_\theta=0,$$

and

$$B_\phi=\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta)-\frac{\partial A_R}{\partial\theta}\right],$$

or

$$B_\phi=-i\frac{\mu_0}{4\pi}I_0\frac{\omega}{c}\frac{\exp(-i\omega[t-R/c])}{R}\sin\theta F(\theta)-\frac{\mu_0}{4\pi}I_0\frac{\omega}{c}\frac{\exp(-i\omega[t-R/c])}{R^2}\left[-\sin\theta F(\theta)+\cos\theta\frac{dF}{d\theta}\right] \tag{7.3.5}$$

This field contains two terms. The first term decreases with distance to the observer like $(\lambda R)^{-1}$. The second term decreases with distance like R^{-2} . This means that for the condition R≫ λ one can ignore the second term because it becomes very small relative to the first term. So in the far field of the antenna (R ≫ λ) one finds

$$B_\phi=-i\frac{\mu_0}{4\pi}I_0\frac{\omega}{c}\frac{\exp(-i\omega[t-R/c])}{R}\sin\theta F(\theta), \tag{7.3.6}$$

and

$$B_\theta=B_R=0.$$

The electric field can be most easily obtained from B by means of the third Maxwell equation (7.2.11.). In free space $\vec{J}_T=0$ so that

$$\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\text{curl}(\vec{B}),$$

or, since the time variation is proportional to exp (−iωt),

$$-i\frac{\omega}{c^2}\vec{E} = \text{curl}(\vec{B}). \tag{7.3.7}$$

The electric field components calculated from Equation (7.3.7) are:

$$\begin{aligned} -i\frac{\omega}{c^2}E_R &= \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi) \\ -i\frac{\omega}{c^2}E_\theta &= -\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi) \\ E_\phi &= 0 \end{aligned} \tag{7.3.8}$$

Notice that the component E_R varies with distance like $1/R^2$, whereas the component E_θ varies with distance like 1/R. For distances such that $R \gg \lambda$ the component E_R becomes very small compared with E_θ and can be ignored. Thus in the **far field limit** $\vec{E}$ has only the component E_θ and $\vec{B}$ has only the component B_ϕ. Notice that $\vec{E}$ and $\vec{B}$ are orthogonal to each other and both are orthogonal to the line joining the observer to the center of the antenna. Also note that from Equation (7.3.8)

$$E_\theta = cB_\phi, \tag{7.3.9}$$

independent of the angle of observation.

In the limit of small angles, θ, the factor sin θF(θ) in Equation (7.3.6) simplifies to $2L\theta\sin\left(\frac{\omega L}{c}\right)$. This means that in the limit θ → 0 the field amplitudes fall off to zero as the observer becomes aligned with the antenna (θ is defined in Figure (7.3.2)). On the other hand, for an observer in the X-Y plane θ = ϕ/2 and

$$\sin\theta F(\theta) = \left(\frac{2c}{\omega}\right)\left[1-\cos\left(\frac{\omega L}{c}\right)\right].$$

The radiation fields in the equatorial plane are non-zero and become particularly large when cos(ωL/c)= 0 or -1. Such an antenna is said to be resonant. The condition ωL/c = π/2 corresponds to the commonly used half-wave antenna for which L = λ/4, where λ is the free space wavelength 2πc/ω. For such a half-wave antenna the angular dependence of the radiation fields becomes

$$\sin\theta F(\theta) = \frac{(2c/\omega)\cos(\pi\cos\theta/2)}{\sin\theta}. \tag{7.3.10}$$

Despite its more complicated appearance, this function is very similar to the sin θ angular variation that characterizes a point electric dipole radiator, as we shall see in the next section.

The present section has demonstrated how one can calculate the strength of a radio signal generated by a typical linear antenna. It also demonstrates that relatively complex fields are a consequence of the presence of the retarded time in the relatively simple formula for the vector potential, Equation (7.2.18).

7.4: An Electric Dipole Radiator

An atom or a molecule in an excited state may develop an oscillating electric dipole moment density, $\vec{P}$. This dipole density oscillates at a frequency that ~ is proportional to the energy difference between the excited state and the ground state, $\Delta\mathrm{E}: \hbar\omega = \Delta\mathrm{E}$. The oscillating dipole moment generates an electromagnetic field that carries off the excited state energy ΔE, and the atom or molecule returns to the ground state. An oscillating dipole moment density constitutes a current density. From Equation (7.2.3)

$$\vec{\mathrm{J}}_T = \frac{\partial\vec{\mathrm{P}}}{\partial \mathrm{t}} = \vec{\mathrm{P}},$$

if $\vec{\mathrm{J}}_f$ and $\vec{M}$ are both zero as shall be assumed here. Also assume that the ~ dipole density has only a z-component as shown in Figure (7.4.3). Then from

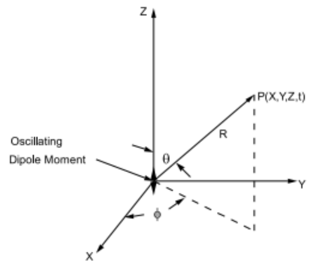


Figure 7.4.3: The co-ordinate system used to describe the fields generated by an oscillating atomic or molecular electric dipole moment.

Equation (7.2.18) the vector potential can be written

$$\mathrm{A}(\vec{\mathrm{R}}, t) = \frac{\mu_0}{4\pi}\frac{1}{\mathrm{R}}\int\int\int_{Vol} \mathrm{d}\tau\dot{\mathrm{P}}_\mathrm{z}(\vec{\mathrm{r}}, t - \mathrm{R}/\mathrm{c}),$$

where $\mathrm{d}\tau$ is the element of volume and the integral is carried out over the volume, Vol, of the atom or molecule. In the above equation it has been assumed that the dimensions of the atom or molecule are so small that all parts are at the same distance from the observer: ie. as $\vec{r}$ in (7.2.18) ranges over the volume, Vol, changes in the distance to the observer can be neglected both in the denominator and in the retarded time. This is the point dipole approximation. The volume integration with this assumption simply gives the total atomic or molecular dipole moment, $\vec{\mathrm{p}}_\mathrm{z}$, and

$$\mathrm{A}_\mathrm{z}(\mathrm{R}, \theta, \phi, t) = \frac{\mu_0}{4\pi}\frac{\dot{\mathrm{p}}_\mathrm{z}}{\mathrm{R}},$$

where $\dot{\mathrm{p}}_\mathrm{z}$ is the total dipole moment evaluated at the retarded time $\mathrm{t_R = t - R/c}$. This vector potential can now be used to calculate the magnetic field, $\vec{\mathrm{B}} = \mathrm{curl}(\vec{\mathrm{A}})$. For this purpose it is convenient to use spherical polar co-ordinates, see Figure (7.4.3):

$$\mathrm{A_R}(\mathrm{R}, \theta, \mathrm{t}) = \frac{\mu_0}{4\pi}\frac{\dot{\mathrm{p}}_\mathrm{z}(\mathrm{t} - \mathrm{R}/\mathrm{c})}{\mathrm{R}}\cos\theta, \tag{7.4.1}$$

$$\mathrm{A}_\theta(\mathrm{R}, \theta, \mathrm{t}) = -\frac{\mu_0}{4\pi}\frac{\dot{\mathrm{p}}_\mathrm{z}(\mathrm{t} - \mathrm{R}/\mathrm{c})}{\mathrm{R}}\sin\theta,$$

$$\mathrm{A}_\phi(\mathrm{R}, \theta, \mathrm{t}) = 0.$$

Notice that the vector potential does not depend on the angle ϕ so that the operator $(\partial/\partial\phi)$ gives zero. From (7.4.1) one finds

$$\mathrm{B_R} = \mathrm{B}_\theta = 0,$$

and

$$\mathrm{B}_\phi = \frac{1}{\mathrm{R}}\left[\frac{\partial}{\partial\mathrm{R}}(\mathrm{RA}_\theta) - \frac{\partial\mathrm{A_R}}{\partial\theta}\right],$$

or

$$B_\phi = \frac{\mu_0}{4\pi}\left[\frac{\ddot{p}_z}{cR} + \frac{\dot{p}_z}{R^2}\right]\sin\theta. \tag{7.4.2}$$

(Remember that $\partial\dot{p}_z/\partial R = -\ddot{p}_z/c$ because the dipole moment must be evaluated at the retarded time, $t_R = t - R/c$). The electric field components can be most easily calculated from the maxwell equation $\operatorname{curl}(\vec{B}) = (1/c^2)\,\partial\vec{E}/\partial t$ since outside the atom or molecule $\vec{J}_T = 0$. The components of $\operatorname{curl}(\vec{B})$ are:

$$\operatorname{curl}(\vec{B})_R = \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi),$$

so that

$$\operatorname{curl}(\vec{B})_R = \frac{\mu_0}{4\pi}\left[\frac{\dot{p}_z}{cR^2} + \frac{\dot{p}_z}{R^3}\right]\cos\theta$$
$$\operatorname{curl}(\vec{B})_\theta = -\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi)$$

so that

$$\operatorname{curl}(\vec{B})_\theta = \frac{\mu_0}{4\pi}\left(\frac{\dddot{p}_z}{c^2R} + \frac{\ddot{p}_z}{cR^2} + \frac{\dot{p}_z}{R^3}\right)\sin\theta. \tag{7.4.3}$$

Finally,

$$\operatorname{curl}(\vec{B})_\phi = 0.$$

The electric field components are therefore given by

$$E_R = \frac{1}{2\pi\epsilon_0}\left[\frac{\dot{p}_z}{cR^2} + \frac{p_z}{R^3}\right]\cos\theta, \tag{7.4.4}$$
$$E_\theta = \frac{1}{4\pi\epsilon_0}\left[\frac{\ddot{p}_z}{c^2R} + \frac{\dot{p}_z}{cR^2} + \frac{p_z}{R^3}\right]\sin\theta,$$
$$E_\phi = 0.$$

The electric field derivatives from $\operatorname{curl}(\vec{B})$ have been integrated with respect to time, and $c^2 = 1/(\mu_0\epsilon_0)$ has been used to eliminate μ_0.

The fields generated by a point dipole fall naturally into two groups:

(a) **The Near Fields.** These are the terms that become dominant as R, the distance from dipole to observer, becomes small. They are

$$B_\phi = \frac{\mu_0}{4\pi}\frac{\dot{p}_z}{R^2}\sin\theta, \tag{7.4.5}$$
$$E_R = \frac{1}{4\pi\epsilon_0}\frac{2p_z\cos\theta}{R^3},$$
$$E_\theta = \frac{1}{4\pi\epsilon_0}\frac{p_z\sin\theta}{R^3}.$$

The magnetic field is just that generated by a static current element according to the law of Biot-Savart, see Equation (4.2.2). The electric field components have the same form as those generated by a static dipole oriented along the z-axis, see Equation (1.2.12). Therefore near the dipole retardation effects are unimportant as one would expect.

(b) **The Far Fields or the Radiation Fields.** Far from the dipole the dominant terms for both the electric and magnetic fields are those that fall off with distance like 1/R.

$$B_\phi = \frac{\mu_0}{4\pi}\frac{\ddot{p}_z}{cR}\sin\theta, \tag{7.4.6}$$

$$
\begin{aligned}
E_R &\to 0 \quad \text{like}\left(1/R^2\right) \\
E_\theta &= \frac{\mu_0}{4\pi}\frac{\ddot{p}_z}{R}\sin\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{\ddot{p}_z}{c^2 R}\sin\theta
\end{aligned}
$$

In this limit $E_\theta = cB_\phi$, and the electric and magnetic fields are orthogonal. Moreover, both the electric and magnetic field are orthogonal to the line joining the observer to the dipole. These radiation fields are said to be transverse fields. Note particularly that the dipole moment is to be calculated at the retarded time, $t_R = t - R/c$, where t is the time of observation. Any fields created by the dipole at a particular instant require a transit time R/c to reach the observer.

7.5: A Point Magnetic Dipole

Consider an oscillating magnetic dipole moment, mz, oriented along the z-axis and located at the origin of co-ordinates similar to the case of the oscillating electric dipole of Figure (7.4.3). If the dipole were static it would generate a vector potential having only a ϕ-component:

$$\mathrm{A}_{\phi}=\frac{\mu_{0}}{4 \pi} \frac{\mathrm{m}_{\mathrm{z}} \sin \theta}{\mathrm{R}^{2}} . \tag{7.5.1}$$

This follows from the general expression for the vector potential generated by a point dipole, Equation (4.3.4)

$$\vec{\mathrm{A}}=\frac{\mu_{0}}{4 \pi} \frac{(\vec{\mathrm{m}} \times \vec{\mathrm{R}})}{\mathrm{R}^{3}} .$$

However, it can be shown that due to the effects of time retardation the equation for the vector potential,(7.5.1) must be modified to read

$$\mathrm{A}_{\phi}=\frac{\mu_{0}}{4 \pi} \sin \theta\left[\frac{\mathrm{m}_{\mathrm{z}}}{\mathrm{R}^{2}}+\frac{\mathrm{m}_{\mathrm{z}}}{\mathrm{cR}}\right] . \tag{7.5.2}$$

The fields derived from this expression for the vector potential, $\vec{\mathrm{B}}=\operatorname{curl}(\vec{\mathrm{A}})$, are

$$\begin{gathered}
\mathrm{B}_{\mathrm{R}}=\frac{\mu_{0}}{4 \pi} 2 \cos \theta\left[\frac{\mathrm{m}_{\mathrm{z}}}{\mathrm{R}^{3}}+\frac{\dot{\mathrm{m}}_{\mathrm{z}}}{\mathrm{cR}^{2}}\right]_{\mathrm{t}_{\mathrm{R}}}, \\
\mathrm{B}_{\theta}=\frac{\mu_{0}}{4 \pi} \sin \theta\left[\frac{\mathrm{m}_{z}}{\mathrm{R}^{3}}+\frac{\dot{\mathrm{m}}_{z}}{\mathrm{cR}^{2}}+\frac{\ddot{\mathrm{m}}_{z}}{\mathrm{c}^{2} \mathrm{R}}\right]_{\mathrm{t}_{\mathrm{R}}}, \\
\mathrm{B}_{\phi}=0=\mathrm{E}_{\mathrm{R}}=\mathrm{E}_{\theta}, \\
\mathrm{E}_{\phi}=-\frac{\mu_{0}}{4 \pi} \sin \theta\left[\frac{\dot{\mathrm{m}}_{\mathrm{z}}}{\mathrm{R}^{2}}+\frac{\ddot{\mathrm{m}}_{\mathrm{z}}}{\mathrm{cR}}\right]_{\mathrm{t}_{\mathrm{R}}} .
\end{gathered} \tag{7.5.3}$$

where $t_R = t - R/c$. Far from the dipole the radiation fields that decrease with distance like (1/R) are given by

$$\mathrm{B}_{\theta}=\frac{\mu_{0}}{4 \pi} \frac{\ddot{\mathrm{m}}_{z}}{\mathrm{c}^{2} \mathrm{R}} \sin \theta, \tag{7.5.4}$$

$$\mathrm{E}_{\phi}=-\frac{\mu_{0}}{4 \pi} \frac{\ddot{\mathrm{m}}_{z}}{\mathrm{cR}} \sin \theta=\mathrm{cB}_{\theta},$$

both evaluated at the retarded time t_R. Just as for the electric dipole far fields $|\vec{\mathrm{E}}|=\mathrm{c}|\vec{\mathrm{B}}|$, and $\vec{E}$ and $\vec{B}$ are orthogonal to each other and to the line joining the position of the observer to the dipole.

7.6: A Moving Point Charge in Vacuum

The charge density corresponding to a point charge is a singular distribution that can be written

$$\rho_f(\vec{r}, t) = q\delta(\vec{r} - \vec{r}_0(t)), \tag{7.6.1}$$

where $\delta(\vec{r})$ is the Dirac delta-function introduced in Chapters (2) and (4), and $\vec{r}_0(t)$ describes the time variation of the position of the particle. The delta function is supposed to be zero for all values of its argument except when the argument is equal to zero; at that point the function becomes infinitely large but in such a manner that its integral is unity. The 1-dimensional δ-function may be thought of as the limit as $\epsilon \to 0$ of a very thin rectangular shape that is ϵ wide and that has an amplitude $1/\epsilon$. The three dimensional δ-function may be envisioned as the product of three 1-dimensional δ-functions. The potential function that is generated by the distribution (7.6.1) can be written using Equation (7.2.16):

$$V(\vec{R}, t) = \frac{q}{4\pi\epsilon_0} \int\int\int_{Space} d\tau \frac{\delta[\vec{r} - \vec{r}_0(t_R)]}{|\vec{R} - \vec{r}|}. \tag{7.6.2}$$

The integrand is very sharply peaked when $\vec{r} = \vec{r}_0(t_R)$ so that it is very tempting to conclude that

$$V(\vec{R}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left|\vec{R} - \vec{r}_0(t_R)\right|}.$$

This equation is **WRONG**, because it ignores the position dependence of the retarded time which appears in the argument of the δ−function. In

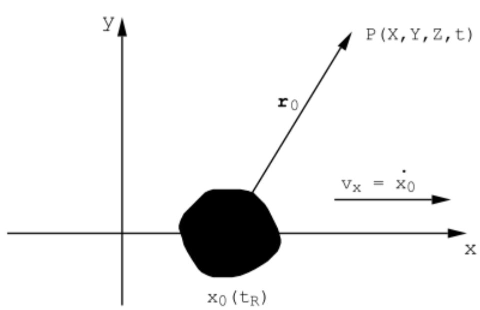


Figure $7.6.i$: A small blob of charge moving along the x-axis with a velocity vx. The contribution to the potential at P at the time of observation, t, comes from the position of the particle at the retarded time $t_R = t - r_0/c$.

order to understand this, suppose for simplicity that a co-ordinate system is chosen so that at the retarded time the particle is moving along the x-axis, i.e. $y_0 = z_0 = 0$ and y,z are not changing with time because the velocity of the particle is directed along x (see Figure (7.6.4)). The integral of Equation (7.6.2) can be written explicitly in cartesian co-ordinates: the result is

$$V(X, Y, Z, t) = \frac{q}{4\pi\epsilon_0} \int\int\int_{-\infty}^{+\infty} dxdydz \frac{\delta[x - x_0(t_R)]\,\delta[y]\delta[z]}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}}.$$

The integrations over y,z are just ordinary integrations over δ−functions that may be carried out at once using

$$\int_{-\infty}^{+\infty} du\, f(u)\delta(u) = f(0).$$

This leaves the integration over x to be carried out;

$$V(X, Y, Z, t) = \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx \frac{\delta[x - x_0(t_R)]}{\sqrt{(X-x)^2 + Y^2 + Z^2}}. \tag{7.6.3}$$

In order to turn (7.6.3) into an ordinary integration over a δ−function it is necessary to change variables so as to get rid of the spatial variation that is contained in the retarded time, t_R. Introduce the new variable

$$u = x - x_0(t_R).$$

Then

$$\mathrm{du}=\mathrm{dx}-\dot{\mathrm{x}}_0\left(\frac{\partial \mathrm{t_R}}{\partial \mathrm{x}}\right)\mathrm{dx},$$

where

$$\mathrm{t_R}=t-\frac{\sqrt{(\mathrm{X}-\mathrm{x})^2+\mathrm{Y}^2+\mathrm{Z}^2}}{\mathrm{c}},$$

so that

$$\frac{\partial \mathrm{t_R}}{\partial \mathrm{x}}=\frac{(\mathrm{X}-\mathrm{x})}{\mathrm{c}\sqrt{(\mathrm{X}-\mathrm{x})^2+\mathrm{Y}^2+\mathrm{Z}^2}}.$$

One finally obtains for the differential du the expression

$$\mathrm{du}=\mathrm{dx}\left(1-\frac{\dot{\mathrm{x}}_0(\mathrm{X}-\mathrm{x})}{\mathrm{c}\sqrt{(\mathrm{X}-\mathrm{x})^2+\mathrm{Y}^2+\mathrm{Z}^2}}\right),$$

and the integral (7.6.3) becomes

$$\mathrm{V}(\mathrm{X},\mathrm{Y},\mathrm{Z},\mathrm{t})=\frac{\mathrm{q}}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\mathrm{du}\frac{\delta(\mathrm{u})}{\left(\sqrt{(\mathrm{X}-\mathrm{x})^2+\mathrm{Y}^2+\mathrm{Z}^2}-\frac{\dot{x}_0(\mathrm{x}-\mathrm{x})}{\mathrm{c}}\right)}.$$

The expression for the potential function has been transformed into a $\delta-$function integration that can be carried out immediately to give

$$\mathrm{V}(\mathrm{X},\mathrm{Y},\mathrm{Z},\mathrm{t})=\frac{\mathrm{q}}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\left(\mathrm{X}-\mathrm{x}_0\right)^2+\mathrm{Y}^2+\mathrm{Z}^2}-\frac{\dot{x}_0(\mathrm{X}-\mathrm{x}_0)}{\mathrm{c}}}\right)_{\mathrm{t_R}},\tag{7.6.4}$$

since for u=0 x=$\mathrm{x_0(t_R)}$. The result, Equation (7.6.4), can be written in a more general and compact form using vector notation:

$$\mathrm{V}(\vec{\mathrm{R}},t)=\frac{\mathrm{q}}{4\pi\epsilon_0}\left(\frac{1}{|\vec{\mathrm{r}}_0|-\frac{\vec{v}\cdot\vec{r}_0}{c}}\right)_{t-r_0/\mathrm{c}},\tag{7.6.5}$$

where $\vec{\mathrm{r}}_0=(\vec{\mathrm{R}}-\vec{\mathrm{r}})$ is the vector that specifies the position of the particle at the retarded time relative to the point of observation at time t, P(X,Y,Z,t) (see Figure (7.4)), and $\vec{v}$ is the particle velocity at the retarded time.

Feynman (loc.cit. section 21-5) has given a very physical description of why the retarded potential contains the complicated denominator of Equation (7.6.5) rather than simply the retarded distance | $\mathrm{r_0}$ |. He explains how the volume integration of (7.6.2) for the potential must explicitly take into account that the contribution to the potential at a fixed time of observation comes from different retarded times for different points in the charge distribution.

Exactly the same arguments apply to the calculation of the vector potential for a moving point charge from Equation (7.2.18). The current density for a point charge moving with a velocity $\vec{v}$ is given by

$$\vec{\mathrm{J}}(\vec{\mathrm{r}},t)=\mathrm{q}\vec{\mathrm{v}}\delta\left(\vec{\mathrm{r}}-\vec{\mathrm{r}}_0(\mathrm{t})\right),$$

where $\vec{\mathrm{r}}_0(\mathrm{t})$ describes the position of the particle at time t. Upon carrying out the integration in Equation (7.2.18) the resulting vector potential is found to be

$$\vec{\mathrm{A}}(\vec{\mathrm{R}},\mathrm{t})=\frac{\mu_0}{4\pi}\left(\frac{\mathrm{q}\vec{\mathrm{v}}}{|\vec{\mathrm{r}}_0|-\frac{\vec{v}\cdot\vec{r}_0}{\mathrm{c}}}\right)_{\mathrm{t}-\mathrm{r}_0/\mathrm{c}},\tag{7.6.6}$$

where $\vec{r}_0$ is the vector drawn from the position of the particle at the retarded time, t_R, to the point of observation at time t. Eqns.(7.6.5) and (7.6.6) are called the Lienard-Wiechert potentials for a point charge. They are consistent with the theory of relativity. The electric and magnetic fields generated by a moving point charge, Chpt.(1), Equations (1.1.9) and (1.1.10), can be deduced from them by means of the relations

$$\vec{B} = \text{curl}(\vec{A}),$$

and

$$\vec{E} = -\overrightarrow{\text{grad}}(V) - \frac{\partial \vec{A}}{\partial t}.$$

In the general case these result in the rather complex equations of Equations (1.1.9) and (1.1.10). However, in the limit $v/c \ll 1$ the fields generated by a moving point charge can be obtained relatively simply from the low velocity limit of the vector potential. Consider a charge q near the origin, at $\vec{r} = (0, 0, \xi)$, and moving along the z-axis with a velocity $v = \dot{\xi}$. In spherical polar co-ordinates one has

$$A_r = \frac{\mu_0}{4\pi} \frac{qv\cos\theta}{\left(r - \frac{vz}{c}\right)} = \frac{\mu_0}{4\pi} \frac{qv\cos\theta}{r\left(1 - \frac{v\cos\theta}{c}\right)}$$
$$A_\theta = -\frac{\mu_0}{4\pi} \frac{qv\sin\theta}{r\left(1 - \frac{v\cos\theta}{c}\right)}.$$

For a slowly moving particle, $v \ll c$, these become

$$A_r = \frac{\mu_0}{4\pi} \frac{qv\cos\theta}{r},$$

and

$$A_\theta = -\frac{\mu_0}{4\pi} \frac{qv\sin\theta}{r},$$

where $A_\phi = 0$, $v = \dot{\xi}$, and A_r, A_θ are to be evaluated at the retarded time $t_R = t - r/c$. The magnetic field is given by $\vec{B} = \text{curl}(\vec{A})$:

$$B_r = B_\theta = 0,$$

and

$$B_\phi = \frac{1}{r} \frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta}(A_r),$$

where $\frac{\partial}{\partial r}$ includes a term $-\partial/c\partial t$ because if r changes by dr the retarded time changes by $dt_R = -dr/c$. Thus

$$B_\phi = \frac{\mu_0}{4\pi} q\sin\theta \left[\frac{v}{r^2} + \frac{\dot{v}}{cr}\right]. \tag{7.6.7}$$

Eqn.(7.6.7) is just the field generated by a point electric dipole at the origin if one writes $p_z = q\xi$, $\dot{p}_z = q\dot{\xi}$, and $\ddot{p}_z = q\ddot{\xi}$; then

$$B_\phi = \frac{\mu_0}{4\pi} \sin\theta \left[\frac{\dot{p}_z}{r^2} + \frac{\ddot{p}_z}{cr}\right], \tag{7.6.8}$$

see Equation (7.4.2).

The electric field can be calculated from

$$\text{curl}(\vec{B}) = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}.$$

Thus

$$\frac{1}{c^2} \frac{\partial E_r}{\partial t} = \frac{\mu_0}{4\pi} 2\cos\theta \left[\frac{qv}{r^3} + \frac{\dot{v}}{cr^2}\right],$$

or

$$\frac{\partial E_r}{\partial t}=\frac{1}{4\pi\epsilon_0}2\cos\theta\left[\frac{\dot{p}_z}{r^3}+\frac{\ddot{p}_z}{cr^2}\right].$$

This last equation can be integrated with respect to the time to obtain

$$E_r=E_0+\frac{2\cos\theta}{4\pi\epsilon_0}\left[\frac{p_z}{r^3}+\frac{\dot{p}_z}{cr^2}\right], \tag{7.6.9}$$

where the constant of integration is simply the static field due to a charge, q, at the origin,

$$E_0=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

The radial electric field component is that of a point charge at the origin plus a field due to a point dipole at the origin.

The transverse electric field component is given by

$$\frac{\partial E_\theta}{\partial t}=\frac{\sin\theta}{4\pi\epsilon_0}\left[\frac{\dot{p}_z}{r^3}+\frac{\ddot{p}_z}{cr^2}+\frac{\dddot{p}_z}{c^2r}\right].$$

This expression can be integrated to give

$$E_\theta=\frac{\sin\theta}{4\pi\epsilon_0}\left[\frac{p_z}{r^3}+\frac{\dot{p}_z}{cr^2}+\frac{\ddot{p}_z}{c^2r}\right]. \tag{7.6.10}$$

For this case the constant of integration is zero because in the static limit the only contribution to the θ-component of the electric field is a dipole term due to the displacement of the charge from the origin by the vanishingly small distance ξ. Eqn.(7.6.10) is just the θ-component of the field generated by a point electric dipole at the origin, Equation (7.4.3). The radiation field terms, the terms that fall off like 1/r, can be written

$$E_\theta=\frac{\sin\theta}{4\pi\epsilon_0}\frac{qa}{c^2r},$$

$$cB_\phi=\frac{\sin\theta}{4\pi\epsilon_0}\frac{qa}{c^2r},$$

where $a=\ddot{\xi}$. These radiation fields can be written as follows in terms of general vector position co-ordinates where the particle is taken to be at the origin:

$$\vec{E}(\vec{r},t)=\frac{q}{4\pi\epsilon_0}\frac{[\vec{r}\times\vec{r}\times\vec{a}]}{c^2r^3}\bigg|_{t_R} \tag{7.6.11}$$

$$c\vec{B}(\vec{r},t)=\frac{q}{4\pi\epsilon_0}\frac{|\vec{a}\times\vec{r}|}{c^2r^2}\bigg|_{t_R},$$

where the acceleration $\vec{a}$ is evaluated at the retarded time $t_R = t - r/c$.

Eqns.(7.6.11) are valid only for a slowly moving charge whose velocity is very much smaller than the velocity of light in vacuum. These radiation fields fall off as the first power of the distance from the observer to the particle.

Eqns.(7.6.9 and 7.6.10) can also be calculated from the formula

$$\vec{E}=-\overrightarrow{\text{grad}}(V)-\frac{\partial\vec{A}}{\partial t},$$

using the low velocity limit for the vector potential along with the expression for the potential function, Equation (7.6.5), expanded to lowest order in the small quantities ξ and $\dot{\xi}/c$:

$$V(\vec{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[1+\frac{\xi\cos\theta}{r}+\frac{\dot{\xi}\cos\theta}{c}\right].$$

CHAPTER OVERVIEW

8: Electromagnetic Fields and Energy Flow

Chapter 7 treated the problem of how electromagnetic fields are generated by time varying charge and current distributions. Electro-magnetic fields transport energy and momentum through space, and this chapter is concerned about how to calculate the energy density contained in those fields and how to calculate the rate of energy transport. Scattering from atoms or molecules is also discussed, as are the generation of the continuous X-ray spectrum produced in an X-ray tube.

Thumbnail: Animation of a half-wave dipole antenna transmitting radio waves, showing the electric field lines. (Public Domain; Chetvorno via Wikipedia)

8.2: Poynting's Theorem

A relation between energy flow and energy stored in the electromagnetic field can be obtained from Maxwell's equations and the vector identity

$$\mathrm{div}(\vec{E}\times\vec{H})=\vec{H}\cdot\mathrm{curl}(\vec{E})-\vec{E}\cdot\mathrm{curl}(\vec{H}). \tag{8.2.1}$$

Multiply the Maxwell equation

$$\mathrm{curl}(\vec{E})=-\frac{\partial\vec{B}}{\partial t}$$

by $\vec{H}$, and multiply

$$\mathrm{curl}(\vec{H})=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}$$

by $\vec{E}$ and subtract to obtain

$$\vec{H}\cdot\mathrm{curl}(\vec{E})-\vec{E}\cdot\mathrm{curl}(\vec{H})=-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{J}_f\cdot\vec{E}-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}. \tag{8.2.2}$$

Using the identity (8.2.1) this may be rewritten

$$-\mathrm{div}(\vec{E}\times\vec{H})=\vec{J}_f\cdot\vec{E}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}.$$

Integrate the latter equation over a volume V bounded by a closed surface S. The volume integral over the divergence can be converted to a surface integral by means of Gauss' theorem:

$$-\int\int\int_V d\tau div(\vec{E}\times\vec{H})=-\iint_S dS(\vec{E}\times\vec{H})\cdot\hat{u}_n,$$

where dS is an element of surface area, and $\widehat{u}_n$ is a unit vector normal to dS. Using Gauss' Theorem one obtains

$$-\iint_S dS(\vec{E}\times\vec{H})\cdot\hat{u}_n=\iiint_V d\tau\left(\vec{J}_f\cdot\vec{E}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right). \tag{8.2.3}$$

Equation (8.2.3) is the statement of Poynting's theorem. Each term in (8.2.3) has the units of a rate of change of energy density. The quantity

$$\vec{S}=\vec{E}\times\vec{H} \tag{8.2.4}$$

is called Poynting's vector; it is a measure of the momentum density carried by the electromagnetic field. Momentum density in the field is given by

$$\vec{g}=\frac{\vec{S}}{c^2};$$

see the Feynman Lectures on Physics, Volume(II), Chapter(27); (R.P.Feynman, R.B.Leighton, and M.Sands, Addison-Wesley, Reading, Mass., 1964).

The surface integral of the Poynting vector, $\vec{S}$, over any closed surface gives the rate at which energy is transported by the electromagnetic field into the volume bounded by that surface. The three terms on the right hand side of Equation (8.2.3) describe how the energy carried into the volume is distributed.

These three terms are:

(1) $\iiint_V d\tau\left(\vec{J}_f\cdot\vec{E}\right)$

This term describes the rate at which mechanical energy in the system defined by the volume V increases due to the mechanical forces exerted on charged particles by the electric field: it describes the conversion of electric and magnetic energy into kinetic energy and heat. This can be understood by considering the force on a charged particle

$$\vec{f}=q(\vec{E}+(\vec{v}\times\vec{B})).$$

The rate at which the electromagnetic field does work on the charged particle is

$$\frac{dW}{dt}=\vec{f}\cdot\vec{v}\equiv q\vec{v}\cdot\vec{E}. \tag{8.2.5}$$

(The magnetic field makes no contribution to the work done on the particle because the magnetic force is perpendicular to the velocity, $\vec{v}$). When summed over all the charges in a volume element, Equation (8.2.5) gives, per unit volume, $\vec{J}_f\cdot\vec{E}$.

(2) $\int\int\int_V d\tau\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)$

This term gives the rate at which the energy stored in the macroscopic electric field increases with time. Its effect can be represented by the rate of increase of an energy density W_E:

$$\frac{\partial W_E}{\partial t}=\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}. \tag{8.2.6}$$

Notice that this term depends upon the properties of the material because it involves the polarization vector through the displacement vector $\vec{D}=\epsilon_0\vec{E}+\vec{P}$.

(3) $\int\int\int_V d\tau\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$

This integral describes the rate of increase of energy stored in the volume V in the form of magnetic energy. It corresponds to a rate of increase of a magnetic energy density WB:

$$\frac{\partial W_B}{\partial t}=\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}. \tag{8.2.7}$$

Notice that this term involves the properties of the matter in the volume V through the presence of the magnetization density, $\vec{M}$, in the definition of $\vec{H}=\left(\vec{B}/\mu_0\right)-\vec{M}$.

Let us apply Poynting's theorem, Equation (8.3), to a spherical surface surrounding the dipole radiator of Chapter(7). Suppose that the radius of the sphere, R, is so large that only the radiation fields have an appreciable amplitude on its surface; recall that the radiation fields fall off with distance like 1/R (see Equations (7.33)) , whereas the other field components fall off like $1/R^2$ or $1/R^3$. For the case of dipole radiation in free space the Poynting vector has only an r-component because $\vec{E}$, $\vec{H}$ are perpendicular to one another and also perpendicular to the direction specified by the unit vector $\hat{u}_r=\vec{r}/r$. In free space $\vec{B}=\mu_0\vec{H}$ and

$$S_r=\frac{E_\theta B_\phi}{\mu_0}=\frac{1}{c\mu_0}\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{d^2p_z}{dt^2}\right)^2_{t_R}\left(\frac{\sin^2\theta}{c^4R^2}\right), \tag{8.2.8}$$

where as usual $t_R = t - R/c$ is the retarded time. Now take $p_z = p_0\cos(\omega t)$ so that

$$\left(\frac{d^2p_z}{dt^2}\right)_{tr}=-\omega^2p_0\cos(\omega t_R),$$

and therefore

$$S_r=\frac{1}{c\mu_0}\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{\omega}{c}\right)^4p_0^2\cos^2\omega(t-R/c)\left(\frac{\sin^2\theta}{R^2}\right). \tag{8.2.9}$$

The time averaged value of the term $\cos^2\omega(t - R/c)$ is 1/2 ; also $c^2 = 1/(\epsilon_0\mu_0)$. These can be used in (8.2.9) to obtain the average rate at which energy is transported through a surface having a radius R:

$$<S_r>=\left(\frac{1}{8\pi}\right)\left(\frac{c}{4\pi\epsilon_0}\right)\left(\frac{\omega}{c}\right)^4\frac{p_0^2\sin^2\theta}{R^2}. \tag{8.2.10}$$

Eqn.(8.2.10) gives the angular distribution of the time-averaged power radiated by an oscillating electric dipole. The power radiated along the direction of the dipole is zero, and the maximum power is radiated in the plane perpendicular to the dipole (see Figure (8.1.1)). The total average power radiated by the dipole can be obtained by integrating (8.2.10) over the surface of the sphere of radius R:

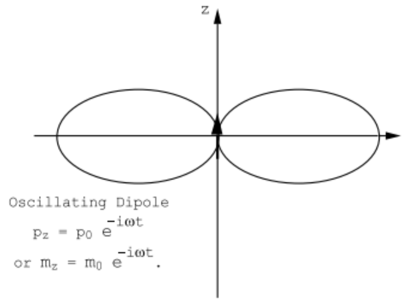


Figure 8.2.1: The pattern of radiated power for an oscillating electric dipole. There is no power radiated along the direction of the dipole, $\vec{p}$ or $\vec{m}$.

$$\int\int_{Sphere} \mathrm{dS} < \mathrm{S_r} >= \int_0^\pi < \mathrm{S_r} > 2\pi \mathrm{R}^2 \sin\theta \mathrm{d}\theta = \frac{\mathrm{c}}{16\pi\epsilon_0}\left(\frac{\omega}{\mathrm{c}}\right)^4 \mathrm{p}_0^2 \int_0^\pi \sin^3\theta \mathrm{d}\theta.$$

But $\int_0^\pi \sin^3\theta \mathrm{d}\theta = 4/3$ so that the total average power radiated by the oscillating electric dipole is given by

$$P_E = \frac{1}{3}\frac{c}{4\pi\epsilon_0}\left(\frac{\omega}{c}\right)^4 p_0^2 \quad Watts. \tag{8.2.11}$$

The rate of energy radiated by the dipole increases very rapidly with the frequency for a fixed dipole moment, $\mathrm{p_0}$.

A similar calculation gives the average rate, $\mathrm{P_M}$, at which energy is radiated by an oscillating magnetic dipole. The far fields generated by an oscillating magnetic dipole are given by

$$\mathrm{B}_\theta = \frac{\mu_0}{4\pi}\left(\frac{\mathrm{d}^2\mathrm{m}_z}{\mathrm{dt}^2}\right)_{\mathrm{t_R}} \frac{\sin\theta}{\mathrm{c}^2\mathrm{R}},$$

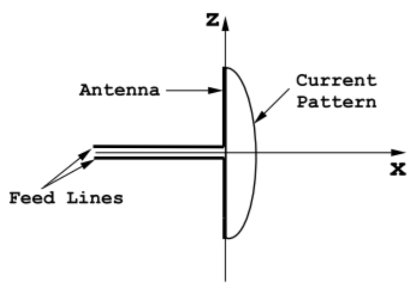


Figure 8.2.2: The schematic diagram of a center-fed, linear, half-wave antenna oriented along the z-axis. The current is zero at the ends of the antenna; these are located at z=-L and at z=+L.

$$\mathrm{E}_\phi = -\mathrm{c}\,\mathrm{B}_\theta,$$

where as usual $\mathrm{t_R} = \mathrm{t} - \mathrm{R/c}$ is the retarded time. For a magnetic dipole whose amplitude is $\mathrm{m_0}$ one finds

$$\mathrm{P_M} = \frac{\mathrm{c}}{3}\frac{\mu_0}{4\pi}\left(\frac{\omega}{\mathrm{c}}\right)^4 \mathrm{m}_0^2 \quad \mathrm{Watts}\,. \tag{8.2.12}$$

8.3: Power Radiated by a Simple Antenna

The radiation fields generated by a simple center-fed linear antenna oriented along the z-axis can be written

$$\begin{aligned} B_\phi &= \frac{\mu_0}{4\pi} I_0 \left(\frac{\omega}{c}\right) \cos(\omega[t-R/c]) \frac{\sin\theta F(\theta)}{R}, \\ E_\theta &= cB_\phi \end{aligned}, \tag{8.3.1}$$

where

$$\sin\theta F(\theta) = \frac{2}{\left(\frac{\omega}{c}\right)\sin\theta}\left[\cos\left(\frac{\omega L\cos\theta}{c}\right) - \cos\left(\frac{\omega L}{c}\right)\right], \tag{8.3.2}$$

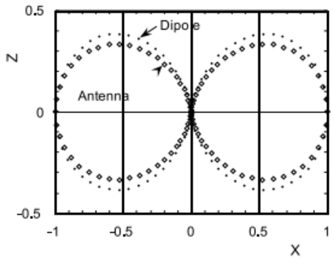


Figure 8.3.3: The radiation pattern for a half-wave antenna compared with the radiation pattern for an oscillating electric dipole. The dipole pattern is indicated by the + symbols.

(see Chapter(7) Equations (7.3.6 and 7.3.4). The simplest resonant antenna is that for which (ωL/c) = π/2. This is a half-wave antenna for which 2L = λ/2; ie. the total length of the antenna is half the free space wavelength λ = 2π(c/ω) = 2π/k. For this half-wave antenna Equation (7.3.4) becomes

$$\sin\theta F(\theta) = \frac{2}{\left(\frac{\omega}{c}\right)\sin\theta}\cos\left(\frac{\pi}{2}\cos\theta\right), \tag{8.3.3}$$

and the current distribution along the antenna becomes

$$I_z(z) = I_0 \sin(\omega t)\cos\left(\frac{\pi z}{2L}\right), \tag{8.3.4}$$

see Figure (8.2.2). The Poynting vector is SR = ($E_\theta B_\varphi/\mu_0$), and since $\vec{E}$ and $\vec{B}$ oscillate in phase the time averaged Poynting vector is given by

$$< S_R >= \frac{|E_\theta||B_\phi|}{2\mu_0},$$

where | E_θ | and | B_φ | are the electric and magnetic field amplitudes. For the particular case of the half-wave antenna one finds using (8.3.3)

$$< S_R >= \frac{1}{8\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}} I_0^2 \frac{\cos^2\left(\frac{\pi\cos\theta}{2}\right)}{R^2\sin^2\theta}. \tag{8.3.5}$$

$Z_0 = \sqrt{\mu_0/\epsilon_0} = 120\pi$ Ohms= 377 Ohms is the impedance of free space. The variation with angle of the radiated power is shown in Figure (8.3.3) where it is compared with the simple $\sin^2\theta$ pattern characteristic of a point dipole. The total average power passing through a sphere of radius R is independent of R and is given by

$$< P_R >= 2\pi R^2 \int_0^\pi < S_R > \sin\theta d\theta,$$

or

$$< P_R >= 30 I_0^2 \int_0^\pi d\theta \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta},$$

since $\sqrt{\mu_0/\epsilon_0} = 120\pi$ Ohms. The integral can be evaluated numerically. The result is

$$< P_R >= 73.13 \left(\frac{I_0^2}{2}\right) \quad \text{Watts}. \tag{8.3.6}$$

The average power dissipated in a resistor, R Ohms, by a sinusoidal current having an amplitude I_0 Amps is $R\left(I_0^2/2\right)$, therefore the ideal halfwave antenna presents an impedance to the power source whose real part is R_A= 73.13 Ohms. The resistance R_A is called the radiation resistance of the antenna. An ideal antenna is one for which the Ohmic resistance of the antenna wire itself is negligible compared with the radiation resistance. The antenna will also present an inductive or capacitive impedance to the generator since energy is stored in the electric and magnetic near fields. A thin wire half-wavelength antenna has an impedance Z = 73.1+i42.5 Ohms; in other words, the impedance contains an inductive component. However, if the antenna is made 0.49λ long, the impedance becomes purely resistive at approximately 73 Ohms. Such an antenna is said to be tuned. The input impedance of a shorter antenna contains a capacitive component; a longer antenna carries an inductive impedance component. Thus the input impedance of a half-wave dipole antenna varies rapidly with frequency. For practical use it is desirable to construct an antenna that (1) radiates most of its energy into a relatively narrow cone, and (2) one that has an input impedance that is relatively insensitive to frequency. These requirements have led to the development of a large variety of antenna configurations. These are described by John D. Kraus in ”Antennas”, McGraw-Hill, New York, 1988.

An antenna can, of course, also be used to detect the power broadcast by an antenna. It is instructive to examine the problem of an antenna used as a receiver. Let the antenna be terminated by a matched load; the resistive part of the load will thus be equal to the antenna radiation resistance, R_A. Let us use the specific example of a half-wave antenna for which $R_A \approx 73$ Ohms. Assume that the receiving antenna is oriented parallel with the transmitting antenna so that the incident electric field vector is oriented along the receiving antenna: if the $\vec{E}$ field is transverse to the antenna no signal will be detected. Usually, the transmitter is so far removed from the receiver that the incident electric field amplitude can be taken as constant over the receiving antenna. Let the amplitude of this incident electric field be E_0. The incident electric field will induce a current distribution on the half-wave antenna that has the form described by Equation (8.3.4) and an amplitude I_A Amps. Assume that the antenna is connected to a detector whose input impedance has been matched to the antenna impedance. The average rate at which power is extracted from the incident radio wave is

$$< P_A >= \frac{1}{2} \int_{-L}^{+L} dz E_0 I_A \cos\left(\frac{\pi z}{2L}\right) = \frac{2L}{\pi} E_0 I_A \quad \text{Watts}. \tag{8.3.7}$$

This means that for a half-wave antenna and a matched load the detector resistance will be 73 Ohms. For this matched receiver one half the incident power will be re-radiated (the current distribution will after all radiate away power at the average rate of $R_A I_A^2/2$ Watts), and half the power will be absorbed by the matched detector, $R_A I_A^2/2$ Watts. Thus the useful power picked up by the antenna and delivered to the detector is

$$P_D = \frac{L}{\pi} E_0 I_A = \frac{R_A}{2} I_A^2. \tag{8.3.8}$$

From this equation one finds $I_A = 2LE_0/(\pi R_A)$, and

$$P_D = \frac{2L^2}{\pi^2} \frac{E_0^2}{R_A} \quad \text{Watts}. \tag{8.3.9}$$

It is useful and interesting to ask ”how large must a disc be so that all the transmitted energy intercepted by the disc is equal to the power P_D delivered to the detector?”. The area of such a disc is called the ”Effective Aperture”, A_E, of the receiving antenna. The amplitude of the time-averaged Poynting vector for an incident wave of amplitude E_0 is

$$< P >= \frac{E_0^2}{2c\mu_0} = \frac{E_0^2}{2Z_0} \quad \text{Watts}/\text{m}^2,$$

where Z_0=377 Ohms. Therefore

$$\frac{E_0^2}{2Z_0} A_E = P_D = \frac{2L^2}{\pi^2} \frac{E_0^2}{R_A},$$

from which

$$A_E = \frac{4}{\pi^2}\frac{Z_0}{R_A}L^2 \quad m^2. \tag{8.3.10}$$

For the half-wave antenna R_A= 73 Ohms and 2L= λ/2, so that

$$A_E = 0.131\lambda^2.$$

In other words, the useful power delivered to the detector is all the incident power contained in a circle whose diameter is 0.4λ, a diameter nearly equal to the length of the antenna!

8.4: A Non-Sinusoidal Time Dependence

Nothing in the calculation of the radiation fields required the time variation of the dipole moment to be sinusoidal. If a charge undergoes an acceleration ~a at the retarded time $t_R = t - R/c$ then the Poynting vector at time t on a surface of radius R will have the radial component

$$S_r = \frac{1}{c\mu_0}\left(\frac{qa\sin\theta}{4\pi\epsilon_0 c^2 R}\right)^2,$$

(see Equations (7.4.5)). This expression can be written

$$S_r = \frac{c}{4\pi}\frac{1}{4\pi\epsilon_0}\left(\frac{q^2a^2\sin^2\theta}{c^4R^2}\right)_{tr}, \tag{8.4.1}$$

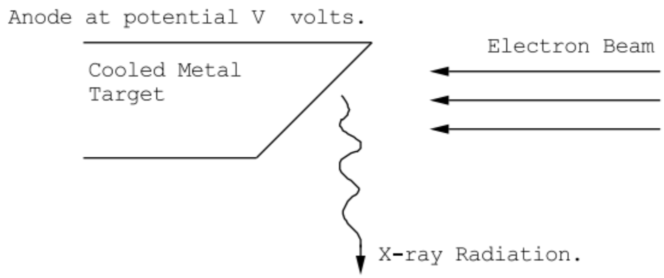


Figure 8.4.4: A schematic diagram of an X-ray tube illustrating the production of the white X-ray spectrum. The electrons undergo a de-acceleration upon striking the metal anode. This de-acceleration is of order a= 1024 m/sec^2 for a typical 20 keV potential drop between the anode and the cathode: this assumes an electron stopping distance of 35×10^{-10} m. During a brief period, $\sim 10^{-16}$ seconds, the electron radiates at the rate of $\sim 5.7\times10^{-6}$ Watts, therefore each electron emits a pulse of radiation containing $\sim 5.7\times10^{-22}$ Joules. The number of electrons that impinge on the anode per second for a beam of 1 mAmp is 6.25×10^{15}. The average power in the X-ray beam will be $(6.25\times10^{15})(5.7\times10^{-22}) = 3.6\times10^{-6}$ Watts. This energy is distributed over a range of frequencies from zero to 4.8×10^{18} Hz ($h\nu_{max}$ =| e | V). This calculation does not include the energy contained in the characteristic X-ray spectrum emitted from the target.

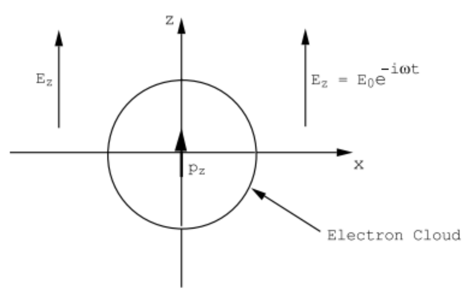


Figure 8.4.5: Schematic diagram of an atom in a time varying electric field. The atom develops a time varying dipole moment that scatters the incident radiation.

and the power integrated over a sphere of radius R is given by

$$P_q = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\left(\frac{q^2a^2}{c^3}\right)_{t_R} \quad \text{Watts}, \tag{8.4.2}$$

where $a(t_R)$ means that the acceleration is measured at the retarded time (t − R/c) if the power is measured at the time t. Eqn.(8.4.2) can be used to understand the production of the continuous X-ray spectrum, refer to Figure (8.4.4). The conversion efficiency for X-ray production is rather small; approximately 10^{-7} of the incident power is converted to continuous spectrum X-ray energy.

8.5: Scattering from a Stationary Atom

Let plane wave radiation propagating in vacuum fall upon a stationary atom located at the origin of co-ordinates. Let the plane wave be polarized with the electric vector directed along z, and let the wave be propagating along x (Figure (8.4.5)):

$$E_z = E_0 \exp i(kx - \omega t). \tag{8.5.1}$$

We will consider only radiation whose wavelength, λ, is much larger than the dimensions of the atom, i.e. much larger than 10^{-10} meters. This condition restricts the frequency of the radiation to $f \leq 3\times10^{17}$ Hz. In this long wavelength limit one can take the electric field to be uniform over the atom. Each charged particle in the atom will be subjected, as a first approximation, to the electric field of Equation (8.5.1). The response of the nucleus to the oscillating electric field can be neglected for purposes of estimating the induced atomic electric dipole moment because the nucleus is so massive compared with the electrons. The main contribution to the induced dipole moment will be due to the response of the cloud of electrons to the applied oscillating electric field. The results of calculations using quantum mechanics shows that the electrons behave like simple harmonic oscillators that follow an equation of motion of the form

$$\frac{d^2z}{dt^2} + \omega_0^2 z = -\frac{|e|}{m} E_0 e^{-i\omega t}, \tag{8.5.2}$$

where ω_0^2 is the square of the natural frequency associated with the electron. (Quantum mechanics must be used to calculate the resonant frequencies associated with the various electron groups in an atom). The steady state solution of Equation (8.5.2) is

$$z(t) = z_0 e^{-i\omega t}, \tag{8.5.3}$$

where

$$z_0 = \frac{\left(\frac{|e|}{m}\right) E_0}{(\omega^2 - \omega_0^2)}. \tag{8.5.4}$$

In Equation (8.5.4) $|e| = 1.602 \times 10^{-19}$ Coulombs, the electron charge, and $m = 9.11 \times 10^{-31}$ kg, the electron mass. The electron develops an oscillating dipole moment, $p_z = -|e|z$, as a result of the motion induced by the forcing electric field. The oscillating dipole will radiate energy at the same frequency as the incident radiation, and in this way energy will be removed from the incident plane wave and scattered in all directions around the z-axis as per the discussion of section (8.2), Equation (8.2.10). If the atom contains many groups of electrons, each group characterized by a characteristic resonant frequency, ωn, then each group will develop a dipole moment as a result of a vibration of the form of Equation (8.5.3). The total dipole moment developed by the atom is obtained by summing the contributions from each electron group:

$$p_z = -\left(\frac{e^2}{m}\right)\left(\sum_n \frac{f_n}{(\omega^2 - \omega_0^2)}\right) E_0 e^{-i\omega t} = p_0 e^{-i\omega t}. \tag{8.5.5}$$

The factors f_n are called the oscillator strengths. Each f_n is a measure of the effective number of electrons in a particular group characterized by the resonant frequency ω_n. The above model does not include damping processes and therefore the response described by Equation (8.5.5) becomes infinite whenever the frequency of the incident radiation becomes equal to one of the resonant frequencies, ω_n. In any actual atomic system the response of the electrons is limited by a number of energy loss mechanisms, including the energy radiated by the oscillating dipole moment, so that at resonance the electronic response becomes large but it does remain finite. The rate at which energy is scattered into the direction specified by θ,φ is given by Equation (8.2.10) of section (8.2) :

$$< S_r >= \frac{1}{8\pi} \frac{c}{4\pi\epsilon_0} \left(\frac{\omega}{c}\right)^4 \frac{p_0^2 \sin^2\theta}{R^2}, \tag{8.5.6}$$

where p_0 is given by Equation (8.5.5). A number of interesting conclusions can be drawn from the above result:

1. No energy is scattered along the direction parallel with the incident electric field;
2. The intensity of the scattered radiation is maximum in the plane perpendicular to the direction of the incident light electric vector, and the scattered light will be linearly polarized;

3. For frequencies that are much less than the lowest atomic resonant frequency the dipole moment, Equation (8.5.5), becomes independent of frequency. Under these conditions the intensity of the scattered radiation increases very rapidly with frequency; it is proportional to ω^4 ;
4. In the high frequency limit, $\omega \gg \omega_{max}$, where ω_{max} is the greatest resonant frequency, the dipole moment amplitude becomes inversely proportional to the square of the frequency so that the intensity of the scattered radiation becomes independent of frequency.

If one observes the result of scattering of visible, unpolarized light from atoms or molecules in a direction that is perpendicular to the direction of propagation of the incident light, it will be found that the scattered light will tend to be blue and it will be linearly polarized, see Figure (8.5.6). The scattered light tends to be blue because in the visible the scattering intensity increases approximately as the fourth power of the frequency, and red light has a lower frequency than blue light. This immediately suggests an explanation for the observation that the sky appears to be blue. It also explains why light from the sky is partially polarized when viewed in a direction perpendicular to the sun.

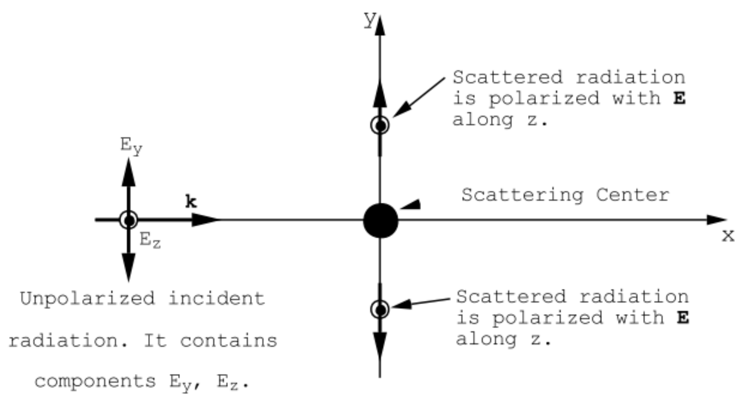


Figure 8.5.6: The production of polarized light by means of scattering. Let the incident light be unpolarized. The light scattered along the y-axis will be completely linearly polarized because only the component of the dipole moment along z can scatter radiation into the y-direction (see Equation (8.30)). The argument can be generalized to show that for any direction in the y-z plane the scattered radiation will be linearly polarized.

The total rate of energy loss from a plane wave due to atomic scattering can be calculated from the integral of (8.5.6) over a sphere having a radius R. The result is (see Equation (8.2.11))

$$P_E = \frac{1}{3}\frac{c}{4\pi\epsilon_0}\left(\frac{\omega}{c}\right)^4 p_0^2 \quad \text{Watts}\,.$$

The rate at which energy is carried to the atom by the incident plane wave can be calculated from the time average of the Poynting vector, Equation (8.2.4), using $H_0 = E_0/c\mu_0$:

$$< S_x >= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 \quad \text{Watts}\,/\text{m}^2.$$

This expression can be combined with Equation (8.2.11) to calculate an effective area for scattering, i.e. an area such that if all the incident power that falls on that area were to be removed from the plane wave, then that energy loss would just equal the radiated power given by Equation (8.2.11). Such an area, which is clearly frequency dependent, is called the **scattering cross-section**; it is often designated by σ_a. Far from resonance atomic cross-sections tend to be of the order of $10^{-37}m^2$ for visible light; i.e. the cross-section is small compared with the atomic area of $\sim 10^{-20}m^2$. For incident light having a wavelength of 10^{-10} m. or less, the calculation of the scattering cross-section becomes complicated because the electric field strength in the incident wave is not constant in amplitude across the atom. However, in the limit of very short wavelengths, i.e. for very high frequencies, the electrons in the atom or molecule behave like independent scattering centers whose motions are uncorrelated. In this very high frequency regime one has $\omega \gg \omega_n$ for all n, and so the dipole amplitude for each electron becomes

$$p_0 = -\left(\frac{e^2}{m\omega^2}\right) E_0. \tag{8.5.7}$$

Each electron behaves as if it were free, and consequently it oscillates with an amplitude given by

$$\mathrm{z}=\frac{|\mathrm{e}|}{\mathrm{m}\omega^{2}}\mathrm{E}_{0}\mathrm{e}^{-i\omega t}.$$

The total power radiated by each electron in the high frequency limit can be calculated from Equation (8.2.11). The result is

$$\mathrm{P}_{\mathrm{e}}=\frac{1}{3}\frac{1}{4\pi\epsilon_{0}}\left(\frac{\mathrm{e}^{4}}{\mathrm{m}^{2}\mathrm{c}^{3}}\right)\mathrm{E}_{0}^{2},$$

which is independent of frequency. This, combined with the equation for the rate of incident energy flow per unit area, can be used to calculate the effective cross-section for each electron in the high frequency limit. The result is

$$\sigma_{e}=\frac{8\pi}{3}\left(\frac{1}{4\pi\epsilon_{0}}\frac{\mathrm{e}^{2}}{\mathrm{mc}^{2}}\right)^{2}=\frac{8\pi}{3}\mathrm{r}_{\mathrm{e}}^{2}, \tag{8.5.8}$$

where the length re = 2.81×10^{-15} meters is called the classical radius of the electron. The frequency independent area (8.5.8) is called the **Thompson cross-section**; it has the numerical value $\sigma_e = 66.2\times10^{-30}$ meters^2 . In the high frequency limit where each electron in the atom scatters independently, the total cross-section is proportional to the total number of electrons in the atom, i.e. it is proportional to the atomic number.

Of course, in general, atoms that scatter light are not stationary; they are moving in a random direction with a speed, V, that is related to the mean thermal energy. This motion results in a Doppler frequency shift that is proportional to the ratio V/c, to first order. The problem of scattering of radiation by a moving atom can be treated by transforming from the laboratory frame to a frame in which the atom is at rest- the rest frame. After having calculated the intensity and distribution of the scattered light one can then perform an inverse transformation back to the laboratory frame. Thermal velocities of atoms at room temperatures are quite small. They are largest for hydrogen atoms, and even in that case the velocity corresponding to 300K is only 2.2×10^{3} meters/sec. Therefore Doppler frequency shifts are of the order of 1 part in 10^5 or smaller.

CHAPTER OVERVIEW

9: Plane Waves I

The use of phasors to describe the propagation of plane waves through space.

Thumbnail: The wavefronts of a plane wave traveling in 3-space. (Public Domain; Quibik via Wikipedia)

9.1: Introduction to Plane Waves

An electric dipole directed along z, located at the origin, and oscillating with the circular frequency ω produces electric and magnetic fields far from the origin that have the form (see equations (7.4.5)):

$$\begin{aligned} \mathrm{E}_{\theta} &= -\frac{\omega^2}{4\pi\epsilon_0}\frac{\mathrm{p}_0\sin\theta}{\mathrm{c}^2\mathrm{R}}\exp(-i\omega[\mathrm{t}-\mathrm{R}/\mathrm{c}]), \\ \mathrm{B}_{\phi} &= \mathrm{c}\mathrm{E}_{\theta} \\ \mathrm{H}_{\phi} &= -\frac{\omega^2}{4\pi}\frac{\mathrm{p}_0\sin\theta}{\mathrm{cR}}\exp(-i\omega[\mathrm{t}-\mathrm{R}/\mathrm{c}]) \end{aligned} \tag{9.1.1}$$

where $\mathrm{p}_{\mathrm{z}} = \mathrm{p}_0\exp(-i\omega[\mathrm{t}-\mathrm{R}/\mathrm{c}])$, and t is the time at which the observer at $\vec{R}$ measures the fields. It must always be kept in mind that the fields are represented by real numbers; the notation of complex numbers is simply a convenient book-keeping device for dealing with sinusoidal functions. The notation $\exp(-i\omega\mathrm{t})$ “the real part of $\exp(-i\omega\mathrm{t})$” i.e. $\cos(\omega\mathrm{t})$. It is particularly important to remember this when calculating the Poynting vector or the energy densities which involve the product of two field amplitudes. For example, the Poynting vector corresponding to the fields of Equations (9.1.1) is given by

$$\mathrm{S}_{\mathrm{r}} = \mathrm{E}_{\theta}\mathrm{H}_{\phi} = \frac{1}{4\pi\epsilon_0}\frac{\omega^4}{4\pi}\frac{\mathrm{p}_0^2\sin^2\theta}{\mathrm{c}^3\mathrm{R}^2}\cos^2(\omega[\mathrm{t}-\mathrm{R}/\mathrm{c}]) \tag{9.1.2}$$

Note that the time factor is not the same as

$$\text{Real}\left(\exp(-2i\omega[\mathrm{t}-\mathrm{R}/\mathrm{c}])\right) = \cos(2\omega[\mathrm{t}-\mathrm{R}/\mathrm{c}]). \tag{9.1.3}$$

The time average of Equation (9.1.3) is zero, whereas the time average of the correct expression, Equation (9.1.2), is given by

$$< \mathrm{S}_{\mathrm{r}} >= \left(\frac{1}{8\pi}\right)\left(\frac{\mathrm{c}}{4\pi\epsilon_0}\right)\left(\frac{\omega}{\mathrm{c}}\right)^4\frac{\mathrm{p}_0^2\sin^2\theta}{\mathrm{R}^2}, \tag{9.1.4}$$

since the time average of the cosine squared function is 1/2. At distances far removed from the dipole radiator the surface of constant R can be approximated locally by a plane perpendicular to $\hat{\mathrm{u}}_{\mathrm{r}}$, a unit vector parallel with $\vec{R}$. This suggests that Maxwell’s equations ought to have plane wave solutions of the form

$$\begin{aligned} \vec{\mathrm{E}}(\vec{\mathrm{r}},\mathrm{t}) &= \vec{\mathrm{E}}_0\exp(i[\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}-\omega\mathrm{t}]), \\ \vec{\mathrm{B}}(\vec{\mathrm{r}},\mathrm{t}) &= \vec{\mathrm{B}}_0\exp(i[\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}-\omega\mathrm{t}]), \end{aligned} \tag{9.1.5}$$

where $\vec{k}$ is a vector whose magnitude is ω/c and whose direction lies along the direction of propagation of the wave, and where $\vec{\mathrm{E}}_0$ and $\vec{\mathrm{B}}_0$ are constant vectors that are perpendicular to each other and to the wave-vector $\vec{k}$ (see Figure (9.1.1)).

Equations (9.1.5) can be written in component form using some convenient co-ordinate system, and using $\text{Real}(\exp(i[\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}-\omega\mathrm{t}])) = \cos(\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}-\omega\mathrm{t})$:

$$\begin{aligned} \mathrm{E}_{\mathrm{x}} &= \mathrm{E}_{0x}\cos(\mathrm{k}_{\mathrm{x}}\mathrm{x}+\mathrm{k}_{\mathrm{y}}\mathrm{y}+\mathrm{k}_{\mathrm{z}}\mathrm{z}-\omega\mathrm{t}), \\ \mathrm{E}_{\mathrm{y}} &= \mathrm{E}_{0y}\cos(\mathrm{k}_{\mathrm{x}}\mathrm{x}+\mathrm{k}_{y}\mathrm{y}+\mathrm{k}_{\mathrm{z}}\mathrm{z}-\omega\mathrm{t}), \\ \mathrm{E}_{\mathrm{z}} &= \mathrm{E}_{0z}\cos(\mathrm{k}_{\mathrm{x}}\mathrm{x}+\mathrm{k}_{y}\mathrm{y}+\mathrm{k}_{\mathrm{z}}\mathrm{z}-\omega\mathrm{t}), \\ \mathrm{B}_{\mathrm{x}} &= \mathrm{B}_{0x}\cos(\mathrm{k}_{\mathrm{x}}\mathrm{x}+\mathrm{k}_{\mathrm{y}}\mathrm{y}+\mathrm{k}_{\mathrm{z}}\mathrm{z}-\omega\mathrm{t}), \\ \mathrm{B}_{\mathrm{y}} &= \mathrm{B}_{0y}\cos(\mathrm{k}_{\mathrm{x}}\mathrm{x}+\mathrm{k}_{\mathrm{y}}\mathrm{y}+\mathrm{k}_{\mathrm{z}}\mathrm{z}-\omega\mathrm{t}), \\ \mathrm{B}_{\mathrm{z}} &= \mathrm{B}_{0z}\cos(\mathrm{k}_{\mathrm{x}}\mathrm{x}+\mathrm{k}_{y}\mathrm{y}+\mathrm{k}_{\mathrm{z}}\mathrm{z}-\omega\mathrm{t}), \end{aligned} \tag{9.1.6}$$

Using these expressions it is easy to show that

$$\mathrm{curl}(\vec{\mathrm{E}}) = -\left(\vec{\mathrm{k}}\times\vec{\mathrm{E}}_0\right)\sin(\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}-\omega\mathrm{t}),$$

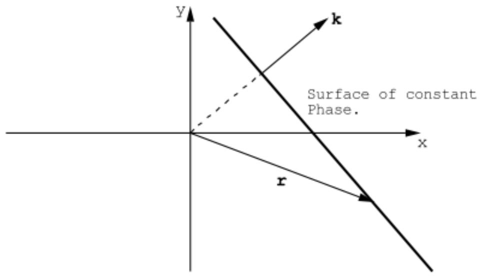


Figure 9.1.1: A plane wave propagating along the direction specified by $\vec{k}$ and for which $|\vec{\mathrm{k}}| = \mathrm{k} = \omega/\mathrm{c}$. For an electromagnetic plane wave in free space for which the fields $\vec{E}$ and $\vec{B}$ satisfy Maxwell's equations, both $\vec{E}$ and $\vec{B}$ lie in the surface of constant phase and are perpendicular to each other.

$$\begin{aligned} \operatorname{div}(\vec{\mathrm{E}}) &= -\left(\vec{\mathrm{k}} \cdot \vec{\mathrm{E}}_0\right) \sin(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}} - \omega \mathrm{t}), \\ \operatorname{curl}(\vec{\mathrm{B}}) &= -\left(\vec{\mathrm{k}} \times \vec{\mathrm{B}}_0\right) \sin(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}} - \omega \mathrm{t}), \\ \operatorname{div}(\vec{\mathrm{B}}) &= -\left(\vec{\mathrm{k}} \cdot \vec{\mathrm{B}}_0\right) \sin(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}} - \omega \mathrm{t}), \end{aligned}$$

In free space Maxwell's equations become

$$\begin{aligned} \operatorname{curl}(\overrightarrow{\mathrm{E}}) &= -\frac{\partial \overrightarrow{\mathrm{B}}}{\partial \mathrm{t}}, \\ \operatorname{curl}(\overrightarrow{\mathrm{B}}) &= \epsilon_0 \mu_0 \frac{\partial \overrightarrow{\mathrm{E}}}{\partial \mathrm{t}}, \\ \operatorname{div}(\overrightarrow{\mathrm{E}}) &= 0, \\ \operatorname{div}(\overrightarrow{\mathrm{B}}) &= 0. \end{aligned} \tag{9.1.7}$$

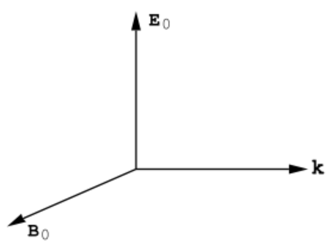


Figure 9.1.2: An electromagnetic plane wave propagating in free space. The electric field vector, $\vec{\mathrm{E}}(\vec{\mathrm{r}}, \mathrm{t}) = \vec{\mathrm{E}}_0 \cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}} - \omega \mathrm{t})$, and the magnetic field vector, $\vec{\mathrm{B}}(\vec{r}, t) = \vec{\mathrm{B}}_0 \cos(\vec{\mathrm{k}} \cdot \vec{r} - \omega t)$, along with the propagation vector, $\vec{k}$, form a right handed orthogonal triad.

Substitution of Equations (9.1.6) into Maxwell's Equations (9.1.7) gives

$$\begin{aligned} \vec{\mathrm{k}} \times \vec{\mathrm{E}}_0 &= \omega \vec{\mathrm{B}}_0, \\ \vec{\mathrm{k}} \times \vec{\mathrm{B}}_0 &= -\epsilon_0 \mu_0 \omega \vec{\mathrm{E}}_0, \\ \vec{\mathrm{k}} \cdot \vec{\mathrm{E}}_0 &= 0, \\ \vec{\mathrm{k}} \cdot \vec{\mathrm{B}}_0 &= 0. \end{aligned} \tag{9.1.8}$$

The last two equations state that for plane wave solutions of Maxwell's equations in free space both the electric and magnetic field vectors must be perpendicular to the direction of propagation specified by the vector $\vec{k}$; i.e. $\vec{\mathrm{E}}_0$ and $\vec{\mathrm{B}}_0$ must be parallel with the surfaces of constant phase. The first two equations of (9.1.8) state that the fields $\vec{\mathrm{E}}_0$ and $\vec{\mathrm{B}}_0$ must be mutually perpendicular; thus the three vectors $\vec{\mathrm{E}}_0$, $\vec{\mathrm{B}}_0$, and $\vec{k}$ form an orthogonal right handed triad. In order to satisfy Maxwell's equations the magnitude of the wave-vector must be given by

$$k^2=\epsilon_0\mu_0\omega^2=\left(\frac{\omega}{c}\right)^2, \tag{9.1.9}$$

and the amplitudes of the electric and magnetic fields must be related by

$$\left|\vec{E}_0\right|=c\left|\vec{B}_0\right|,$$

see Figure (9.1.2). Notice that E and B oscillate in phase: ie. they have exactly the same sinusoidal dependence on space and on time. These relations are the same as those which were earlier associated with the wave produced by an oscillating dipole, Equations (7.4.5).

In free space the displacement vector, $\vec{D}$, is related to the electric field by $\vec{D}=\epsilon_0\vec{E}$ so that the time rate of change of the energy density stored in the electric field, Equation (8.2.6), becomes

$$\frac{\partial W_E}{\partial t}=\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\epsilon_0 E^2}{2}\right). \tag{9.1.10}$$

Using (9.1.10), the energy density stored in the electric field of a plane wave is given by

$$W_E=\frac{\epsilon_0 E_0^2}{2}\cos^2(\vec{k}\cdot\vec{r}-\omega t),\quad \text{Joules}/m^3,$$

This energy density oscillates in both space and time, in particular at a fixed point in space the energy density periodically vanishes. However, the average energy density measured at any point in space is independent of both position and time:

$$<W_E>=\frac{\epsilon_0}{4}E_0^2,\quad \text{Joules}/m^3, \tag{9.1.11}$$

Similarly, the time rate of change of the energy density stored in the magnetic field is given by (8.7)

$$\frac{\partial W_B}{\partial t}=\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right). \tag{9.1.12}$$

Therefore one can write

$$W_B=\frac{B^2}{2\mu_0}=\frac{B_0^2}{2\mu_0}\cos^2(\vec{k}\cdot\vec{r}-\omega t)\quad \text{Joules}/m^3.$$

The time averaged energy density stored in the magnetic field is independent of position and since $B=E/c$ is given by

$$<W_B>=\frac{B_0^2}{4\mu_0}=\frac{E_0^2}{4\mu_0c^2}=\frac{\epsilon_0E_0^2}{4}=<W_E>\quad \text{Joules}/m^3. \tag{9.1.13}$$

The average energy density stored in the magnetic field is exactly the same, in free space, as the average energy density stored in the electric field. The total time averaged energy density stored in the electromagnetic field is

$$<W>=<W_E>+<W_B>=\frac{\epsilon_0E_0^2}{2},\quad \text{Joules}/m^3. \tag{9.1.14}$$

The average rate at which energy in the electromagnetic field is transported across a unit area normal to the direction of propagation, i.e. normal to $\vec{k}$, can be obtained by multiplying Equation (9.1.14) by the speed of light: this rate is also just the time average of the Poynting vector

$$<S>=c\frac{\epsilon_0E_0^2}{2}=\frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E_0^2,\quad \text{Watts}/m^2. \tag{9.1.15}$$

The quantity $Z_0=\sqrt{\mu_0/\epsilon_0}$ has the units of a resistance; it is called the impedance of free space, and $Z_0=377$ Ohms. From the equations for the space and time variation of a plane wave, Equations (9.1.6), it follows that for a fixed time the electric and magnetic fields vary in space with a period along the direction of $veck$ given by $2\pi/|\vec{k}|$. By definition, this spatial period is the wavelength, λ, therefore $|\vec{k}|=2\pi/\lambda$. Similarly, at a fixed position in space the fields oscillate in time with the period $2\pi/\omega$; by definition, this period, T, is the inverse of the frequency,f, therefore $\omega = 2\pi \mathrm{f}$. In order to satisfy Maxwell's equations, the frequency and wavelength of a plane wave are related by Equation (9.1.9)

$$\omega = \mathrm{c}|\vec{\mathrm{k}}|;$$

this can be written in the more familiar form $\mathrm{f}\lambda = \mathrm{c}$.

9.2: Phasors

It is very convenient to represent sinusoidal functions i.e. sines and cosines, by complex exponential functions when dealing with linear differential equations such as Maxwell's equations. For example

$$y = A\exp[i(kx - \omega t)]$$

means

$$y = \text{Real Part}(A\exp[i(kx - \omega t)]) = A\cos(kx - \omega t)$$

if A is a real number, or if A= a+ib is a complex number

$$\begin{aligned} y &= \text{Real Part}((a+ib)\exp[i(kx - \omega t)]), \\ &= a\cos(kx - \omega t) - b\sin(kx - \omega t). \end{aligned} \tag{9.2.1}$$

A complex amplitude represents a phase shift. Since

$$\cos(\alpha + \beta) = \cos\beta\cos\alpha - \sin\beta\sin\alpha, \tag{9.2.2}$$

Equation (9.2.1) can be written

$$y = \sqrt{a^2 + b^2}\cos(kx - \omega t + \beta),$$

where

$$\sin\beta = \frac{b}{\sqrt{a^2+b^2}}$$

and

$$\cos\beta = \frac{a}{\sqrt{a^2+b^2}}, \tag{9.2.3}$$

or

$$\tan\beta = \frac{b}{a}. \tag{9.2.4}$$

In phasor notation

$$y = \sqrt{a^2+b^2}\exp[i(kx - \omega t + \beta)] = (\sqrt{a^2+b^2}\exp i\beta)\exp[i(kx - \omega t)].$$

The prefactor $(\sqrt{a^2+b^2}\exp i\beta)$ is just the polar representation of the complex number (a+ib).

Derivatives are particularly convenient in the complex phasor notation because the derivative of an exponential function gives back the same exponential function multiplied by a constant (usually a complex number).

One must be careful when calculating energy densities or when calculating the Poynting vector using the phasor notation because the Real Part of the product of two complex exponentials is not the same as the product of the two Real sinusoidal functions that appear in the product. There is, however, a trick which is useful. Consider a plane wave propagating along z and which can be described by

$$\begin{aligned} E_x &= E_0 e^{i(kz - \omega t + \phi_1)} \\ H_y &= H_0 e^{i(kz - \omega t + \phi_2)} \end{aligned} \tag{9.2.5}$$

These electric and magnetic fields are not in phase because ϕ_1 and ϕ_2 are different, and therefore this plane wave is not propagating in free space. It corresponds to a wave propagating in a medium characterized by a complex dielectric constant as will be discussed in a later chapter. Now calculate the time average of the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, using Equations (9.2.5). It is asserted that **the time average of the product of two phasors can be obtained as one-half of the real part of the product of one phasor with the complex conjugate of the other phasor.**

Thus

$$< S_z >= \frac{1}{2}\mathrm{Real}\left(E_x H_y^*\right) = \frac{1}{2}\mathrm{Real}(E_x^* H_y), \tag{9.2.6}$$

where E_x^* means the complex conjugate of E_x, and H_y^* means the complex conjugate of H_y. Using Equation (9.2.5) in Equation (9.2.6) one obtains

$$< S_z >= \frac{1}{2}\mathrm{Real}(E_0 H_0 \exp i\,(\phi_1 - \phi_2)) = \frac{E_0 H_0}{2}\cos(\phi_1 - \phi_2), \tag{9.2.7}$$

since E_0, H_0 are taken to be real amplitudes. Eqn.(9.2.6) can be checked by writing the fields (9.2.5) in real form:

$$S_z = E_0 H_0 \cos(kz - \omega t + \phi_1)\cos(kz - \omega t + \phi_2),$$

or, using Equation (9.2.2),

$$\begin{aligned} S_z = E_0 H_0\, &(\cos\phi_1 \cos(kz - \omega t) - \sin\phi_1 \sin(kz - \omega t)) \\ &\times (\cos\phi_2 \cos(kz - \omega t) - \sin\phi_2 \sin(kz - \omega t))\,, \end{aligned}$$

or upon an explicit multiplication

$$\begin{aligned} S_z =\ & E_0 H_0 \left(\cos\phi_1 \cos\phi_2 \cos^2(kz - \omega t) - \cos\phi_2 \sin\phi_1 \sin(kz - \omega t)\cos(kz - \omega t)\right) \\ & - E_0 H_0 \left(\sin\phi_2 \cos\phi_1 \sin(kz - \omega t)\cos(kz - \omega t) - \sin\phi_1 \sin\phi_2 \sin^2(kz - \omega t)\right) \end{aligned}$$

Upon taking the time averages one obtains

$$< S_z >= \frac{E_0 H_0}{2}(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)\,.$$

This equation can be written compactly as

$$< S_z >= \frac{E_0 H_0}{2}\cos(\phi_1 - \phi_2),$$

in agreement with the result Equation (9.2.7) obtained using the prescription (9.2.6).

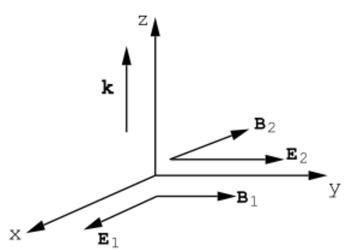


Figure 9.2.3: Two coherent plane waves having orthogonal polarizations, and propagating along the z-direction. Each wave is characterized by the same circular frequency, ω, and the same wave-vector, $\vec{k}$, where $k_z = |\vec{k}| = \omega/c$. Let the fields in wave number (2) be shifted in phase by ϕ radians relative to the fields in wave number (1).

9.3: Elliptically Polarized Plane Waves

It may happen that two plane waves corresponding to the same frequency are propagating in the same direction, but they may have electric fields that are oriented in different directions and which may be shifted in phase relative to one another. For example, consider the plane waves of circular frequency ω and propagating along z as shown in Figure (9.2.3).

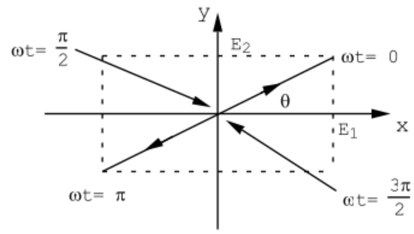


Figure 9.3.4: The sum of two orthogonally polarized plane waves that have the same frequency and wave-vector. Case(a). ϕ = 0, the two electric fields are in phase: $E_x = E_1 \cos \omega t$ and $E_y = E_2 \cos \omega t$.

Let wave no.1 be polarized with its electric vector along the x-axis;

$$\mathrm{E_x} = \mathrm{E_1} \exp[i(\mathrm{kz} - \omega \mathrm{t})]$$

and

$$\mathrm{B_y} = \frac{\mathrm{E_1}}{\mathrm{c}} \exp[i(\mathrm{kz} - \omega \mathrm{t})].$$

Let wave no.2 be polarized with its electric vector along the y-axis:

$$\mathrm{E_y} = \mathrm{E_2} \exp[i(\mathrm{kz} - \omega \mathrm{t} + \phi)]$$

and

$$\mathrm{B_x} = -\frac{\mathrm{E_2}}{\mathrm{c}} \exp[i(\mathrm{kz} - \omega \mathrm{t} + \phi)].$$

Note that the fields in wave number(2) are shifted in phase by ϕ radians relative to the fields in wave number(1). Now make a diagram that displays the time variation of the total electric field at a fixed point in space; for simplicity, take z=0. There are a number of interesting cases:

- Case(a). ϕ = 0. The two electric fields are in phase. This is an ordinary plane wave in which the electric vector is oriented at an angle with respect to the co-ordinate axes, Figure (9.3.4).
- Case(b). $\phi = \pi/2$. The two electric fields are in quadrature i.e. they are 90° out of phase. The tip of the electric vector traces out an elliptical pattern as a function of time, Figure (9.3.5). The sense of rotation of the electric vector is such that a nut on a right handed screw thread would advance along the +z axis; this radiation is said to be right hand elliptically polarized. For the special case in which $E_1 = E_2$ the tip of the electric vector traces out a circle; such radiation is said to be right hand circularly polarized.
- Case(c). $\phi = 3\pi/2$. In this case the electric fields are in quadrature, as for Case(b), but the sense of rotation of the electric vector is in the opposite direction, fig(9.3.6). This radiation is said to be left hand elliptically polarized. When $E_1 = E_2$ the radiation is left hand circularly polarized.
- Case(d). $\phi = \pi/4$. The phase shift in this case is equal to 45° and is less

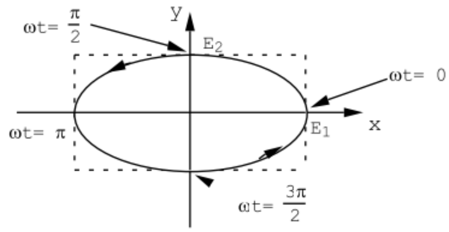


Figure 9.3.5: The sum of two orthogonally polarized plane waves that have the same frequency and wave-vector. Case(b). $\phi = \pi/2$, the two electric fields are out of phase by $\pi/2$ radians: $E_x = E_1 \cos \omega t$ and $E_y = E_2 \sin \omega t$.

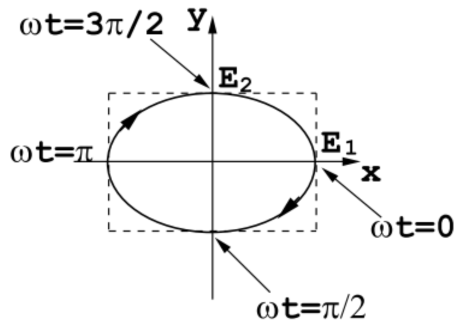


Figure 9.3.6: The sum of two orthogonally polarized plane waves that have the same frequency and wave-vector. Case(c). ϕ = 3 π/2, the two electric fields are out of phase by 3π/2 radians: $E_x = E_1 \cos \omega t$ and $E_y = -E_2 \sin \omega t$.

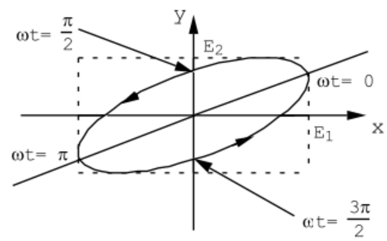


Figure 9.3.7: The sum of two orthogonally polarized plane waves that have the same frequency and wave-vector. Case(d). ϕ = π/4, the two electric fields are out of phase by π/4 radians: $E_x = E_1 \cos \omega t$ and $E_y = \left(E_2/\sqrt{2}\right)(\cos \omega t + \sin \omega t)$.

than 90° , Figure (9.3.7). The electric fields are given by

$$E_x = E_1 \cos \omega t,$$
$$E_y = \frac{E_2}{\sqrt{2}}(\cos \omega t + \sin \omega t).$$

The tip of the electric vector traces out an elliptical path as time goes on. The sense of the rotation is such that a nut on a right handed screw would advance along the positive z-axis; the radiation is right hand elliptically polarized. However, the principle axes of the ellipse are not parallel with the x,y co-ordinate axes. As the phase angle between the two electric vectors, ϕ, increases from zero, the ellipticity of the radiation increases and the principle axes of the ellipse rotate until they coincide with the co-ordinate axes at ϕ = π/2. Upon further increase in phase angle, the ellipticity decreases until the radiation becomes linearly polarized again for ϕ = π: the plane of polarization is rotated 90° relative to the plane of polarization illustrated in Figure (9.3.4). Further increases in the phase angle produces left hand elliptically polarized radiation.

The production of elliptically polarized radiation requires the superposition of two plane waves whose frequencies are identical, whose phases are correlated, and whose electric vectors are not co-linear. Such radiation is produced only by special sources. Visible radiation from a hot filament or from a hot plasma is usually unpolarized. The light emitted from such a source consists of a superposition of pulses each of which is quite short on a human time scale, $\sim 10^{-8}$ secs., but quite long compared with the period of the radiation, $\sim 10^{-14}$ to 10^{-15} secs. Each pulse is emitted from an atomic dipole oscillator that has been set into motion by thermal agitation. The pulses from the various atoms are uncorrelated in phase; moreover, the dipole moments on the individual atoms are oriented at random and so the orientation of the electric vector of the emitted light is also oriented at random.

Unpolarized light consists of a collection of many pulses in which the orientation of the electric vector from pulse to pulse is random. A polaroid filter can be used to produce linearly polarized light from such an ensemble of randomly polarized pulses. It works because a polaroid filter preferentially absorbs light whose electric vector is parallel with a particular direction, i.e. the polaroid material exhibits anisotropic absorption. The light that gets through the filter consists of those pulses for which the electric vector is mainly oriented along the poorly absorbing axis of the crystals which make up a polaroid filter.

Visible radiation emitted from a gas laser source is usually plane polarized and coherent because the dipole moments on the radiating atoms in the laser plasma tube are parallel to one another and are locked in phase by the standing optical wave in the laser cavity. The whole ensemble of radiating atoms behaves like one enormous extended dipole source. The particular orientation of the electric field is determined by the Brewster windows that are used on the ends of the laser plasma tube. The optical gain provided by the laser plasma tube depends upon the orientation of the Brewster windows.

There exists a class of anisotropic materials such that the velocity of radiation depends upon the orientation of the electric vector relative to crystalline axes. Suitable thicknesses of such crystals can be used to introduce a controlled phase shift between two orthogonal components of the electric field. In that way it is possible to convert linearly polarized light to elliptically polarized light, and vice versa.

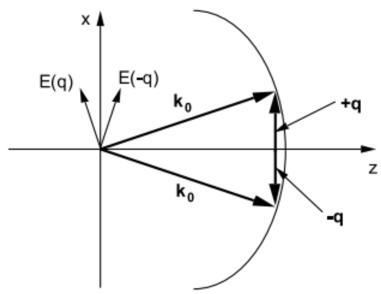


Figure 9.3.8: The construction of a beam of radiation having a finite size in the direction perpendicular to the direction of the beam propagation. Such a beam can be constructed from the superposition of a large number of plane waves that are propagating at small angles to the direction of the beam.

9.4: Gaussian Light Beams

It is impossible to generate an unbounded plane wave, of course. Nevertheless, the concept of unbounded plane waves is a very useful one because a finite beam of radiation can be described as the superposition of plane waves having different amplitudes and phases and propagating in slightly different directions, see Figure (9.3.8). To simplify matters let us assume that the amplitude function, A(p,q), is symmetric in p,q: ie. A(-p,-q)= A(p,q). This simplification allows one to construct a beam in which the electric field is polarized along a particular direction in the plane- along the x-direction, say. Eqn.(9.4.1) illustrates how such a beam could be constructed:

$$E_x(x,y,z,t)=\int\int_{-\infty}^{\infty} dpdq\, A(p,q)\exp(i[px+qy+kz-\omega t]), \tag{9.4.1}$$

where

$$p^2+q^2+k^2=\left(\frac{\omega}{c}\right)^2=k_0^2,$$

Eqn.(9.4.1) is an example of a **Fourier Integral**. The amplitude function A(p,q) can be chosen to give the required beam profile in the x-y plane at some plane z=constant; it is convenient to choose this plane to be at z=0. The beam profile at any other position z can be obtained using the integral (9.4.1). As an example of how this works let us treat a specific case for which the mathematics can be easily worked out. Suppose that at z=0 the beam cross-section can be described as a plane wave whose amplitude falls off exponentially along x and y:

$$E_x(x,y,0)=E_0\exp\left(\frac{-\left(x^2+y^2\right)}{w_0^2}\right). \tag{9.4.2}$$

A time dependence exp $(-i\omega t)$ is assumed, but this factor will be suppressed in the following. The output beam from a typical gas laser, a He-Ne laser for example, exhibits the spatial variation (9.4.2) at the output mirror with w_0 approximately equal to 1 mm. Such a beam profile is called a Gaussian beam profile. The spatial Fourier integral in (9.4.1) can be inverted for z=0 to obtain

$$A(p,q)=\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} dxdy\, E_x(x,y,0)\exp(-i[px+qy]). \tag{9.4.3}$$

Using the Gaussian spatial variation of Equation (9.4.2) one finds

$$A(p,q)=\frac{E_0 w_0^2}{4\pi}\exp\left(-\frac{w_0^2}{4}\left(p^2+q^2\right)\right). \tag{9.4.4}$$

The Fourier transform of a Gaussian function is another Gaussian function: see section(9.4.1).

Notice that the amplitude function (9.4.4) becomes very small if p^2 or q^2 is greater than $4/w_0^2$: : this means that the waves in the bundle describing the radiation beam that have transverse components p,q much larger than $\pm 2/w_0$ can be neglected. In a typical case the laser beam radius is ∼1 mm so that the amplitude A(p,q) becomes small for | p |, | q | larger than 2×10^3 m^{-1} . But at optical frequencies $\lambda\sim(1/2)\times 10^{-6}$ m so that $k_0\sim 2\pi/\lambda\sim 4\pi\times 10^6$ m^{-1} . Thus the important values of the transverse components p,q of the plane waves that make up the beam are very small compared with the total wavevector k_0. The longitudinal component of the wave-vector, the z-component k, is given by

$$k^2=\left(\frac{\omega}{c}\right)^2-p^2-q^2=k_0^2-p^2-q^2.$$

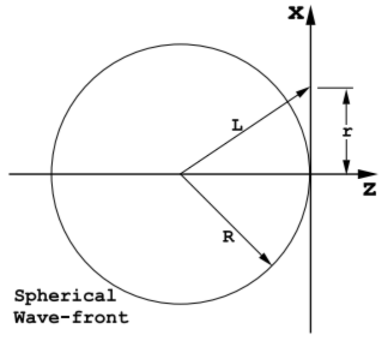


Figure 9.4.9: The variation of the phase across the x-y plane for a spherical wave is described by $\exp\left(\frac{ik_0}{2R}\left(x^2+y^2\right)\right)$. The phase increases with distance, r, in the plane because the distance from the center of curvature to an off-axis point on the plane is larger than the radius of curvature R.

But $\left(p^2+q^2\right)/k_0^2$ is much less than unity so that one can write

$$k \approx k_0 - \frac{\left(p^2+q^2\right)}{2k_0}. \tag{9.4.5}$$

Now using the approximation Equation (9.4.5) in Equation (9.4.1) investigate the beam profile at some arbitrary value of z:

$$E_x(x,y,z) = \frac{E_0 w_0^2}{4\pi} \int\int_{-\infty}^{\infty} dpdq \exp\left(-\frac{w_0^2}{4}\left[p^2+q^2\right]\right). \tag{9.4.6}$$
$$\cdot \exp(i[px+qy]) \exp\left(-i\frac{\left[p^2+q^2\right]z}{2k_0}\right) \exp(ik_0 z)$$

The integrals in Equation (9.4.6) can be evaluated to obtain

$$E_x(x,y,z) = \frac{E_0}{\sqrt{1+(z/z_R)^2}} \exp\left(\frac{ik_0}{2\tilde{q}}\left(x^2+y^2\right)\right) \exp(i\left[k_0 z-\psi\right]), \tag{9.4.7}$$

where

$$\tilde{q} = z - iz_R, \tag{9.4.8}$$

$$z_R = \frac{k_0 w_0^2}{2} = \frac{\pi w_0^2}{\lambda}, \tag{9.4.9}$$

and

$$\tan(\psi) = z/z_R. \tag{9.4.10}$$

See Section(9.4.2) for the details of the calculation. The variable $\tilde{q}$ is called the complex radius of curvature of the beam. This nomenclature stems from the description of a spherical wave-front, Figure (9.4.9) as will be explained in the next paragraph. The length z_R is called the Rayleigh range.

A spherical wave-front exhibits a phase variation across a plane perpendicular to the direction of propagation given by

$$\exp\left(\frac{ik_0}{2R}\left(x^2+y^2\right)\right),$$

where R is the radius of curvature. A comparison of this expression with Equation (9.4.7) shows why $\tilde{q}$ is called the complex radius of curvature. One can separate the reciprocal of the complex radius of curvature into its real and imaginary parts:

$$\frac{1}{\tilde{q}} = \frac{1}{z-z_R} = \frac{z+iz_R}{z^2+z_R^2}. \tag{9.4.11}$$

The real part of Equation (9.4.11) gives the real radius of curvature of the wave-front:

$$\frac{1}{R}=\frac{z}{z^2+z_R^2},$$

or

$$\mathrm{R}=\mathrm{z}+\frac{\mathrm{z}_\mathrm{R}^2}{\mathrm{z}}. \tag{9.4.12}$$

The radius of curvature is infinite at z=0 corresponding to a plane wave-front. For $z \gg z_R$ the radius of curvature approaches the distance z.

When Equation (9.4.11) is introduced into the expression for the electric field, Equation (9.4.7), the imaginary part of $1/\tilde{q}$ gives rise to a Gaussian spatial variation

$$\exp\left(\frac{-\mathrm{k}_0\mathrm{z}_\mathrm{R}\left(\mathrm{x}^2+\mathrm{y}^2\right)}{2\left(\mathrm{z}^2+\mathrm{z}_\mathrm{R}^2\right)}\right)=\exp\left(-\frac{\left(\mathrm{x}^2+\mathrm{y}^2\right)}{\mathrm{w}^2}\right), \tag{9.4.13}$$

where

$$\mathrm{w}^2=\mathrm{w}_0^2\left[1+\left(\frac{\mathrm{z}}{\mathrm{z}_\mathrm{R}}\right)^2\right]. \tag{9.4.14}$$

This means that as one moves along the beam the radius of the beam slowly increases and becomes greater by $\sqrt{2}$ at $z = z_R$: ie. at one Rayleigh range removed from the minimum beam radius, or beam waist.

The beam radius at the output mirror, the position of the minimum beam radius, is usually $\mathrm{w}_0 \cong 1\mathrm{mm}$ for a typical gas laser operating in the visible. For a wavelength of $\lambda = 5 \times 10^{-7}$ meters the Rayleigh range for such a laser is z_R= 6.28 meters. Therefore the beam diameter will have expanded by only $\sqrt{2}$ = 1.41 at a distance of 6.28 meters from the laser output mirror.

Interested readers can learn more about Gaussian beams and Gaussian beam optics in the book ”An Introduction to Lasers and Masers” by A.E. Siegman, McGraw-Hill, New York, 1971; chapter 8.

9.4.1 The Fourier Transform of a Gaussian.

From Equations (9.4.2) and (9.4.3) one has

$$\mathrm{A}(\mathrm{p},\mathrm{q})=\frac{\mathrm{E}_0}{4\pi^2}\int\int_{-\infty}^{\infty}\mathrm{dxdy}\exp\left(-\left[\frac{\mathrm{x}^2}{\mathrm{w}_0^2}+i\mathrm{px}\right]\right)\exp\left(-\left[\frac{\mathrm{y}^2}{\mathrm{w}_0^2}+i\mathrm{qy}\right]\right). \tag{9.4.15}$$

These integrals separate into the product of two integrals having an identical form

$$\mathrm{I}=\int_{-\infty}^{\infty}\mathrm{dx}\exp\left(-\left[\frac{\mathrm{x}^2}{\mathrm{w}_0^2}+i\mathrm{px}\right]\right). \tag{9.4.16}$$

It is useful to complete the square in the exponent of (9.4.16) in order to proceed:

$$\frac{1}{w_0^2}\left[x^2+ipw_0^2x\right]=\frac{1}{w_0^2}\left[x+\frac{ipw_0^2}{2}\right]^2+\frac{p^2w_0^2}{4}. \tag{9.4.17}$$

Eqn.(9.4.16) can now be re-written in terms of a new variable

$$\mathrm{u}=\mathrm{x}+i\frac{\mathrm{pw}_0^2}{2},$$

and

$$\mathrm{du}=\mathrm{dx}.$$

Thus the integral I, Equation (9.4.16), becomes

$$I=\exp\left(-\frac{p^2 w_0^2}{4}\right)\int_{-\infty}^{\infty} du \exp\left(-\frac{u^2}{w_0^2}\right)=w_0\sqrt{\pi}\exp\left(-\frac{p^2 w_0^2}{4}\right). \tag{9.4.18}$$

Using this result the amplitude function, Equation (9.4.15), becomes Equation (9.4.4)

$$A(p,q)=\frac{E_0}{4\pi}w_0^2\exp\left(-\frac{w_0^2}{4}\left[p^2+q^2\right]\right).$$

9.4.2 Integrals that are Required in the Fourier Transform, Equation (9.26).

The integrals required to calculate the Fourier transform of the electric field in Equation (9.4.6) have the form

$$I=\int_{-\infty}^{\infty} dp \exp\left(-\frac{w_0^2 p^2}{4}+ipx-i\frac{p^2 z}{2k_0}\right). \tag{9.4.19}$$

The exponent in the exponential function can be written in the form

$$\text{Exponent}=-\frac{w_0^2}{4}\left(p^2-\frac{4ipx}{w_0^2}+\frac{2ip^2 z}{w_0^2 k_0}\right),$$

or

$$\text{Exponent}=-\frac{w_0^2}{4}\left(1+\frac{2iz}{w_0^2 k_0}\right)\left[p^2-\frac{4ipx}{w_0^2\left(1+\frac{2iz}{w_0^2 k_0}\right)}\right]. \tag{9.4.20}$$

Upon completing the square in Equation (9.4.20) this becomes

$$\begin{aligned}\text{Exponent} &=-\frac{w_0^2}{4}\left(1+\frac{2iz}{w_0^2 k_0}\right)\\ &\cdot\left(\left[p-\frac{2ix}{\left(w_0^2+\frac{2iz}{k_0}\right)}\right]^2+\frac{4x^2}{\left(w_0^2+\frac{2iz}{k_0}\right)^2}\right),\end{aligned}$$

or

$$\begin{aligned}\text{Exponent}=-\left(\frac{w_0^2}{4}+\frac{iz}{2k_0}\right)\left[p-\frac{2ix}{\left(w_0^2+\frac{2ix}{k_0}\right)}\right]^2\\ -\frac{x^2}{\left(w_0^2+\frac{2iz}{k_0}\right)}\end{aligned} \tag{9.4.21}$$

Introduce the new variable

$$u=p-\frac{2ix}{\left(w_0^2+\frac{2iz}{k_0}\right)},$$

with

$$du=dp,$$

then

$$I=\exp\left(\frac{-x^2}{\left(w_0^2+\frac{2iz}{k_0}\right)}\right)\int_{-\infty}^{\infty} du \exp\left(-\left[\frac{w_0^2}{4}+\frac{iz}{2k_0}\right]u^2\right),$$

and carrying out the integration

$$I=\frac{\sqrt{\pi}}{\sqrt{\frac{w_0^2}{4}+\frac{iz}{2k_0}}}\exp\left(\frac{ik_0x^2}{2\left[z-ik_0w_0^2/2\right]}\right). \tag{9.4.22}$$

Using the above result, Equation (9.4.22), in Equation (9.4.6) for the electric field amplitude gives

$$E_x(x,y,z)=\frac{E_0w_0^2}{4\left(\frac{w_0^2}{4}+\frac{iz}{2k_0}\right)}\exp\left(\frac{ik_0\left[x^2+y^2\right]}{2\tilde{q}}\right)\exp(ik_0z), \tag{9.4.23}$$

where

$$\tilde{q}=z-i\frac{k_0w_0^2}{2}. \tag{9.4.24}$$

The quantity $\tilde{q}$ is the complex radius of curvature of the wave-front.

It is further useful to define a distance called the Rayleigh range, z_R:

$$z_R=\frac{k_0w_0^2}{2}=\frac{\pi w_0^2}{\lambda}. \tag{9.4.25}$$

At the waist of the beam the complex radius of curvature is purely imaginary

$$\tilde{q}_0=-iz_R.$$

The prefactor in Equation (9.4.23) can be written

$$\begin{aligned}\frac{E_0w_0^2}{4\left(\frac{w_0^2}{4}+\frac{iz}{2k_0}\right)}&=\frac{E_0}{\left(1+\frac{iz}{z_R}\right)}\\&=\frac{E_0\left[1-iz/z_R\right]}{\left(1+\left[z/z_R\right]^2\right)}\\&=\frac{E_0}{\sqrt{1+\left(z/z_R\right)^2}}\exp(-i\psi)\end{aligned}$$

where

$$\tan(\psi)=z/z_R.$$

Finally,

$$E_x(x,y,z)=\frac{E_0}{\sqrt{1+\left(z/z_R\right)^2}}\exp\left(\frac{ik_0}{2\tilde{q}}\left[x^2+y^2\right]\right)\exp(i\left[k_0z-\psi\right]), \tag{9.4.26}$$

and

$$\tilde{q}=z-iz_R,$$

with

$$z_R=\frac{\pi w_0^2}{\lambda}.$$

CHAPTER OVERVIEW

10: Plane Waves II

An investigation of the behaviour of plane waves incident on a plane interface between two media having different optical properties.

Thumbnail: The wavefronts of a plane wave traveling in 3-space. (Public Domain; Quibik via Wikipedia)

10.1: Normal Incidence

Consider a plane interface at z=0 that separates vacuum on the left (z < 0) from a half-space on the right containing an isotropic material, see Figure (10.1). It is assumed that the relation between $\vec{D}$ and $\vec{E}$ in this material linear, i.e. $\vec{\mathrm{D}}=\epsilon(\omega) \vec{\mathrm{E}}$, where the dielectric constant $\epsilon(\omega)$ depends upon the frequency, ω. The dielectric constant, ϵ, can be represented by a complex number meaning that there is a phase shift between the vectors $\vec{D}$ and $\vec{E}$. It is often useful to write $\epsilon(\omega)=\epsilon_{\mathrm{r}}(\omega) \epsilon_{0}$ where ϵ_r is the relative dielectric constant. The relative dielectric constant, ϵ_r, is a dimensionless, complex number.

Let the material in the right half-space be non-magnetic so that its permeability can be taken to be the same as the permeability of free space, μ_0. A plane wave of the form

$$\mathrm{E}_{\mathrm{x}}(\mathrm{z}, \mathrm{t})=\mathrm{E}_{0} \exp (i[\mathrm{kz}-\omega \mathrm{t}]) \tag{10.1.1}$$

falls upon the interface. A disturbance will be set up in the material to the right of the boundary and we may reasonably suppose that it will also have the form of a plane wave;

$$\mathrm{E}_{\mathrm{x}}(\mathrm{z}, \mathrm{t})=\mathrm{A} \exp \left(i\left[\mathrm{k}_{\mathrm{m}} \mathrm{z}-\omega \mathrm{t}\right]\right). \tag{10.1.2}$$

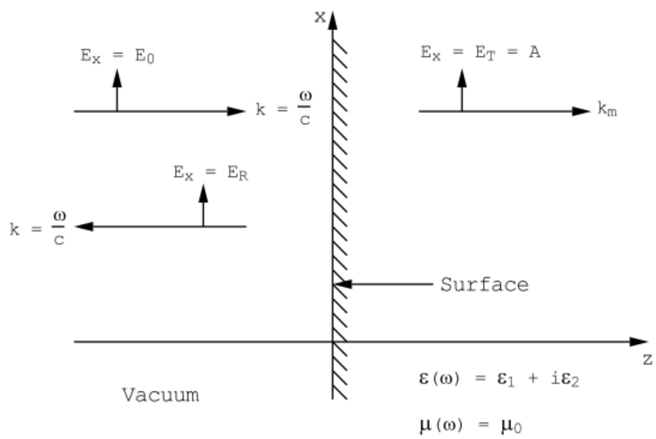


Figure 10.1.1: A plane wave, $\mathrm{E}_{\mathrm{x}}=\mathrm{E}_{0} \exp (i[\mathrm{kz}-\omega \mathrm{t}])$ is incident from vacuum on a material characterized by a dielectric constant ϵ at the circular frequency ω. The wave falls upon the surface at normal incidence. The amplitude of the reflected wave is E_R, and the amplitude of the transmitted wave is E_T.

The plane wave propagating in the material (z > 0) must have the same frequency as the incident wave because the response of the material is driven by the incident electric field at the circular frequency ω. However, its wavevector need not be the same as for free space; it must be chosen so as to satisfy Maxwell's equations. The amplitude of the wave in the material must be chosen so as to satisfy boundary conditions on the surface of discontinuity between the material and vacuum at z=0.

In the material (z > 0) Maxwell's equations can be written

$$\operatorname{curl}(\vec{\mathrm{E}})=i \omega \mu_{0} \vec{\mathrm{H}}. \tag{10.1.3}$$

It is assumed that there is no free current density, $\vec{J}_{\mathrm{f}}=0$, so that $\operatorname{curl}(\vec{\mathrm{H}})$ simplifies to

$$\operatorname{curl}(\vec{\mathrm{H}})=-i \omega \vec{\mathrm{D}}=-i \omega \epsilon_{\mathrm{r}} \epsilon_{0} \vec{\mathrm{E}}. \tag{10.1.4}$$

It is also assumed that there is no free charge density in the material so that

$$\operatorname{div}(\vec{\mathrm{D}})=0. \tag{10.1.5}$$

In the material we assume that $\vec{\mathrm{B}}=\mu_{0} \vec{\mathrm{H}}$ and therefore

$$\operatorname{div}(\vec{\mathrm{H}})=0. \tag{10.1.6}$$

In writing these equations use has been made of the definitions from linear response theory in which it is assumed that the polarization per unit volume is a linear function of the electric field strength:

$$\vec{\mathrm{P}}(\omega)=\epsilon_0\chi_{\mathrm{E}}(\omega)\vec{\mathrm{E}},$$

and

$$\vec{\mathrm{D}}=\epsilon_0\vec{\mathrm{E}}+\vec{\mathrm{P}},$$

so that

$$\vec{\mathrm{D}}=\left[1+\chi_{\mathrm{E}}(\omega)\right]\epsilon_0\vec{\mathrm{E}}=\epsilon_{\mathrm{r}}\epsilon_0\vec{\mathrm{E}}. \tag{10.1.7}$$

The relative dielectric function $\epsilon_{\mathrm{r}}(\omega)$ will, in general, be a complex number because the response of the material, $\vec{P}$, to a driving electric field, $\vec{E}$, is not in phase with the electric field. In the above equations the time dependence of the fields, exp (−iωt) , has been explicitly used. The divergence of $\vec{D}$ is zero because it has been explicitly assumed that the material is uncharged. If the electric field is taken to have only an x-component, and to be propagating along z as shown in Figure (10.1.1), then its curl simplifies to give (from Equation (10.1.3))

$$\frac{\partial\mathrm{E}_{\mathrm{x}}}{\partial\mathrm{z}}=i\omega\mu_0\mathrm{H}_{\mathrm{y}}; \tag{10.1.8}$$

it follows from this that the magnetic field has only a y-component. Similarly from Equation (10.1.4) one finds

$$\frac{\partial\mathrm{H}_{\mathrm{y}}}{\partial\mathrm{z}}=i\omega\epsilon_{\mathrm{r}}\epsilon_0\mathrm{E}_{\mathrm{x}}. \tag{10.1.9}$$

Both $\vec{E}$ and $\vec{H}$ in the plane wave of Equation (10.1.2) automatically satisfy the condition that their divergences are zero because they are transverse waves; thus Equations (10.1.5) and (10.1.6) are satisfied. From Equations (10.1.8) and (10.1.9) one can obtain

$$\frac{\partial^2\mathrm{E}_{\mathrm{x}}}{\partial^2\mathrm{z}}=-\epsilon_{\mathrm{r}}\epsilon_0\mu_0\omega^2\mathrm{E}_{\mathrm{x}}=-\epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\mathrm{E}_{\mathrm{x}}. \tag{10.1.10}$$

It follows that a wave in the material will satisfy Maxwell’s equations providing that

$$\mathrm{k}_{\mathrm{m}}^2=\epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2. \tag{10.1.11}$$

This means that there are two waves in the material that can be used to satisfy Maxwell’s equations:

$$\mathrm{k}_{\mathrm{m}}=+\left(\frac{\omega}{\mathrm{c}}\right)\sqrt{\epsilon_{\mathrm{r}}}=\left(\frac{\omega}{\mathrm{c}}\right)(\mathrm{n}+i\kappa), \tag{10.1.12}$$

and

$$\mathrm{k}_{\mathrm{m}}=-\left(\frac{\omega}{\mathrm{c}}\right)\sqrt{\epsilon_{\mathrm{r}}}=-\left(\frac{\omega}{\mathrm{c}}\right)(\mathrm{n}+i\kappa), \tag{10.1.13}$$

where

$$\epsilon_{\mathrm{r}}=(\mathrm{n}+i\kappa)^2=\left(\mathrm{n}^2-\kappa^2\right)+2i\mathrm{n}\kappa, \tag{10.1.14}$$

and n and κ are defined by Equation (10.1.14).

If the parameter κ is greater than zero the wave-vector (10.1.12) represents a wave whose amplitude decays to the right since the constant A in Equation (10.1.2) is multiplied by the factor

$$\exp\left(-\left(\frac{\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right).$$

On the other hand, the wave-vector (10.1.13) represents a wave whose amplitude increases to the right in proportion to

$$\exp\left(+\left(\frac{\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right).$$

This wave which grows towards the interior of the material clearly cannot be appropriate for the present problem because it would imply that the wave was being amplified by its passage through the passive medium in the right half-space of Figure (10.1.1). It can be concluded that the wave in the material for z≥0 must have the form

$$\mathrm{E_x}=\mathrm{A}\exp\left(-\left(\frac{\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right)\exp\left(i\left(\frac{\mathrm{n}\omega}{\mathrm{c}}\mathrm{z}-\omega\mathrm{t}\right)\right), \tag{10.1.15}$$

and from either of equations (10.1.8) or (10.1.9)

$$\mathrm{H_y}=\sqrt{\frac{\epsilon_0}{\mu_0}}(\mathrm{n}+i\kappa)\mathrm{A}\exp\left(-\left(\frac{\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right)\exp\left(i\left(\frac{\mathrm{n}\omega}{\mathrm{c}}\mathrm{z}-\omega\mathrm{t}\right)\right). \tag{10.1.16}$$

Notice that the ratio of $\mathrm{H_y}$ to $\mathrm{E_x}$ is different from the vacuum case:

$$\frac{\mathrm{H_y}}{\mathrm{E_x}}=(\mathrm{n}+i\kappa)\sqrt{\frac{\epsilon_0}{\mu_0}}, \tag{10.1.17}$$

as opposed to

$$\frac{\mathrm{H_y}}{\mathrm{E_x}}=\sqrt{\frac{\epsilon_0}{\mu_0}}=\frac{1}{137}\quad Ohm^{-1}$$

for free space.

The average energy density stored in the electric field is given by

$$\mathrm{W_E}=\frac{\vec{\mathrm{E}}\cdot\vec{\mathrm{D}}}{2}$$

from Poynting's Theorem and the fact that $\vec{D}$ is proportional to $\vec{E}$, see Chapter(8). The average energy density stored in the electric field is given by

$$<\mathrm{W_E}>=\frac{1}{4}\mathrm{Real}\left(\vec{\mathrm{E}}\cdot\vec{\mathrm{D}}^*\right)=\frac{1}{4}\mathrm{Real}\left(\epsilon_{\mathrm{r}}\epsilon_0\mathrm{E}^2\right)$$

or

$$<\mathrm{W_E}>=\frac{1}{4}\epsilon_0\left(\mathrm{n}^2-\kappa^2\right)|\mathrm{A}|^2\exp\left(-\left(\frac{2\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right)\quad\mathrm{Joules}\,/m^3. \tag{10.1.18}$$

The average energy density stored in the magnetic field is given by

$$\begin{aligned}<\mathrm{W_B}>&=\frac{\mu_0}{4}\mathrm{Real}(\mathrm{HH}^*)\\&=\frac{\epsilon_0}{4}\left(\mathrm{n}^2+\kappa^2\right)|\mathrm{A}|^2\exp\left(-\left(\frac{2\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right)\quad\mathrm{Joules}/m^3\end{aligned} \tag{10.1.19}$$

The sum of these two energy densities is

$$<\mathrm{W}>=<\mathrm{W_E}>+<\mathrm{W_B}>=\frac{\epsilon_0\mathrm{n}^2}{2}\quad|\mathrm{A}|^2\exp\left(-\left(\frac{2\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right)\quad\mathrm{Joules}/m^3. \tag{10.1.20}$$

The energy density decays towards the interior of the material as one would expect.

The Poynting vector, $\vec{\mathrm{S}}=\vec{\mathrm{E}}\times\vec{\mathrm{H}}$, has only a z-component

$$\begin{aligned}<\mathrm{S_z}>&=\frac{1}{2}\mathrm{Real}\left(\mathrm{E_x}\mathrm{H_y^*}\right)\\&=\frac{\mathrm{n}}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\mathrm{A}|^2\exp\left(-\left(\frac{2\omega}{\mathrm{c}}\right)\kappa\mathrm{z}\right)\quad\mathrm{Watts}\,/m^2\end{aligned} \tag{10.1.21}$$

or

$$<\mathrm{S_z}>=\frac{\mathrm{c}}{\mathrm{n}}<\mathrm{W}>\quad\mathrm{Watts}\,/\mathrm{m}^2. \tag{10.1.22}$$

The energy flow in the wave takes place with the velocity c/n. The number n is called the index of refraction. Under some circumstances the index of refraction may be less than 1. In that case the phase velocity in the material exceeds the velocity of light in vacuum. It appears at first sight that a phase velocity greater than the speed of light in vacuum must violate one of the postulates of the theory of relativity. However, no information can be transmitted using a wave of constant amplitude stretching over all time from $t=-\infty$ to $t=\infty$. In order to transmit a message one must modulate the amplitude, or the frequency, of the wave. Any such modulation is propagated with the group velocity; it can be shown that the group velocity is always less than the speed of light in vacuum.

Having determined the wave-vector of the disturbance generated in the material filled half-space by the incident electromagnetic wave, it remains to calculate the amplitude of this disturbance at z=0. In order to find the amplitude A it is necessary to apply appropriate boundary conditions on E_x and H_y on the interface plane z=0.

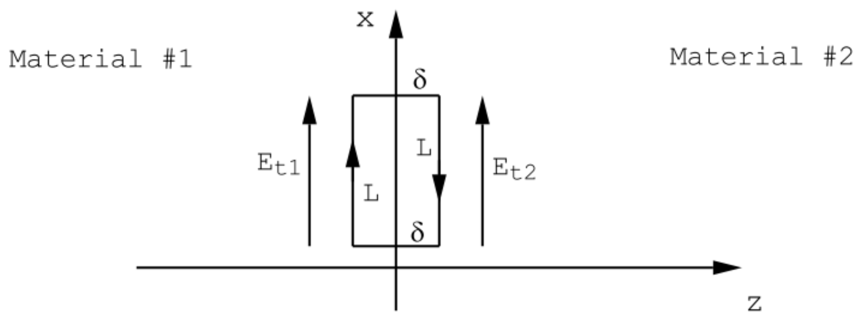


Figure 10.1.2: The Maxwell equation $\mathrm{curl}(\vec{E}) = -\partial\vec{B}/\partial t$ requires the tangential components of $\vec{E}$ to be continuous across any interface. See the text.

10.2: Boundary Conditions

10.2.1 The Tangential Components of the Electric Field.

Apply Stokes' theorem to the Maxwell equation

$$\mathrm{curl}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t}$$

and the small loop whose sides are L long and δ long as shown in Figure (10.1.2):

$$\oint \vec{E} \cdot d\overrightarrow{L} = -\frac{\partial}{\partial t} \int \int_{\text{Area}} \vec{B} \cdot d\vec{A}.$$

One then takes the limit as the sides δ shrink to zero. The line integral of the electric field gives

$$\oint \vec{E} \cdot d\vec{L} = (E_{t1} - E_{t2})\, L,$$

where E_{t1} is the field component parallel with L in material number 1 (vacuum in this case) and E_{t2} is the electric field component parallel with L in material number 2. The flux of the magnetic field through the loop goes to zero as δ goes to zero, therefore

$$(E_{t1} - E_{t2}) = 0$$

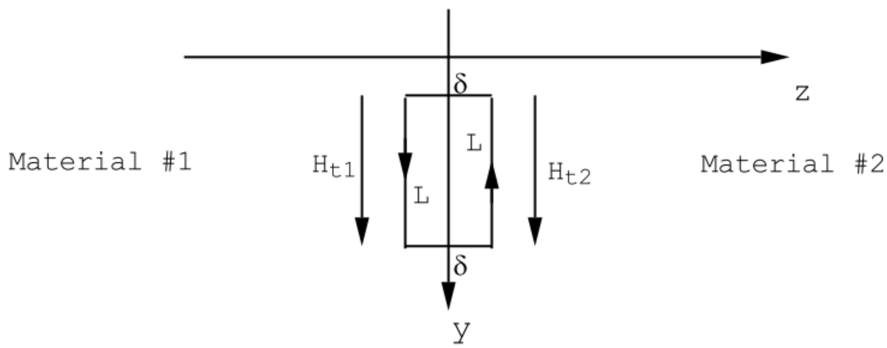


Figure 10.2.3: The Maxwell equation $\mathrm{curl}(\vec{H}) = -\partial\vec{D}/\partial t$ requires the tangential components of $\vec{H}$ to be continuous across any interface. See the text.

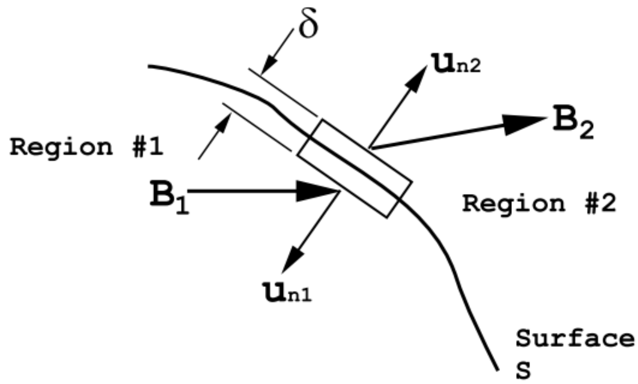


Figure 10.2.4: The Maxwell equation $\operatorname{div}(\vec{B}) = 0$ requires the normal component of $\vec{B}$ to be continuous across any interface. See the text.

or

$$E_{t1} = E_{t2}. \tag{10.2.1}$$

At the boundary between two materials the transverse components of $\vec{E}$ must be continuous.

10.2.2 The Tangential Components of the Magnetic Field.

Apply Stokes' theorem to a small loop as shown in fig(10.2.3):

$$\operatorname{curl}(\vec{H}) = \frac{\partial \vec{D}}{\partial t},$$

where it has been assumed that there are no free currents in either material, and no surface free current density on the interface between material number(1) and material number(2). Therefore

$$\oint_C \vec{H} \cdot d\vec{L} = \frac{\partial}{\partial t} \int\int_{Area} \vec{D} \cdot \overrightarrow{dS}.$$

Upon taking the limit as δ shrinks to zero the surface integral over $\vec{D}$ gives nothing and

$$(H_{t1} - H_{t2})\, L = 0,$$

that is

$$H_{t1} = H_{t2}. \tag{10.2.2}$$

The transverse components of the magnetic field $\vec{H}$ must be continuous across the boundary between two materials.

10.2.3 The Normal Component of the Field B.

The normal component of the magnetic field $\vec{B}$ must be continuous across **any** interface as a consequence of the Maxwell equation $div(\vec{B}) = 0$; see Figure (10.2.4). In Figure (10.2.4) Gauss' theorem is applied to a small pill-box that spans an arbitrary surface. The height of the pill-box, δ, is taken to be so small that any contributions to the surface integral from the sides of the box can be neglected. The continuity of the normal component of $\vec{B}$ is then forced by the requirement that the surface integral of $\vec{B}$ over the pill-box be zero:

$$B_{n1} = B_{n2}. \tag{10.2.3}$$

10.3: Application of the Boundary Conditions to a Plane Interface

Returning to the problem of a wave incident on a plane interface as shown in Figure (10.1.1), one could satisfy the boundary condition on $vecE$ by choosing the amplitude of the wave transmitted into the right half-space to be A= E_0 for z=0, where E_0 is the amplitude of the incident wave. This choice would, however, produce a discontinuity in H_y at the boundary because the ratio of H_y/E_x is different in the region z_0. In order to match both E_x and H_y inside and outside the boundary it is necessary to assume that the oscillating dipoles in the material to the right of z=0 give rise to a reflected wave, so that for z<0, in the vacuum in this example, one has

$$\mathrm{E_x} = \mathrm{E_0}\exp(i[\mathrm{kz}-\omega \mathrm{t}]) + \mathrm{E_R}\exp(-i[\mathrm{kz}+\omega \mathrm{t}]), \tag{10.3.1}$$

and, since

$$\mathrm{H_y} = \frac{1}{i\omega\mu_0}\frac{\partial \mathrm{E_x}}{\partial \mathrm{z}},$$

$$\mathrm{H_y} = \sqrt{\frac{\epsilon_0}{\mu_0}}\left(\mathrm{E_0}\exp(i[\mathrm{kz}-\omega \mathrm{t}]) - \mathrm{E_R}\exp(-i[\mathrm{kz}+\omega \mathrm{t}])\right), \tag{10.3.2}$$

where k = ω/c. In Equation (10.3.1) E_R is the amplitude of the reflected wave, as yet undetermined. Notice the change in sign of the space part of the reflected wave phasor; this sign change is required because the reflected wave must propagate towards the left i.e. towards z=−∞. The expression for the magnetic field H_y is obtained from applying Maxwell's equation (10.1.3) to Equation (10.3.1). From Equations (10.3.1) and (10.3.2) one obtains on the vacuum side of the interface at z=0

$$\begin{aligned} \mathrm{E_x}(0) &= (\mathrm{E_0}+\mathrm{E_R})\exp(-i\omega \mathrm{t}) \\ \mathrm{H_y}(0) &= \sqrt{\frac{\epsilon_0}{\mu_0}}(\mathrm{E_0}-\mathrm{E_R})\exp(-i\omega \mathrm{t}). \end{aligned} \tag{10.3.3}$$

On the material side of the interface at z=0 one has

$$\begin{aligned} \mathrm{E_x}(0) &= \mathrm{A}\exp(-i\omega \mathrm{t}) \\ \mathrm{H_y}(0) &= \sqrt{\frac{\epsilon_0}{\mu_0}}(\mathrm{n}+i\kappa)\mathrm{A}\exp(-i\omega \mathrm{t}). \end{aligned} \tag{10.3.4}$$

Apply the boundary conditions that E_x and H_y must be continuous through the boundary at z=0 to obtain

$$\mathrm{E_0}+\mathrm{E_R} = \mathrm{A}$$

and

$$\mathrm{E_0}-\mathrm{E_R} = (\mathrm{n}+i\kappa)\mathrm{A}.$$

These two equations can be readily solved:

$$\mathrm{T} = \frac{\mathrm{A}}{\mathrm{E_0}} = \frac{2}{(1+\mathrm{n}+i\kappa)}, \tag{10.3.5}$$

$$\mathrm{R} = \frac{\mathrm{E_R}}{\mathrm{E_0}} = \left(\frac{1-(\mathrm{n}+i\kappa)}{1+(\mathrm{n}+i\kappa)}\right). \tag{10.3.6}$$

Optical parameters n and κ are listed in Table(10.3.1) for green light and for a number of common materials. Metals are quite opaque at optical frequencies as can be seen from the Table. For example, at a wavelength of 0.5145 microns (a standard Argon ion laser line) the optical electric field amplitude in copper falls to 1/e of its initial value in a distance δ = λ/2πκ, or δ = λ/16.3 = 31.5 × 10^{-9} meters. The attenuation of the fields in glass or in water at frequencies corresponding to visible light is very small, see Table(10.1). The attenuation coefficient, proportional to κ, is extremely sensitive to the presence of small amounts of impurities. Very pure glasses have been developed for use in optical fibres in which the length over which the field amplitudes have decayed by e^{-1} is in excess of 1 km.

It is of interest to calculate the absorption coefficient associated with the plane interface of Figure (10.1.1). This is the time-averaged rate at which energy flows into the surface divided by the time-averaged rate at which the incident wave carries power towards the surface. It can be calculated in two ways:

(1) As the difference between the time-averaged Poynting vectors for the incident and reflected waves divided by the incident wave Poynting vector. For the incident wave

$$<S_{z0}>=\frac{E_0^2}{2\mu_0 c}=\frac{E_0^2}{2Z_0}.$$

For the reflected wave

$$<S_{zr}>=\frac{E_R^2}{2Z_0}.$$

Material	n	κ	$\epsilon_r=(n+i\kappa)^2$ $=(n^2-\kappa^2)+i(2n\kappa)$
Copper[a]	1.19	2.60	-5.34+i 6.19
Silver[a]	0.05	3.27	-10.69+i 0.327
Gold[a]	0.73	2.02	-3.55+i 2.95
Iron[b]	2.83	2.90	-0.40+i16.41
Cobalt[b]	1.95	3.65	-9.52+i 14.24
Nickel[b]	1.84	3.38	-8.04 +i 12.44
Crown Glass	1.525	$\sim 10^{-8}$	2.33
H_2O	1.333@ $\lambda=0.589\ \mu m$	$< 10^{-8}$	1.78

Table 10.3.1: Optical constants for some selected materials at a wavelength of 0.5145 microns (514.5 nm). This wavelength is a standard Argon ion laser green line. It corresponds to a frequency of f= 5.827 × 10^{14} Hz. A time dependence exp (−iωt) has been assumed. (a) P.B. Johnson and R.W. Christy, Phys.Rev.**B6**, 4370 (1972). (b) P.B. Johnson and R.W. Christy, Phys.Rev.**B9**, 5056 (1974).

In these last two equations $Z_0=\sqrt{\mu_0/\epsilon_0}=377$ Ohms is the impedance of free space. Therefore, the absorption coefficient is given by

$$\alpha=\frac{(<S_{z0}>-<S_{zr}>)}{<S_{z0}>}=1-\left|\frac{E_R}{E_0}\right|^2,$$

or, using Equation (10.3.5) for the reflection coefficient

$$\alpha=\frac{4n}{(1+n)^2+\kappa^2}, \tag{10.3.7}$$

(2) From the ratio of the time averaged Poynting vector just inside the material at z=0 to the incident wave Poynting vector.

$$<S_z>=\frac{1}{2}\text{Real}\left(H_y^* E_x\right)$$

$$<S_z>=\frac{1}{2}\text{Real}\left(\sqrt{\frac{\epsilon_0}{\mu_0}}(n-i\kappa)A^2\right)=\frac{n\left|A^2\right|}{2Z_0}.$$

But from Equation (10.3.5)

$$\left|A^2\right|=\frac{4E_0^2}{(1+n)^2+\kappa^2},$$

and therefore the absorption coefficient is given by the same expression as was obtained above

$$\alpha=\frac{<S_z>}{<S_{z0}>}=\frac{4n}{(1+n)^2+\kappa^2}.$$

10.4: Reflection from a Metal at Radio Frequencies

The response of a metal is completely dominated by its dc conductivity, σ_0, for frequencies less than $\sim 10^{12}$ Hz (1 THz). The relaxation time for the charge carriers in a good metal at ∼300K is of order $\tau = 10^{-14}$ seconds. That means that the dc conductivity can be meaningfully used for frequencies up to approximately 10^{12} Hz. In order to understand why the response of the unbound charge carriers dominates the response of the bound electrons at low frequencies consider the Maxwell equation

$$\operatorname{curl}(\mathrm{H})=\vec{\mathrm{J}}_f+\frac{\partial \vec{\mathrm{D}}}{\partial \mathrm{t}},$$

or in the low frequency limit

$$\operatorname{curl}(\mathrm{H})=\sigma_0 \vec{\mathrm{E}}+\epsilon \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}}.$$

The term $\sigma_0 \vec{\mathrm{E}}$ in the above equation takes into account the response of the unbound electrons: the last term takes into account the bound electrons. The response of the bound electrons at low frequencies is of order $\epsilon_0 \omega$, therefore one can compare these two terms by comparing σ_0 with $\omega\epsilon_0$. For copper at room temperature $\sigma_0 = 6.45 \times 10^7$ /Ohm-m. At 10^{12} Hz $\omega \epsilon_0=\left(2 \pi \times 10^{12}\right) / 36 \pi \times 10^9=55.6 / \mathrm{Ohm}-\mathrm{m}$. It is clear that for frequencies up to 10^{12} Hz the contribution of the bound electrons in copper is completely negligible compared with the contribution from the unbound charges. In this low frequency limit, and for an electric field polarized along x and propagating along z, Maxwell's equations can be written

$$\begin{aligned} \frac{\partial \mathrm{E}_{\mathrm{x}}}{\partial \mathrm{z}} &= i \omega \mu_0 \mathrm{H}_{\mathrm{y}} \\ \frac{\partial \mathrm{H}_{\mathrm{y}}}{\partial \mathrm{z}} &= -\sigma_0 \mathrm{E}_{\mathrm{x}}. \end{aligned} \tag{10.4.1}$$

These follow from the relations

$$\operatorname{curl}(\vec{\mathrm{E}})=-\frac{\partial \vec{\mathrm{B}}}{\partial \mathrm{t}},$$

and

$$\operatorname{curl}(\vec{\mathrm{H}})=\sigma_0 \vec{\mathrm{E}}.$$

From Equation (10.4.1) one obtains

$$\frac{\partial^2 \mathrm{E}_{\mathrm{x}}}{\partial \mathrm{z}^2}=-i \omega \sigma_0 \mu_0 \mathrm{E}_{\mathrm{x}}. \tag{10.4.2}$$

For a plane wave solution of the form

$$\mathrm{E}_{\mathrm{x}}=\mathrm{A} \exp (i[\mathrm{kz}-\omega \mathrm{t}])$$

Equation (10.4.2) requires that

$$\mathrm{k}^2=i \omega \sigma_0 \mu_0,$$

or

$$\mathrm{k}=\sqrt{\frac{\omega \sigma_0 \mu_0}{2}}(1+i), \tag{10.4.3}$$

and from Equation (10.4.1)

$$\frac{E_x}{H_y}=\frac{\omega \mu_0}{k}=\sqrt{\frac{\omega \mu_0}{2 \sigma_0}}(1-i). \tag{10.4.4}$$

The wave in the metal is clearly very heavily damped because the distance over which the electric field amplitude decays to 1/e of its initial value is approximately equal to the wavelength. This decay distance at 1 GHz for copper at room temperature is $\sqrt{2 / \omega \sigma_0 \mu_0}=\delta=1.98 \times 10^{-6}$. Radiation at 1 GHz does not penetrate very far into copper!

The wave impedance of copper at 1 GHz and at room temperature is given by

$$\mathrm{Z}=\frac{\mathrm{E_x}}{\mathrm{H_y}}=\left(7.82\times 10^{-3}\right)(1-i)\quad \text{Ohms}\ ,$$

compared with Z_0= 377 Ohms for free space. This means that the electric field amplitude in the metal is very small compared with the electric field amplitude of the incident wave. At the interface between vacuum and the metal one must construct electric and magnetic field amplitudes so that the tangential components of $\vec{E}$ and $\vec{H}$ are continuous across the surface: the normal component of $vecB$ is automatically continuous across the surface because the wave falls on the metal at normal incidence. These boundary conditions give

$$\begin{aligned}&\mathrm{E_0}+\mathrm{E_R}=\mathrm{A}\\&\frac{1}{Z_0}(E_0-E_R)=\frac{A}{Z},\end{aligned}$$

or

$$\mathrm{E_0}-\mathrm{E_R}=\frac{\mathrm{Z_0 A}}{\mathrm{Z}}.$$

The resulting wave amplitude at the metal surface, z=0, is

$$\mathrm{A}=\frac{2\mathrm{ZE_0}}{\mathrm{Z}+\mathrm{Z_0}}\cong\frac{2\mathrm{Z}}{\mathrm{Z_0}}\mathrm{E_0}. \tag{10.4.5}$$

The amplitude of the reflected wave is given by

$$\mathrm{E_R}=\left(\frac{\mathrm{Z}-\mathrm{Z_0}}{\mathrm{Z}+\mathrm{Z_0}}\right)\mathrm{E_0},$$

or

$$\frac{\mathrm{E_R}}{\mathrm{E_0}}\cong -1+\frac{2\mathrm{Z}}{\mathrm{Z_0}},$$

because $(Z/Z_0) \ll 1$.

Notice that for our example of copper at room temperature, and for a frequency of 1GHz, the magnitude of the reflected electric field amplitude is the same as the incident electric field amplitude to within $\sim 10^{-4}$, but the reflected electric field is 180° out of phase with the incident electric field so that the two fields cancel at the metal surface. The electric field in the metal is very small; approximately A= E_0/25000. On the other hand, the magnetic field amplitude at the metal surface is very nearly twice the magnetic field amplitude in the incident wave. In the metal at z=0

$$\mathrm{H_y}=\frac{\mathrm{A}}{\mathrm{Z}}=\frac{2\mathrm{E_0}}{\mathrm{Z}+\mathrm{Z_0}}\cong 2\frac{\mathrm{E_0}}{\mathrm{Z_0}},$$

whereas the magnetic field amplitude in the incident wave is given by E_0/Z_0.

One can speak of a perfectly conducting metal, one for which the conductivity approaches infinity. For such a perfectly conducting metal the electric field decays away in zero depth: a surface current sheet is set up that perfectly shields the metal from the electric field in the incident wave. The magnitude of the current sheet can be obtained by applying Stokes' theorem to the relation $\mathrm{curl}(\vec{\mathrm{H}})=\vec{\mathrm{J}}_f$ integrated over a small loop that spans the metal surface as shown in Figure (10.4.5). One has

$$\int int_{\text{Area}}\,\mathrm{curl}(\vec{\mathrm{H}})\cdot\overrightarrow{\mathrm{dS}}=\int\int_{\text{Area}}\vec{\mathrm{J}}_f\cdot\overrightarrow{\mathrm{dS}},$$

where Area=δL. But from Stokes' theorem

$$\oint_C\vec{\mathrm{H}}\cdot\overrightarrow{\mathrm{dL}}=\int\int_{Area}\vec{\mathrm{J}}_f\cdot\overrightarrow{\mathrm{dS}}=\mathrm{J_s L}, \tag{10.4.6}$$

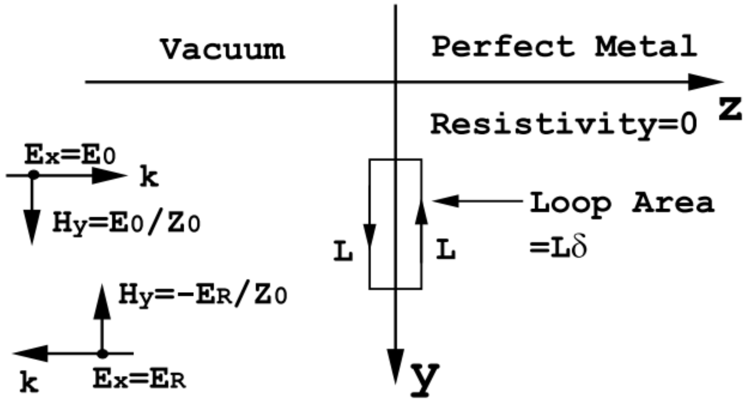


Figure 10.4.5: Diagram to aid in the calculation of the surface current density that shields the interior of a perfectly conducting metal from incident electric and magnetic fields.

where J_s is the surface current density in Amps/m, and L is the length of the loop. Inside the metal $H_y = 0$ so from (10.37) one obtains

$$J_s = H_y(0), \tag{10.4.7}$$

where $H_y(0)$ is the magnetic field amplitude at the vacuum/metal interface, and $H_y(0) = 2E_0/Z_0$.

For a perfect metal the wave impedance approaches zero, $Z = E_x/H_y$ and $Z \rightarrow 0$, so that in this limit the electric field has a node at the metal surface. For a perfect metal the boundary condition on the electric field at the interface becomes

$$E_t = 0,$$

where Et is the tangential component of the electric field.

It is straight forward to calculate the absorption coefficient for a metal surface from Equation (10.4.4) and from the amplitude A Equation (10.4.5):

$$\alpha = \frac{< S_z(\text{metal at } z = 0) >}{S_z(\text{incident})} = \frac{4c}{\omega Z_0^2}|Z|^2\sqrt{\frac{\omega\sigma_0\mu_0}{2}},$$

or

$$\alpha = \frac{2\omega}{c}\sqrt{\frac{2}{\sigma_0\omega\mu_0}} = \frac{2\omega\delta}{c}, \tag{10.4.8}$$

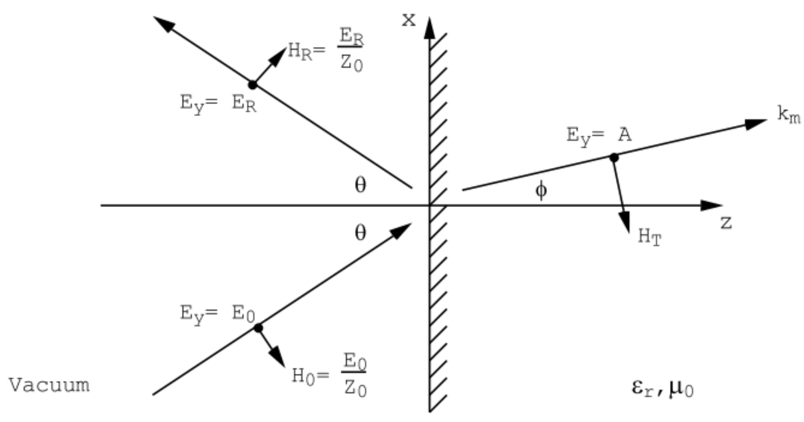


Figure 10.4.6: An S-polarized plane wave incident at the angle θ on the plane interface between vacuum and an isotropic medium characterized by material parameters ϵ_r and μ_0. The electric vector in the incident wave is perpendicular to the plane of incidence.

where $\delta=\sqrt{\frac{2}{\omega\sigma_0\mu_0}}$ is the characteristic length for attenuation of the fields in the metal.

10.5: Oblique Incidence

When a plane wave falls upon the plane interface between two media the incident and reflected wave-vectors define the plane of incidence, see Figures (10.4.6) and (10.5.7). The direction of the electric field vector in the incident wave may make an arbitrary angle with the plane of incidence. The general case may be treated as the sum of two special cases: an electric vector perpendicular to the plane of incidence (called s-polarized light from the German word for perpendicular, ”senkrecht”), and an electric vector which lies in the plane of incidence (p-polarized light).

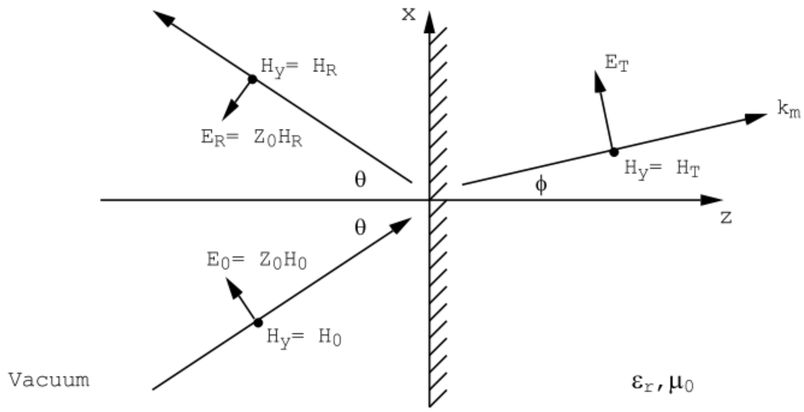


Figure 10.5.7: A P-polarized plane wave incident at the angle θ on the plane interface between vacuum and an isotropic medium characterized by material parameters ϵ_r and μ_0. The electric vector in the incident wave is parallel with the plane of incidence.

10.5.1 S-polarized Waves.

Consider first S-polarized waves, Figure (10.4.6). The incident wave electric vector can be described by the equation

$$E_y = E_0 \exp(i[x(k\sin\theta) + z(k\cos\theta) - \omega t]), \tag{10.5.1}$$

where k = ω/c because this wave is incident on the interface from vacuum. Eventually one is going to have to ensure that the tangential components of the electric and magnetic fields are continuous across the interface, and these boundary conditions must hold at any particular time at all points on the interface. This requirement means that all the waves in this problem, both inside the material and on the vacuum side of the interface, must have the same spatial dependence on the co-ordinates which lie in the interface plane. For the present example, Figure (10.6), the incident wave varies with the in-plane co-ordinate like

$$\exp(ixk\sin\theta) = \exp(ix\omega\sin\theta/c),$$

therefore this same factor must appear both in the reflected wave and in the transmitted wave that is generated in the region z>0. Since the reflected wave-vector has the same magnitude as the incident wave-vector, k = ω/c as determined by Maxwell’s equations, and since its x-component of the wavevector must be the same as for the incident wave, it follows that **the angle of reflection must be the same as the angle of incidence** as is shown in Figure (10.4.6). The electric vector of the reflected wave is given by

$$E_y = E_R \exp(i[xk\sin\theta - zk\cos\theta - \omega t]). \tag{10.5.2}$$

(Note the change in the sign of the z-component of k). The magnetic field vector in the incident wave must be perpendicular both to the electric field vector and to the wave-vector:

$$\begin{aligned} H_x^{(i)} &= -H_0\cos\theta\exp(i[xk\sin\theta + zk\cos\theta - \omega t]) \\ H_z^{(i)} &= H_0\sin\theta\exp(i[x, k\sin\theta + zk\cos\theta - \omega t]) \end{aligned} \tag{10.5.3}$$

where $H_0 = E_0/Z_0$, and $Z_0 = c\mu_0 = \sqrt{\mu_0/\epsilon_0} = 377$ Ohms. The magnetic field vector in the reflected wave must simultaneously be orthogonal to the reflected wave electric vector and also to the wave-vector:

$$
\begin{aligned}
H_x^{(R)} & =H_R \cos \theta \exp (i[x k \sin \theta-z k \cos \theta-\omega t]) \\
H_z^{(R)} & =H_R \sin \theta \exp (i[x k \sin \theta-z k \cos \theta-\omega t])
\end{aligned} \tag{10.5.4}
$$

where $H_R = E_R/Z_0$. Eqns.(10.5.3and 10.5.4) satisfy Maxwell's equations for the vacuum.

The electric field in the transmitted wave will be polarized along y because the material in the region z≥0 is assumed to be linear and isotropic so that a y-directed incident electric field will generate a y-directed transmitted electric field:

$$
E_y=A \exp (i[x k \sin \theta]) \exp (i[z k_z-\omega t]). \tag{10.5.5}
$$

The Maxwell equation $\operatorname{curl}(\vec{H})=-\frac{\partial \vec{B}}{\partial t}=i \omega \mu_0 \vec{H}$ becomes

$$
\begin{aligned}
\frac{\partial E_y}{\partial z} & =-i \omega \mu_0 H_x, \\
\frac{\partial E_y}{\partial x} & =i \omega \mu_0 H_z .
\end{aligned} \tag{10.5.6}
$$

The Maxwell equation $\operatorname{curl}(\vec{H})=\frac{\partial \vec{D}}{\partial t}=-i \epsilon_r \epsilon_0 \omega \vec{E}$ becomes

$$
\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=-i \omega \epsilon_r \epsilon_0 E_y . \tag{10.5.7}
$$

Combine Equations (10.5.6) and (10.5.7) to obtain

$$
\frac{\partial^2 E_y}{\partial z^2}+\frac{\partial^2 E_y}{\partial x^2}=-\left(\frac{\omega}{c}\right)^2 \epsilon_r E_y . \tag{10.5.8}
$$

This equation requires that

$$
k_z^2+k^2 \sin ^2 \theta=\epsilon_r\left(\frac{\omega}{c}\right)^2,
$$

or, since k = ω/c,

$$
k_z^2=\left[\epsilon_r-\sin ^2 \theta\right]\left(\frac{\omega}{c}\right)^2 .
$$

The z-component of the transmitted wave-vector must therefore be calculated from

$$
k_z=\sqrt{\left[\epsilon_r-\sin ^2 \theta\right]}\left(\frac{\omega}{c}\right), \tag{10.5.9}
$$

where the imaginary part of k_z must be chosen to be positive in order that the wave (10.5.5) be damped out as the wave travels along the z -direction. The wave-vector component k_z will in general be a complex number corresponding to the fact that the relative dielectric constant, $\epsilon_r=\epsilon_R+i \epsilon_I$, is a complex number: here ϵ_R and ϵ_I I are both real numbers. A complex index of refraction can be defined for the case of oblique incidence by setting

$$
k_z=\left(n_\theta+i \kappa_\theta\right)\left(\frac{\omega}{c}\right). \tag{10.5.10}
$$

the parameters n_θ and κ_θ are explicit functions of the angle of incidence. The electric field transmitted into the material on the right of z=0 will be given by

$$
E_y=A \exp (i[x k \sin \theta]) \exp \left(-\kappa_\theta \omega z / c\right) \exp \left(i\left[\frac{n_\theta \omega z}{c}-\omega t\right]\right), \tag{10.5.11}
$$

and from Equations (10.5.6) the magnetic field components are given by

$$
\begin{aligned}
H_x= & -\frac{\left(n_\theta+i \kappa_\theta\right)}{Z_0} . \\
& \cdot A \exp (i[x k \sin \theta]) \exp \left(-\kappa_\theta \omega z / c\right) \exp \left(i\left[\frac{n_\theta \omega z}{c}-\omega t\right]\right), \\
H_z= & \frac{\sin \theta}{Z_0} A \exp (i[x k \sin \theta]) \exp \left(-\kappa_\theta \omega z / c\right) \exp \left(i\left[\frac{n_\theta \omega z}{c}-\omega t\right]\right),
\end{aligned} \tag{10.5.12}
$$

where $Z_0 = c\mu_0 = 377$ Ohms. The planes of **constant amplitude** are parallel with the plane interface. The planes of constant phase are tilted at an angle ϕ with respect to the interface plane. The wave-vector in the material, which is perpendicular to the planes of **constant phase**, has components that are given by

$$k_x = \left(\frac{\omega}{c}\right)\sin\theta,$$

and

$$\text{Real}(k_z) = n_\theta\left(\frac{\omega}{c}\right),$$

therefore the tilt angle ϕ illustrated in Figure (10.4.6) can be calculated from

$$\tan\phi = \frac{\sin\theta}{n_\theta}. \tag{10.5.13}$$

In cases for which the dielectric constant can be taken to be real, i.e. negligible losses, one has

$$k_m = \sqrt{\left[k^2\sin^2\theta + k_z^2\right]} = \sqrt{\epsilon_r}\left(\frac{\omega}{c}\right).$$

Then

$$\sin\phi = \frac{k\sin\theta}{\sqrt{\epsilon_r}(\omega/c)} = \frac{\sin\theta}{\sqrt{\epsilon r}}.$$

This is just Snell's law:

$$\sin\theta = \sqrt{\epsilon_r}\sin\phi. \tag{10.5.14}$$

For this case a real index of refraction can be defined for the medium, $n = \sqrt{\epsilon_r}$, and the phase velocity of the wave in the medium is c/n; the refracted wave propagates in the direction specified by the angle ϕ obtained from Snell's law. In the more general case of a lossy medium the angle between the surfaces of constant phase and the boundary surface must be calculated from Equation (10.5.13).

At z=0 the tangential components of $\vec{E}$ and $\vec{H}$ must be continuous across the interface and this condition determines the amplitudes of the reflected and transmitted waves. One finds

$$E_0 + E_R = A, \tag{10.5.15}$$

and

$$-H_0\cos\theta + H_R\cos\theta = -\frac{(n_\theta + i\kappa_\theta)}{Z_0}A,$$

or, since $H_0 = E_0/Z_0$ and $H_R = E_R/Z_0$

$$-E_0 + E_R = -\frac{(n_\theta + i\kappa_\theta)}{\cos\theta}A. \tag{10.5.16}$$

The parameters n_θ and κ_θ are defined by equations (10.5.9) and (10.5.10). The two equations, (10.5.15) and (10.5.16), can be solved for the amplitudes E_R and A in terms of the incident wave amplitude E_0.

$$\begin{aligned}\frac{A}{E_0} &= \frac{2\cos\theta}{\left[\cos\theta + (n_\theta + i\kappa_\theta)\right]}, \\ \frac{E_R}{E_0} &= \left(\frac{\cos\theta - (n_\theta + i\kappa_\theta)}{\cos\theta + (n_\theta + i\kappa_\theta)}\right),\end{aligned} \tag{10.5.17}$$

where, it will be recalled,

$$(n_\theta + i\kappa_\theta) = \sqrt{\epsilon_r - \sin^2\theta},$$

and the sign must be chosen so that $\kappa_\theta > 0$.

10.5.2 P-polarized Waves.

Arguments for P-polarized light are similar to those for S-polarized light. However, for P-polarized radiation the magnetic field is polarized perpendicular to the plane of incidence, Figure (10.5.7). The incident wave can be written

$$\begin{aligned} \mathrm{H_y^{inc}} &= \mathrm{H_0}\exp(i[\mathrm{xk}\sin\theta + \mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{E_x^{inc}} &= \mathrm{Z_0 H_0}\cos\theta\exp(i[\mathrm{xk}\sin\theta + \mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{E_z^{R}} &= -\mathrm{Z_0 H_R}\sin\theta\exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega \mathrm{t}]), \end{aligned} \tag{10.5.18}$$

and for the reflected wave:

$$\begin{aligned} \mathrm{H}_y^{\mathrm{R}} &= \mathrm{H_R}\exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{E_x^{R}} &= -\mathrm{Z_0 H_R}\cos\theta\exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{E_z^{R}} &= -\mathrm{Z_0 H_R}\sin\theta\exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega \mathrm{t}]). \end{aligned} \tag{10.5.19}$$

Inside the material, z≥0, which is assumed to be characterized by a complex relative dielectric constant ϵ_r, one finds from

$$\begin{aligned} \mathrm{curl}(\vec{\mathrm{H}}) &= -i\omega\epsilon_\mathrm{r}\epsilon_0\vec{\mathrm{E}}, \\ \frac{\partial \mathrm{H_y}}{\partial \mathrm{z}} &= i\omega\epsilon_\mathrm{r}\epsilon_0\mathrm{E_x}, \\ \frac{\partial \mathrm{H_y}}{\partial \mathrm{x}} &= -i\omega\epsilon_\mathrm{r}\epsilon_0\mathrm{E_z}, \end{aligned} \tag{10.5.20}$$

and from

$$\begin{aligned} \mathrm{curl}(\vec{\mathrm{E}}) &= -\frac{\partial \vec{\mathrm{B}}}{\partial \mathrm{t}} = i\omega\mu_0\vec{\mathrm{H}}, \\ \frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{E_z}}{\partial \mathrm{x}} &= i\omega\mu_0\mathrm{H_y}. \end{aligned} \tag{10.5.21}$$

Equations (10.5.20) and (10.5.21) can be combined to give

$$\frac{\partial^2 \mathrm{H}_y}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{H}_y}{\partial \mathrm{z}^2} = -\epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 \mathrm{H_y}. \tag{10.5.22}$$

The solution of Equation (10.5.22) can be written

$$\mathrm{H}_y = \mathrm{H_T}\exp(i\,[\mathrm{xk}\sin\theta + \mathrm{zk_z} - \omega \mathrm{t}]), \tag{10.5.23}$$

where

$$\mathrm{k}^2\sin^2\theta + \mathrm{k_z^2} = \epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2. \tag{10.5.24}$$

In these Equations $\mathbf{k} = (\omega/\mathbf{c})$. Equation (10.5.24) for k_z is the same as that which was obtained for the case of an incident S-polarized wave. Solving for k_z one obtains:

$$\mathrm{k}_z = \sqrt{[\epsilon_r - \sin^2\theta]}\left(\frac{\omega}{\mathrm{c}}\right),$$

or

$$\mathrm{k_z} = (\mathrm{n}_\theta + i\kappa_\theta)\left(\frac{\omega}{\mathrm{c}}\right),$$

where

$$\mathrm{n}_\theta + i\kappa_\theta = \sqrt{[\epsilon_\mathrm{r} - \sin^2\theta]},$$

and the sign of the square root must be chosen so as to make the imaginary part of k_z positive in order to describe an optical disturbance that is attenuated as z increases.

From the form of the magnetic field, Equation (10.5.23), and from the Maxwell Equations (10.5.20), it follows that

$$\begin{aligned} E_x &= \frac{(n_\theta + i\kappa_\theta)}{\epsilon_r} Z_0 H_T \exp(ixk\sin\theta)\exp(-\kappa_\theta \omega z/c)\exp\left(i\left[\frac{n_\theta \omega z}{c} - \omega t\right]\right), \\ E_z &= -\frac{\sin\theta}{\epsilon_r} Z_0 H_T \exp(ixk\sin\theta)\exp(-\kappa_\theta \omega z/c)\exp\left(i\left[\frac{n_\theta \omega z}{c} - \omega t\right]\right). \end{aligned} \tag{10.5.25}$$

And from the boundary conditions at z=0 (continuity of the tangential components of $\vec{E}$ and $\vec{H}$) one finds:

$$H_0 + H_R = H_T,$$

$$Z_0 H_0 \cos\theta - Z_0 H_R \cos\theta = \frac{(n_\theta + i\kappa_\theta)}{\epsilon_r} Z_0 H_T,$$

or

$$H_0 - H_R = \frac{(n_\theta + i\kappa_\theta)}{\epsilon_r \cos\theta} H_T.$$

These two equations can be solved to obtain

$$\begin{aligned} \frac{H_T}{H_0} &= \frac{2\epsilon_r \cos\theta}{(\epsilon_r \cos\theta + (n_\theta + i\kappa_\theta))}, \\ \frac{H_R}{H_0} &= \left(\frac{\epsilon_r \cos\theta - (n_\theta + i\kappa_\theta)}{\epsilon_r \cos\theta + (n_\theta + i\kappa_\theta)}\right), \end{aligned} \tag{10.5.26}$$

where $(n_\theta + i\kappa_\theta) = \sqrt{[\epsilon_r - \sin^2\theta]}$ and $\kappa_\theta > 0$.

Notice that $\text{div}(\vec{D}) = 0$ for both the S- and P-polarized waves. This is obvious for the S-polarized light because the electric field has only a y-component and this component does not depend upon the y co-ordinate, Equation (10.5.11). For P-polarized radiation, from Equations (10.5.25),

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0,$$

so that $\text{div}(\vec{E}) = 0$ and, since $\vec{D} = \epsilon_r \epsilon_0 \vec{E}$, so also $\text{div}(\vec{D}) = 0$. There are no free charges set up in the material for either S- or P-polarized radiation. The condition $\text{div}(\vec{D}) = 0$ can also be deduced directly from the Maxwell's equation

$$\text{curl}(\vec{H}) = \frac{\partial \vec{D}}{\partial t} = -i\omega \vec{D},$$

because the divergence of any curl is zero. It is easy to show by direct calculation that the normal component of the magnetic field $\vec{B}$ is continuous across the surface of the dielectric material for both S- and P-polarized radiation.

10.5.3 Oblique Incidence on a Lossless Material.

For a material in which the losses are very small so that the imaginary part of the dielectric constant can be neglected, a real index of refraction can be defined by

$$n = \sqrt{\epsilon_r}.$$

For S-polarized radiation the reflection and transmission coefficients, Equations (10.5.17), become

$$\begin{aligned} R_S &= \frac{E_R}{E_0} = \left(\frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi}\right), \\ T_S &= \frac{E_T}{E_0} = \left(\frac{2\cos\theta}{\cos\theta + n\cos\phi}\right), \end{aligned} \tag{10.5.27}$$

where $\sin\phi = \sin\theta/\mathrm{n}$.

For P-polarized radiation, and $\mathbf{n} = \sqrt{\epsilon_\mathbf{r}}$ a real number, the reflection and transmission coefficients (10.5.26) become

$$\mathrm{R_P} = \frac{\mathrm{H_R}}{\mathrm{H_0}} = \left(\frac{\mathrm{n}\cos\theta - \cos\phi}{\mathrm{n}\cos\theta + \cos\phi}\right), \tag{10.5.28}$$

$$\mathrm{T_P} = \frac{\mathrm{H_T}}{\mathrm{H_0}} = \left(\frac{2\mathrm{n}\cos\theta}{\mathrm{n}\cos\theta + \cos\phi}\right),$$

where, as above, sin ϕ = sin θ/n and $\mathrm{n} = \sqrt{\epsilon_\mathrm{r}}$. The relation

$$\mathrm{n}_\theta = \sqrt{\mathrm{n}^2 - \sin^2\theta} = \mathrm{n}\cos\phi$$

has also been used.

10.6: Example- Copper

The real and imaginary parts $(\mathrm{n}_{\theta}+i\kappa_{\theta})=\sqrt{\epsilon_{\mathrm{r}}-\sin^{2}\theta}$ have been plotted in Figure (10.6.8) as a function of the angle of incidence, θ, for room temperature copper and for a wavelength of λ= 0.5145 microns (see Table(10.1)). As can be seen from the figure, the angular dependence of the indices n_{θ}, κ_{θ} is not very pronounced. For a lossy material such as copper that has a complex dielectric constant the reflectivity, E_R/E_0, is complex; that is, the phase shift between the incident wave and reflected wave electric vectors is neither 0° (in phase) nor 180° (out of phase). The real and imaginary parts of the reflectivity have been plotted in Figure (10.6.9) as a function of the angle of incidence for S-polarized 0.5145 micron light incident on room temperature copper; the absolute value of the reflectivity has been plotted in Figure (10.6.10).

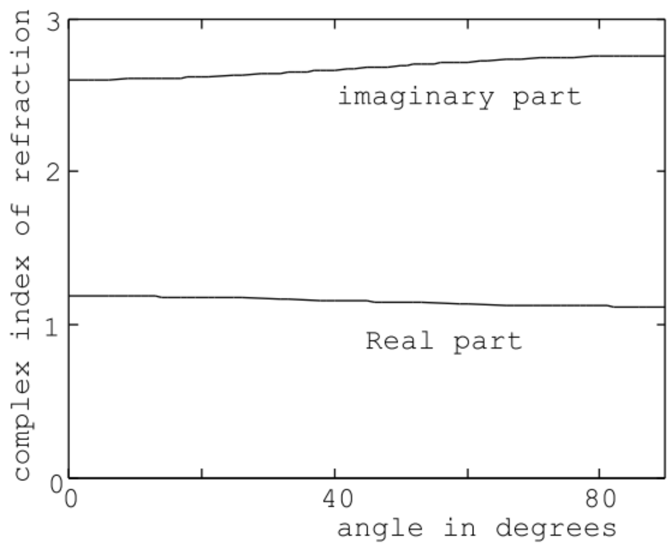


Figure 10.6.8: The dependence of the complex index of refraction, n_{θ} + iκ_{θ}, upon angle of incidence of the incident wave calculated for copper at room temperature and for an incident wavelength of λ = 0.5145 µm. The normal component of the wave-vector in copper is given by $k_z=\left(\frac{\omega}{c}\right)(n_{\theta}+i\kappa_{\theta})$. At this wavelength the relative dielectric constant for copper is ϵ_r= (-5.34+i6.19), n=1.19, and κ=2.60.

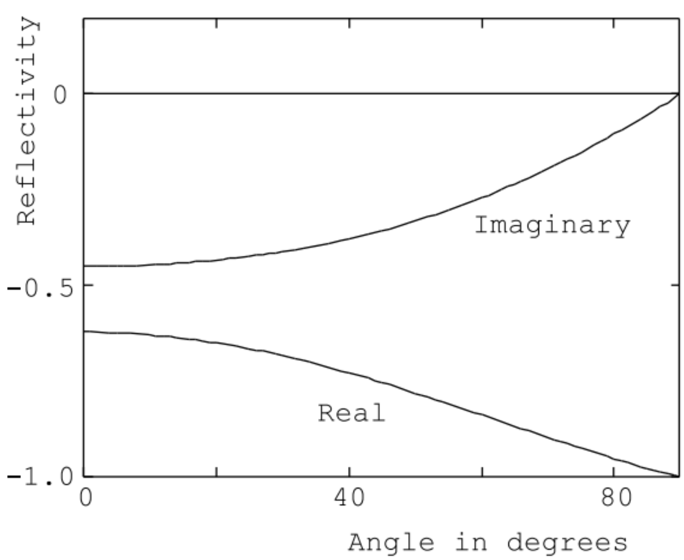


Figure 10.6.9: The real and imaginary part of the reflectivity of copper, E_R/E_0, as a function of angle of incidence for a wavelength λ= 0.5145 µm and S-polarized radiation. Copper at room temperature; ϵ_r= (-5.34 + i6.19), n=1.19, and κ=2.60.

Similarly, the real and the imaginary parts of the ratio H_R/H_0 have been plotted in Figure (10.6.11) as a function of the angle of incidence for P-polarized 0.5145 micron light incident on copper; the absolute value of this ratio is shown in Figure (10.6.12). The reflection coefficient for P-polarized radiation is given by $R_P = E_R/E_0$ but this is very closely related to the ratio H_R/H_0 because $E_0 = Z_0H_0$ and $E_R = -Z_0H_R$, where Z_0= 377 Ohms, the impedance of free space. Notice that the real part of the reflectivity for P-polarized light vanishes at an angle of incidence of approximately 69° ; the phase of the reflected light at that angle is shifted by 90° relative to the incident light. The phase shift between reflected and incident light is much less pronounced for S-polarized light; approximately 15° for an angle of incidence of 69° .

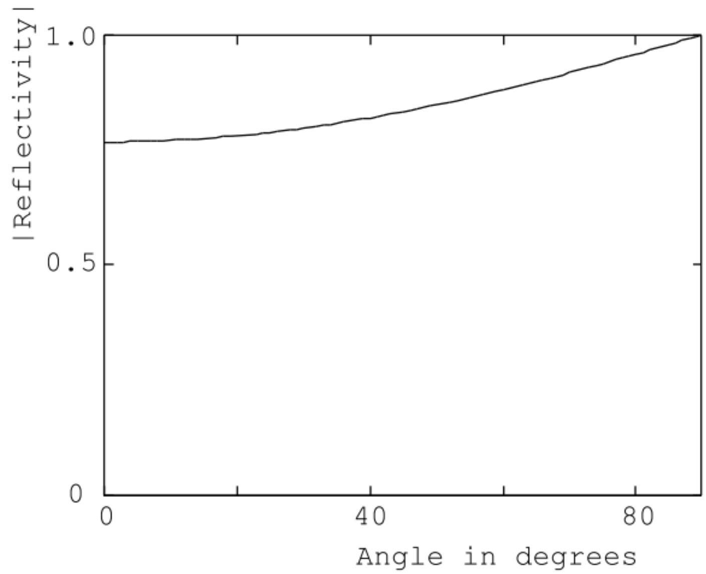


Figure 10.6.10: The absolute value of the reflectivity of copper, | E_R/E_0 |, as a function of angle of incidence for a wavelength λ= 0.5145 µm and S-polarized radiation. Copper at room temperature; ϵ_r=(-5.34 + i6.19), n=1.19, and κ=2.60.

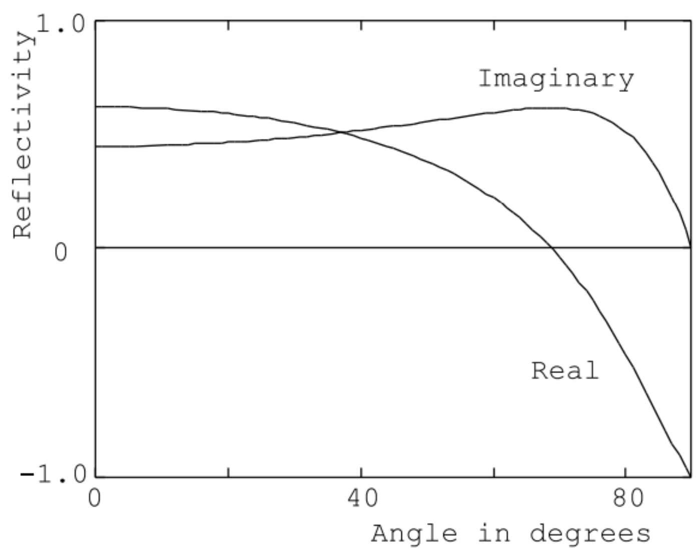


Figure 10.6.11: The real and imaginary parts of the complex ratio HR/H0 for copper as a function of angle of incidence for a wavelength λ= 0.5145 µm and for P-polarized radiation. Copper at room temperature; ϵ_r=(-5.34 + i6.19), n=1.19, and κ=2.60.

10.7: Example- Crown Glass

The dependence of the reflectivity on angle of incidence for a typical nondissipative material, Crown glass, is shown in Figure (10.7.13) and in Figure (10.7.14) for a wavelength of 0.5145 microns, see Table(10.3.1). Fig.(10.7.13) shows the variation of the ratio E_R/E_0 (Equation (10.5.27)) as a function of the angle of incidence for S-polarized light.

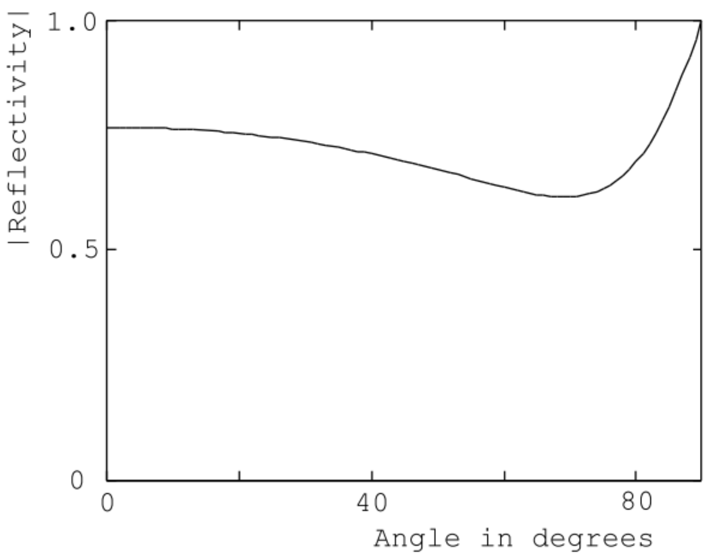


Figure 10.7.12: The absolute value of the reflectivity of copper, $| E_R/E_0 |$, as a function of angle of incidence for a wavelength λ= 0.5145 μm and for P-polarized radiation. Copper at room temperature; ϵ_r=(-5.34 + i6.19), n=1.19, and κ=2.60.

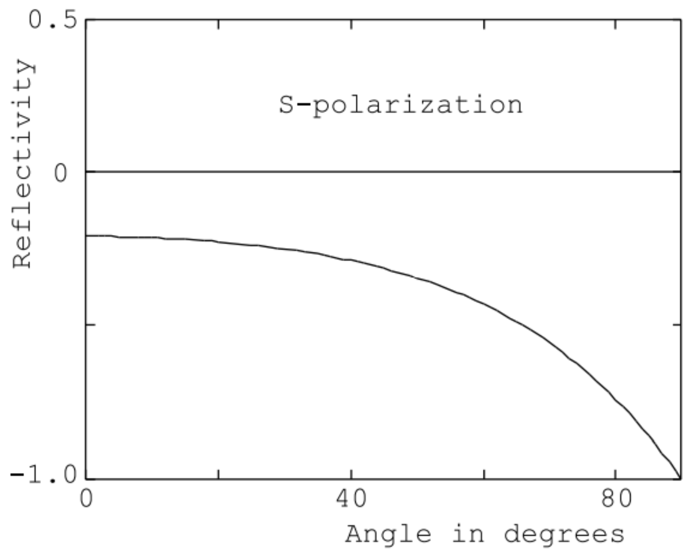


Figure 10.7.13: The reflectivity, E_R/E_0, as a function of the angle of incidence for Crown glass at a wavelength λ= 0.5145 μm and for S-polarized radiation. The reflectivity is real because the dielectric constant is real corresponding to very small losses in the glass. n=1.525=$\sqrt{\epsilon_r}$.

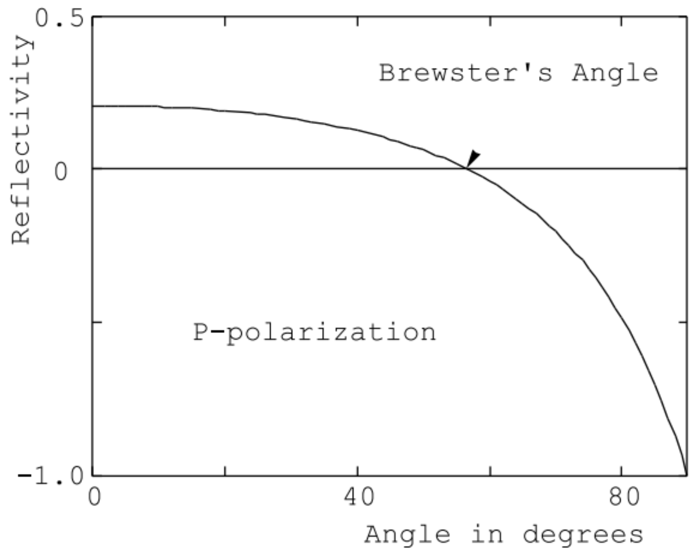


Figure 10.7.14: The reflectivity, H_R/H_0, as a function of the angle of incidence for Crown glass at a wavelength λ= 0.5145 μm and for P-polarized radiation. The reflectivity is real because the dielectric constant is real corresponding to very small losses in the glass. n=1.525=$\sqrt{\epsilon_r}$.

Fig.(10.7.14) shows the variation of the ratio HR/H0 (Equation (10.5.28)) as a function of the angle of incidence for P-polarized light. Notice that the angular dependence of the reflectivity for S-polarized light is qualitatively similar to that of copper. Of course, the reflectivity of Crown glass has only a real part because there are negligible losses in the glass and therefore the reflected electric field is 180◦ out of phase with the incident electric field. The angular variation of the reflection coefficient for P-polarized light is more interesting because the reflectivity goes to zero at approximately 57° , Figure (10.7.14). This angle is called Brewster's angle. Unpolarized light incident on a lossless dielectric material at Brewster's angle yields reflected light that is entirely S-polarized, ie. the electric vector in the reflected beam is oriented perpendicular to the plane of incidence. Before the advent of polaroid filters a stack of glass plates with the light incident at Brewster's angle was used to produce plane polarized light. Brewster's angle windows are used on each end of the plasma tubes in gas lasers.The light emitted from such a laser is plane polarized because the gain of the system is greater for P-polarized light than it is for S-polarized radiation due to the greater reflection losses at the plasma tube windows for S-polarized light.

It can be shown that at Brewster's angle the reflected light and the light transmitted into the dielectric medium form an angle of exactly 90° . This is a handy device for remembering how to determine Brewster's angle. The demonstration that the angle between the reflected beam and the transmitted beam is 90° for light incident on a dielectric medium from vacuum depends upon the two relations

$$\mathrm{n}\cos\theta = \cos\phi,$$

(from Equation (10.5.28)) and

$$\sin\theta = n\sin\phi$$

from Snell's law. Using the fact that the sum of the squares of the sin and cos functions is identically equal to 1 the above relations can be manipulated to give

$$\sin\theta = \frac{n}{\sqrt{n^2+1}} = \cos\phi,$$

and

$$\cos\theta = \frac{1}{\sqrt{\mathrm{n}^2+1}} = \sin\phi.$$

It follows from these relations that θ and ϕ are complementary angles, and that tan θ= n.

10.8: Metals at Radio Frequencies

We are interested in the practical case of metals at room temperature and frequencies less than 1000 GHz so that the metallic response to an electric field may be characterized by its dc conductivity, σ_0. We are also interested in the general case of radiation at oblique incidence. In this relatively low frequency regime the conduction current density in a metal is much larger than the displacement current density; i.e. for a time dependence $\sim$ exp (−iωt) one finds that $\sigma_0 \gg \omega\epsilon_r\epsilon_0$ in the Maxwell equation

$$\mathrm{curl}(\vec{\mathrm{H}}) = \sigma_0\vec{\mathrm{E}} - i\omega\epsilon_r\epsilon_0\vec{\mathrm{E}}.$$

The relevant Maxwell's equations for low frequency fields in a non-magnetic metal, $\mu \cong \mu_0$, become

$$\mathrm{curl}(\vec{\mathrm{E}}) = -\mu_0\frac{\partial\vec{\mathrm{H}}}{\partial \mathrm{t}} \tag{10.8.1}$$

and

$$\mathrm{curl}(\vec{\mathrm{H}}) = \sigma_0\vec{\mathrm{E}}. \tag{10.8.2}$$

Take the curl of (10.68) and use a time variation $\sim$ exp (−iωt) to obtain

$$\mathrm{curlcurl}(\vec{\mathrm{E}}) = i\omega\mu_0\sigma_0\vec{\mathrm{E}},$$

or

$$-\nabla^2\vec{\mathrm{E}} + graddiv(\vec{\mathrm{E}}) = i\omega\mu_0\sigma_0\vec{\mathrm{E}}. \tag{10.8.3}$$

However, the divergence of any curl of a vector is equal to zero, and consequently div($\vec{E}$) = 0 from Equation (10.8.2). It follows that for a metal at low frequencies the electric field components must satisfy the equation

$$\nabla^2\mathrm{E}_\alpha = -i\omega\mu_0\sigma_0\mathrm{E}_\alpha, \tag{10.8.4}$$

where α stands for each of the three cartesian components x,y, or z.

The solution of the problem of a plane wave incident at an oblique angle on a plane metallic surface proceeds just as for the general case of oblique incidence discussed in section(10.5). Two cases are of interest: (1) S-polarization in which the electric vector of the incident wave is parallel with the plane interface, see Figure (10.4.6), and (2) P-polarization in which the electric vector of the incident wave lies in the plane of incidence and the magnetic vector therefore lies parallel with the interface, see Figure (10.5.7).

10.8.1 S-polarization.

Using the co-ordinate system of Figure (10.4.6) the fields in the metal can be written

$$\begin{aligned} \mathrm{E_y} &= \mathrm{E_T}\exp(i\,[\mathrm{xk}\sin\theta + \mathrm{zk_z} - \omega \mathrm{t}]), \\ \mathrm{H_x} &= -\frac{\mathrm{k_z}}{\omega\mu_0}\mathrm{E_T}\exp(i\,[\mathrm{xk}\sin\theta + \mathrm{zk_z} - \omega \mathrm{t}]), \\ \mathrm{H_z} &= \frac{\sin\theta}{\mathrm{Z_0}}\mathrm{E_T}\exp(i\,[\mathrm{xk}\sin\theta + \mathrm{zk_z} - \omega \mathrm{t}]), \end{aligned} \tag{10.8.5}$$

where $Z_0 = c\mu_0$= 377 Ohms, and k = ω/c. The wave-vector component k_z in the metal must be chosen so that E_y satisfies Equation (10.8.4), i.e.

$$\mathrm{k_z^2} = i\omega\mu_0\sigma_0 - \left(\frac{\omega\sin\theta}{\mathrm{c}}\right)^2.$$

Apparently the wave-vector component k_z depends upon the angle of incidence of the driving incident plane wave. This dependence is illusory because $\mu_0\sigma_0$ is much larger than w/c^2 : for copper at 100 GHz w/c^2 = 7×10^{-6} whereas $\mu_0\sigma_0$= 81. For the range of frequencies and conductivities that are of interest here the term in $\sin^2\theta$ is negligible compared with the term proportional to the conductivity, and for any angle of incidence one may use

$$\mathrm{k_2^2} = i\omega\mu_0\sigma_0,$$

and

$$k_z = \sqrt{\frac{\omega\mu_0\sigma_0}{2}}(1+i) = \frac{(1+i)}{\delta}, \tag{10.8.6}$$

where δ is a length that is inversely proportional to the square root of the frequency. It is handy to remember that δ= 2μm for copper at 1 GHz and at room temperature.

At the metal-vacuum interface the tangential components of $\vec{E}$ and $\vec{H}$ must be continuous through the surface. These boundary conditions at z=0 result in two equations for the two unknown electric field amplitudes E_R and E_T; E_R is the amplitude of the wave reflected from the metal surface, and E_T is the amplitude of the electric field transmitted into the metal. The solutions of these equations are

$$\frac{E_R}{E_0} = \left(\frac{(\omega/c)\cos\theta - k_z}{(\omega/c)\cos\theta + k_z}\right), \tag{10.8.7}$$
$$\frac{E_T}{E_0} = \left(\frac{2(\omega/c)\cos\theta}{(\omega/c)\cos\theta + k_z}\right).$$

The wave-vector k_z is very large compared with (ω/c) cos θ so that if one divides the equations in (10.74) by k_z top and bottom the reflection and transmission coefficients can be expressed as a power series expansion in the small parameter ω cos θ/(ck_z): for example

$$\frac{E_R}{E_0} = \frac{-1+\frac{\omega\cos\theta}{ck_z}}{1+\frac{\omega\cos\theta}{ck_z}} \cong -1 + \left(\frac{2\omega\cos\theta}{ck_z}\right).$$

In terms of $(1/k_z) = \frac{\delta}{2}(1-i)$ one finds

$$\frac{E_R}{E_0} \cong -1 + \left(\frac{\delta\omega\cos\theta}{c}\right)(1-i), \tag{10.8.8}$$

$$\frac{E_T}{E_0} \cong \left(\frac{\delta\omega\cos\theta}{c}\right)(1-i). \tag{10.8.9}$$

The rate at which energy is carried through the surface per meter squared to be dissipated as Joule heat in the metal is given by the time average of the Poynting vector at z=0.

$$< S_z >= \frac{1}{2}\text{Real}(-E_y H_x^*)$$

so that

$$< S_z >= \frac{1}{2}\text{Real}\left(E_T\left(\frac{k_z^*}{\omega\mu_0}\right)E_T^*\right) = \frac{1}{2\delta\omega\mu_0}|E_T|^2,$$

or

$$< S_z >= \frac{\delta\omega}{c}\cos^2\theta\left(\frac{E_0^2}{Z_0}\right). \tag{10.8.10}$$

The time-averaged rate at which the incident wave transports energy in the z-direction is given by the z-component of the incident wave Poynting vector:

$$< S_0 >= \frac{1}{2}\text{Real}(-E_y H_x^*) = \frac{E_0^2}{2Z_0}\cos\theta. \tag{10.8.11}$$

The absorption coefficient associated with the metal surface is given by

$$\alpha = \frac{< S_z >}{< S_0 >} = 2\left(\frac{\delta\omega}{c}\right)\cos\theta, \tag{10.8.12}$$

where $\delta=\sqrt{2/(\omega\mu_0\sigma_0)}$. The absorption coefficient is very small and increases with frequency like $\sqrt{\omega}$, and decreases in proportion with the increase of the square root of the conductivity. Notice that at the surface of the metal the magnetic field components H_x in the incident and reflected waves add in phase so that at z=0

$$H_x=-H_0\cos\theta-H_R\cos\theta,$$

or

$$H_x=\frac{E_0\cos\theta}{Z_0}\left(-2+\frac{\delta\omega\cos\theta}{c}(1-i)\right). \tag{10.8.13}$$

Since $\delta\omega/c=2\pi(\delta/\lambda)$ is very small one makes very little error by taking the parallel component of the magnetic field at the metal surface to be just **twice the parallel magnetic field component of the incident wave**. In the limit of infinite conductivity the parameter $\delta\rightarrow 0$, the electric field in the metal becomes zero, and the component H_x at the metal surface has twice the amplitude of H_x in the incident wave. The component H_z also becomes zero at the metal surface in the limit of infinite conductivity, so that the normal component of $\vec{B}$, $B_z=\mu_0H_z$, is continuous across the vacuum-metal interface as is required by the Equation div($\vec{B}$) = 0.

10.8.2 P-polarization.

The magnetic vector of the incident wave is parallel with the metal surface, Figure (10.5.7). For this case the waves in the metal are described by

$$\begin{aligned}
H_y &= H_T\exp(i[xk\sin\theta+zk_z-\omega t]), \\
E_x &= -\frac{ik_z}{\sigma_0}H_T\exp(i[xk\sin\theta+zk_z-\omega t]), \\
E_z &= \frac{i\omega\sin\theta}{c\sigma_0}H_T\exp(i[xk\sin\theta+zk_z-\omega t]),
\end{aligned} \tag{10.8.14}$$

where

$$k_z^2=i\omega\mu_0\sigma_0-\left(\frac{\omega}{c}\right)^2\sin^2\theta\cong i\omega\mu_0\sigma_0$$

and therefore

$$k_z=\sqrt{\frac{\omega\mu_0\sigma_0}{2}}(1+i)=\frac{(1+i)}{\delta}. \tag{10.8.15}$$

The boundary conditions on H_y and on E_x at the interface z=0 (continuity of the tangential components of $\vec{E}$ and $\vec{H}$), plus a bit of algebra, readily gives the results

$$\begin{aligned}
\frac{H_R}{H_0} &= \frac{\left(\cos\theta-\left(\frac{\omega\delta}{2c}\right)(1-i)\right)}{\left(\cos\theta+\left(\frac{\omega\delta}{2c}\right)(1-i)\right)}, \\
\frac{H_T}{H_0} &= \frac{2\cos\theta}{\left(\cos\theta+\left(\frac{\omega\delta}{2c}\right)(1-i)\right)}, \\
\frac{E_x}{H_0} &= \frac{Z_0\left(\frac{\omega\delta}{c}\right)\cos\theta(1-i)}{\left(\cos\theta+\left(\frac{\omega\delta}{2c}\right)(1-i)\right)}.
\end{aligned} \tag{10.8.16}$$

In the above expressions Z_0= 377 Ohms, the impedance of free space. The ratio $\omega\delta/c=2\pi\delta/\lambda$ is very small, approximately 4×10^{-5} for copper at 1 GHz and 300K. It therefore follows that

$$\frac{H_R}{H_0}\cong 1$$

and

$$\frac{H_T}{H_0} \cong 2,$$

and

$$\frac{E_x}{H_0} \cong Z_0 \left(\frac{\omega\delta}{c}\right)(1-i) \sim 0.$$

In fact, for a perfect metal, one for which the conductivity becomes infinitely large, the length parameter, δ, goes to zero and the electric field does not penetrate into the metal.

The rate at which energy is carried into the metal surface at z=0 is given by

$$< S_z >= \frac{1}{2}\text{Real}\left(E_x H_y^*\right) = \left(\frac{2\omega\delta}{c}\right)\frac{Z_0}{2}\left|H_0\right|^2 \quad \text{Watts}/m^2.$$

The rate at which energy is carried to the surface by the incident wave is given by

$$< S_0 >= \frac{1}{2}\text{Real}\left(E_x H_y^*\right) = \frac{Z_0}{2}\cos\theta\left|H_0\right|^2 \quad \text{Watts}/m^2.$$

It follows that the absorption coefficient associated with the metal surface is

$$\alpha = \frac{< S_z >}{< S_0 >} = \frac{2\omega\delta}{c\cos\theta}. \tag{10.8.17}$$

Equation (10.83) is only valid if cos θ ≫ (ωδ/c). In the opposite limit, for angles very near to π/2 so that cos θ ≪ (ωδ/c), it can be shown that

$$\alpha \to 4\cos\theta/(\omega\delta/c),$$

so that the absorption coefficient goes to zero as the angle of incidence approaches π/2.

CHAPTER OVERVIEW

11: Transmission Lines

Thumbnail: 3-phase high-voltage lines. (CC BY-SA 3.0; Jeffrey G. Katz via Wikipedia)

11.1: Introduction

Consider a plane wave propagating along the z-direction in vacuum, and polarized with its electric vector along the x-axis: its magnetic field vector must be directed along the y-axis. Now introduce two infinitely conducting metal planes which block off all of space except the region between x= +a and x= -a, see Figure (11.1.1). The boundary conditions at x= ±a that must be satisfied by the electric and magnetic fields are

1. the tangential components of E must be zero;
2. the normal component of H must be zero**.**

This latter condition is a consequence of the Maxwell equation

$$\operatorname{div}(\overrightarrow{\mathrm{B}})=0$$

which requires the normal component of $\vec{B}$ to be continuous through an interface, coupled with the requirement that both the electric and magnetic fields are zero inside a perfect conductor: recall from Chapter(10) that in the limit of infinite conductivity the skin depth of a metal goes to zero. **Notice that the above two boundary conditions are satisfied by the plane wave.** The plane wave solutions of Maxwell's equations

$$\mathrm{E_x}=\mathrm{E_0}\exp(i[\mathrm{kz}-\omega\mathrm{t}]), \tag{11.1.1}$$
$$\mathrm{H_y}=\mathrm{H_0}\exp(i[\mathrm{kz}-\omega\mathrm{t}]), \tag{11.1.2}$$

can be used to describe the propagation of electromagnetic energy between two conducting planes. Energy is transported at the speed of light just as it is for a plane wave in free space. Notice that if it is attempted to close in the radiation with conducting planes at y= ±b the boundary conditions E_x = 0 and H_y = 0 cannot be satisfied on the planes y= ±b. Waves can be transmitted through such hollow pipes but the radiation bounces from wall to wall in a complex pattern that will be studied later. It will be shown that waves cannot be transmitted through a hollow pipe if the frequency is too low; there exists a lower frequency cut-off. However, a pair of parallel conducting planes unbounded in one transverse direction can transmit waves at all frequencies. In practice infinite planes are inconvenient, so one uses either strip-lines or co-axial cables, see Figures (11.1.2) and (11.2.3).

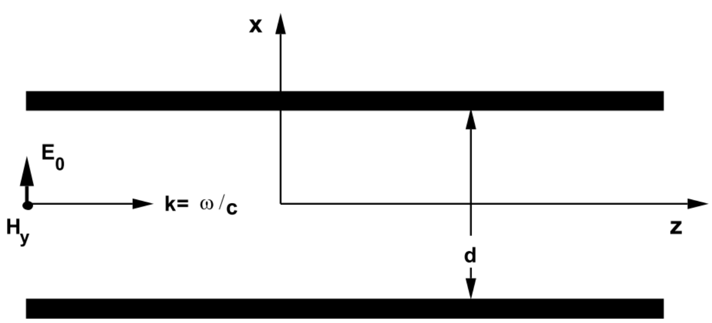


Figure 11.1.1: A plane wave propagating between two perfectly conducting planes. $\mathrm{E_x}=\mathrm{E_0}\exp(i[\mathrm{kz}-\omega\mathrm{t}]), \mathrm{H_y}=(\mathrm{E_0}/\mathrm{Z_0})\exp(i[\mathrm{kz}-\omega\mathrm{t}])$.

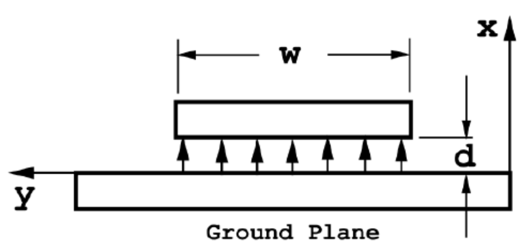


Figure 11.1.2: A strip-line. $\mathrm{E_x}=\mathrm{E_0}, \mathrm{V}=\mathrm{E_0 d}. \mathrm{H_y}=\mathrm{E_0}/\mathrm{Z_0}, \mathrm{I}=\mathrm{wJ_s}=\mathrm{w}\,(\mathrm{E_0}/\mathrm{Z_0})$

11.2: Strip-lines

See Figure (11.1.2). The electric field has only an x-component, if edge effects are neglected, and in the first approximation this component is independent of position across the width of the strip, i.e. E_x is independent of y. Similarly, the magnetic field has only a y-component and this is independent of x and y. From the Maxwell equation

$$\operatorname{curl}(\vec{E})=-\frac{\partial\vec{B}}{\partial t}=-\mu_0\frac{\partial\vec{H}}{\partial t}$$

one has

$$\frac{\partial E_x}{\partial z}=-\mu_0\frac{\partial H_y}{\partial t}. \tag{11.2.1}$$

From the Maxwell equation

$$\operatorname{curl}(\vec{H})=\frac{\partial\vec{D}}{\partial t}=\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

one finds

$$-\frac{\partial H_y}{\partial z}=\epsilon_0\frac{\partial E_x}{\partial t}. \tag{11.2.2}$$

Equations (11.2.1) and (11.2.2) can be combined to obtain

$$\begin{gathered}\left(\frac{\partial^2 E_x}{\partial z^2}\right)=-\mu_0\left(\frac{\partial^2 H_y}{\partial z\partial t}\right)=\epsilon_0\mu_0\left(\frac{\partial^2 E_x}{\partial t^2}\right),\\ \left(\frac{\partial^2 H_y}{\partial z^2}\right)=-\epsilon_0\left(\frac{\partial^2 E_x}{\partial z\partial t}\right)=\epsilon_0\mu_0\left(\frac{\partial^2 H_y}{\partial t^2}\right).\end{gathered} \tag{11.2.3}$$

The first of equations (11.2.3) can be satisfied by any function of the form

$$E_x(z,t)=F(z-ct)+G(z+ct), \tag{11.2.4}$$

where $c=1/\sqrt{\epsilon_0\mu_0}$ is the speed of light. This statement can be checked by carrying out the differentiations of Equations (11.2.3):

$$\frac{\partial E_x}{\partial z}=\frac{\partial F}{\partial z}+\frac{\partial G}{\partial z},\quad \frac{\partial E_x}{\partial t}=-c\frac{\partial F}{\partial z}+c\frac{\partial G}{\partial z}$$

and

$$\frac{\partial^2 E_x}{\partial z^2}=\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 G}{\partial z^2},\quad \frac{\partial^2 E_x}{\partial t^2}=c^2\frac{\partial^2 F}{\partial z^2}+c^2\frac{\partial^2 G}{\partial z^2}.$$

Therefore indeed one finds that

$$\frac{\partial^2 E_x}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}=\epsilon_0\mu_0\frac{\partial^2 E_x}{\partial t^2},$$

for any arbitrary functions F,G! This means that pulses having any time variation can be transmitted down the line with no distortion. In the real world the pulses do become distorted as a consequence of frequency dependent losses in the line, but for the time being we have only to do with ideal systems that are composed of perfect conductors and lossless dielectrics and such ideal systems transmit pulses without attenuation and with no distortion. The electric field $E_x = F(z-ct)$ corresponds to a pulse propagating along the strip-line in the positive z-direction with the speed of light c. The magnetic field associated with this electric field pulse can be obtained from Equation (11.2.1) or (11.2.2):

$$\frac{\partial H_y}{\partial z}=-\epsilon_0\frac{\partial E_x}{\partial t}=\epsilon_0 c\frac{\partial F}{\partial z},$$

therefore

$$H_y = \epsilon_0 c F(z - ct) = \frac{1}{Z_0} F(z - ct).$$

In other words, $H_y = E_x/Z_0$, where $Z_0 = 1/(c\epsilon_0)$=377 Ohms, the impedance of free space.

The electric field $E_x = G(z + ct)$ corresponds to a pulse propagating in the negative z-direction with the speed of light c. The corresponding magnetic field pulse is given by

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t} = -\epsilon_0 c \frac{\partial G}{\partial z},$$

or

$$H_y = -\epsilon_0 c G(z + ct) = -E_x/Z_0.$$

Note that the sign of the magnetic field component is opposite for the forward propagating and the backwards propagating pulses.

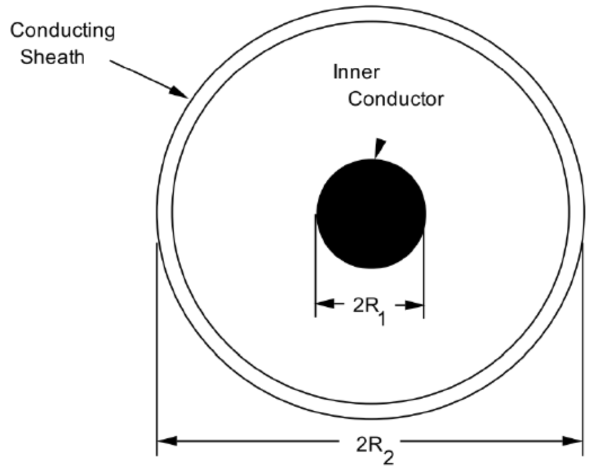


Figure 11.2.3: Cross-section through a co-axial cable. The electric field has a radial component, E_r , and the magnetic field has only the component H_θ independent of the angle θ.

It is usually more convenient to describe pulses on a strip-line or on a coaxial cable in terms of voltages and currents rather than in terms of electric and magnetic fields. The potential difference between the two conducting planes in the strip-line is $V = E_x d$, where d is the separation between the planes, Figure (11.1.2). On the other hand, surface currents must flow on the perfectly conducting metal planes in order to reduce the magnetic and electric fields to zero inside the metal. This surface current density can be calculated by the application of Stoke' Theorem to the Maxwell equation

$$\text{curl}(\vec{H}) = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

for a small loop that spans the metal surface, as shown in Figure (11.2.4).

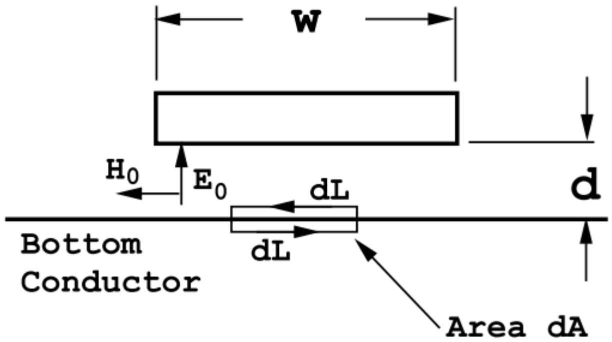


Figure 11.2.4: A diagrammatic representation of a strip-line in order to illustrate the relation between the electric field, E_x and the potential difference between the two conductors, and to illustrate the connection between the magnetic field, H_y and the surface current density. The potential difference is $V=E_0d$, and the bottom conductor is positive with respect to the upper conductor for the fields shown in the figure. The bottom conductor carries a total current $I=J_zw=H_0w$ Amps.

In the limit as the area of the loop, dA, shrinks to zero the term $\partial\vec{D}/\partial t$ gives nothing so that

$$\int\int_{Area} \mathrm{dS}\,\mathrm{curl}(\vec{\mathrm{H}})\cdot\hat{u}_n = \oint_C \vec{\mathrm{H}}\cdot\overrightarrow{\mathrm{d}\vec{\mathrm{L}}} = \mathrm{J}_2\mathrm{dL},$$

where J_z is the surface current density and $\hat{u}_n$ is a unit vector perpendicular to dA. Therefore

$$\mathrm{H}_0\mathrm{dL} = \mathrm{J}_z\mathrm{dL}$$

so that

$$\mathrm{J}_z = \mathrm{H}_0.$$

This current density flows along +z in the bottom conductor. The total current carried by the bottom conductor is just proportional to the width of the active region on the strip-line:

$$\mathrm{I} = \mathrm{J}_z\mathrm{w} = \mathrm{H}_0\mathrm{w} \quad \text{Amps.}\,.$$

The potential difference between the two conductors is

$$\mathrm{V} = \mathrm{E}_0\mathrm{d} \quad \text{Volts}\,,$$

and the bottom plane is positive with respect to the upper plane for the fields shown in the figure.

The characteristic impedance of the line, Z_0 = V/I Ohms, is given by

$$\mathrm{Z}_0 = \frac{\mathrm{E}_0\mathrm{d}}{\mathrm{H}_0\mathrm{w}} = \frac{\mathrm{d}}{\mathrm{w}}\sqrt{\mu_0/\epsilon_0}, \tag{11.2.5}$$

because for a forward propagating wave

$$\mathrm{H}_0 = \epsilon_0\mathrm{c}\mathrm{E}_0 = \sqrt{\epsilon_0/\mu_0}\mathrm{E}_0.$$

The conductors in a practical strip-line are usually separated by a nonmagnetic and non-conducting dielectric material characterized by a magnetic permeability μ_0 and a dielectric constant ϵ. The above equations are still applicable to such a strip-line providing that dielectric losses can be neglected: one has only to replace ϵ_0 by ϵ.

In terms of potential difference and current, the picture that emerges is the one illustrated in Figure (11.2.5). A voltage pulse of arbitrary shape propagates along the line with a velocity v that is determined by the properties of the dielectric spacer (here assumed to be lossless). For a non-magnetic spacer material having a dielectric constant ϵ this velocity is given by

$$\mathrm{v} = \frac{1}{\sqrt{\epsilon\mu_0}}. \tag{11.2.6}$$

The voltage pulse is accompanied by a current pulse of the same shape as the voltage pulse. The scaling factor between current and voltage is the characteristic impedance of the line. The characteristic impedance depends upon the strip-line geometry:for the plane strip-line of Figure (11.2.4) it is given by

$$Z_0 = \frac{d}{w}\sqrt{\frac{\mu_0}{\epsilon}} = \frac{d}{w}\frac{1}{\epsilon v}. \tag{11.2.7}$$

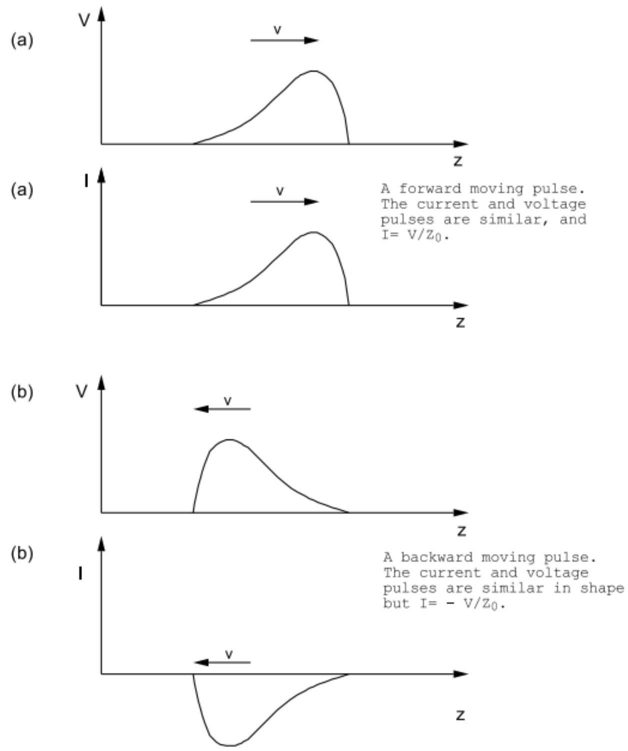


Figure 11.2.5: Voltage and current pulses on a transmission line.

For a forward moving pulse

$$V = +Z_0 I, \tag{11.2.8}$$

and for a backward moving pulse

$$V = -Z_0 I, \tag{11.2.9}$$

where V is the potential difference in Volts and I is the current on the transmission line in Amps.

Maxwell's equations, (11.2.3), can be rewritten in terms of the potential difference, V, and the current on the line, I. Since E_x is proportional to V the first of equations (11.2.3) becomes

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}. \tag{11.2.10}$$

Similarly, since H_y is proportional to the current I, the second of equations(11.3) becomes

$$\frac{\partial^2 I}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 I}{\partial t^2}. \tag{11.2.11}$$

These telegraph line equations were derived by Lord Kelvin in 1855 (this was before Maxwell's equations had been discovered) by treating the transmission line as a repeating series of inductances shunted by capacitors. See Electromagnetic Theory by J.A.Stratton, McGraw-Hill, New York, 1941, section 9.20, Figure (103); for a lossless line R=0 and G=∞.

11.3: Co-axial Cables

Cylindrical co-ordinates are appropriate for the problem of a co-axial cable, Figure (11.2.3). The relevant Maxwell's equations become

$$\operatorname{curl}(\vec{\mathrm{E}})=-\mu_{0} \frac{\partial \vec{\mathrm{H}}}{\partial \mathrm{t}},$$

and

$$\operatorname{curl}(\vec{\mathrm{H}})=\epsilon \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}},$$

where ϵ is a real number for a lossless line. Look for solutions of these equations in which, by analogy with a strip-line curved around on itself, the electric field has only a radial component, E_r , that is independent of angle, and the magnetic field has only an angularly independent component H_θ:

$$\frac{\partial \mathrm{E}_{\mathrm{r}}}{\partial \mathrm{z}}=-\mu_{0} \frac{\partial \mathrm{H}_{\theta}}{\partial \mathrm{t}}, \tag{11.3.1}$$

$$\frac{\partial \mathrm{H}_{\theta}}{\partial \mathrm{z}}=-\epsilon \frac{\partial \mathrm{E}_{\mathrm{r}}}{\partial \mathrm{t}}. \tag{11.3.2}$$

In addition, take $E_z = 0$ because the tangential components of the electric field must be zero at the perfectly conducting walls of the co-axial cable. But if $E_z = 0$ it follows from Maxwell's equations that

$$\operatorname{curl}(\vec{\mathrm{H}})_{z}=0=\frac{1}{\mathrm{r}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{rH}_{\theta}\right).$$

This implies that

$$\mathrm{H}_{\theta}=\frac{\mathrm{a}(\mathrm{z}, \mathrm{t})}{\mathrm{r}}, \tag{11.3.3}$$

where a(z,t) is a function of time and of position along the cable. Similarly, from $\operatorname{div}(\vec{E}) = 0$ one has

$$\frac{1}{\mathrm{r}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{rE}_{\mathrm{r}}\right)=0,$$

and this is satisfied by

$$\mathrm{E}_{\mathrm{r}}=\frac{\mathrm{b}(\mathrm{z}, \mathrm{t})}{\mathrm{r}}. \tag{11.3.4}$$

By combining the Maxwell Equations (11.3.1) the electric and magnetic fields, Equations (11.3.3) and (11.3.4), must satisfy

$$\frac{\partial^{2} \mathrm{E}_{\mathrm{r}}}{\partial \mathrm{z}^{2}}=-\mu_{0} \frac{\partial^{2} \mathrm{H}_{\theta}}{\partial \mathrm{z} \partial \mathrm{t}}=\epsilon \mu_{0} \frac{\partial^{2} \mathrm{E}_{\mathrm{r}}}{\partial \mathrm{t}^{2}}, \tag{11.3.5}$$

$$\frac{\partial^{2} \mathrm{H}_{\theta}}{\partial \mathrm{z}^{2}}=-\epsilon \frac{\partial^{2} \mathrm{E}_{\mathrm{r}}}{\partial \mathrm{z} \partial \mathrm{t}}=\epsilon \mu_{0} \frac{\partial^{2} \mathrm{H}_{\theta}}{\partial \mathrm{t}^{2}}. \tag{11.3.6}$$

These have the same form as the strip-line equations (11.2.3). It follows from these equations,and from the requirements (11.3.3) and (11.3.4), that the general solution for the electric field can be written

$$\mathrm{E}_{\mathrm{r}}(\mathrm{z}, \mathrm{t})=\frac{\mathrm{F}(\mathrm{z}-\mathrm{vt})}{\mathrm{r}}+\frac{\mathrm{G}(\mathrm{z}+\mathrm{vt})}{\mathrm{r}}, \tag{11.3.7}$$

where F(u) and G(u) are arbitrary functions of their arguments, and where

$$\mathrm{v}=\frac{1}{\sqrt{\epsilon \mu_{0}}}.$$

The corresponding general solution for the magnetic field is

$$H_{\theta}(z,t)=\epsilon v\left(\frac{F(z-vt)}{r}-\frac{G(z+vt)}{r}\right). \tag{11.3.8}$$

The above electric and magnetic fields satisfy the wave equations (11.3.5), they satisfy Equations (11.3.1), and they have the form required by Equations (11.3.3and 11.3.4).

Instead of the electric field strength, the state of the electric field in the cable can be specified by the potential difference between the inner and outer conductors:

$$V=\int_{R_1}^{R_2}E_r dr=F(z-vt)\int_{R_1}^{R_2}\frac{dr}{r}=F(z-vt)\ln\left(\frac{R_2}{R_1}\right)$$

for a forward propagating wave. Note that the inner conductor is positive with respect to the outer conductor. The corresponding current on the inner conductor is given by

$$I=J_z\left(2\pi R_1\right)=H_{\theta}\left(R_1\right)\left(2\pi R_1\right)=\epsilon v\left(2\pi R_1\right)\frac{F(z-vt)}{R_1},$$

so that

$$I=2\pi\epsilon vF(z-vt).$$

The current flows flows towards +z for the current on the inner conductor; the current flows towards minus z on the outer conductor. That is, on the outer conductor

$$I=-2\pi R_2H_{\theta}\left(R_2\right)=-2\pi\epsilon vF(z-vt).$$

so that the net current flow through a section of the cable is zero. The characteristic impedance of the cable is given by

$$Z_0=\frac{V}{I}=\frac{1}{2\pi\epsilon v}\ln\left(\frac{R_2}{R_1}\right)$$

or

$$Z_0=\frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon}}\ln\left(\frac{R_2}{R_1}\right). \tag{11.3.9}$$

The potential difference, V, is proportional to the electric field, E_r , and the current, I, is proportional to the magnetic field, H_{θ}, therefore from Equations (11.3.5) the voltage and current satisfy the wave equations

$$\frac{\partial^2V}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2V}{\partial t^2}, \tag{11.3.10}$$

$$\frac{\partial^2 I}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 I}{\partial t^2}, \tag{11.3.11}$$

where $v^2 = 1/(\epsilon\mu_0)$. For a forward propagating pulse having the form

$$V(z,t)=F(z-vt)$$

the corresponding current pulse is described by

$$I(z,t)=\frac{1}{Z_0}F(z-vt)=\frac{V(z,t)}{Z_0}, \tag{11.3.12}$$

where the characteristic impedance for a co-axial cable is given by Equation (11.18). For a backward propagating potential pulse of the form

$$V(z,t)=G(z+vt)$$

the corresponding current pulse is described by

$$I(z,t)=-\frac{1}{Z_0}V(z,t)=-\frac{G(z+vt)}{Z_0}. \tag{11.3.13}$$

In the above equations F(z-vt) and G(z+vt) are arbitrary functions of their arguments.

11.4: Transmission Lines in General

Relations similar to Equations (11.3.8,11.3.9, and 11.3.10) are valid for a transmission line constructed of arbitrarily shaped conductors. In the general case it is convenient to describe the properties of a lossless line in terms of the inductance per unit length, L Henries/m, and the capacitance per unit length,

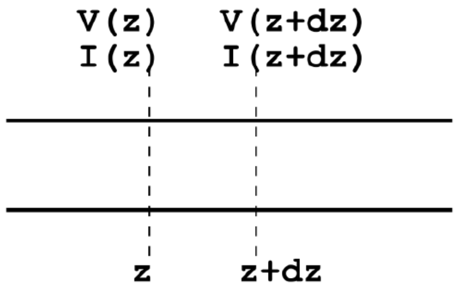


Figure 11.4.6: Schematic diagram of a general transmission line. The line is characterized by an inductance per unit length, L Henries/m, and a capacitance per unit length, C Farads/m.

C Farads/m. One can write the appropriate transmission line equations using ordinary circuit theory. If the current on the line changes with time there will be a voltage drop in going from z to z+dz along the line, see Figure 11.4.6). This drop in potential is due to the line inductance. One can write

$$dV = -Ldz\left(\frac{\partial I}{\partial t}\right).$$

Thus it follows that

$$\frac{\partial V}{\partial z} = -L\left(\frac{\partial I}{\partial t}\right). \tag{11.4.1}$$

If the potential difference between the two conductors on the transmission line changes with time the current at z+dz will be a little smaller than the current at z because some current is shunted through the capacitive coupling between the electrodes. Therefore

$$dI = -Cdz\left(\frac{\partial V}{\partial t}\right),$$

or

$$\frac{\partial I}{\partial z} = -C\left(\frac{\partial V}{\partial t}\right). \tag{11.4.2}$$

Notice the similarity in form between Equations (11.4.1 and ???) and Equations (11.2.1 and 11.2.2). The above two equations can be combined to give

$$\begin{aligned} \frac{\partial^2 V}{\partial z^2} &= -L\frac{\partial^2 I}{\partial t\partial z} = LC\frac{\partial^2 V}{\partial t^2} \\ \frac{\partial^2 I}{\partial z^2} &= -C\frac{\partial^2 V}{\partial t\partial z} = LC\frac{\partial^2 I}{\partial t^2} \end{aligned} \tag{11.4.3}$$

Notice that these equations have the same form as do Equations (11.3.8). It follows from this similarity that

$$LC = \frac{1}{v^2}$$

or

$$v^2 = \frac{1}{LC}. \tag{11.4.4}$$

The velocity of propagation along the transmission line is independent of the line geometry and is determined only by the dielectric constant and the permeability of the medium that carries the electric and magnetic fields that characterize the propagating disturbance. For a uniform medium

$$v^2 = \frac{1}{\epsilon\mu}.$$

It follows from this, plus Equation (11.4.4), that

$$LC = \epsilon\mu, \tag{11.4.5}$$

a relation that is not a priori obvious. It also follows from Equations (11.4.1 and ???) that the characteristic impedance of the line is given by

$$Z_0 = Lv = \frac{1}{Cv} = \sqrt{\frac{L}{C}}. \tag{11.4.6}$$

The characteristic impedance does depend upon the geometry of the transmission line.

It is perhaps worth emphasizing the physical picture of the manner in which a pulse of charge is propagated along a transmission line, see Figure (11.4.7). In Figure (11.4.7) the upper electrode has a positive potential with respect to the lower electrode in the region where the charge patch is located. The electric field is terminated by a patch of charge on each metal electrode surface: this charge is of one sign on the upper electrode and of opposite sign on

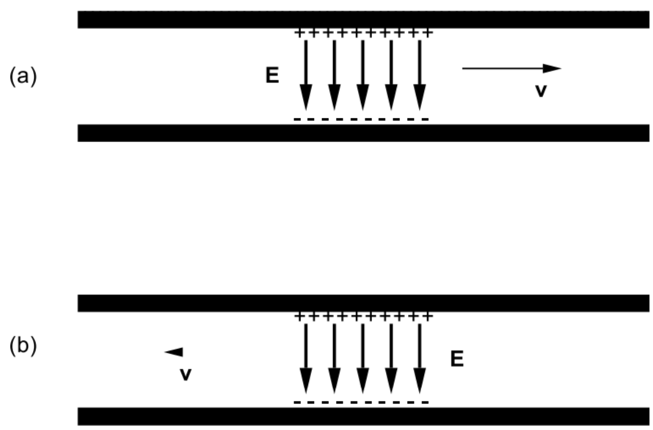


Figure 11.4.7: Pulses propagating along a transmission line. (a) A forward moving pulse. (b) A backward moving pulse.

the other electrode as shown schematically in Figure (11.4.7). The charge patches move down the line with the velocity v characteristic of the wave velocity in the medium between the electrodes. At any section of the line the current on either electrode is zero until the charge patch arrives. Current is the rate at which charge is transported past a particular point, therefore the current at some point on the electrode is the product of the velocity and the charge density per unit length of line. It is clear from Figure (11.4.7(a)), which depicts a positive voltage pulse moving to the right, that the current in the upper electrode will be positive, whereas the current pulse carried by the lower electrode will be negative because negative charge is flowing from left to right. Similarly, if the pulse is moving from right to left the current flow in the upper conductor will be negative because positive charges are flowing in the negative z direction. At the same time, the current in the bottom electrode is positive because negative charges are flowing from right to left. For a positive voltage pulse moving from left to right one adopts the convention that the associated current pulse is positive. For a positive voltage pulse moving from right to left one adopts the convention that the associated current pulse is negative.

11.5: A Terminated Line

Any discontinuity in the properties of a transmission line results in a reflected pulse and a transmitted pulse whose amplitudes are smaller than the original pulse amplitude. In order to see how this comes about, consider a few simple cases

(i) Two pulses of identical shape, but mirror images, propagate towards one another on an infinite line, Figure (11.5.8(a)). The experiment is set up so that the pulses collide at z=0. Maxwell's equations are linear, and we assume that the dielectric and magnetic response of the material of which the line is made is also linear. For linear response, the total potential difference at any point is just the sum of the potential differences associated with the two pulses; similarly, the current at any point on the line is the sum of the currents in the individual pulses. In particular, at z=0 where the two pulses collide the current is zero! The pulses pass through each other without interacting. During the overlap time, the voltage at z=0 will be twice as large as it would be for a single pulse. The system of two pulses shown in Figure (11.5.8) satisfies the boundary conditions at z=0 for an open line, i.e. I=0. It can be deduced from this that an observer placed to the left of the point z=0 could not tell the difference between an experiment in which a single pulse is injected into a line that is open (i.e. terminated by an open circuit) at z=0 or an experiment in which two mirror image pulses are injected into an infinite line from opposite directions.

Another way of obtaining this result starts from the general expressions for the voltage and current on a transmission line,

$$\mathrm{V(z,t)=F(z-vt)+G(z+vt)},$$

and

$$\mathrm{I(z,t)=F(z-vt)/Z_0-G(z+vt)/Z_0}.$$

The current must be zero for all times at an open circuit, i.e. at that point on the line

$$\mathrm{I=0=(F-G)/Z_0}.$$

It follows from this that at the open circuit F=G for all times, and therefore the reflected pulse at any time, G(z+vt), must be the mirror image of the incident pulse F(z-vt) where the mirror is located at the position of the open circuit.

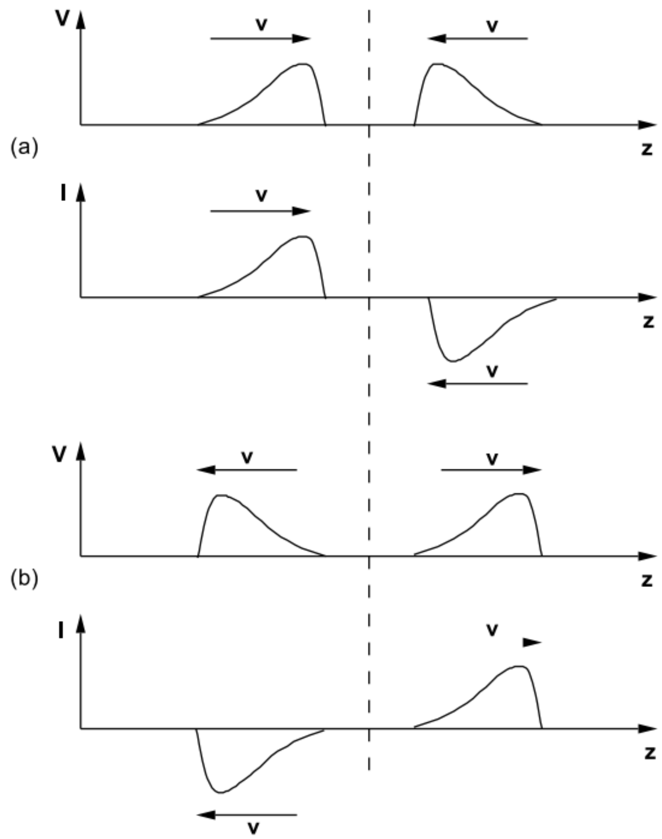


Figure 11.5.8: Pulses on an infinite transmission line. (a) Before the collision. (b) After the collision.

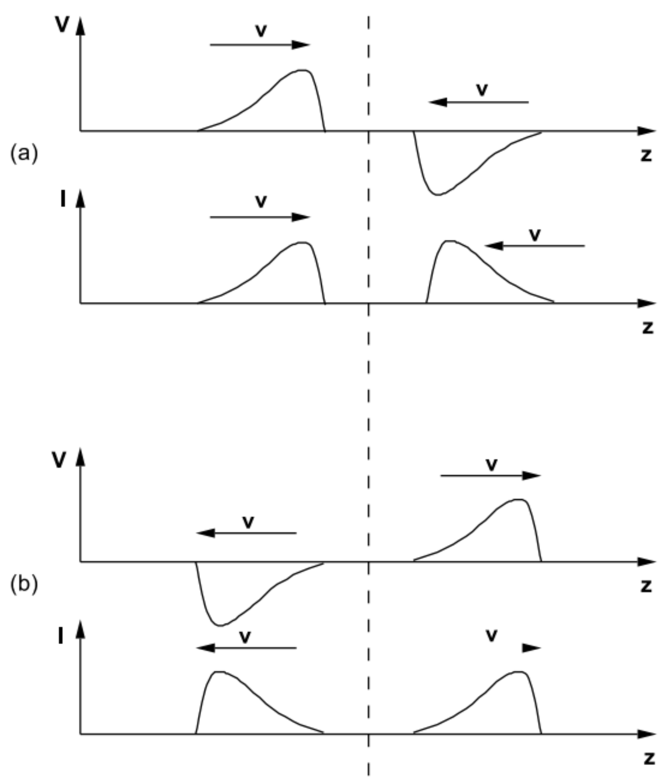


Figure 11.5.9: Pulses on an infinite transmission line. (a) Before the collision. (b) After the collision.

(ii) Two pulses of identical shape, but one pulse is a mirror image of the other and is inverted as shown in Figure (11.5.9), are launched towards one another on an infinite line. They collide at z=0. Because the system is a linear one the pulses simply pass through one another without interacting in any way. In this case the potential difference at z=0 always remains equal to zero because the two voltage pulses cancel one another. On the other hand, the two current pulses add, so that at z=0 the current becomes twice as large as it would be for the passage of a single pulse. Thus these two counter propagating pulses satisfy the boundary conditions at z=0 required for a short circuit, i.e. at z=0 one has R=V/I=0. We can deduce from this thought experiment that a pulse is inverted upon reflection from the end of a shorted line.

The principle illustrated by these two examples can be extended to cover the case of a line terminated by an arbitrary resistance R Ohms. Let the incident pulse have an amplitude V_0 Volts. Let the amplitude of the reflected pulse be V_R Volts. The corresponding currents are $I_0 = V_0/Z_0$ and $I_R = -V_R/Z_0$; the latter current is negative because the pulse is propagating from right to left. Suppose that the pulses collide at z=0. At z=0 one has, by superposition,

$$\mathrm{V} = \mathrm{V}_0 + \mathrm{V}_\mathrm{R}$$

and

$$\mathrm{I} = \mathrm{I}_0 + \mathrm{I}_\mathrm{R} = \frac{\mathrm{V}_0}{\mathrm{Z}_0} - \frac{\mathrm{V}_\mathrm{R}}{\mathrm{Z}_0}.$$

But we require R=V/I at the point z=0. Therefore

$$\mathrm{R} = \mathrm{Z}_0\left(\frac{\mathrm{V}_0 + \mathrm{V}_\mathrm{R}}{\mathrm{V}_0 - \mathrm{V}_\mathrm{R}}\right).$$

This equation can be solved in order to find the amplitude of the reflected pulse and the reflection coefficient, $\rho = V_R/V_0$:

$$\rho = \left(\frac{\mathrm{V}_\mathrm{R}}{\mathrm{V}_0}\right) = \frac{((\mathrm{R}/\mathrm{Z}_0) - 1)}{((\mathrm{R}/\mathrm{Z}_0) + 1)}. \tag{11.5.1}$$

The two cases explicitly treated above are contained in Equation (11.5.1) as limiting cases: if $R \to \infty$ then $\rho \to +1$, and the voltage pulse is reflected without a change in amplitude and with no change in sign; if $R \to 0$ then $\rho \to -1$, and the voltage pulse is reflected without a change in amplitude but the pulse is inverted. On the other hand, if $R = Z_0$ there is no reflected pulse because $\rho = 0$: the pulse is completely absorbed by the terminating resistance. A line terminated by a resistance equal to the characteristic impedance of the line looks like an infinite line to the generator.

This method for dealing with a discontinuity on the line can be extended to treat a capacitive or an inductive termination. At a capacitor one requires

$$\mathrm{Q} = \mathrm{CV}$$

or

$$\mathrm{I} = \frac{\mathrm{dQ}}{\mathrm{dt}} = \mathrm{C}\frac{\mathrm{dV}}{\mathrm{dt}}.$$

But $I = I_0 + I_R$ and $V = V_0 + V_R$, so that

$$\mathrm{I}_0 + \mathrm{I}_\mathrm{R} = \frac{\mathrm{V}_0}{\mathrm{Z}_0} - \frac{\mathrm{V}_\mathrm{R}}{\mathrm{Z}_0} = \mathrm{C}\left(\frac{\mathrm{dV}_0}{\mathrm{dt}} + \frac{\mathrm{dV}_\mathrm{R}}{\mathrm{dt}}\right).$$

This relation gives a differential equation from which $V_R(t)$ can be calculated from the known time dependence of the initial pulse $V_0(t)$:

$$\frac{\mathrm{dV}_\mathrm{R}}{\mathrm{dt}} + \left(\frac{1}{\mathrm{CZ}_0}\right)\mathrm{V}_\mathrm{R} = -\frac{\mathrm{dV}_0}{\mathrm{dt}} + \left(\frac{1}{\mathrm{CZ}_0}\right)\mathrm{V}_0(\mathrm{t}). \tag{11.5.2}$$

As an example, consider an incident rectangular pulse whose time dependence is shown in Figure (11.5.10). The derivative of a rectangular pulse consists of two very sharp impulses; one impulse is associated with the leading edge at t=0 where $dV/dt = V_0\delta(t)$, and the other impulse is associated with the trailing edge at t=T seconds where $dV/dt = -V_0\delta(t - T)$. In the time interval from t=0 to t=T the reflected pulse amplitude at the termination is given by (from Equation (11.5.2))

$$V_R(t) = V_0 - 2V_0 \exp(-t/\tau),$$

where $\tau = CZ_0$. At t=0 the reflected voltage pulse has the amplitude $V_R = -V_0$: this makes sense because the initially uncharged capacitor looks like a short circuit. The capacitor charges at a rate determined by the time constant $\tau = CZ_0$ until the voltage across it reaches the value $2V_0$: it then looks like an open circuit because it can accept no more charge. When the capacitor is fully charged and becomes equivalent to an open circuit the reflected pulse amplitude becomes equal to the incident pulse amplitude and the potential drop across the capacitor is $V = V_0 + V_R = 2V_0$. (It has been assumed that

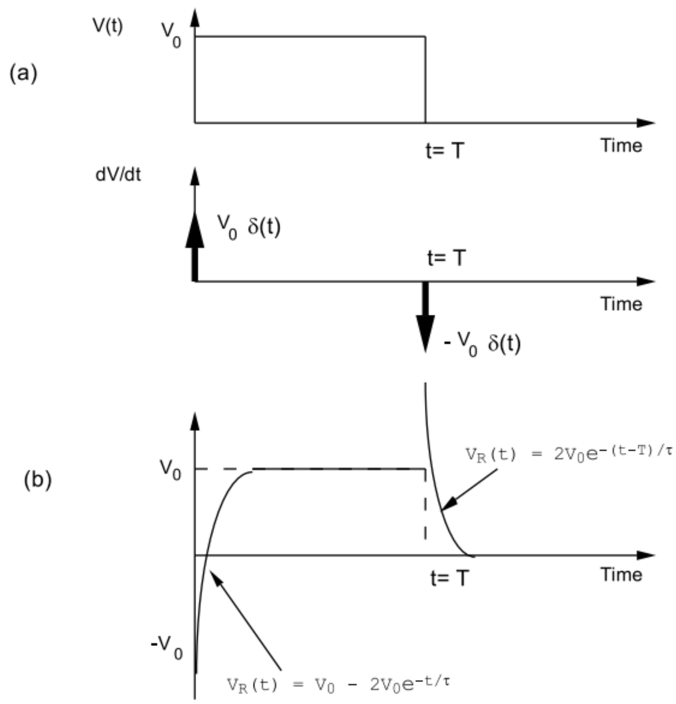


Figure 11.5.10: The reflection of a rectangular pulse from the end of a transmission line terminated by a capacitor C Farads. (a) The input pulse plus it's derivative. (b) The reflected voltage pulse. The input pulse duration is assumed to be long compared with the decay constant $\tau = CZ_0$.

the width of the pulse, T, is much longer than the time constant $\tau = CZ_0$). At the end of the incident pulse, t=T, the capacitor, charged to a potential difference of $2V_0$ Volts, just discharges into the line with a time constant $\tau = CZ_0$, and therefore

$$V_R(t) = 2V_0 \exp(-[t-T]/\tau)$$

for $t \geq T$.

Similar arguments can be used to discuss a line terminated by an inductor whose resistance is much smaller than the characteristic impedance, Z_0. The potential drop across an inductor is related to the current passing through it by the relation

$$V = L\left(\frac{dI}{dt}\right). \tag{11.5.3}$$

But at z=0 on the cable where the pulses overlap

$$I = I_0 + I_R = \frac{V_0}{Z_0} - \frac{V_R}{Z_0},$$

and

$$V = V_0 + V_R,$$

where $V_0(t)$ is the amplitude of the incident pulse and $V_R(t)$ is the corresponding amplitude of the reflected pulse. From the boundary condition Equation (11.5.3) one obtains

$$V_0(t) + V_R(t) = \frac{L}{Z_0}\left(\frac{dV_0}{dt} - \frac{dV_R}{dt}\right),$$

or

$$\frac{dV_R}{dt} + \frac{V_R}{\tau} = \left(\frac{dV_0}{dt}\right) - V_0/\tau, \tag{11.5.4}$$

where τ = L/Z_0. The time variation of the incident pulse at z=0 is known so that the differential Equation (11.5.4) can be solved for the time variation of the reflected pulse using the condition that $V_R(t)$ is identically zero for times before the incident pulse arrives at the termination. Consider, as an example, the rectangular pulse of duration T, shown in Figure (11.5.11), where the length of the pulse T is much longer than the time constant τ = L/Z_0. The derivative of the incident pulse is zero everywhere except at t=0 where $dV_0/dt = V_0\delta(t)$, and at t=T where $dV_0/dt = -V_0\delta(t-T)$. These impulses produce steps in

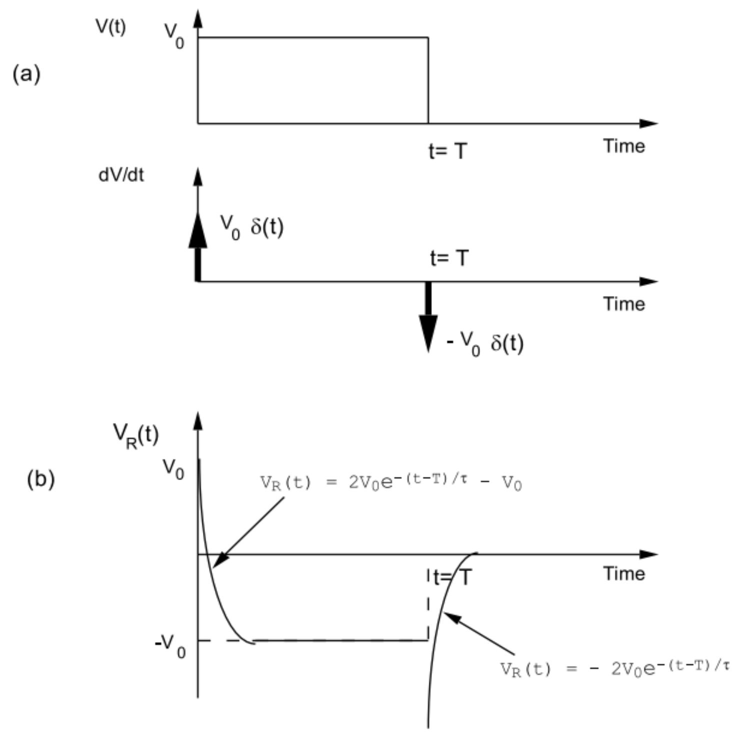


Figure 11.5.11: Reflection of a rectangular pulse from the end of a transmission line terminated by an inductance L Henries. (a) The input pulse plus its derivative. (b) The reflected voltage pulse, V_R. It has been assumed that the length of the input pulse is much greater than the time constant τ = L/Z_0.

the reflected voltage, V_R, of $+V_0$ at t=0 and of $-V_0$ at t=T. The reflected pulse amplitude at t=0 is $+V_0$ because initially the inductor looks like an open circuit since there is no current flow. Eventually the current through the inductor builds up to a steady state value and the voltage drop across the inductor becomes zero; at this point the inductor looks like a short circuit so that the reflected voltage pulse amplitude is $V_R = -V_0$. The time variation of the reflected voltage pulse is

$$V_R(t) = 2V_0 \exp(-t/\tau) - V_0,$$

for $0 < t \leq T$. Immediately after t=T seconds the driving pulse has become zero, and so the energy stored in the inductor decays into the transmission line at a rate determined by the inductance and the characteristic impedance, Z_0:

$$V_R(t) = -2V_0 \exp(-[t-T]/\tau),$$

for $t \geq T$. The reflected pulse has a negative voltage because because the collapse of the magnetic field in the inductor coil operates to maintain a positive current flow. In a reflected pulse a positive current flow is associated with a negative voltage.

The above methods can be extended to treat a transmission line terminated by an arbitrary impedance.

11.6: Sinusoidal Signals on a Terminated Line

Let a transmission line having a characteristic impedance Z_0 be used to connect a sinusoidal signal generator to a load, Z_L, as shown in Figure (11.6.12). In phasor notation the generator voltage may be written

$$V_G(t) = V_0 \exp[i\omega t]; \tag{11.6.1}$$

this corresponds to a real time variation cos ωt. A positive sign has been used in the phasor exponential in accord with the usual engineering convention for the description of alternating current circuits. The potential difference at any point along the line will consist of a forward propagating wave plus a backward propagating wave due to a reflection from the load. Recall that the current and potential waves must be functions of (z-vt) and (z+vt) in order to satisfy Maxwell's equations (11.3.8). The forward propagating voltage wave

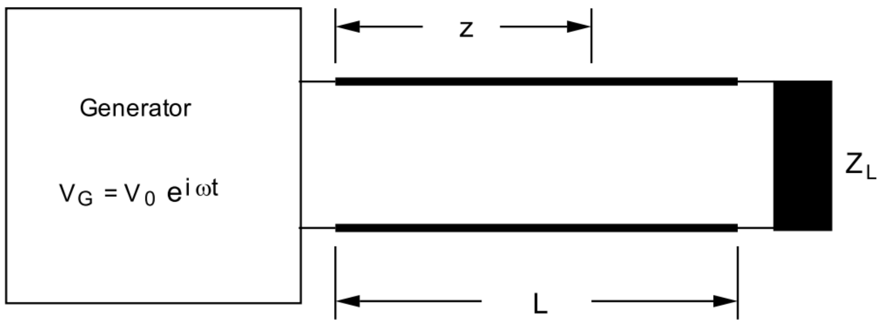


Figure 11.6.12: A transmission line of length L meters is used to connect a sinusoidal generator to a load impedance Z_L Ohms.

must therefore have the form

$$V_f(z,t) = a \exp\left(-i\frac{\omega}{v}[z - vt]\right) = a \exp\left(-i\frac{\omega z}{v}\right) \exp(i\omega t),$$

and the reflected wave must have the form

$$V_r(z,t) = b \exp\left(i\frac{\omega}{v}[z + vt]\right) = b \exp\left(i\frac{\omega Z}{v}\right) \exp(i\omega t),$$

where a,b are constants that must be determined from the generator potential and from the boundary conditions at the load, i.e. Z_L = V/I. It is customary to write k = ω/v, where ω = 2πf and where v is the velocity of a pulse on the cable. The potential difference at any point along the line is given by

$$V(z,t) = (a\exp(-ikz) + b\exp(+ikz))\exp(+i\omega t), \tag{11.6.2}$$

and the current is given by

$$I(z,t) = \left(\left(\frac{a}{Z_0}\right)\exp(-ikz) - \left(\frac{b}{Z_0}\right)\exp(+ikz)\right)\exp(+i\omega t). \tag{11.6.3}$$

The expression (11.6.3) for the current follows from Equation (11.6.2) for the voltage combined with the transmission line characteristic impedances for forward and backward propagating waves, Equations (11.2.8) and (11.2.9). At the generator, assumed to be located at z=0, one has

$$V_0 = a + b.$$

At the load, assumed to be at z=L, one has

$$V/I = Z_L,$$

or

$$\left(\frac{a\exp(-ikL) + b\exp(+ikL)}{a\exp(-ikL) - b\exp(+ikL)}\right) = \left(\frac{Z_L}{Z_0}\right).$$

The above two equations can be solved to give

$$\begin{aligned}a&=\frac{\left(\frac{z_L}{Z_0}+1\right)V_0}{\left(\left(\frac{z_L}{Z_0}+1\right)+\left(\frac{z_L}{Z_0}-1\right)\exp(-2ikL)\right)}\\ b&=\frac{\left(\frac{Z_L}{Z_0}-1\right)V_0\exp(-2ikL)}{\left(\left(\frac{Z_L}{Z_0}+1\right)+\left(\frac{Z_L}{Z_0}-1\right)\exp(-2ikL)\right)}.\end{aligned}\tag{11.6.4}$$

The impedance as seen by the generator can be obtained from the voltage and the current at z=0:

$$\mathrm{Z_G}=\frac{\mathrm{V(z=0)}}{\mathrm{I(z=0)}}=\frac{\mathrm{V_0}}{[\mathrm{a-b}]}\mathrm{Z_0},$$

or

$$\frac{\mathrm{Z_G}}{\mathrm{Z_0}}=\frac{\left(\frac{\mathrm{z_L}}{\mathrm{Z_0}}+1\right)+\left(\frac{\mathrm{Z_L}}{\mathrm{Z_0}}-1\right)\exp(-2i\mathrm{kL})}{\left(\left(\frac{\mathrm{z_L}}{\mathrm{Z_0}}+1\right)-\left(\frac{\mathrm{Z_L}}{\mathrm{Z_0}}-1\right)\exp(-2i\mathrm{kL})\right)}.$$

Or, introducing the reduced impedance $z_G = Z_L/Z_0$ and the new variable Γ, where

$$\Gamma=\left(\frac{\frac{\mathrm{Z_L}}{\mathrm{Z_0}}-1}{\frac{\mathrm{Z_L}}{\mathrm{Z_0}}+1}\right)=\left(\frac{\mathrm{z_L}-1}{\mathrm{z_L}+1}\right),\tag{11.6.5}$$

one finds

$$\mathrm{z_G}=\left(\frac{1+\Gamma\exp(-2i\mathrm{kL})}{1-\Gamma\exp(-2i\mathrm{kL})}\right)=\left(\frac{\exp(+i\mathrm{kL})+\Gamma\exp(-i\mathrm{kL})}{\exp(+i\mathrm{kL})-\Gamma\exp(-i\mathrm{kL})}\right).\tag{11.6.6}$$

In the above development it has been assumed that the cable is lossless.

The expression for the load on the generator, Equation (11.6.6), is rather complicated but it should be clear that the impedance as seen from the generator may be quite different from the load impedance especially if the length of the cable, L, is comparable to, or larger than, the wavelength of the disturbance on the cable, λ, where

$$\mathrm{k}=\frac{2\pi}{\lambda}=\omega/\mathrm{v}.$$

A few concrete examples may help to form a picture of how a cable can be used to transform a load impedance.

11.6.1 Case(1). A Shorted Cable.

For this case Z_L = 0 and Γ= -1 from Equation (11.6.5) so that

First of all, notice that the impedance as seen from the generator is not in general equal to zero: in fact, when the cable length is such that kL=π/2, 3π/2, 5π/2, etc. the generator appears to be attached to an open circuit! If cable losses are taken into account (see below), the load on the generator will be finite at these lengths but large compared with the characteristic impedance providing that the line is not too long. The condition kL=π/2 corresponds to a cable that is a quarter wavelength long, L=λ/4. Secondly, if the impedance as seen from the generator is not zero (i.e. L not a multiple of a half-wavelength) or infinite (L an odd multiple of a quarter wavelength) it appears to be a pure reactance if the cable is lossless. This makes sense since a lossless cable and a lossless load cannot absorb any energy from the generator. For example, if kL=π/4, L=λ/8, the impedance at the generator is Z_G = +iZ_0, and therefore the generator looks into a purely inductive load.

11.6.2 Case(2). An Open-ended Cable.

For this case ZL = ∞ and therefore Γ = +1 from Equation (11.6.5). The reduced impedance at the generator terminals is given by

$$\mathrm{z_G}=\frac{1+\exp(-2i\mathrm{kL})}{1-\exp(-2i\mathrm{kL})}=-i\cot\mathrm{kL}.$$

The generator load appears to be an open circuit if the length of the cable is a multiple of a half-wavelength. A cable whose length is an odd multiple of a quarter wavelength presents a short circuit to the generator. For other cable lengths the generator would appear to be connected to a capacitor or an inductor according to whether cot kL was positive or negative.

11.6.3 Case(3). The Cable is Terminated by the Characteristic Impedance.

For this case $\Gamma = 0$ from Equation (11.6.5) and therefore the impedance at the generator is $z_G = 1$, or $Z_G = Z_0$. T**he load across the generator is independent of the length of the cable.**

11.6.4 Case(4). A Purely Inductive Load.

Let the inductor be such that $Z_L = iL\omega$ is equal in magnitude to the characteristic impedance, Z_0. Then $z_L = Z_L/Z_0$ and therefore

$$\Gamma=\frac{i-1}{i+1}=+i.$$

The normalized load on the generator, from Equation (11.6.6), is

$$\mathrm{z_G}=\frac{\mathrm{Z_G}}{\mathrm{Z_0}}=\left(\frac{1+\sin 2\mathrm{kL}+i\cos 2\mathrm{kL}}{1-\sin 2\mathrm{kL}-i\cos 2\mathrm{kL}}\right).$$

In the limit as $k_L \rightarrow \pi/4$ ($L \rightarrow \lambda/8$) the generator appears to be attached to an open circuit. However, for a quarter wavelength line, $kL = \pi/2$, the generator load appears to be due to a pure capacitance such that $1/C\omega = Z_0$. For $kL=3\pi/4$, $L = 3\lambda/8$, the generator looks into a short circuit. Finally for a half-wavelength cable the generator sees a an inductive reactance such that $L\omega = Z_0$. As the cable length is increased further the whole cycle is repeated.

11.6.5 Case(5). A Purely Capacitive Load.

Let the capacitor be such that $Z_L = -i/C\omega$ has the magnitude of the characteristic impedance, Z_0. For this case $z_L = Z_L/Z_0 = -i$, and

$$\Gamma=\frac{z_L-1}{z_L+1}=-i.$$

The normalized impedance as seen by the generator is given by

$$\mathrm{z_G}=\frac{\mathrm{Z_G}}{\mathrm{Z_0}}=\left(\frac{1-\sin 2\mathrm{kL}-i\cos 2\mathrm{kL}}{1+\sin 2\mathrm{kL}+i\cos 2\mathrm{kL}}\right).$$

This case is very similar to that of the cable terminated by an inductor. For $kL=\pi/4$, $L=\lambda/8$, the generator is short circuited because $z_G = 0$. For a quarter wavelength line, $kL=\pi/2$, the generator looks into a pure inductance, $z_G= +i$. For $kL=3\pi/4$, $L=3\lambda/8$, one finds that $z_G \rightarrow \infty$ so that the generator appears to be looking into an open circuit. Finally, for a half-wavelength line, $kL=\pi$, the same effect is obtained as if the load were connected directly across the generator. The whole cycle is repeated as the cable length is increased.

11.6.6 Summary

The following conclusions can be drawn from the above examples:

1. A cable acts like an impedance transformer.
2. A lossless cable whose length is an integral number of half-wavelengths long effectively places the load directly across the generator terminals, i.e. $Z_G = Z_L$.
3. A lossless cable whose length is an odd multiple of a quarter wavelength acts like an impedance inverter. For this case exp $(-2ikL) = -1$ so that from Equation (11.6.6)

$$\mathrm{z_G}=\frac{1-\Gamma}{1+\Gamma}\equiv 1/\mathrm{z_L}.$$

The formula for the impedance at the generator terminals in terms of the load impedance and the length of the cable that connects the load to the generator, Equation (11.6.6), is very complicated. Graphical methods have been developed for determining the load on the generator given the load impedance, ZL, and the characteristics of the transmission line. A very common method is based on the use of a Smith chart: it is described in detail in the book ”Microwave Measurements” by E.L.Ginzton, McGraw-Hill, New

York, 1957; section 4.9. The need for a graphical technique has become very much less pressing now that digital computers have become readily available.

11.7: The Slotted Line

A slotted line is a section of a co-axial cable that uses air as a dielectric medium and in which a narrow slot has been cut along the length of the outer conductor; the slot is sufficiently narrow so that its presence does not appreciably affect the electric field distribution between the conducting cylinders which form the walls of the cable. A thin pin is inserted through the slot and is used to pick up a signal that is a measure of the electric field strength in the co-axial cable. This pin is mounted on a carriage whose position along the slotted line can be accurately measured. A picture and a sketch of a slotted line can be found in Ginzton's book (see his Figures (5.11) and (5.12)). The signal picked up by the probe pin is usually rectified by means of a high frequency diode and it is the resulting dc signal that is measured. The dc signal provides a measure of the amplitude of the potential difference at any point along the slotted line. If the signal picked up is very small, less than ∼1 mV say, the dc signal provides a measure of the time averaged square of the potential difference between the outer and inner conductors; in many instances the signal picked up is greater than 10 mV, and in such cases the high frequency diodes commonly used for such measurements produce a dc signal that is proportional to the root mean square of the potential difference between the outer and inner conductors. If the slotted line is used to connect a generator operating at a fixed frequency with a load that is different from the characteristic impedance of the line, the rectified probe signal will be found to exhibit a sinusoidal variation between a maximum signal and a minimum signal as the probe is moved along the line. The ratio of the maximum signal to the minimum signal as the probe is moved along the slotted line provides a measure of the ratio of the forward wave amplitude to the reflected wave amplitude, i.e. the amplitudes a and b of Equation (11.6.2) that describes the position dependence of the voltage along the cable.

From Equation (11.6.2)

$$\mathrm{V}(\mathrm{z}, \mathrm{t})=\mathrm{a} \exp (-i \mathrm{kz})\left(1+\frac{\mathrm{b}}{\mathrm{a}} \exp (2 i \mathrm{kz})\right) \exp (i \omega \mathrm{t}),$$

where, from Equations (11.6.4)

$$\left(\frac{\mathrm{b}}{\mathrm{a}}\right)=\left(\frac{\frac{\mathrm{Z}_{\mathrm{L}}}{\mathrm{Z}_{0}}-1}{\frac{\mathrm{Z}_{\mathrm{L}}}{\mathrm{Z}_{0}}+1}\right) \exp (-2 i \mathrm{kL}).$$

But by definition, Equation (11.6.5),

$$\Gamma=|\Gamma| \exp (i \theta)=\left(\frac{\frac{\mathrm{Z}_{\mathrm{L}}}{\mathrm{Z}_{0}}-1}{\frac{\mathrm{Z}_{\mathrm{L}}}{\mathrm{Z}_{0}}+1}\right),$$

or, in terms of the normalized impedance

$$\mathrm{z}_{\mathrm{L}}=\frac{\mathrm{Z}_{\mathrm{L}}}{\mathrm{Z}_{0}}$$

one has

$$\Gamma=\left(\frac{\mathrm{z}_{\mathrm{L}}-1}{\mathrm{z}_{\mathrm{L}}+1}\right)=|\Gamma| \exp (i \theta). \tag{11.7.1}$$

The ratio (b/a) can be written

$$\left(\frac{b}{a}\right)=|\Gamma| \exp (-i[2 k L-\theta]),$$

from which one obtains

$$\mathrm{V}(\mathrm{z}, \mathrm{t})=\mathrm{a} \exp (-i \mathrm{kz})(1+|\Gamma| \exp (i[2 \mathrm{k}(\mathrm{z}-\mathrm{L})+\theta])) \exp (i \omega \mathrm{t}),$$

and

$$\mathrm{V}^{*}(\mathrm{z}, \mathrm{t})=\mathrm{a}^{*} \exp (+i \mathrm{kz})(1+|\Gamma| \exp (-i[2 \mathrm{k}(\mathrm{z}-\mathrm{L})+\theta])) \exp (-i \omega \mathrm{t}),$$

where V* (z,t) is the complex conjugate of the potential function V(z,t). The time averaged value of the square of the voltage is given by

$$< V^2 >= \frac{1}{2}\mathrm{Real}(VV^*),$$

or

$$< V^2 >= \frac{|a|^2}{2}\left(1+|\Gamma|^2+2|\Gamma|\cos(2k[z-L]+\theta)\right). \tag{11.7.2}$$

The maximum value of $< V^2 >$, or of $\sqrt{< V^2 >}$, occurs at those positions z such that (2k[z − L] + θ) = 2πn where n is an integer. These maxima are spaced a half-wavelength apart; the variation of the pick-up signal as the probe is moved along the slotted line provides a direct measure of the wavelength of the radiation. The maximum value of the root mean square potential difference is

$$(\sqrt{< V^2 >})_{\max} = \frac{|a|}{\sqrt{2}}(1+|\Gamma|).$$

The minimum value of the pick-up signal occurs at those positions such that (2k[z − L]+θ) = πm where m is an odd integer: these minima are also spaced one half-wavelength apart. At a minimum the root mean square voltage is

$$(\sqrt{< V^2 >})_{\min} = \frac{|a|}{\sqrt{2}}(1-|\Gamma|).$$

The ratio of these two voltages is called the ”Voltage Standing Wave Ratio” and is usually designated by VSWR:

$$\mathrm{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|}. \tag{11.7.3}$$

This expression can be inverted to give

$$|\Gamma| = \left(\frac{\mathrm{VSWR}-1}{\mathrm{VSWR}+1}\right). \tag{11.7.4}$$

The Voltage Standing Wave Ratio, which can be easily measured by means of a slotted line, provides information about the load impedance through the absolute value of the parameter $\Gamma = (z_L - 1)/(z_L + 1)$. In order to determine the phase of the load impedance it is also necessary to measure the phase of the parameter Γ: from the definition of Γ

$$z_L = \frac{Z_L}{Z_0} = \left(\frac{1+\Gamma}{1-\Gamma}\right). \tag{11.7.5}$$

Thus a knowledge of the amplitude and phase of Γ serves to determine the amplitude and phase of the load impedance, Z_L. The phase of Γ can be obtained from the position of the voltage maximum or minimum on the slotted line; it is preferable to use the position of the minimum because the position of a minimum signal can be measured much more accurately than the position of a maximum signal.

The structure of the relationship between the complex number Γ and the complex load impedance, $z_L = Z_L/Z_0$, is such that for a load impedance

having an inductive component the phase angle θ must lie between 0 and π radians, whereas for a load impedance having a capacitive component the phase angle θ must lie between 0 and −π radians. As an example consider a purely inductive load such that $z_L = +i\beta$. For this case

$$\Gamma = \frac{-1+i\beta}{1+i\beta}.$$

The numerator of Γ may be written

$$N = \sqrt{1+\beta^2}\,\exp(i[\pi-\phi]),$$

where tan ϕ = β. The denominator of Γ may be written

$$D = \sqrt{1+\beta^2}\,\exp(i\phi),$$

where, as above, tan ϕ = β. Thus for this example

$$\Gamma = \exp(i[\pi - 2\phi]),$$

where for β → 0 θ → ϕ, and for β very large θ → 0. This case is a special one but using complex algebra it can be shown that θ must lie between 0 and ϕ radians for any load having an inductive component. Consider the position of a minimum in the slotted line voltage corresponding to an inductive load. From Equation (11.7.2) one of the minima occurs at position z when

$$2\mathrm{k}(\mathrm{z}-\mathrm{L})+\theta=\pi,$$

or since k = 2ϕ/λ, the minimum occurs at

$$\mathrm{z}=\mathrm{L}+\frac{(\pi-\theta)}{4\pi}\lambda.$$

The position of this particular minimum ranges from z=L for θ = π to z = L + (λ/4) for θ = 0. Clearly one cannot measure the position of this minimum because it lies outside the slotted line (z>L). However, the pattern described by Equation (11.7.2) repeats itself every half wavelength along the slotted line. Therefore one has only to measure the position of the voltage minimum relative to a position, z_2, located exactly one half wavelength from the end of the slotted line. This position, z_2, can be found simply by locating the position of the appropriate minimum signal when the load impedance is replaced by a short circuit. The condition for the position of a minimum voltage with the load in place can then be written

$$2\mathrm{k}\left(\mathrm{z}-\mathrm{z}_{2}\right)+\theta=\pi$$

and therefore

$$\theta=\pi-\frac{4\pi}{\lambda}\left(\mathrm{z}-\mathrm{z}_{2}\right), \tag{11.7.6}$$

where z > z_2 for a load having an inductive component.

For a load having a capacitive component consider the minimum corresponding to

$$2\mathrm{k}(\mathrm{z}-\mathrm{L})+\theta=-\pi.$$

This minimum in the standing wave voltage occurs at

$$\mathrm{z}=\mathrm{L}-\frac{(\pi+\theta)}{4\pi}\lambda.$$

Since θ lies between 0 and $-\pi$ radians for this case the position of the minimum varies from z = L − λ/4 to z=L. In other words the position of the minimum is shifted towards the generator. This minimum is accessible on the slotted line but it is more convenient to measure the position of that particular minimum that is located near z_2, the position that is a half wavelength removed from the end of the slotted line. The condition that describes the position of the minimum that lies within λ/4 of z_2 is given by

$$2\mathrm{k}\left(\mathrm{z}-\mathrm{z}_{2}\right)+\theta=-\pi,$$

and therefore

$$\theta=-\pi+\frac{4\pi}{\lambda}\left(\mathrm{z}_{2}-\mathrm{z}\right), \tag{11.7.7}$$

where for an impedance having a capacitive component z_2 > z.

Recapitulation

In order to determine an unknown impedance using a slotted line one must obtain the amplitude of the parameter Γ from the voltage standing wave ratio, Equation (11.7.4), as well as the phase of Γ from the position of a voltage minimum on the slotted line. The phase of Γ, θ, can be determined from the position of the minimum, z, relative to a position, z_2, located exactly λ/2 from the end of the slotted line. The position of z_2 is determined by the position of the appropriate minimum when the slotted line is terminated with a short circuit. With the slotted line terminated by the unknown impedance one looks for a voltage minimum located within λ/4 of the shorted position z_2. If the position of this minimum is displaced from z_2 towards the **load** then that impedance has an inductive component and the phase angle θ is to be calculated using Equation (11.7.6). If the position of the minimum is displaced from z_2 towards the generator then the load has a capacitive component and the phase angle θ is to be calculated using Equation (11.7.7).

11.8: Transmission Line with Losses

The voltage and current on a lossless transmission line must satisfy the following equations:

$$\begin{aligned}\frac{\partial^2 \mathrm{V}}{\partial \mathrm{z}^2}=\epsilon \mu_0 \frac{\partial^2 \mathrm{V}}{\partial \mathrm{t}^2}, \\ \frac{\partial^2 I}{\partial z^2}=\epsilon \mu_0 \frac{\partial^2 I}{\partial t^2}.\end{aligned} \tag{11.8.1}$$

These are a direct consequence of Maxwell's equations. Strictly speaking, they are only correct providing that ϵ, the dielectric constant, is truly a constant and therefore independent of frequency. Even the best of dielectric insulating materials exhibit some losses that are frequency dependent: in many cases the imaginary part of the dielectric constant is proportional to the frequency. For a time dependence exp (iωt) the above two equations, (11.8.1), become

$$\begin{aligned}\frac{\partial^2 V}{\partial z^2}=-\epsilon \mu_0 \omega^2 V, \\ \frac{\partial^2 I}{\partial z^2}=-\epsilon \mu_0 \omega^2 I,\end{aligned} \tag{11.8.2}$$

where ϵ may have real and imaginary parts, both of which will depend upon the frequency. Solutions of Equations (11.8.2) that are harmonic in space, i.e. V and I are proportional to exp (−ikz), must be described by a wave-vector k that satisfies the condition

$$\mathrm{k}^2=\epsilon \mu_0 \omega^2,$$

In the presence of dielectric losses ϵ will in general be a complex quantity, and therefore so also must the wave-vector be complex :

$$\mathrm{k}=\pm \omega \sqrt{\epsilon \mu_0} \tag{11.8.3}$$

so that

$$\mathrm{k}=\pm\left(\mathrm{k}_1-\mathrm{ik}_2\right). \tag{11.8.4}$$

The general solutions of the wave equations (11.8.2) for the voltage and current on the transmission line in the presence of a lossy dielectric can be written

$$\begin{aligned}\mathrm{V}(\mathrm{z}, \mathrm{t})=\left[\mathrm{a} \exp \left(-\mathrm{k}_2 \mathrm{z}\right) \exp \left(-i \mathrm{k}_1 \mathrm{z}\right)+\mathrm{b} \exp \left(\mathrm{k}_2 \mathrm{z}\right) \exp \left(i \mathrm{k}_1 \mathrm{z}\right)\right] \cdot \exp (i \omega \mathrm{t}), \\ \mathrm{I}(\mathrm{z}, \mathrm{t})=\frac{1}{\mathrm{Z}_0}\left[\mathrm{a} \exp \left(-\mathrm{k}_2 \mathrm{z}\right) \exp \left(-i \mathrm{k}_1 \mathrm{z}\right)-\mathrm{b} \exp \left(\mathrm{k}_2 \mathrm{z}\right) \exp \left(i \mathrm{k}_1 \mathrm{z}\right)\right] \exp (i \omega \mathrm{t}),\end{aligned} \tag{11.8.5}$$

where $(k_2/k_1) \ll 1$ for a high quality cable, and k_1, k_2 are the real and imaginary parts of the wave-vector. Notice that k_2 must be positive in order that the amplitude of the forward propagating wave decays with distance.

In actual fact, part of the energy loss as a wave propagates down a transmission line is due to Ohmic losses in the skin-depth of the conductors: i.e. the metal electrodes do possess a finite conductivity and therefore there are energy losses due to the shielding currents that flow in them. It can be easily shown, using the methods of Chapter(10), that the rate of energy loss in each conductor per unit area of surface is given by

$$<\mathrm{S}_{\mathrm{n}}>=\frac{1}{2} \sqrt{\frac{\omega \mu_0}{2 \sigma_0}}\left|\mathrm{H}_0\right|^2 \quad \text { Watts } / m^2.$$

$< S_n >$ is the time averaged Poynting vector component corresponding to energy flow into the conductor surface, σ_0 is the dc conductivity of the metal wall, H_0 is the magnetic field strength at the conductor surface, and $\omega = 2\pi f$ is the circular frequency. This energy loss must be added to the energy loss in the dielectric material. The conductor losses can be taken into account by increasing the imaginary part of the wave-vector, k_2, in Equations (11.8.5). One can write

$$\begin{aligned}\mathrm{V}(\mathrm{z}, \mathrm{t})=\left[\mathrm{a} \exp (-\alpha \mathrm{z}) \exp \left(-i \mathrm{k}_1 \mathrm{z}\right)+\mathrm{b} \exp (\alpha \mathrm{z}) \exp \left(+i \mathrm{k}_1 \mathrm{z}\right)\right] \exp (i \omega t) \\ \mathrm{I}(\mathrm{z}, \mathrm{t})=\frac{1}{\mathrm{Z}_0}\left[\mathrm{a} \exp (-\alpha \mathrm{z}) \exp \left(-i \mathrm{k}_1 \mathrm{z}\right)-\mathrm{b} \exp (\alpha \mathrm{z}) \exp \left(+i \mathrm{k}_1 \mathrm{z}\right)\right] \exp (i \omega \mathrm{t})\end{aligned} \tag{11.8.6}$$

where α is an empirical parameter whose frequency dependence can be measured for a particular cable. The constants a,b in (11.8.6) must be adjusted to satisfy the boundary condition at the position of the load; i.e. at the load Z_L= V/I. For a cable having a characteristic impedance Z_0 that connects a generator at z=0 with a load at z=L this condition requires

$$\frac{Z_L}{Z_0}=\left[\frac{a\exp(-\alpha L)\exp(-ik_1 L)+b\exp(\alpha L)\exp(ik_1 L)}{a\exp(-\alpha L)\exp(-ik_1 L)-b\exp(\alpha L)\exp(ik_1 L)}\right],$$

from which

$$\frac{b}{a}=\left[\frac{Z_L-1}{Z_0}-1\right]\exp(-2\alpha L)\exp(-2ik_1 L).$$

Using the previous notation $z_L = Z_L/Z_0$, and $z_G = Z_G/Z_0$, and

$$\Gamma=\frac{z_L-1}{z_L+1}=|\Gamma|\exp(i\theta),$$

one finds

$$z_G=\left[\frac{1+\Gamma\exp(-2\alpha L)\exp(-2ik_1 L)}{1-\Gamma\exp(-2\alpha L)\exp(-2ik_1 L)}\right]. \tag{11.8.7}$$

Equation (11.8.7) shows that the impedance seen by the generator approaches the characteristic impedance of the cable if the load is connected to the generator through a cable that is long compared with the attenuation length $(1/\alpha)$.

Characteristics for a few representative co-axial cables are listed in Table(11.8.1), and their attenuation lengths at a number of frequencies are listed in Table(11.8.2). The length of cable for which the amplitude of a voltage pulse is attenuated to (1/e)= 0.368 of its original amplitude is given by $(1/\alpha)$. For example, this attenuation length is 9.8 meters for RG-8 cable at 5 GHz.

The attenuation parameter, α, for the cables listed in Table(11.8.2) are observed to be approximately proportional to $\sqrt{\omega}$, and this suggests that most of the losses in these cables is due to eddy currents in the conductors.

Cable	Outer Diam. in inches	Characteristic Impedance,Ohms	Velocity m/sec	Capacitance Farads/m
RG-8	0.405	52	1.98×10^8	96.8×10^{-12}
RG-58U	0.195	53	1.98×10^8	93.5×10^{-12}
RG-59U	0.242	75	1.98×10^8	68.9×10^{-12}
RG-62U	0.242	93	2.52×10^8	44.3×10^{-12}
RG-174U	0.100	50	1.98×10^8	98.4×10^{-12}

Table 11.8.1: Characteristics of some commonly used commercial co-axial cables. The dielectric material between the conductors is polyethylene. The data was taken from the 1985/86 catalogue of RAE Industrial Electronics Ltd., Vancouver, BC.

Cable	1.0	10.0	50.0	100	200	400	1000	3000	5000
RG-8	5.67 $\times10^{-4}$	2.08 $\times10^{-3}$	4.91 $\times10^{-3}$	7.18 $\times10^{-3}$	1.02 $\times10^{-2}$	1.55 $\times10^{-2}$	0.030	0.060	0.102
RG-58U	1.25 $\times10^{-3}$	4.72 $\times10^{-3}$	1.19 $\times10^{-2}$	1.74 $\times10^{-2}$	2.61 $\times10^{-2}$	3.97 $\times10^{-2}$	6.61 $\times10^{-2}$	0.142	0.227
RG-59U	1.25 $\times10^{-3}$	4.16 $\times10^{-3}$	9.07 $\times10^{-3}$	1.28 $\times10^{-2}$	1.85 $\times10^{-2}$	2.64 $\times10^{-2}$	4.53 $\times10^{-2}$	0.100	0.196
RG-62U	9.4 $\times10^{-4}$	3.21 $\times10^{-3}$	7.18 $\times10^{-3}$	1.02 $\times10^{-2}$	1.44 $\times10^{-2}$	2.00 $\times10^{-2}$	3.29 $\times10^{-2}$	0.070	0.113
RG-174U	8.69 $\times10^{-3}$	1.47 $\times10^{-2}$	2.49 $\times10^{-2}$	3.36 $\times10^{-2}$	4.53 $\times10^{-2}$	6.61 $\times10^{-2}$	1.13 $\times10^{-1}$	0.242	0.374

Table $11.8.2$: Frequency dependence of the attenuation parameter α for some selected co-axial cables. V(z) = V_0 exp $(-\alpha z)$. Frequencies in MHz. The data is taken from the 1985/86 catalogue of RAE Industrial Electronics Ltd., Vancouver, BC.

CHAPTER OVERVIEW

12: Waveguides

A study of the propagation of electromagnetic waves through rectangular and circular hollow conducting pipes.

Thumbnail: Waveguide flange UBR320 for microwaves. (Public Domain; Catslash via Wikipedia)

12.1: Simple Transverse Electric Modes

Consider two infinite plane waves of circular frequency ω oscillating in phase, and such that their propagation vectors lie in the x-z plane and make the angles θ with the z-axis: one wave has a positive x-component of wavevector, the other has a negative x-component of wave-vector, as illustrated in Figure 12.1.1. Explicit expressions for the electric and magnetic field components of these waves for the case in which the electric field in each wave has the same amplitude and is polarized along the y-direction are as follows:

Wave Number (1)

$$\begin{aligned} \mathrm{E_{y1}} &= \mathrm{E_0} \exp(i\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{H_{x1}} &= -\frac{\mathrm{E_0}}{\mathrm{Z}}\cos\theta \exp(i\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{H_{z1}} &= \frac{\mathrm{E_0}}{\mathrm{Z}}\sin\theta \exp(i\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega \mathrm{t}]), \end{aligned}$$

Wave Number (2)

$$\begin{aligned} \mathrm{E_{y2}} &= \mathrm{E_0} \exp(-i\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{H_{x2}} &= -\frac{\mathrm{E_0}}{\mathrm{Z}}\cos\theta \exp(-i\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega \mathrm{t}]), \\ \mathrm{H_{z2}} &= -\frac{\mathrm{E_0}}{\mathrm{Z}}\sin\theta \exp(-i\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega \mathrm{t}]). \end{aligned}$$

In writing these equations it has been assumed that the waves are propagating in a medium characterized by a real dielectric constant $\epsilon = \epsilon_r \epsilon_0$, and a magnetic permeability μ_0. The wave-vector is $\mathrm{k} = \sqrt{\epsilon_r}(\omega/\mathrm{c})$, and the wave impedance is $\mathrm{Z} = \sqrt{\mu_0/\epsilon} = \mathrm{Z_0}/\sqrt{\epsilon_r}$ Ohms, where Z_0 = 377 Ohms. The above fields satisfy Maxwell's equations. One can now introduce two perfectly conducting infinite planes that lie parallel with the xz plane and which are separated by an arbitrary spacing, b. The plane waves of Figure 12.1.1 still satisfy Maxwell's equations between the conducting surfaces: they also satisfy the required boundary conditions on the electric and magnetic fields. In the first place, there is only one electric field component, E_y, and it is normal to the conducting planes, consequently the tangential component of $\vec{E}$ is zero on the perfectly conducting surfaces as is required. In the second place, the magnetic field components lie parallel with the conducting planes so that the normal component of $\vec{H}$ is zero at the perfectly conducting planes as is required by the considerations discussed in Chpt.(10). The total electric field at any point in the space between the two conducting planes is given by

$$\mathrm{E_y} = \mathrm{E_{y1}} + \mathrm{E_{y2}},$$

or

$$\mathrm{E_y} = 2\mathrm{E_0}\cos(\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega\mathrm{t}]). \tag{12.1.1}$$

The components of the magnetic field are given by

$$\mathrm{H_x} = -\frac{2\mathrm{E_0}\cos\theta}{\mathrm{Z}}\cos(\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega\mathrm{t}]), \tag{12.1.2}$$

and

$$\mathrm{H_z} = +i\frac{2\mathrm{E_0}\sin\theta}{\mathrm{Z}}\sin(\mathrm{xk}\sin\theta)\exp(i[\mathrm{zk}\cos\theta - \omega\mathrm{t}]). \tag{12.1.3}$$

Notice that E_y and H_x are both zero, independent of z, on the planes defined by (xk sin θ) = ±π/2, ±3π/2, ±5π/2, etc, i.e. on the planes

$$\mathrm{x} = \frac{1}{\mathrm{k}\sin\theta}\left(\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \text{ etc.}\right). \tag{12.1.4}$$

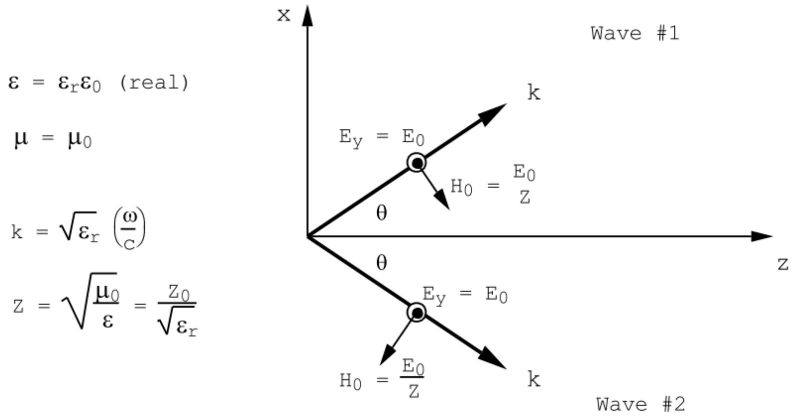


Figure 12.1.1: Two plane waves having the same frequency, oscillating in phase, and propagating in the x-z plane at an angle θ with respect to the z-axis.

This means that the wave defined by Equations (12.1.1),(12.1.2), and (12.1.3) can propagate along the hollow rectangular pipe bounded by perfectly conducting planes spaced b apart along the y-direction, and spaced a apart along the x-direction where a= m π/(k sin θ), and where m is an odd integer, and yet satisfy the boundary conditions imposed by the presence of the perfectly conducting surfaces. The distribution of the electric and magnetic fields across the section of the wave-guide formed by the intersection of the four conducting planes is shown in Figure 12.1.2 for the mode corresponding to k sin θ = π/a.

The width of the wave-guide along the x-direction, a, determines the propagation angle for waves that satisfy the boundary condition Ey = 0 on x=±a/2:

$$\mathrm{k}\sin\theta=\mathrm{m}\frac{\pi}{\mathrm{a}}=\sqrt{\epsilon_{\mathrm{r}}}\left(\frac{\omega}{\mathrm{c}}\right)\sin\theta.$$

The component of the propagation vector parallel with the wave-guide axis, along z, is given by

$$\mathrm{k_g}=\mathrm{k}\cos\theta=\sqrt{\epsilon_{\mathrm{r}}}\left(\frac{\omega}{\mathrm{c}}\right)\cos\theta.$$

The sum of the squares of these two components must be equal to the square

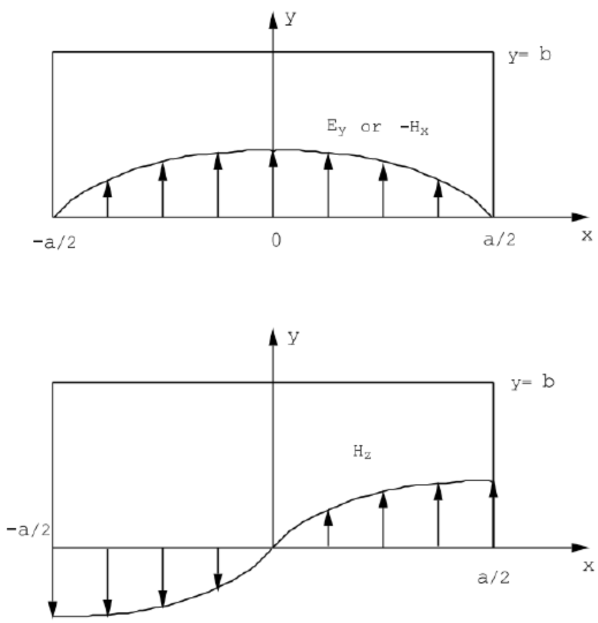


Figure 12.1.2: The lowest frequency transverse electric mode (a TE mode) for a rectangular wave-guide whose cross-sectional dimensions are a and b, where a is greater than b.

of the wave-vector k, where $\mathrm{k}=\sqrt{\epsilon_{\mathrm{r}}}\omega/\mathrm{c}$:

$$k^2 = k_g^2 + k^2 \sin^2\theta = \epsilon_r \left(\frac{\omega}{c}\right)^2,$$

from which

$$k_g^2 + m^2 \left(\frac{\pi}{a}\right)^2 = \epsilon_r \left(\frac{\omega}{c}\right)^2, \tag{12.1.5}$$

where m is an odd integer.

The most important wave-guide mode is that for which m=1, the mode illustrated in Figure (12.1.2). In most applications the wave-guide is filled with air for which ϵ_r = 1. For this m=1 mode, and assuming that (\epsilon_{r}\)=1.0, the fields are given by

$$\begin{aligned} E_y &= A\cos\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z - \omega t]), \\ H_x &= -\frac{A}{Z_0}\left(\frac{ck_g}{\omega}\right)\cos\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z - \omega t]), \\ H_z &= i\frac{A}{Z_0}\left(\frac{c\pi}{\omega a}\right)\sin\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z - \omega t]), \end{aligned} \tag{12.1.6}$$

where $Z_0 = c\mu_0 = \sqrt{\mu_0/\epsilon_0}$. The mode of Equations (12.1.6), Figure 12.1.2, is called a **transverse electric mode**, or a TE mode, because the electric field has no component along the guide axis, i.e. no component along the direction of propagation of the wave-guide mode. Notice that the ratio E_y/H_x = Z_G is independent of position inside the wave-guide; in particular, it is independent of position across the wave-guide cross-section. The magnetic field H_x is equivalent to a surface current density $J_y^s = H_x$ Amps/m (from $\text{curl}(\vec{H}) = \vec{J}_f$), and E_y has the units of Volts/m. The wave impedance Z_G = E_y/H_x therefore has the units of Ohms: it plays a role for wave-guide problems that is similar to the role played by the characteristic impedance for transmission line problems. The analogy between transmission lines and wave-guides is discussed in a very clear manner in the article " The Elements of Wave Propagation using the Impedance Concept" by H.G.Booker, Electrical Engineering Journal, volume 94, pages 171-202, 1947.

The Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, associated with the TE_{10} mode, Equation (12.1.6), has two components:

$$S_x = E_y H_z,$$

and

$$S_z = -E_y H_x.$$

The time averaged value of S_x is zero; this corresponds to the fact that no energy, on average, is transported across the guide from one side to the other. There is a non-zero time average for the z-component of the Poynting vector corresponding to energy flow along the guide:

$$< S_z >= -\frac{1}{2}\text{Real}(E_y H_x^*) = \frac{1}{2}\frac{|A|^2}{Z_0}\left(\frac{ck_g}{\omega}\right)\cos^2\left(\frac{\pi x}{a}\right). \tag{12.1.7}$$

It is useful to integrate the time-averaged value of the Poynting vector over the cross-sectional area of the wave-guide in order to obtain the rate at which energy is carried past a particular section of the guide. A simple integration gives

$$b\int_{-a/2}^{a/2} < S_z > dx = \frac{ab}{4}\frac{|A|^2}{Z_0}\left(\frac{ck_g}{\omega}\right) \quad \text{Watts}. \tag{12.1.8}$$

The time-averaged energy density associated with a wave-guide mode is given by

$$< W >= \frac{1}{2}\text{Real}\left(\frac{\vec{E}\cdot\vec{D}^*}{2} + \frac{\vec{H}\cdot\vec{B}^*}{2}\right),$$

or

$$< W >= \frac{1}{4}\text{Real}\left(\epsilon E_y E_y^* + \mu_0 H_x H_x^* + \mu_0 H_z H_z^*\right). \tag{12.1.9}$$

For the fundamental TE_{10} mode, Equations (12.1.6), one obtains

$$<W>=\frac{\epsilon_0}{4}|A|^2\cos^2\left(\frac{\pi x}{a}\right)+\frac{\mu_0}{4}\frac{|A|^2}{Z_0^2}\left[\left(\frac{ck_g}{\omega}\right)^2\cos^2\left(\frac{\pi x}{a}\right)+\left(\frac{c\pi}{a\omega}\right)^2\sin^2\left(\frac{\pi x}{a}\right)\right]. \tag{12.1.10}$$

Integrate this energy density across a section of the guide in order to obtain the average energy per unit length of the wave-guide:

\[

$$\begin{aligned} b\int_{-a/2}^{a/2}<W>dx &=\frac{ab}{8}|A|^2\left(\epsilon_0+\frac{\mu_0}{Z_0^2}\left[\left(\frac{ck_g}{\omega}\right)^2+\left(\frac{c\pi}{a\omega}\right)^2\right]\right) \\ &=\frac{ab}{4}\epsilon_0|A|^2 \quad \text{Joules}/m, \end{aligned} \tag{12.1.11}$$

where we have used $\mu_0/Z_0^2=\epsilon_0$, and for a waveguide filled with air

$$\left(\frac{ck_g}{\omega}\right)^2+\left(\frac{c\pi}{a\omega}\right)^2=\epsilon_r=1.0.$$

The velocity with which energy is transported down the guide is called the group velocity, v_g. The group velocity must have a value such that its product with the energy density per unit length of guide, Equation (12.1.11), gives the rate at which energy is transported past a wave-guide section, Equation (12.1.8): i.e.

$$v_g\frac{ab\epsilon_0}{4}|A|^2=\frac{ab}{4}\frac{|A|^2}{Z_0}\left(\frac{ck_g}{\omega}\right).$$

Thus

$$v_g=c\left(\frac{ck_g}{\omega}\right) \quad m/\text{sec}. \tag{12.1.12}$$

It is easy to verify by direct differentiation of Equation (12.1.5) that this velocity is also given by the relation

$$v_g=\frac{\partial\omega}{\partial k_g}. \tag{12.1.13}$$

Equation (12.1.13) is valid for an arbitrary relative dielectric constant: from (12.1.5)

$$v_g=\frac{c}{\epsilon_r}\left(\frac{ck_g}{\omega}\right). \tag{12.1.14}$$

The phase velocity, v_{phase}, on the other hand is obtained from the condition

$$k_g z-\omega t=\text{ constant }.$$

That is z must increase at the rate

$$v_{phase}=\frac{dz}{dt}=\frac{\omega}{k_g}$$

in order to remain on a crest as the wave propagates along the guide. As the guide wave-vector, k_g, approaches zero the phase velocity may become very large- much larger than the velocity of light in vacuum. This occurs because the phase velocity measures the rate of propagation down the guide of two intersecting wave fronts as these waves bounce back and forth across the guide (see Figure (12.1.3)). This intersection velocity clearly becomes infinitely

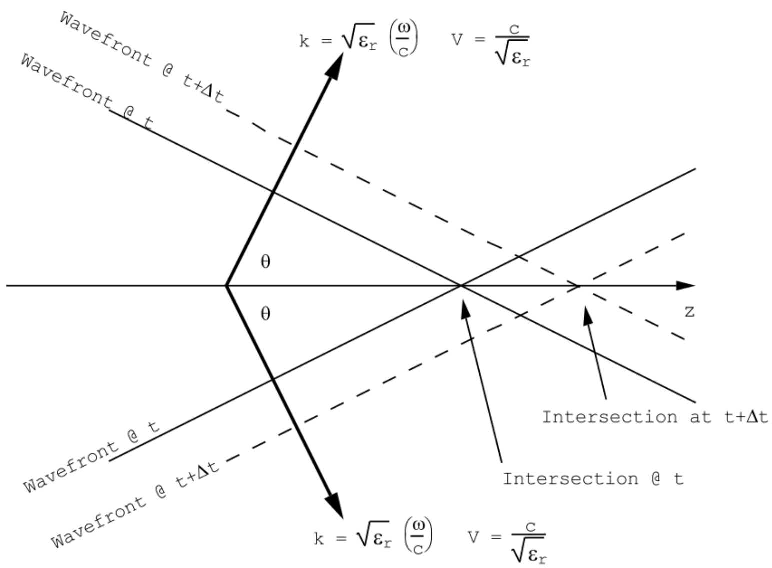


Figure 12.1.3: The phase velocity along z is the velocity with which the intersection between two wavefronts propagates along the wave-guide. This velocity, $v_{phase} = \omega/kg$, becomes very large as $\theta \rightarrow \pi/2$.

large in the limit as the wavefronts become parallel with the guide axis, i.e. in the limit as the guide wave-number, kg, goes to zero. The velocity of energy transport down the guide, the group velocity, goes to zero as θ approaches $\pi/2$, the condition corresponding to waves that simply bounce forth and back along the x-direction between the perfectly conducting planes at $x = \pm a/2$. The group velocity, the velocity with which information can be transmitted down the guide, is always less than the velocity of light in vacuum.

The frequency at which the group velocity goes to zero can be calculated from Equation (12.1.5) by setting $k_g = 0$, since the group velocity is proportional to kg from (12.1.14):

$$\omega_m = \frac{c\pi}{a\sqrt{\epsilon_r}}.$$

The wave-guide is a high pass filter that will transmit energy for frequencies larger than the cut-off frequency ω_m. For $\epsilon_r = 1$ and a=1 cm, the cut-off frequency is $\omega_m = 9.42 \times 10^{10}$ radians/sec. corresponding to a frequency of f=15 GHz.

It should be clear from the above construction that Equations (12.1.6) represents the solution of Maxwell's equations for the TE_{10} mode that carries energy in the positive z-direction. The TE_{10} mode that carries energy in the negative z-direction is described by

$$\begin{aligned} E_y &= B\cos\left(\frac{\pi x}{a}\right)\exp(-i\,[k_g z + \omega t]), \\ H_x &= \frac{B}{Z_0}\left(\frac{ck_g}{\omega}\right)\cos\left(\frac{\pi x}{a}\right)\exp(-i\,[k_g z + \omega t]), \\ H_z &= i\frac{B}{Z_0}\left(\frac{c\pi}{a\omega}\right)\sin\left(\frac{\pi x}{a}\right)\exp(-i\,[k_g z + \omega t]), \end{aligned} \tag{12.1.15}$$

where B is an arbitrary amplitude (**NOT** the magnetic field!).

In order to answer the question of what happens if the frequency is less than the cut-off frequency, ω_m, it is best to start from Maxwell's equations. Consider the case for which there is only a y-component of electric field. From

$$\text{curl}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t} = i\omega\mu_0 \vec{H},$$

for a time dependence $\sim \exp(-i\omega t)$, and for the permeability of free space, one obtains

$$i\omega\mu_0 H_x = -\frac{\partial E_y}{\partial z}, \tag{12.1.16}$$

and

$$i\omega\mu_0 \mathrm{H_z} = \frac{\partial \mathrm{E_y}}{\partial \mathrm{x}}. \tag{12.1.17}$$

From

$$\mathrm{curl}(\overrightarrow{\mathrm{H}}) = \frac{\partial \overrightarrow{\mathrm{D}}}{\partial \mathrm{t}} = -i\omega\epsilon \overrightarrow{\mathrm{E}}$$

one obtains

$$\frac{\partial \mathrm{H_x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{H_z}}{\partial \mathrm{x}} = -i\omega\epsilon \mathrm{E_y}.$$

The above equations can be combined to give a single second order equation for E_y:

$$\frac{\partial^2 \mathrm{E_y}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{E_y}}{\partial \mathrm{z}^2} = -\epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 \mathrm{E_y}.$$

For an electric field having the form

$$\mathrm{E_y} = \mathrm{A}\cos\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(i\,[\mathrm{k_g z} - \omega \mathrm{t}])$$

it follows that

$$\left(\frac{\pi}{\mathrm{a}}\right)^2 + \mathrm{k_g^2} = \epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2,$$

or

$$\mathrm{k_g^2} = \epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 - \left(\frac{\pi}{\mathrm{a}}\right)^2.$$

For a frequency less than the cut-off frequency corresponding to $\omega_m = c\pi/a\sqrt{\epsilon_r}$ the square of the wave-vector k_g becomes negative and therefore its square root becomes pure imaginary. A pure imaginary wave-vector

$$\mathrm{k_g} = \pm i\alpha,$$

where α is a real number, corresponds to a disturbance that decays away exponentially along the guide either to the right or to the left. For example, kg = +iα gives a disturbance of the form

$$\mathrm{E_y} = \mathrm{A}\cos\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(-\alpha \mathrm{z})\exp(-i\omega \mathrm{t}), \tag{12.1.18}$$

with magnetic field components (from Equation (12.1.16) and Equation (12.1.17))

$$\mathrm{H_x} = -i\frac{1}{\mathrm{Z_0}}\left(\frac{\mathrm{c}\alpha}{\omega}\right)\mathrm{A}\cos\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(-\alpha \mathrm{z})\exp(-i\omega \mathrm{t}), \tag{12.1.19}$$

and

$$\mathrm{H_z} = i\frac{1}{\mathrm{Z_0}}\left(\frac{\mathrm{c}\pi}{\omega \mathrm{a}}\right)\mathrm{A}\sin\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(-\alpha \mathrm{z})\exp(-i\omega \mathrm{t}), \tag{12.1.20}$$

Using these components, it is easy to show that the time-averaged z-component of the Poynting vector, $S_z = -E_yH_x$, is exactly equal to zero. The average energy density stored in the fields is not zero:

$$<\mathrm{W_E}>= \frac{\epsilon}{4}|\mathrm{A}|^2\cos^2\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(-2\alpha \mathrm{z}), \tag{12.1.21}$$

and

$$<\mathrm{W_B}>= \frac{\mu_0}{4}\frac{|\mathrm{A}|^2}{\mathrm{Z_0^2}}\exp(-2\alpha \mathrm{z})\left(\left(\frac{\mathrm{c}\alpha}{\omega}\right)^2\cos^2\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right) + \left(\frac{\mathrm{c}\pi}{\omega \mathrm{a}}\right)^2\sin^2\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\right). \tag{12.1.22}$$

These expressions correspond to the energy density stored in the electric field, (12.1.21), and to the energy density stored in the magnetic field, (12.1.22). If a source of energy oscillating at a frequency less than the cut-off frequency is introduced into a wave-guide at some point, the resulting electromagnetic fields will remain localized around the source, and the effective load on the source will be purely reactive for a wave-guide whose walls are perfectly conducting. In the case of a real guide whose walls have some finite resistivity, the load on a source oscillating at a frequency which is less than the cut-off frequency will appear to be partly resistive but mainly reactive.

12.2: Higher Order Modes

The wave-guide modes discussed above are very simple ones because they presumed that there was no spatial variation of the fields along the y-direction. There exist wave-guide solutions of Maxwell's equations that involve spatial variations along all three axes: these higher order modes correspond to the co-ordinated propagation of plane waves whose wave-vectors make an oblique angle with the guide axis so that they are repeatedly reflected from all four walls. These modes divide naturally into two classes:

a. Transverse Electric (TE) Modes;
b. Transverse Magnetic (TM) Modes.

A transverse electric mode is one in which there is no component of the electric field parallel to the direction of propagation. A transverse magnetic mode is one in which there is no component of the magnetic field parallel to the direction of propagation. For both classes of modes one seeks solutions of Maxwell's equations that correspond to waves travelling down the waveguide; i.e. all of the field components are required to be proportional to the phasor

$$\exp(i\,[\mathrm{k_g z}-\omega \mathrm{t}]).$$

Furthermore, it is convenient at this point to change the description of the wave-guide co-ordinate system so that the origin is located at one corner of the hollow rectangular pipe as shown in Figure (12.2.4): in the new system the walls of the guide are formed by the intersection of the planes x=0,a and y=0,b. For a time variation of the form exp (−iωt) Maxwell's equations become

$$\begin{aligned}\operatorname{curl}(\overrightarrow{\mathrm{E}}) &= i\omega\mu_0\overrightarrow{\mathrm{H}},\\ \operatorname{curl}(\overrightarrow{\mathrm{H}}) &= -i\omega\epsilon\overrightarrow{\mathrm{E}}.\end{aligned} \tag{12.2.1}$$

The divergence of any curl is zero, and therefore the electric and the magnetic fields satisfy the conditions

$$\begin{aligned}div(\overrightarrow{\mathrm{E}}) &= 0,\\ \operatorname{div}(\overrightarrow{\mathrm{H}}) &= 0.\end{aligned} \tag{12.2.2}$$

Note that the equations for $\vec{E}$ and $\vec{H}$ are very similar. This symmetry between the equations for $\vec{E}$ and $\vec{H}$ can be exploited to generate a second set of solutions to Maxwell's equations from a primary set of fields that satisfy Maxwell's equations. This works as follows: suppose that one has found the fields $\vec{E}_1$ and $\vec{H}_1$ that satisfy Equations (12.2.1). Now consider a second set of fields

$$\begin{aligned}\overrightarrow{\mathrm{E}}_2 &= \mathrm{Z}\overrightarrow{\mathrm{H}}_1,\\ \overrightarrow{\mathrm{H}}_2 &= -\overrightarrow{\mathrm{E}}_1/\mathrm{Z}.\end{aligned} \tag{12.2.3}$$

where $\mathrm{Z}=\sqrt{\mu_0/\epsilon}$ is the wave impedance for a medium characterized by a permeability μ_0 and a dielectric constant ϵ. Substitute these new fields into Equations (12.2.1) to obtain

$$\operatorname{curl}\left(\overrightarrow{\mathrm{E}}_2\right) = i\omega\mu_0\overrightarrow{\mathrm{H}}_2.$$

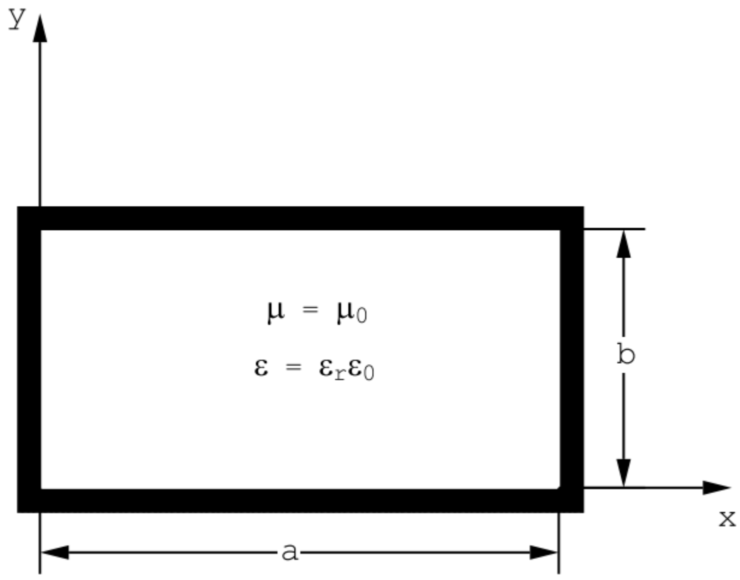


Figure 12.2.4: A rectangular wave-guide formed by conducting walls at x=0, x=a, y=0, and y=b. The lossless material inside the guide is characterized by a real dielectric constant ϵ, and a permeability μ_0.

Upon the substitution (12.2.3) this becomes

$$\operatorname{curl}\left(\overrightarrow{\mathrm{H}}_1\right)=-\frac{i\omega\mu_0}{\mathrm{Z}^2}\overrightarrow{\mathrm{E}}_1=-i\omega\epsilon\overrightarrow{\mathrm{E}}_1,$$

and this by hypothesis satisfies Maxwell's Equations (12.2.1). Similarly, from (12.2.1) one has

$$\operatorname{curl}\left(\overrightarrow{\mathrm{H}}_2\right)=-i\omega\epsilon\overrightarrow{\mathrm{E}}_2.$$

Upon substitution of Equations (12.2.3) one finds

$$\operatorname{curl}\left(\overrightarrow{\mathrm{E}}_1\right)=i\omega\mu_0\overrightarrow{\mathrm{H}}_1,$$

so that the new fields, $\vec{E}_2$ and $\vec{H}_2$ satisfy both of Equations (12.2.1). Clearly Equations (12.2.2) are satisfied since $\vec{E}_2$ and $\vec{H}_2$ are proportional to $\vec{E}_1$ and $\vec{H}_1$. It follows that the prescription of Equation (12.2.3) can be used to generate a second, different, set of solutions for Maxwell's equations from a primary set of solutions. This procedure can often be used to avoid a great deal of computational tedium.

12.2.1 TM Modes.

In order to proceed with the rectangular wave-guide problem it is convenient to use the vector potential $\vec{A}$, and the scalar potential, V, where

$$\begin{aligned}\overrightarrow{\mathrm{H}}&=\operatorname{curl}(\overrightarrow{\mathrm{A}}),\\ \overrightarrow{\mathrm{E}}&=-\operatorname{grad}(\mathrm{V})-\mu_0\frac{\partial\overrightarrow{\mathrm{A}}}{\partial\mathrm{t}}.\end{aligned}\tag{12.2.4}$$

The choice

$$\mathrm{A}_\mathrm{z}=\mathrm{A}(\mathrm{x},\mathrm{y})\exp(i\left[\mathrm{k_g z}-\omega\mathrm{t}\right]),\tag{12.2.5}$$

plus $A_x = A_y = 0$ will guarantee that the z-component of the magnetic field, $\vec{H}$, is zero: in other words, this choice of vector potential will generate only TM modes, and

$$\mathrm{H_x} = \frac{\partial \mathrm{A_z}}{\partial \mathrm{y}}$$
$$\mathrm{H_y} = -\frac{\partial \mathrm{A_z}}{\partial \mathrm{x}}$$

For a time dependence exp (−iωt), and using (12.2.4) in Maxwell's equations (12.2.1), one finds

$$\mathrm{curlcurl}(\overrightarrow{\mathrm{A}}) = \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\overrightarrow{\mathrm{A}} + i\omega\epsilon\,\mathrm{grad}(\mathrm{V}).$$

But in cartesian co-ordinates

$$\mathrm{curlcurl}(\overrightarrow{\mathrm{A}}) = -\nabla^2\overrightarrow{\mathrm{A}} + \mathrm{graddiv}(\overrightarrow{\mathrm{A}}),$$

so that

$$\nabla^2\overrightarrow{\mathrm{A}} + \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\overrightarrow{\mathrm{A}} = \mathrm{grad}(\mathrm{div}(\overrightarrow{\mathrm{A}}) - i\omega\epsilon\mathrm{V}).$$

As explained in Chapter(7), one can set

$$i\omega\epsilon\mathrm{V} = \mathrm{div}(\overrightarrow{\mathrm{A}}),$$

so that for this problem where there are no driving charges or currents one finds

$$\nabla^2\overrightarrow{\mathrm{A}} + \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\overrightarrow{\mathrm{A}} = 0.$$

In particular, if $\vec{A}$ has only a z-component one finds

$$\nabla^2\mathrm{A_z} + \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\mathrm{A_z} = 0, \tag{12.2.6}$$
$$\textit{and}$$
$$i\omega\epsilon\mathrm{V} = \frac{\partial \mathrm{A_z}}{\partial \mathrm{z}}.$$

We require solutions that propagate along z: ie solutions that are proportional to exp ($\mathrm{ik_g z}$). Thus write

$$\mathrm{A_z}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) = \mathrm{A}(\mathrm{x},\mathrm{y})\exp(i\,[\mathrm{k_g z} - \omega\mathrm{t}]),$$

for which A(x, y) must satisfy

$$\frac{\partial^2\mathrm{A}}{\partial\mathrm{x}^2} + \frac{\partial^2\mathrm{A}}{\partial\mathrm{Y}^2} - \mathrm{k_g^2 A} + \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\mathrm{A} = 0. \tag{12.2.7}$$

This equation is solved by products of sines and cosines:

$$\mathrm{A}(\mathrm{x},\mathrm{y}) = \text{ constant }(\sin(\mathrm{px}) \text{ or } \cos(\mathrm{px}))(\sin(\mathrm{qy}) \text{ or } \cos(\mathrm{qy})),$$

where

$$\mathrm{p}^2 + \mathrm{q}^2 + \mathrm{k_g^2} = \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2. \tag{12.2.8}$$

The particular combination of sines and cosines required must be chosen so that $\vec{H}$ satisfies the boundary condition that the normal component of $\vec{H}$ vanishes at the wave-guide walls. Using the magnetic field components calculated from Equation (12.2.4) and the co-ordinate system of Figure (12.2.4), it can be readily concluded that we require

$$\mathrm{A}(\mathrm{x},\mathrm{y}) = \mathrm{A_0}\sin\left(\frac{\mathrm{m}\pi\mathrm{x}}{\mathrm{a}}\right)\sin\left(\frac{\mathrm{n}\pi\mathrm{y}}{\mathrm{b}}\right), \tag{12.2.9}$$

so that

$$\begin{aligned} H_x &= \frac{\partial A}{\partial y} = \left(\frac{n\pi}{b}\right) A_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \\ H_y &= -\frac{\partial A}{\partial x} = -\left(\frac{m\pi}{a}\right) A_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \end{aligned} \tag{12.2.10}$$

where m,n are integers, and Equation (12.2.8) becomes

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_g^2 = \epsilon_r\left(\frac{\omega}{c}\right)^2. \tag{12.2.11}$$

Notice that A_z=0 at the walls of the wave-guide. E_z is proportional to A_z so that if A_z=0 on the walls of the guide then the tangential component E_z will also vanish on the walls of the wave-guide as is required by the boundary conditions on the tangential components of E.

The electric field components can be most easily calculated from the second of Equations (12.2.1),

$$\begin{aligned} \operatorname{curl}(\overrightarrow{H}) &= -i\epsilon\omega\overrightarrow{E}. \\ E_x &= \frac{-i}{\epsilon\omega}\left(\frac{\partial H_y}{\partial z}\right), \\ E_y &= \frac{+i}{\epsilon\omega}\left(\frac{\partial H_x}{\partial z}\right), \\ E_z &= \frac{i}{\epsilon\omega}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]. \end{aligned}$$

The resulting electric field components are (dropping the factor exp $(i[k_g z - \omega t])$):

$$\begin{aligned} E_x &= -\left(\frac{m\pi}{a}\right)\left(\frac{k_g}{\epsilon\omega}\right) A_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \\ E_y &= -\left(\frac{n\pi}{b}\right)\left(\frac{k_g}{\epsilon\omega}\right) A_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \\ E_z &= \left(\frac{i}{\epsilon\omega}\right)\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] A_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right). \end{aligned} \tag{12.2.12}$$

Notice that these electric field components satisfy the requirement that the tangential components of E must vanish at the walls of the wave-guide. The field components Equations (12.2.10) and (12.2.12) correspond to the TM_{mn} mode.

For a propagating wave the value of k_g^2 calculated from (12.2.11) must be positive. This introduces a cut-off frequency, ω_m, such that k_g =0. This cut-off frequency is given by

$$\epsilon_r\left(\frac{\omega_m}{c}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{12.2.13}$$

For given interior dimensions of the wave-guide there is a lower limit to the frequency for which a particular mode may be propagated along the waveguide. For example, a popular X-band wave-guide has interior dimensions a= 2.286 cm and b= 1.016 cm. For this guide the TM_{11} mode can be propagated only for frequencies greater than 16.15 GHz if ϵ_r= 1. There are no TM modes corresponding to m=0 or n=0 since the fields are zero if m=0 or if n=0 because A(x,y)=0 from Equation (12.2.9). Thus the lowest TM frequency that can be propagated down the above guide is 16.15 GHz.

Non-propagating TM modes do exist for frequencies less than the cutoff frequency. If $\omega < \omega_m$ then k_g calculated from (12.2.8) is negative. This means that kg is a purely imaginary number, $k_g = i\beta$ say. The phasor exp $(i[k_g z - \omega t])$ becomes exp $(-\beta z)$ exp $(-i\omega t)$ corresponding to a disturbance that decays to a small amplitude over a distance $z \sim (1/\beta)$.

12.2.2 TE Modes.

There exists another group of modes for which $E_z = 0$; these are the TE modes. Using the symmetry relations Equations (12.2.3) and the magnetic fields (12.2.10) one might guess that the TE mode electric fields ought to be given by (the factor exp $(i[k_g z - \omega t])$ is suppressed)

$$\begin{aligned}E_x &= Z\left(\frac{n\pi}{b}\right) A_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\\ E_y &= -Z\left(\frac{m\pi}{a}\right) A_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).\end{aligned}$$

These electric fields satisfy Maxwell's equations but they do not satisfy the boundary condition that the tangential components of $\vec{E}$ must vanish at the wave-guide walls: ie. E_x=0 at y=0,b and E_y=0 at x=0,a (see Figure (12.2.4)). However, the following equations for the electric field components do satisfy the required boundary conditions:

$$\begin{aligned}E_x &= E_1 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\ E_y &= E_2 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\\ E_z &= 0.\end{aligned}$$

These field components vanish on the wave-guide walls. The electric field must also satisfy the Maxwell equation div($\vec{E}$) = 0. This condition requires

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{y} = 0.$$

It follows that

$$\left(\frac{m\pi}{a}\right) E_1 = -\left(\frac{n\pi}{b}\right) E_2.$$

Using this relation between E_1 and E_2 the electric field components corresponding to the TE modes in a rectangular wave-guide have the form:

$$\begin{aligned}E_x &= \left(\frac{n\pi}{b}\right) E_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\ E_y &= -\left(\frac{m\pi}{a}\right) E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\\ E_z &= 0,\end{aligned} \tag{12.2.14}$$

where E_0 is a constant, and the factor exp (i[k_gz − ωt]) has again been suppressed. The magnetic field components corresponding to Equations (12.2.14) can be calculated from Faraday's law: iωμ0 $\vec{H}$ = curl($\vec{E}$). The resulting field components are

$$\begin{aligned}H_x &= \frac{+i}{\omega\mu_0}\frac{\partial E_y}{\partial z}\\ &= \frac{k_g}{\omega\mu_0}\left(\frac{m\pi}{a}\right) E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),\\ H_y &= \frac{-i}{\omega\mu_0}\frac{\partial E_x}{\partial z}\\ &= \frac{k_g}{\omega\mu_0}\left(\frac{n\pi}{b}\right) E_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),\\ H_z &= \frac{-i}{\omega\mu_0}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\\ &= \frac{i}{\omega\mu_0} E_0\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).\end{aligned} \tag{12.2.15}$$

Eqns.(12.2.14) and (12.2.15) satisfy Maxwell's equations and also the boundary conditions that the tangential components of $\vec{E}$ and the normal components of $\vec{H}$ vanish on the wave-guide walls. The TE_{10} mode discussed in section(12.1) corresponds to m=1,n=0: for this mode E_x = 0 and H_y = 0. Referring to the co-ordinate system of Figure (12.2.4) the field components for the TE_{10} mode are:

$$\begin{aligned} E_y &= A\sin\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z-\omega t]), \\ H_x &= -\frac{k_g}{\omega\mu_0}A\sin\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z-\omega t]), \\ H_z &= -\frac{i}{\omega\mu_0}\left(\frac{\pi}{a}\right)A\cos\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z-\omega t]), \end{aligned} \tag{12.2.16}$$

where A is a constant and

$$k_g^2=\epsilon_r\left(\frac{\omega}{c}\right)^2-\left(\frac{\pi}{a}\right)^2. \tag{12.2.17}$$

The cut-off frequency for this mode, corresponding to k_g=0, is given by

$$\epsilon_r\left(\frac{\omega}{c}\right)^2=\left(\frac{\pi}{a}\right)^2.$$

For the popular X-band wave-guide used above for illustrative purposes one has a=2.286 cm and b=1.016 cm. For this guide, and ϵ_r = 1, the cut-of

m	n	TE_{mn}(GHz)	TM_{mn}(GHz)
1	0	6.557	*No* TM *Mode*
0	1	14.753	*No* TM *Mode*
1	1	16.145	16.145
2	0	13.114	*No* TM *Mode*
2	1	19.740	19.740
0	2	29.507	*No* TM *Mode*
1	2	30.227	30.227
2	2	32.290	32.290
3	0	19.671	*No* TM *Mode*
3	1	24.589	24.589
3	2	35.463	35.463
3	3	48.435	48.435

Table 12.2.1: Cut-off frequencies for the lowest transverse electric (TE) modes and the lowest transverse magnetic (TM) modes in X-band waveguides (RG52/U or WR90 brass guides). The internal dimensions of X-band waveguides are a= 0.900 inches = 2.286 cm, and b= 0.400 inches = 1.016 cm. The external dimensions of the guide are 1.00 x 0.50 inches. The cut-off frequencies were calculated for ϵ_r = 1 using $\left(\frac{\omega_{mn}}{c}\right)^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$.

frequency for the TE_{10} mode is 6.56 GHz. Cut-off frequencies for various modes in this X-band wave-guide are listed in Table(12.2.1). Notice that only one mode, the TE_{10} mode, can be propagated along this wave-guide for frequencies between 6.6 and 13.1 GHz.

12.3: Waveguide Discontinuities

Any discontinuity in the dielectric constant, in the permeability, or any discontinuity in the dimensions of a waveguide will result in reflected waves. For simplicity, we will discuss the case in which the waveguide will support the propagation of only the TE_{10} mode. A forward propagating TE_{10} mode that encounters an obstacle in the waveguide will, in general, be partially reflected and partially transmitted. This can be illustrated for a particularly simple obstacle; let a very thin sheet of material of thickness d and characterized by a conductivity σ be placed across the guide as illustrated in Figure (12.3.5).

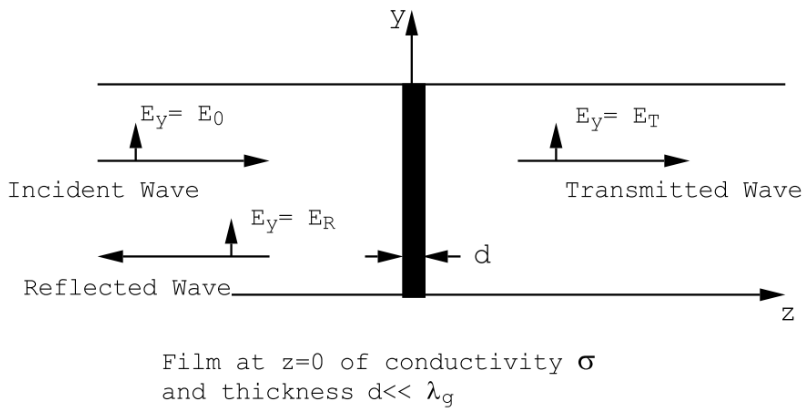


Figure 12.3.5: A TE_{10} wave incident on a conducting diaphragm placed across the waveguide.

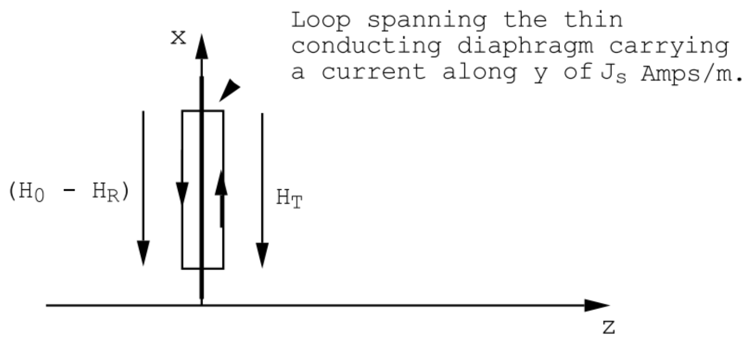


Figure 12.3.6: A TE_{10} wave incident on a conducting diaphragm placed across the waveguide. The magnetic fields on each side of the diaphragm are related by the condition curl($\vec{H}$) = J_s .

The thickness of the sheet, d, is assumed to be very thin compared with the skindepth $\delta=\sqrt{2/(\omega\sigma\mu_0)}$, see Chpt.(10), Section(10.4). The thickness d is also assumed to be very small compared with the wavelength $\lambda_g = 2\pi/k_g$. The tangential component of the electric field in this case, E_y, must be continuous across the diaphragm

$$\mathrm{E_T}=\mathrm{E_0}+\mathrm{E_R}. \tag{12.3.1}$$

The electric field at the conducting diaphragm generates a current density along y whose magnitude is $J_y = \sigma E_T$. When integrated across the film thickness d this current density produces an equivalent surface current sheet having the strength

$$\mathrm{J_s}=\sigma d\mathrm{E_T} \quad \mathrm{Amps}\,/m.$$

This current sheet causes a discontinuity in the tangential component of $\vec{H}$ from Amp`ere's law; i.e. the magnetic field components, H_x, on either side of the film are related by

$$\int\int_{Loop}\mathrm{curl}(\overrightarrow{\mathrm{H}})\cdot\mathrm{d}\,\overrightarrow{\mathrm{S}}=\oint_C\overrightarrow{\mathrm{H}}\cdot\mathrm{d}\,\overrightarrow{\mathrm{L}}=\text{total current through the Loop}. \tag{12.3.2}$$

The loop in question is illustrated in Figure (12.3.6). Condition (12.3.2) results in the equation

$$H_0 - H_R - H_T = J_s = \sigma d E_T,$$

or using the waveguide impedance, $Z_G = \omega\mu_0/k_g$,

$$\frac{(E_0 - E_R - E_T)}{Z_G} = \sigma d E_T,$$

or

$$E_0 - E_R = [1 + Z_G \sigma d]\, E_T. \tag{12.3.3}$$

Equations (12.3.1) and (12.3.3) can be solved to obtain the reflected and transmitted wave amplitudes:

$$R = \frac{E_R}{E_0} = -\frac{Z_G \sigma d}{(2 + Z_G \sigma d)}, \qquad T = \frac{E_T}{E_0} = \frac{2}{(2 + Z_G \sigma d)}. \tag{12.3.4}$$

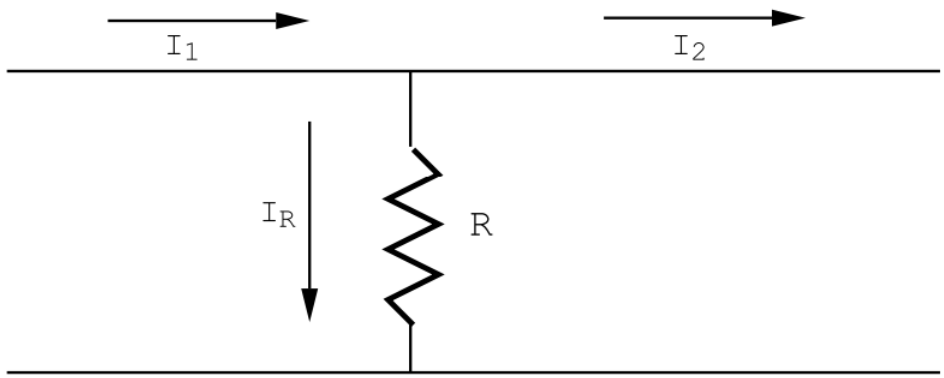


Figure 12.3.7: A transmission line, characteristic impedance Z_0 Ohms, shunted by a resistance of R Ohms. The incident wave amplitude is V_0 Volts; the reflected wave amplitude is V_R Volts; the transmitted wave amplitude is V_T Volts. The reflection coefficient is $R = V_R/V_0 = -(Z_0/R)/\,[2 + (Z_0/R)]$. The transmission coefficient is $T = V_T/V_0 = 2/\,[2 + (Z_0/R)]$.

These reflection and transmission coefficients are the same as would be calculated for the problem of a resistor, $R=1/\sigma d$, placed across a transmission line whose characteristic impedance is Z_G, Figure (12.3.7). The close analogy between waveguide problems and transmission line problems has been stressed in the article by H.G.Booker,"The Elements of Wave Propagation using the Impedance Concept", Electrical Engineering Journal, Volume 94,pages 171- 198, 1947. The analogy holds for more general impedances. For an incident TE_{10} mode a thin wire placed across the waveguide parallel with the electric field has the same reflection and transmission coefficients as an inductive reactance placed across a transmission line, Figure (12.3.8). Similarly, the thin metal diaphragms illustrated in Figure (12.3.9) give rise to reflection and transmission coefficients for the waveguide that are the same as would be produced by the circuits on the right hand side of Figure (12.3.9) if they were to be placed across a transmission line.

A thin metal diaphragm containing a hole such as that shown in Figure (12.3.9c) can be used to produce a resonant cavity, Figure (12.3.10). The resonant structure is formed by the section of waveguide that is contained between the diaphragm and a waveguide short circuit; the waveguide short simply consists of a metal plate that completely closes off the guide. The hole in the diaphragm need not be rectangular as shown in Figure (12.3.9): it is often convenient

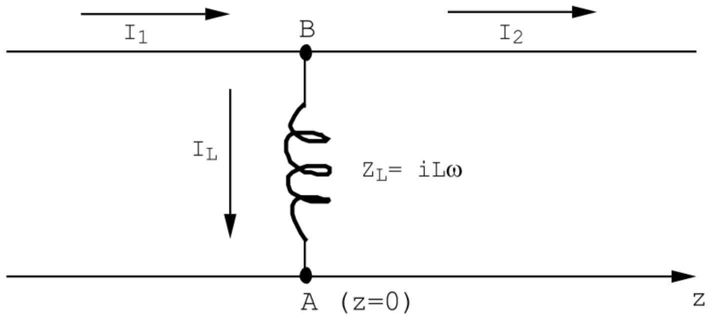


Figure 12.3.8: A transmission line, characteristic impedance Z_0 Ohms, shunted by an inductive impedance of Z_L Ohms. The incident wave amplitude is V_0 Volts; the reflected wave amplitude is V_R Volts; the transmitted wave amplitude is V_T Volts. The reflection coefficient is $R = V_R/V_0 = i(Z_0/L\omega)/\ [2 - (iZ_0/L\omega)]$. The transmission coefficient is $T = V_T/V_0 = 2/\ [2 - (iZ_0/L\omega)]$.

to use a round hole. At a frequency such that the length of the cavity, L, is very nearly equal to an integral number of half-wavelengths the reflection from the diaphragm becomes smaller than its value for neighboring frequencies. The reflectivity exhibits a dip when plotted as a function of frequency, Figure (12.3.11). At resonance a standing wave is set up in the cavity that has nodes at the diaphragm and at the shorted end. Energy is fed into the cavity through the coupling hole, and the fields inside the cavity become so large that the energy dissipated in the cavity walls is equal to the energy carried into the cavity through the coupling hole. The Q, or quality factor, for the cavity is defined to be

$$Q = 2\pi\left(\frac{\text{Energy Stored}}{\text{Energy Loss per Cycle}}\right). \tag{12.3.5}$$

The quality factor for a microwave cavity commonly exceeds 1000 and is often found to be as large as 10,000. The electric and magnetic fields in a properly coupled microwave cavity range in amplitude between 30 and 100 times as large as the amplitudes of the incident wave that feeds the cavity. At resonance the cavity appears to be a pure resistance. The value

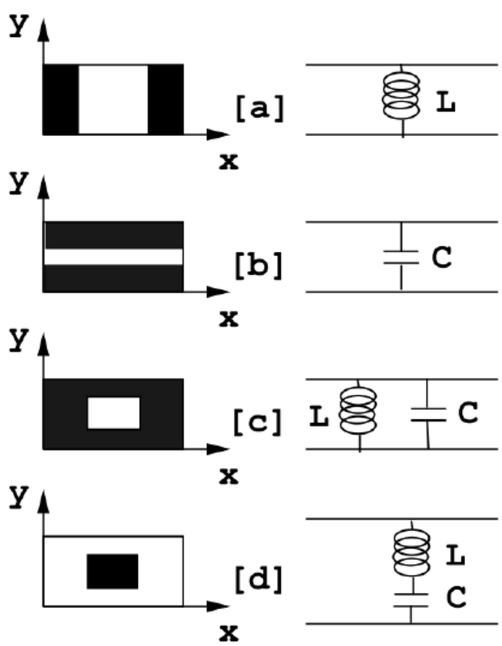


Figure 12.3.9: Thin, perfectly conducting metal diaphragms placed across a waveguide and their equivalent shunt circuits placed across a transmission line. Shaded areas are metal, unshaded areas are unobstructed.

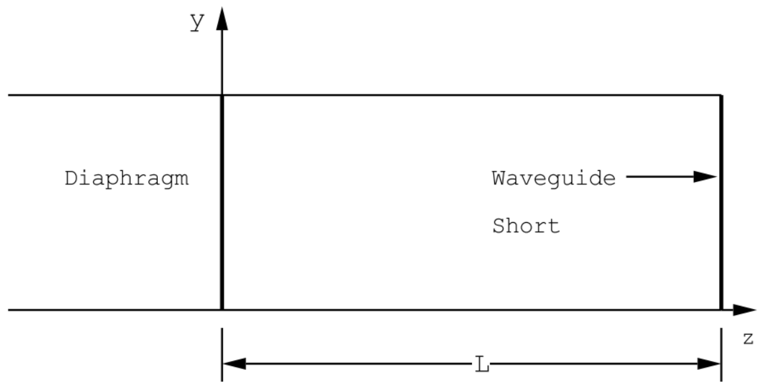


Figure 12.3.10: A microwave cavity resonator. Resonance occurs when L = $n\lambda_g/2$ where $\lambda_g = 2\pi/kg$ and n is an integer.

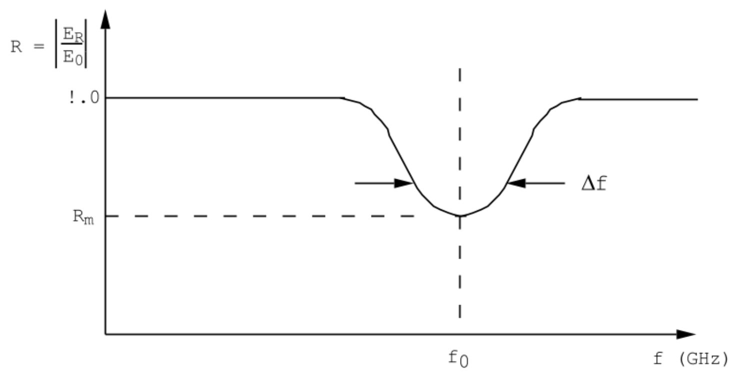


Figure 12.3.11: The absolute value of the reflection coefficient as a function of frequency for a microwave signal having an amplitude of E_0 Volts/m incident on a microwave cavity. The reflected wave amplitude is E_R Volts/m. The quality factor of the cavity is given by $Q=f_0/\Delta f$ where Δf is the frequency interval between the frequencies for which the change in reflectivity is 0.7 of the maximum change.

of that resistance depends upon the size of the coupling aperture. For a particular orifice size, a size that depends upon the cavity losses and which must be determined by trial and error, the cavity absorbs all of the energy that is incident upon it; the cavity forms a matched load. Under matched conditions the reflectivity of the cavity is very sensitive to changes in the cavity losses, and this configuration is often used to measure the magnetic field dependence of the absorption of microwave energy by electron spins (electron spin resonance absorption or ESR).

The fields in the microwave cavity form standing waves at resonance. For a TE_{10} mode

$$\begin{aligned} E_y &= E_0 \sin\left(\frac{\pi x}{a}\right) \sin(k_{gz}) \exp(-i\omega t), \\ H_x &= +i\left(\frac{k_g}{\omega\mu_0}\right) E_0 \sin\left(\frac{\pi x}{a}\right) \cos(k_g z) \exp(-i\omega t), \\ H_z &= -i\left(\frac{\pi}{\omega\mu_0 a}\right) E_0 \cos\left(\frac{\pi x}{a}\right) \sin(k_g z) \exp(-i\omega t), \end{aligned} \tag{12.3.6}$$

The zero for z in Figure (12.3.10) is located at the diaphragm. The length of the cavity must be chosen so that $k_gL = m\pi$, where m is an integer, so that the electric field vanishes on the cavity end walls. Notice that the electric and magnetic fields are 90 out of phase. This means that the energy stored in the cavity swings forth and back between energy stored in the electric field and energy stored in the magnetic field: the total energy is independent of the time. For this cavity resonator, and for m odd, the electric field is large in the central region, i.e. at x=a/2, z=L/2, and in this region the magnetic fields are small. Conversely, the magnetic fields are large at the cavity walls where the electric field is small. The energy losses due to eddy currents flowing in the cavity walls can be estimated using the considerations described in the next section.

12.4: Energy Losses in the Waveguide Walls

When a metal is exposed to a time-varying magnetic field eddy currents are induced which flow so as to shield the interior of the metal from the magnetic field. Let the strength of the magnetic field at the metal surface be H_0, and let the field be oriented along the y-direction. The magnetic field decays towards the interior of the metal, Chapter(10), section(10.4), according to the formula

$$\mathrm{H_y}(\xi)=\mathrm{H_0}\exp(i[\mathrm{k}\xi-\omega \mathrm{t}]),$$

where ξ measures distance into the metal along the normal to the surface; the metal surface is assumed to lie in the y-z plane. The wave-vector, k, is given by

$$\mathrm{k}=\sqrt{\frac{\omega\sigma\mu_0}{2}}(1+i).$$

The electric field that generates the shielding currents in the metal is orthogonal to the magnetic field and parallel with the metal surface:

$$\mathrm{E_z}(\xi)=\left(\frac{i\mathrm{k}}{\sigma}\right)\mathrm{H_0}\exp(i[\mathrm{k}\xi-\omega \mathrm{t}]).$$

The Poynting vector at the metal surface is directed into the metal; its time average is given by

$$<\mathrm{S}_\xi>=\frac{|\mathrm{H_0}|^2}{2}\sqrt{\frac{\omega\mu_0}{2\sigma}}\quad Watts/m^2. \tag{12.4.1}$$

This energy is converted into heat in the metal wall. This Joule heat must, of course, be supplied by the microwaves propagating along the guide, and results in a gradual decrease in signal strength. The resistivity of brass is typically $\rho = 8\times10^{-8}$ Ohm-meters corresponding to $\sigma = 1/\rho = 1.25\times10^7$ per Ohm-m at room temperature. The rate of energy loss to a brass waveguide wall at room temperature and for a frequency of 10 GHz is

$$<\mathrm{S}_\xi>=0.028|\mathrm{H_0}|^2\quad \mathrm{Watts}\ /m^2. \tag{12.4.2}$$

Since energy is lost to the waveguide walls the average energy moving down the guide must decrease with distance, and therefore the amplitude of the wave must decrease. For the TE_{10} mode one has (using the co-ordinate system of Figure (12.2.4))

$$\begin{aligned}
\mathrm{H_z}&=\mathrm{H_0}\cos\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(i[\mathrm{kgz}-\omega\mathrm{t}]), \\
\mathrm{H_x}&=-i\left(\frac{\mathrm{k_g}}{\pi/\mathrm{a}}\right)\mathrm{H_0}\sin\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(i[\mathrm{kgz}-\omega\mathrm{t}]), \\
\mathrm{E_y}&=i\left(\frac{\omega\mu_0}{\pi/\mathrm{a}}\right)\mathrm{H_0}\sin\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right)\exp(i[\mathrm{kgz}-\omega\mathrm{t}]).
\end{aligned} \tag{12.4.3}$$

The average rate at which energy is transported past a waveguide crosssection can be calculated from $<\mathrm{S_z}>=(1/2)\,\mathrm{Real}(-\mathrm{E_y}\mathrm{H_x^*})$ and this must be integrated over the area of the guide:

$$P_z=\frac{dE}{dt}=\left(\frac{ab}{4}\right)\frac{\omega\mu_0 k_g}{(\pi/a)^2}H_0^2\quad Watts. \tag{12.4.4}$$

Let us apply these ideas to calculate the rate at which a microwave signal propagating in a brass X-band waveguide decays with distance. The energy loss per second per unit length of guide due to eddy current losses in the narrow sides of the guide is given by

$$\frac{\mathrm{d}<\mathrm{P_1}>}{\mathrm{dz}}=(2\mathrm{b})\mathrm{H_0^2}(0.028)\quad \mathrm{Watts}\,; \tag{12.4.5}$$

these losses are due to the component H_z whose amplitude at the walls is H_0. The factor two arises because there are contributions from two walls; the factor 0.028 comes from Equation (12.4.2). The contribution from the energy losses at the broad sides of the waveguide are more complicated since there are two magnetic field components H_x and H_z, and both components must be averaged over the x spatial dependence:

$$\frac{d<P_2>}{dz}=(2a)\left(\frac{H_0^2}{2}\right)\left[1+\left(\frac{k_g}{\pi/a}\right)^2\right](0.028)\ \text{Watts.}\ . \tag{12.4.6}$$

Thus the power passing through the guide cross-section must decrease in the distance dz by an amount that is given by the sum of Equations (12.4.5) and (12.4.6):

$$dP=(0.028)H_0^2\left[2b+a\left(1+\left(\frac{k_g}{\pi/a}\right)^2\right)\right]dz. \tag{12.4.7}$$

By differentiating Equation (12.4.4) one obtains

$$dP=\left(\frac{ab}{2}\right)\left(\frac{\omega\mu_0 k_g}{(\pi/a)^2}\right)H_0\left(\frac{dH_0}{dz}\right)dz. \tag{12.4.8}$$

Equating Equations (12.4.7) and (12.4.8) gives an equation for the rate of change of the wave amplitude, H_0, with distance

$$\frac{dH_0}{dz}=-H_0\left(\frac{0.056(\pi/a)^2}{\omega\mu_0 k_g}\right)\left[\frac{2}{a}+\frac{1}{b}\left(1+\left(\frac{k_g}{\pi/a}\right)^2\right)\right].$$

This can be re-written in the form

$$\frac{dH_0}{dz}=-\gamma H_0, \tag{12.4.9}$$

where for X-band waveguide (a=2.29 cm, b=1.02 cm) and for a frequency of 10 GHz kg = 158 m^{-1} and $\gamma = 2.67 \times 10^{-2}$ per meter. Equation (12.4.9) implies that the amplitude of a wave propagating along a waveguide falls off exponentially

$$H_0=A\exp(-\gamma z); \tag{12.4.10}$$

the amplitude decreases by 1/e after having travelled a distance of z= 1/γ meters. This distance is 37.5 meters for X-band waveguide at 10 GHz.

12.5: Circular Waveguides

The details are different, but the modes sustained by a circular wave-guide have much in common with the rectangular wave-guide modes. They may, for example, be classified as **transverse electric modes** (TE modes) in which there is no component of electric field along the guide axis, or as transverse magnetic modes (TM modes) in which there is no component of the magnetic field along the guide axis.

12.5.1 TM Modes.

A vector potential whose transverse components are zero but for which A_z is not zero will generate transverse magnetic modes because H_z is necessarily zero since curl($\vec{A}$) has a zero z-component. A_z must satisfy the wave equation (12.2.6) in order that the fields generated by A_z satisfy Maxwell's equations:

$$\nabla^2 A_z + \epsilon_r \left(\frac{\omega}{c}\right)^2 A_z = 0. \tag{12.5.1}$$

It is convenient to use cylindrical polar co-rdinates (r,θ,z) because of the cylindrical symmetry implied by the shape of a cylindrical wave-guide. For a wave travelling along the z-direction, one can write

$$A_z = A(r,\theta)\exp(i\,[k_g z - \omega t]).$$

From now on the factor $\exp(i[k_g z - \omega t])$ **will be understood and not written out explicitly**. Using cylindrical polar co-ordinates Equation (12.5.1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \left[\epsilon_r\left(\frac{\omega}{c}\right)^2 - k_g^2\right] A = 0.$$

or setting

$$k_c^2 = \epsilon_r\left(\frac{\omega}{c}\right)^2 - k_g^2, \tag{12.5.2}$$

and multiplying through by r^2

$$r\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{\partial^2 A}{\partial \theta^2} + k_c^2 r^2 A = 0. \tag{12.5.3}$$

Now let the amplitude $A(r,\theta)$ be written as the product of a function F(r) that depends only on the radius r and the function cos (mθ), where m is an integer. The constant m must be an integer so that $A(r,\theta)$ will be single valued in angle: ie. A(r,0) must be equal to A(r,2πm). The use of the function cos(mθ) is arbitrary. We could just as well use sin (mθ) or a function of the form $f(\theta) = a\cos(m\theta) + b\sin(m\theta)$, where a and b are constants. All of these choices have in common that $d^2f/d\theta^2 = -m^2 f$. The various choices of a,b simply amount to a choice of the orientation of the wave-guide mode pattern with respect to the axis θ = 0.

The equation for the radial function, $F(r)$, becomes

$$r\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \left(k_c^2 r^2 - m^2\right) F = 0. \tag{12.5.4}$$

This equation for F(r) can be put in the standard form of Bessel's equation by the introduction of a change of variable:

$$x = k_c r, \tag{12.5.5}$$

then Equation (12.5.4) becomes

$$x\frac{d}{dx}\left(x\frac{dF}{dx}\right) + \left(x^2 - m^2\right) F = 0.$$

The solutions of this equation that remain finite at r=0 are

$$F(x) = J_m(x),$$

where the $J_m(x)$ are Bessel's functions of integer order because m is an integer. See "Schaum's Outline Series, Mathematical Handbook" by Murray R. Spiegel, McGraw-Hill, New York, 1968, Chapter 24. The required form of the vector potential is

$$A(r,\theta) = A_0 J_m(k_c r)\cos(m\theta), \tag{12.5.6}$$

where A_0 is a constant, and k_c is given by Equation (12.5.2).

The magnetic field components are obtained from $\vec{H}$ = curl($\vec{A}$):

$$\begin{aligned} H_r &= \frac{1}{r}\left(\frac{\partial A_z}{\partial\theta}\right) = \frac{-m}{r}A_0 J_m(k_c r)\sin(m\theta), \\ H_\theta &= -\left(\frac{\partial A_z}{\partial r}\right) = -k_c A_0 \dot{J}_m(k_c r)\cos(m\theta), \\ H_z &= 0, \end{aligned} \tag{12.5.7}$$

(these are all multiplied by the factor exp ($i[k_g z - \omega t]$), of course). The notation $\dot{J}_m(x)$ means the derivative of the Bessel function with respect to the argument x.

The electric field components can be calculated from $\mathrm{curl}(\overrightarrow{H}) = -i\omega\epsilon\overrightarrow{E}$:

$$\begin{aligned} E_r &= \frac{-i}{\epsilon\omega}\left(\frac{\partial H_\theta}{\partial z}\right) = -\frac{k_g}{\epsilon\omega}k_c A_0 \dot{J}_m(k_c r)\cos(m\theta), \\ E_\theta &= \frac{i}{\epsilon\omega}\left(\frac{\partial H_r}{\partial z}\right) = \frac{k_g}{\epsilon\omega}\frac{mA_0}{r}J_m(k_c r)\sin(m\theta), \\ E_z &= \frac{i}{\epsilon\omega}\frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right) \\ &= \frac{-i}{\epsilon\omega}\frac{A_0}{r^2}\left(k_c^2 r^2 \ddot{J}_m + k_c r\dot{J}_m - m^2 J_m\right)\cos(m\theta). \end{aligned} \tag{12.5.8}$$

The expression for E_z can be simplified because $J_m(k_c r)$ must satisfy the differential equation(12.5.4), therefore

$$\left(k_c^2 r^2 \ddot{J}_m + k_c r\dot{J}_m + k_c^2 r^2 J_m - m^2 J_m\right) = 0.$$

Using this expression E_z becomes

$$E_z = i\frac{k_c^2}{\epsilon\omega}A_0 J_m(k_c r)\cos(m\theta). \tag{12.5.9}$$

The fields of Equations (12.5.7) and (12.5.8) satisfy Maxwell's equations. They must also satisfy the boundary conditions E_θ=0, E_z=0, and H_r=0 at the walls of the wave-guide. Let the inner radius of the wave-guide be R meters. The boundary conditions can be met if $J_m(k_c R)$=0. This condition fixes allowable values for k_c and therefore fixes k_g through Equation (12.5.2)

$$k_g^2 = \epsilon_r\left(\frac{\omega}{c}\right)^2 - k_c^2. \tag{12.5.10}$$

Table(12.5.2) lists the four lowest roots of the equation $J_m(x)$=0 for Bessel's functions with m=0,1,2 and 3. These roots determine the wave-vector, k_g. In particular, they determine the minimum frequency for which energy can be propagated down the wave-guide. The cut-off frequencies correspond to k_g=0, and are given by

$$\frac{\omega}{c} = \frac{k_c}{\sqrt{\epsilon_r}}. \tag{12.5.11}$$

To take a concrete example, suppose that R=1cm =0.01m. The lowest TM mode corresponds to m=0 and to the first root of the Bessel's function J_0: this is called the TM_{01} mode. For this case

$$\begin{aligned} E_r &= -\frac{k_g}{\epsilon\omega} k_c A_0 \dot{J}_0(k_c r), \\ E_\theta &= 0, \\ E_z &= i\frac{k_c^2}{\epsilon\omega} A_0 J_0(k_c r), \\ H_r &= 0, \\ H_\theta &= -k_c A_0 \dot{J}_0(k_c r), \\ H_z &= 0, \end{aligned} \tag{12.5.12}$$

Here k_c = 2.4048/R = 240.48 per meter. This corresponds to a cut-off frequency of 11.48 GHz for ϵ_r =1. The TM_{01} mode pattern is shown in Figure (12.5.12(b)).

	J_m	$\dot{J}_m$
m=0	2.4048 5.5200 8.6537 11.7915	3.8317 7.0156 10.1735 13.3237
m=1	3.8317 7.0156 10.1735 13.3237	1.8412 5.3314 8.5363 11.7060
m=2	5.1356 8.4172 11.6198 14.7960	3.0542 6.7061 9.9695 13.1704
m=3	6.3802 9.7610 13.0152 16.2235	4.2012 8.0152 11.3459 14.5858

Table 12.5.2: The values of x corresponding to the roots of the equations $J_m(x)=0$ and $\dot{J}_m(x)=0$ for the first four Bessel's functions.

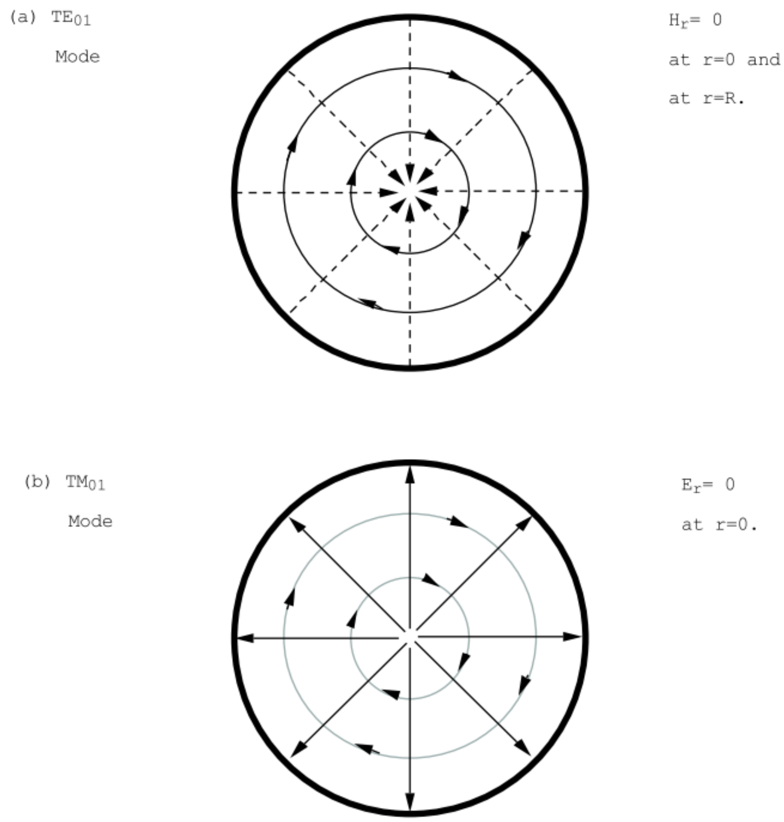


Figure 12.5.12: Electric and magnetic field distributions for the TE_{01} and TM_{01} modes in a circular wave-guide. The dashed lines represent the magnetic field, the full lines the electric field.

12.5.2 TE Modes.

Using the symmetry relations (12.2.3) one can write down the electric field components corresponding to transverse electric modes directly from Equations (12.5.7):

$$\begin{aligned} E_r &= \frac{mE_0}{r} J_m(k_c r)\sin(m\theta), \\ E_\theta &= k_c E_0 J_m(k_c r)\cos(m\theta), \\ E_z &= 0, \end{aligned} \tag{12.5.13}$$

where, as before,

$$k_c^2 = \epsilon_r\left(\frac{\omega}{c}\right)^2 - k_g^2.$$

The magnetic field components can be calculated from $\operatorname{curl}(\vec{E}) = i\omega\mu_0 \vec{H}$:

$$\begin{aligned} H_r &= \frac{i}{\omega\mu_0}\frac{\partial E_\theta}{\partial z} = -\frac{k_g k_c}{\omega\mu_0} E_0 \dot{J}_m(k_c r)\cos(m\theta), \\ H_\theta &= \frac{-i}{\omega\mu_0}\frac{\partial E_r}{\partial z} = \frac{k_g}{\omega\mu_0}\frac{mE_0}{r} J_m(k_c r)\sin(m\theta), \\ H_z &= \frac{-i}{\omega\mu_0 r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right] \\ &= \frac{ik_c^2}{\omega\mu_0} E_0 J_m(k_c r)\cos(m\theta). \end{aligned} \tag{12.5.14}$$

In Equations (12.5.14) the factor exp $(i[k_g z - \omega t])$ has been suppressed. The simple form for H_z has been obtained using the fact that $J_m(k_c r)$ must satisfy Equation (12.5.4), the differential equation for the radial function F(r). In order to satisfy the boundary conditions E_θ=0 and H_r=0 at the wave-guide walls k_cR must be set equal to one of the roots of the equation $\dot{J}_m(k_c r) = 0$, where R is the inner radius of the wave-guide. The lowest four roots of $\dot{J}_m(x) = 0$ have been listed in Table(12.5.2) for the first four Bessel's functions. The lowest cut-off frequency occurs for the first root of $\dot{J}_1(x)$: this mode is called the TE_{11} mode. The cut-off frequency for the TE_{11} mode is 8.79 GHz for ϵ_r=1 and R= 1cm. Compare this with the cut-off frequency for the TM_{01} mode, 11.48 GHz. Thus, over the frequency interval 8.79 to 11.48 GHz an air-filled circular pipe having an inner radius of R=1cm can support only a single mode, the TE_{11} mode. The TE_{11} mode pattern is shown in Figure (12.5.13).

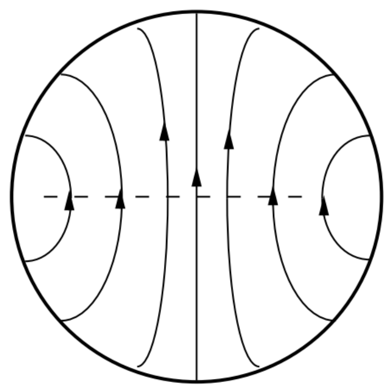


Figure 12.5.13: Electric field lines for the TE_{11} mode in a cylindrical Waveguide. The electric field lines must be normal to the walls at r=R, where R is the inner radius of the wave-guide. The magnetic field lines are orthogonal to the electric field lines, and H_r=0 at r=R.

The TE_{01} mode is of particular interest; the mode pattern is shown in Figure (12.5.12(a)). This mode is very useful for the construction of high-Q cavities of variable frequency. The length of the cavity can be altered by means of a sliding piston. No currents need flow across the gap between the piston and the walls of the cylinder for the TE_{01} mode: the current lines on the face of the piston are similar to the electric field lines shown in Figure (12.5.12(a)) and are concentric circles. Even if the piston does not make good electrical contact with the cavity walls the field lines in the TE_{01} mode remain unperturbed by any small gap between the piston and the cylinder walls. This mode is often used to construct microwave frequency meters.

Wave-guide modes are discussed in detail in the book "Electron Spin Resonance" by Charles P. Poole, 2cd Edition, John Wiley and Sons, New York, 1983.

CHAPTER OVERVIEW

13: Problem and Answers

13.1: Chapter 1

Problem (1.1).

Two charges, each q=+1.6x10^{-19} Coulombs, are located at (0,0,a) and at (0,0,-a) where a=1.0x10^{-9} meters.

(a) Calculate the electric field at the origin (0,0,0).

(Answ: the field is zero.)

(b) Calculate the electric field at (a,a,a).

(Answ: **E**= (6.07,6.07,1.96)x10^8 Volts/m.)

(c) An electron, q=-1.6x10^{-19} Coulombs, flies through the point (a,a,a) with the velocity **v**= v_0(1,2,3) where v_0= 10^5 m/sec. What forces are exerted on the electron due to the two stationary charges?

(Answ: **F**= q**E**= (-9.71,-9.71,-3.14)x10^{-11} Newtons. There is no magnetic force.)

Problem (1.2).

At a certain moment a moving proton, q=+1.6x10^{-19} Coulombs, is located at (0,0,a) with velocity components v_0(1,1,0) where a=10^{-9} m. and v_0=10^5 m/sec. At the same moment a moving electron, q=-1.6x10^{-19} Coulombs, is located at (a,a,a) with velocity components (0,10^6,0) m/sec.

(a) Calculate the electric and magnetic fields at the position of the electron due to the proton.

(Answ: **E**= (E_0,E_0,0) where E_0= 5.09x10^8 V/m. and **B**= (0,0,0) because $\mathbf{v_p}$x**E** =0.)

(b) Calculate the force on the electron due to the electric field of the proton.

(Answ: **F**=(-F_0,-F_0,0) where F_0= |q|E_0 =8.14x10^{-11} N.)

(c) Calculate the force on the electron due to the magnetic field of the proton.

(Answ: **F**= $\mathbf{v}_{\text{electron}}$x**B** = 0 N.)

(d) Calculate the electric and magnetic forces on the proton due to the fields generated by the electron.

Answ: The electric field at the position of the proton, **R**=(0,0,a), due to the electron at **r**=(a,a,a) is given by

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\quad\left(-1.6\times 10^{-19}\right)\frac{\rho}{\rho^3},$$

where ρ= **R**-**r** = (-a,-a,0)= - a(1,1,0), where a= 10^{-9} m.

Therefore

$$\mathbf{E}=\left(5\cdot 09\times 10^{8}\right)(1,1,0).$$

The magnetic field at the position of the proton due to the motion of the electron is given by c^2**B**= **v**x**E**, where the velocity of the electron is **v**= 10^6(0,1,0) m/sec. c^2**B**= (5.09x10^{14})(0,0,-1) so **B**= (0.566x10^{-2})(0,0,-1) Teslas.

The force on the proton due to the electric field is $\mathbf{F_E}$= 8.15x10^{-11}(1,1,0) N. The force on the proton due to the magnetic field is $\mathbf{F_M}$= q($\mathbf{v_p}$x**B**) = 0.906x10^{-16}(-1,1,0) N.

Problem (1.3).

A particle having a velocity **V**=$v_1\mathbf{u_x}$ carries a charge q_1 C and is located at the origin. A second particle, charge q_2, is located at **r**= a$\mathbf{u_x}$ + b$\mathbf{u_y}$ + c$\mathbf{u_z}$, and it has a velocity $\mathbf{V_2}$=$v_y\mathbf{u_y}$.

(a) Show that the force on charge #2 due to the magnetic field generated by charge #1 is $\mathbf{F}_{21}=\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^3}bv_1v_y\mathbf{u}_x$.

(b) Show that the force on charge #1 due to the magnetic field generated by charge #2 is $\mathbf{F}_{12}=-\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^3}av_1V_y\mathbf{u}_y$. Notice that $\mathbf{F_{21}}$ does not equal -$\mathbf{F_{12}}$ so that Newton's law of the equality of forces of action and reaction is not obeyed in this case.

Answer (1.3).

(a) The electric field at the position of particle #2 due to particle #1 is

$$\mathbf{E_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}_1}{\mathrm{r}^3}(\mathrm{a}, \mathrm{b}, \mathrm{c}).$$

The magnetic field at the position of particle #2 due to the motion of particle #1 is given by

$$c^2\mathbf{B_{21}} = \mathbf{v}_1 \times \mathbf{E}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}_1\mathrm{v}_1}{\mathrm{r}^3}(0, -\mathrm{c}, \mathrm{b}),$$

or

$$\mathbf{B_{21}} = \frac{\mu_0}{4\pi}\frac{\mathrm{q}_1\mathrm{v}_1}{\mathrm{r}^3}(0, -\mathrm{c}, \mathrm{b}).$$

The magnetic force on particle #2 due to its motion is

$$\mathbf{F_{2M}} = q_2\left(\mathbf{v_2} \times \mathbf{B_{21}}\right) = \frac{\mu_0}{4\pi}\frac{q_1 q_2 v_1 v_y}{r^3} \quad (b, 0, 0).$$

(b) The electric field at the position of particle #1 due to particle #2 is

$$\mathbf{E_{12}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q2}}{\mathrm{r}^3}(\mathrm{a}, \mathrm{b}, \mathrm{c}).$$

The magnetic field at the position of particle #1 due to the motion of particle #2 is given by

$$c^2\mathbf{B_{12}} = \mathbf{v}_2 \times \mathbf{E_{12}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}_2\mathrm{v}_\mathrm{y}}{\mathrm{r}^3}(\mathrm{c}, 0, -\mathrm{a}),$$

or

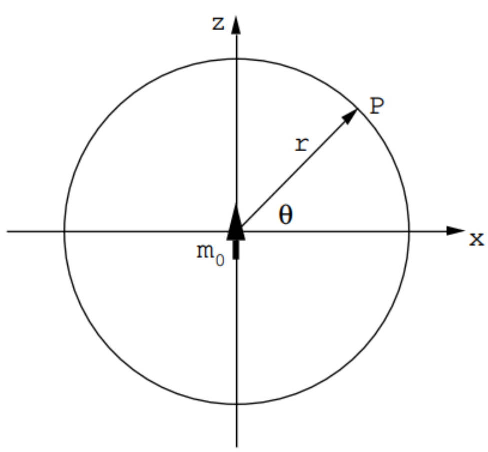
$$\mathbf{B_{12}} = \frac{\mu_0}{4\pi}\frac{\mathrm{q}_2\mathrm{v}_\mathrm{y}}{\mathrm{r}^3}(\mathrm{c}, 0, -\mathrm{a}).$$

The magnetic force on particle #1 due to its motion is

$$\mathbf{F_{1M}} = \mathrm{q}_2\left(\mathbf{v_1} \times \mathbf{B_{12}}\right) = -\frac{\mu_0}{4\pi}\frac{\mathrm{q}_1\mathrm{q}_2\mathrm{v}_1\mathrm{v}_\mathrm{y}}{\mathrm{r}^3}(0, \mathrm{a}, 0).$$

Problem (1.4).

An electron carries a magnetic moment of $|\mathbf{m}_0|$=9.27x10^{-24} Joules/Tesla= 1 Bohr magneton. Suppose that this magnetic moment is oriented along the z-axis as shown in the figure.



(a) At what angle θ is the field measured by an observer at P a maximum?

(Answ: θ= ±π/2.)

(b) If r= 1 micron (10^{-6}m.) what is the magnitude and direction of this maximum field?

(Answ: $|\mathbf{B}_{max}|$= 18.54x10^{-13} Teslas directed along +z).

(c) What is the minimum magnetic field? At what angle θ does it occur, and what is the direction of the field?

(Answ: $|\mathbf{B}_{\text{min}}|$= 9.27x10-13 Teslas directed along -z. The observer is at θ=0 or π.)

Answer

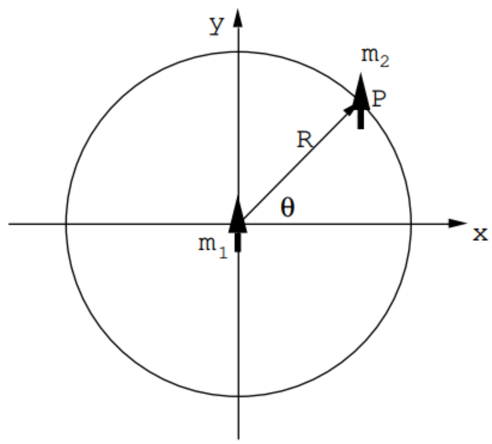
$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(3\frac{m_0 z\mathbf{r}}{r^5}-\frac{m_0\mathbf{u_z}}{r^3}\right) \text{ therefore } \mathrm{B_x}=\frac{\mu_0\mathrm{m_0}}{4\pi}\frac{3\mathrm{xz}}{\mathrm{r^5}}, \quad \mathrm{B_y}=\frac{\mu_0\mathrm{m_0}}{4\pi}\frac{3\mathrm{yz}}{\mathrm{r^5}}, \quad \mathrm{B}_z=\frac{\mu_0\mathrm{m_0}}{4\pi}\left(\frac{3\mathrm{z}^2}{\mathrm{r^5}}-\frac{1}{\mathrm{r^3}}\right).$$

$\mathrm{B}^2=\left(\frac{\mu_0\mathrm{m_0}}{4\pi\mathrm{r}^3}\quad\left(1+\frac{3\mathrm{z}^2}{\mathrm{r}^2}\right)\right)$ so B^2 is a maximum at x=0, y=0, z=r. B^2 is a minimum at z=0. $B_{\text{min}}=\frac{\mu_0 m_0}{4\pi}\frac{1}{r^3}$. $B_{\text{max}}=2B_{\text{min}}$.

Problem (1.5).

The energy of interaction between two magnetic dipoles is given by - $\mathbf{m_1}\bullet\mathbf{B_2}$ or by - $\mathbf{B_1}\bullet\mathbf{m_2}$ where $\mathbf{B_1}$ is the field generated at the position of dipole #2 by dipole #1, and $\mathbf{B_2}$ is the field at dipole #1 generated by dipole #2. Let these two magnetic dipoles be separated by a constant distance R= 10^{-6}m (1 µm).

(a) Assume that the two dipoles are forced to remain parallel as shown in the figure. At what angle θ is the interaction energy a minimum? What is this minimum energy?



(Answ: θ= ± π/2, $\mathrm{U_{min}}=-2\frac{\mu_0\mathrm{m_1m_2}}{4\pi\mathrm{R}^3}$.

(b) Assume that θ=0 in the figure, but that the two dipoles are free to rotate in the x-y plane. Let $m_1|_x = m_1\cos\alpha_1$ and $m_1|_y = m_1\sin\alpha_1$. Similarly let $m_2|_x = m_2\cos\alpha_2$ and $m_2|_y = m_2\sin\alpha_2$. What will be the minimum energy configuration, and what will be the minimum energy?

(Answ: $\alpha_1 = \alpha_2 = 0$ or π. $\mathrm{U_{min}}=-2\frac{\mu_0\mathrm{m_1m_2}}{4\pi\mathrm{R}^3}$.)

Answer

(a) $\mathbf{B_1}=\frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m_1}\bullet\mathbf{r})\mathbf{r}}{\mathrm{r}^5}-\frac{\mathbf{m_1}}{\mathrm{r}^3}\right), \quad \mathrm{x}=\mathrm{R}\cos\theta, \quad \mathrm{y}=\mathrm{R}\sin\theta, \quad \mathrm{z}=0$

$\mathrm{B_{1x}}=\frac{\mu_0}{4\pi}\frac{\mathrm{m_1}}{\mathrm{R}^3}3\sin\theta\cos\theta, \quad \mathrm{B_{1y}}=\frac{\mu_0}{4\pi}\frac{\mathrm{m_1}}{\mathrm{R}^3}\left(3\sin^2\theta-1\right)$,

therefore

$$U=-\mathbf{B_1}\bullet\mathbf{m_2}=-\frac{\mu_0}{4\pi}\frac{m_1m_2}{R^3}\quad\left(3\sin^2\theta-1\right).$$

This expression is clearly a minimum when sinθ= ±1.

(b) When θ=0 and $\mathbf{r}$=(R,0,0) one finds

$\mathrm{B_{1x}}=\frac{\mu_0}{4\pi\mathrm{R}^3}2\mathrm{m_1}\cos\alpha_1, \quad \mathrm{B_{1y}}=-\frac{\mu_0}{4\pi\mathrm{R}^3}\mathrm{m_1}\sin\alpha_1$,

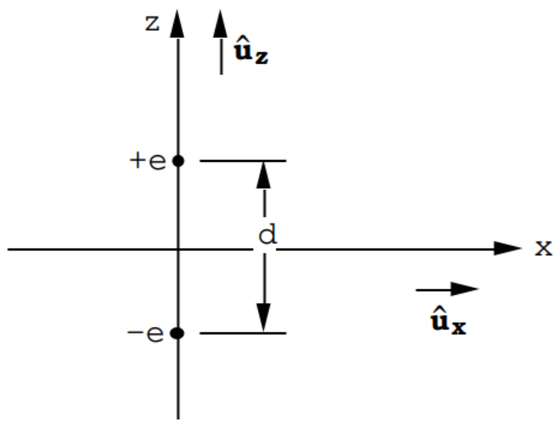
therefore

$$\mathrm{U}=-\mathbf{m_2}\bullet\mathbf{B_1}=-\frac{\mu_0\mathrm{m_1m_2}}{4\pi\mathrm{R}^3}\quad(2\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2)\ .$$

This expression clearly has a minimum when $\cos\alpha_1=\cos\alpha_2=1$ and $\sin\alpha_1=\sin\alpha_2=0$, ie. when $\alpha_1=\alpha_2= 0$ or π.

Problem (1.6)

A proton and an electron are separated by 10^{-12} m = d as shown in the sketch.



(a) Calculate the strength of the electric field 1 micron (= 10^{-6} m) distant from a proton.

(b) Calculate the strength of the electric field a = 1 micron from the above pt dipole at $\mathbf{r} = \mathrm{a}\hat{\mathbf{u}}_z$. What is the direction of this electric field?

(c) Calculate the strength of the electric field a distance a = 1 micron from the dipole at the point $\mathbf{r} = -\mathrm{a}\hat{\mathbf{u}}_z$. What is the direction of this electric field?

(d) Calculate the strength and direction of the electric field at $\mathbf{r} = \mathrm{a}\hat{\mathbf{u}}_x$ where a = 1 micron.

(e) Calculate the strength and direction of the electric field for the above dipole at $\mathbf{r} = \frac{1}{\sqrt{2}} a\left(\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z\right)$ and a= 1 micron.

N.B. $\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z$ are unit vectors along x,y,z.

Answer (1.6)

a) $|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}$

$$\frac{1}{4\pi\varepsilon_0} = 9\times 10^9 \mathrm{e} = 1.6\times 10^{-19}\mathrm{Coulombs}$$

$$\mathrm{r} = 10^{-6}\mathrm{m}$$

$\therefore |\mathbf{E}| = \frac{(9\times 109)\left(1.6\times 10^{-19}\right)}{10^{-12}} = \underline{1440}\mathrm{Volts/m}$.

b) For a point dipole $\mathbf{E}_\mathrm{d} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{\mathrm{r}^5} - \frac{\mathbf{p}}{\mathrm{r}^3}\right]$

In this case **p** and **r** are both along z and hence parallel

$$\therefore |\mathbf{E}_\mathrm{d}| = \frac{1}{4\pi\varepsilon_0}\left(\frac{2\mathrm{p}}{\mathrm{r}^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{\mathrm{e}}{\mathrm{r}^2}\right)\left(\frac{2\mathrm{d}}{\mathrm{r}}\right)$$

$$= \frac{(2)\left(10^{-12}\right)}{10^{-6}} = \left(2\times 10^{-6}\right)\times \text{part (a)}$$

So $|\mathbf{E}_\mathrm{d}|$ = 2.88 x 10^{-3} Volts/m and directed along z.

c) For this part **p.r** = -e d a

since p = e d $\hat{\mathbf{u}}_z$

and $\mathbf{r} = -\mathrm{a}\hat{\mathbf{u}}_z$

therefore $(\mathbf{p}\cdot\mathbf{r})\mathbf{r} = ea^2 d\hat{\mathbf{u}}_z$

So $\frac{(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5} = \frac{ed}{a^3}\hat{\mathbf{u}}_z$

and $|\mathbf{E}\mathrm{d}| = \frac{1}{4\pi\varepsilon_0}\left|\left(\frac{3\mathrm{ed}}{\mathrm{a}^3}\hat{\mathbf{u}}_z - \frac{\mathrm{ed}}{\mathrm{a}^3}\hat{\mathbf{u}}_z\right)\right| = \frac{1}{4\pi\varepsilon_0}\frac{2\mathrm{ed}}{\mathrm{a}^3}$

or exactly the same as part (b). The electric field is also directed along z, just as in part (b).

(d) Here **p.r**=0 because **p** is directed along z whereas **r** is directed along x.

Therefore

$$\begin{aligned}\mathbf{E}_{\mathrm{d}} &= \frac{-1}{4\pi\varepsilon_0} \quad \mathbf{p}/\mathrm{r}^3 \\ &= -\left(\frac{\mathrm{e}}{4\pi\varepsilon_0}\right), \frac{1}{\mathrm{a}^2}(\mathrm{d}/\mathrm{a})\hat{\mathbf{u}}_{\mathrm{z}}\end{aligned}$$

i.e. directed along –z and half as large as the electric field for a point along the dipole axis and a meters from the dipole.

∴|**E**d| = 1.44 x 10^{-3} Volts/m

e) $\mathbf{p} = \mathrm{ed}\hat{\mathbf{u}}_z \quad \therefore \mathbf{p}\cdot\mathbf{r} = \frac{\mathrm{eda}}{\sqrt{2}}$

$\mathbf{r} = \frac{a}{\sqrt{2}} \quad (\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_z)$

$\therefore \quad \mathbf{E}\mathrm{d} = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\mathrm{ed}}{(\sqrt{2})(\sqrt{2})}\frac{\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_z}{\mathrm{a}^3} - \frac{\mathrm{ed}\hat{\mathbf{u}}_z}{\mathrm{a}^3}\right]$

$$\begin{aligned}&\mathbf{E}_{\mathrm{d}} = \frac{e}{4\pi\varepsilon_0}\frac{\mathrm{d}}{\mathrm{a}^3}\left[\frac{3}{2} \quad (\hat{\mathbf{u}}_{\mathrm{x}} + \hat{\mathbf{u}}_z) - \hat{\mathbf{u}}_z\right] \\ &= \frac{\mathrm{e}}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{a}^2}\right)\left(\frac{\mathrm{d}}{2\mathrm{a}}\right)(3\hat{\mathbf{u}}_{\mathrm{x}} + \hat{\mathbf{u}}_z)\end{aligned}$$

So **E**d is directed 18.4° from the xy plane and has the magnitude |**E**d| = (1.58 x 10^{-6}) (1440) = 2.28 x 10^{-3} Volts/m.

Problem (1.7).

Show that the magnetic field at the center of a uniformly magnetized sphere containing a small hole at the center is zero. Uniform magnetization means **M** is constant. Without loss of generality, one can take the magnetization to be directed along the z-axis, ie **M**= $M_0\mathbf{u}_\mathbf{z}$.

(Hint: Add up all the contributions to the field at the center due to volume elements at a distance r from the center. In polar co-ordinates $d\tau = r^2 dr \sin\theta d\theta\, d\phi$, and d**m** = $M_0 d\tau \mathbf{u}_z$.)

Answer (1.7).

If **r**= -x$\mathbf{u}_\mathbf{x}$ - y$\mathbf{u}_\mathbf{y}$ - z$\mathbf{u}_\mathbf{z}$ then **m•r** = - $M_0 d\tau z$ (remember that **r** is the vector drawn from the magnetic moment to the point of observation).

$\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right)$, so that

$B_x = \frac{\mu_0}{4\pi}\frac{M_0 d\tau}{r^5} \quad (3xz), \quad B_y = \frac{\mu_0}{4\pi}\frac{M_0 d\tau}{r^5} \quad (3yz)$,

$B_z = \frac{\mu_0}{4\pi}\frac{M_0 d\tau}{r^5} \quad (2z^2 - x^2 - y^2)$,

Convert to polar co-ordinates and integrate over θ from 0 to π, and over ϕ from 0 to 2π. All field components integrate to zero.

Problem (1.8)

The fields generated at the position **r** from a slowly moving, spinless, point charge are given by $\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$ and $c\mathbf{B} = \frac{\mathbf{v}}{c}\times\mathbf{E}$. Consider a particle moving in a circular orbit whose position at time t is given by $\mathbf{a} = a\cos\omega t\hat{\mathbf{u}}_\mathbf{x} + \sin\omega t \quad \hat{\mathbf{u}}_y$.

(a) Show that the time averaged electric field seen by an observer at $\mathbf{R} = \mathrm{x}\hat{\mathbf{u}}_\mathbf{x} + \mathrm{Y}\hat{\mathbf{u}}_\mathbf{y} + \mathrm{z}\hat{\mathbf{u}}_\mathbf{z}$ is given by $<\mathbf{E}_p> = \frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{R}}{R^3}\right)$ to terms of order $(a/R)^2$.

(b) Show that to lowest order in (a/R) the magnetic field observed at **R** is given by

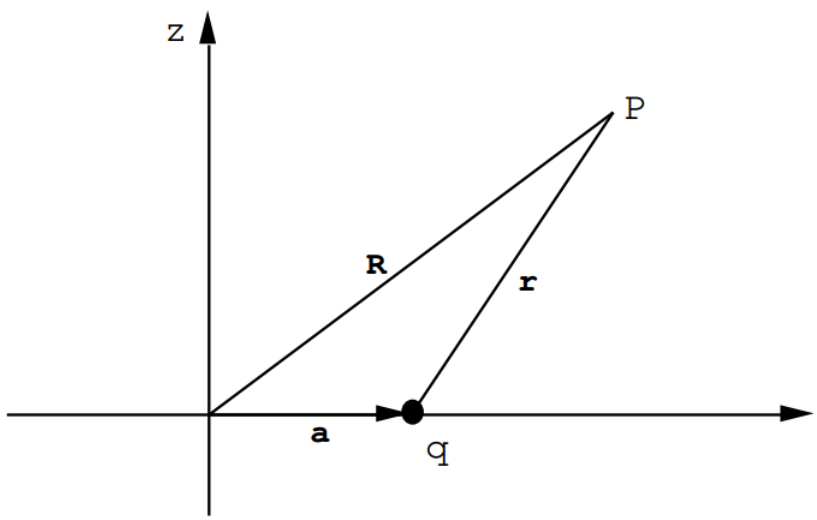
$$< \mathbf{B}_{\mathrm{p}} >= \frac{1}{\mathrm{c}^2} \frac{1}{4\pi\varepsilon_0} \left[\frac{3z\mathbf{R}}{\mathrm{R}^5} - \frac{\hat{\mathbf{u}}_{\mathrm{z}}}{\mathrm{R}^3} \right] \quad \left(\frac{\mathrm{qa}^2\omega}{2} \right)$$

or since $\mathbf{m} = \left(\frac{\mathrm{qa}^2\omega}{2}\right) \hat{\mathbf{u}}_{\mathrm{z}}$ is the magnetic moment ($|\mathbf{m}|$= $\mathrm{I}\pi\mathrm{a}^2$ where I is the current in Amps)

$$\langle \mathbf{B}_{\mathrm{p}} \rangle = \frac{\mu_0}{4\pi} \quad \left[\quad 3\left(\frac{\mathbf{m}\cdot\mathbf{R}}{\mathrm{R}^5} \right) \mathbf{R} - \frac{\mathbf{m}}{\mathrm{R}^3} \right]$$

and $c^2 = \frac{1}{\varepsilon_0\mu_0}$.

Answer (1.8)



We have **r** + **a** = **R**

∴ **r** = **R** - **a**

where $\mathbf{R} = X\hat{\mathbf{u}}_{\mathrm{x}} + \mathrm{Y}\hat{\mathbf{u}}_{\mathrm{y}} + \mathrm{z}\hat{\mathbf{u}}_{\mathrm{z}}$

and $\mathbf{a} = \mathrm{a}\cos\omega\mathrm{t} \quad \hat{\mathbf{u}}_{\mathrm{x}} + \mathrm{a}\sin\omega\mathrm{t}\hat{\mathbf{u}}_{\mathrm{y}}$

$\therefore \quad r^2 = (X - a\cos\omega t)^2 + (Y - a\sin\omega t)^2 + Z^2$

$r^2 = X^2 + Y^2 + Z^2 + a^2 - 2aX\cos\omega t - 2aY\sin\omega t$

or $r^2 = R^2 \left[1 + \left(\frac{a}{R}\right)^2 - \frac{2aX}{R^2}\cos\omega t - \frac{2aY}{R^2}\sin\omega t\right]$.

Keep only the lowest terms in $\left(\frac{a}{R}\right)$:

$r \cong R\left[1 - \frac{2aX}{R^2}\cos\omega t - \frac{2aY}{R^2}\sin\omega t\right]^{1/2}$

So $\frac{1}{r^3} \cong \frac{1}{R^3}\left[1 - \frac{2aX}{R^2}\cos\omega t - \frac{2aY}{R^2}\sin\omega t\right]^{-3/2}$

$\frac{1}{r^3} \cong \frac{1}{R^3}\left[1 + \frac{3aX}{R^2}\cos\omega t + \frac{3aY}{R^2}\sin\omega t\right]$

$\mathbf{E}_{\mathrm{p}} = \frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}}{r^3}\right) = \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{R}-\mathbf{a})}{\mathrm{R}^3}\left[1 + \frac{3aX}{R^2}\cos\omega t + \frac{3aY}{R^2}\sin\omega t\right]$

Now multiply out and take time averages.

$< \cos\omega t >=< \sin\omega t >=< \sin\omega t \quad \cos\omega t >= 0$

$< \cos^2\omega t >=< \sin^2\omega t >= 1/2$

Notice that all terms proportional to a average to zero.

(a) $\therefore \quad \langle\mathbf{E}_{\mathrm{p}}\rangle \cong \frac{q}{4\pi\varepsilon_0} \quad \left(\frac{\mathbf{R}}{R^3}\right)$. The correction terms are of order $\left(\frac{a}{R}\right)^2$.

(b) $\mathbf{B}_{\mathrm{p}} = \frac{1}{\mathrm{c}^2} \quad (\mathbf{V} \quad \times \mathbf{E})$

$\mathbf{v} = \frac{\mathbf{da}}{dt} = -a\omega\sin\omega t \quad \hat{\mathbf{u}}_x + a\omega\cos\omega t \quad \hat{\mathbf{u}}_y$

$\therefore \quad \mathbf{B}_p \cong \frac{1}{c^2}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{[(\mathbf{v}\times\mathbf{R})-(\mathbf{v}\times\mathbf{a})]}{\mathbb{R}^3}\left\{1+\frac{3aX}{R^2}\cos\omega t+\frac{3aY}{R^2}\sin\omega t\right\}$

$\mathbf{v}\times\mathbf{R} = (a\omega Z \quad \cos\omega t) \quad \hat{\mathbf{u}}_x + (a\omega z\sin\omega t)\hat{\mathbf{u}}_y - [\mathrm{a}\omega\mathrm{Y}\sin\omega\mathrm{t} + \mathrm{a}\omega\mathrm{X}\cos\omega\mathrm{t}]\hat{\mathbf{u}}_{\mathrm{z}}$

$\mathbf{v}\times\mathbf{a} = -\mathrm{a}^2\omega\hat{\mathbf{u}}_z$

Multiply out the terms in $\mathbf{B}_{\mathrm{p}}$ and take time averages. The result is

$\langle\mathbf{B}_{\mathrm{p}}\rangle = \frac{1}{\mathrm{c}^2}\left(\frac{\mathrm{q}}{4\pi\varepsilon_{\mathrm{O}}}\right)\frac{1}{\mathrm{R}^3}\left\{\left(\frac{3\mathrm{a}^2\omega\mathrm{XZ}}{2\mathrm{R}^2}\right)\hat{\mathbf{u}}_{\mathrm{x}} + \left(\frac{3\mathrm{a}^2\omega\mathrm{YZ}}{2\mathrm{R}^2}\right)\hat{\mathbf{u}}_{\mathrm{y}} - -\left(\frac{3a^2\omega}{2R^2}\right) \quad (X^2+Y^2) \quad \hat{\mathbf{u}}_z + a^2\omega\hat{\mathbf{u}}_z\right\}$

add and subtract $\frac{3a^2\omega}{2R^2}z^2\hat{\mathbf{u}}_z$ to obtain

$<\mathbf{B}_p> \frac{1}{c^2}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{1}{R^3}\left\{\left(\frac{3a^2\omega}{2R^2}\right) \quad Z\mathbf{R} - \left(\frac{a^2\omega}{2}\right)\hat{\mathbf{u}}_z\right\}$.

Now $\frac{1}{c^2} = \varepsilon_0\mu_0$ and $\mathbf{m} = \frac{\mathrm{qa}^2\omega}{2}\hat{\mathbf{u}}_{\mathrm{z}}$

Therefore $\langle\mathbf{B}_p\rangle = \left(\frac{\mu_0}{4\pi}\right)\left[\frac{3(\mathbf{m}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3}\right]$.

Problem (1.9)

Given the following scalar functions, V, expressed in cylindrical polar co-ordinates. For each function calculate

(1) the components of grad V

(2) ∇^2 V

(a) V = r Cosθ

(b) V = ln r

(c) $V = \frac{\mathrm{Cos}\,\theta}{r}$

(d) $V = \frac{\cos n\theta}{r^n}$, where n is an integer either positive or negative.

Answer (1.9)

$\operatorname{grad} V = \left(\frac{\partial V}{\partial r}\right)\hat{\mathbf{u}}_r + \frac{1}{r}\left(\frac{2V}{\partial\theta}\right)\hat{\mathbf{u}}_\theta + \left(\frac{\partial V}{\partial z}\right)\hat{\mathbf{u}}_z$

$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2}$

(a) V= rCosθ.

$\operatorname{grad} V|_r = \cos\theta$

$\operatorname{grad} V|_\theta = -\sin\theta$

These correspond to a unit vector along x!

∇^2V = 0

(b) V= lnr.

$\operatorname{grad} V|_r = \frac{1}{r}$

$\operatorname{grad} V|_\theta = 0$

∇^2V = 0

(c) $V = \frac{\cos\theta}{r}$

$\operatorname{grad} V|_r = -\frac{\cos\theta}{r^2}$

$\operatorname{grad} V|_\theta = -\frac{\sin\theta}{r^2}$

∇^2V = 0

(d) $\mathrm{V} = \frac{\cos n\theta}{\mathrm{r}^{\mathrm{n}}}$.

$\operatorname{gradv}|_r = -\frac{n\cos n\theta}{r^{n+1}}$

$\operatorname{grad} V|_\theta = -\frac{\mathrm{nsin}(n\theta)}{r^{n+1}}$

∇^2V = 0 for any n.

Problem (1.10)

Given the following scalar functions V expressed in spherical polar co-ordinates. For each function calculate

(1) the components of grad V

(2) ∇^2V

(a) V = r Cosθ

(b) $V = \frac{\operatorname{Cos}\theta}{r^2}$

(c) V = $r^2(3\mathrm{Cos}^2\theta - 1)$

(d) $V = \frac{\left(3\cos^2\theta - 1\right)}{r^3}$

(e) $V = \frac{\cos n\theta}{r^n}$ where n is a positive integer.

Answer (1.10)

$$\nabla \mathrm{V} \quad = \left(\frac{\partial \mathrm{V}}{\partial \mathrm{r}}\right)\hat{\mathbf{u}}_{\mathrm{r}} + \frac{1}{\mathrm{r}}\left(\frac{\partial \mathrm{V}}{\partial \theta}\right)\hat{\mathbf{u}}_\theta + \frac{1}{\mathrm{r}\sin\theta}\left(\frac{\partial \mathrm{V}}{\partial \phi}\right)\hat{\mathbf{u}}_\phi$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

(a) V = rCosθ

$\operatorname{grad} V|_r = \cos\theta$

$\operatorname{gradv}|_\theta = -\sin\theta$

$\operatorname{grad} V|_\phi = 0$

These correspond to a constant field $\hat{\mathbf{u}}_z$.

∇^2V = 0.

(b) $V = \frac{\operatorname{Cos}\theta}{r^2}$

$\operatorname{grad} V|_r = -\frac{2\cos\theta}{r^3}$

$\operatorname{grad} V|_\theta = -\frac{\sin\theta}{r^3}$

$\operatorname{grad} V|_\phi = 0$

Corresponds to a dipole field.

∇^2V = 0.

(c) $\mathrm{V} = \mathrm{r}^2 \quad \left(3\cos^2\theta - 1\right) \quad \operatorname{grad} \mathrm{V}|_{\mathrm{r}} = 2\mathrm{r}\left(3\cos^2\theta - 1\right)$

$\operatorname{gradv}|_\theta = -6r\sin\theta\cos\theta$

$\operatorname{grad} V|_\phi = 0$

∇^2V = 0

(d) $V = \frac{3\cos^2\theta - 1}{r^3}$

$\operatorname{gradv}|_r = -\frac{3}{r^4}\left(3\cos^2\theta - 1\right)$

$\operatorname{grad} V|_\theta = -\frac{6}{r^4}\sin\theta\cos\theta$

$\operatorname{grad} V|_\phi = 0$

$\nabla^2 V = 0$

(e) $V = \frac{\cos n\theta}{r^n}$

$\operatorname{grad} V|_r = -\frac{n\cos(n\theta)}{r^{n+1}}$

$\operatorname{grad} V|_\theta = -\frac{n\sin(n\theta)}{r^{n+1}}$

$\operatorname{grad} V|_\phi = 0$

$\nabla^2 V = -\frac{n}{\sin\theta r^{n+2}}(\sin\theta\cos(n\theta) + \cos\theta\sin(n\theta))$.

Problem (1.11)

Calculate the vector field **B** = curl**A** for the following fields, **A**.

(a) In cylindrical polar co-ordinates

$$A_T = 0$$
$$A\theta = 0$$
$$A_z = -\frac{\mu_0 I}{2\pi}\ln r$$

(b) In cylindrical polar co-ordinates

$$A_r = 0$$
$$A\theta = \frac{B_O r}{2}$$
$$A_Z = 0$$

(c) $\mathbf{A} = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right)$, where $\mathbf{m} = m_0\hat{\mathbf{u}}_z$.

Show that in spherical polar co-ordinates if $\mathbf{m} = m_0\hat{\mathbf{u}}_z$ then $A_r = A_\theta = 0$ and $A_\phi = \frac{\mu_0}{4\pi}\frac{m_0\sin\theta}{r^2}$. This can be used to calculate curl **A**.

Answer (1.11)

(a) $\mathbf{B} = \operatorname{curl}\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = \begin{vmatrix} \frac{1}{r}\frac{\partial A_z}{\partial\theta} \\ -\frac{\partial A_z}{\partial r} \\ 0 \end{vmatrix}$

But $\frac{A_Z}{\partial\theta} = 0 \therefore \quad B_r = 0$

$B_\theta = \frac{\mu_0 I}{2\pi r}$

$B_z = 0$

The field due to a current I Amps flowing along a long wire oriented along z.

(b) $\mathbf{B} = \operatorname{curl}\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & rA_\theta & 0 \end{vmatrix} = \begin{vmatrix} -\frac{\partial A_\theta}{\partial z} \\ \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} \end{vmatrix}$

But $\frac{\partial A_\theta}{\partial z} = 0$ and $A_\theta = \frac{B_0 r}{2}$

$$\therefore \quad B_r = 0$$
$$B_\theta = 0$$
$$B_z = B_0$$

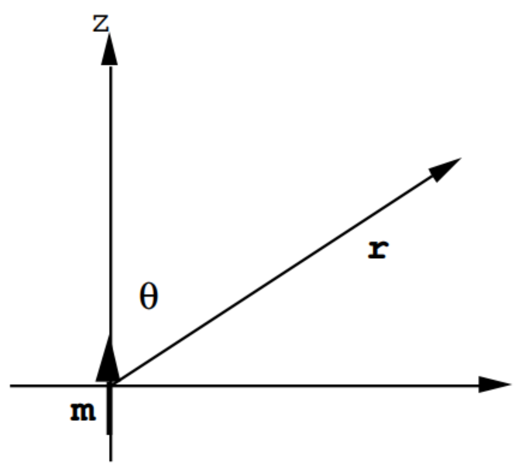
This is the field inside an infinitely long solenoid.

(c) $\mathbf{A} = \frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{r})}{r^3}$

If $\mathbf{m} = \mathrm{m_O}\hat{\mathbf{u}}_z$ this generates the field due to a magnetic dipole.

For $\mathbf{m} = m_0\hat{\mathbf{u}}_z$ $(\mathbf{m}\times\mathbf{r})$ is a vector in the ϕ direction having the magnitude $m_o r\sin\theta$.

$$\mathrm{A_r} = 0$$
$$\mathrm{A}_\theta = 0$$
$$\mathrm{A}_\phi = \frac{\mu_0}{4\pi}\frac{\mathrm{m_0}}{\mathrm{r}^2}\sin\theta, \quad \text{(see the figure)}$$



$$\mathbf{B} = \operatorname{curl}\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & r\sin\theta\hat{\mathbf{u}}\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix} = \begin{vmatrix} \frac{1}{r^2\sin\theta} & \frac{\partial(r\sin\theta A_\phi)}{\partial\theta} \\ -\frac{1}{\mathrm{r}\sin\theta} & \partial(\sin\theta A_\phi) \\ & 0 \end{vmatrix}$$

$$\therefore \mathrm{B_r} = \frac{\mu_0}{4\pi}\frac{2\mathrm{m_0}\cos\theta}{\mathrm{r}^3}, \quad \mathrm{B}_\theta = \frac{\mu_0}{4\pi}\frac{\mathrm{m_0}\sin\theta}{\mathrm{r}^3}, \quad \mathrm{B}_\phi = 0 .$$

Problem (1.12)

A water molecule is planar but the angle between the two oxygen-hydrogen bonds is 105° as shown in the sketch.

(a) If the charge on the oxygen is twice the electronic charge i.e. -2|e| and the charge on each hydrogen is q_H = +|e|, calculate the dipole moment of the molecule assuming an O-H bond length of 5 x 10^{-10} m. [The measured dipole moment is p = 6.17 x 10^{-30} Coulomb-m].

(b) If all of the dipoles in a cubic meter of water were aligned what would be the resulting density of electric dipoles |P|?

Use p = 6.17 x 10^{-30} cm.

Answer (1.12)

(a) The dipole moment is p = qd. In the H_2O molecule q = 2|e| = (2)(1.60 x 10^{-19}) Coulombs or q = 3.2 x 10^{-19} C

The distance $d = b\cos\left(\frac{105}{2}\right)$ where b = 5 x 10^{-10} m is the bond length; d = 3.04 x 10^{-10} m

∴ p = (3.2)(3.04) x 10^{-29} Coulomb m = <u>9.74 x 10^{-29}</u> Cm

Compared with experiment this is too large by ~ 15.7 times.

(b) The molar volume of H_2O is 18 c.c.

∴ No. of moles in 1 m^3 = 10^6/18 = 5.56 x 10^4 moles.

∴ No. of molecules in 1 m^3 = (6.02 x 10^{23})(5.56 x 10^4) = 3.34 x 10^{28} molecules.

∴ |**P**| = (3.34 x 10^{28})(6.17 x 10^{-30}) = <u>0.21 Coulombs/m^2</u>.

(This is very large--in fact H_2O has no permanent dipole moment because the molecules are oriented at random).

Problem (1.13)

An iron atom in metallic iron carries a magnetic moment of 2.2 Bohr magnetons. (1 Bohr magneton, μ_B, is $\mu_B = 9.27 \times 10^{-24}$ Amp m^2 (= Joules/Tesla)). The density of iron is 7.87 gms/cc and its molecular weight is 55.85 gms. If all of the atomic moments were aligned parallel what would be the magnetization per unit volume of iron? Compare this value with the observed internal magnetic field of saturated iron at room temperature $|\mathbf{B}| = \mu_o|\mathbf{M}|$ = 2.15 Teslas = 2.15 Webers/m^2.

Answer (1.13)

The molar volume of iron is $\frac{55.85}{7.87} = 7.10\text{cc}$.

The number of atoms in /m^3 is

$\mathrm{N} = \left(6.02 \times 10^{23}\right)\left(\frac{10^6}{7.10}\right) = 0.848 \times 10^{29}\text{atoms}/\text{m}^3$.

The magnetization/m^3 $|\mathbf{M}|$ = (N)(2.2) μ_B

$|\mathbf{M}|$ = 0.173×10^7 Amps/m.

This would give an internal field $|\mathbf{B}| = \mu_o |\mathbf{M}|$ of $|\mathbf{B}| = (4\pi \times 10^{-7})(0.173 \times 10^7)$ = 2.17 Teslas.

This means that at room temperature the fraction of aligned spins in iron is $\frac{2\cdot 15}{2\cdot 17} = 0.989$ i.e. Very nearly completely aligned!

Problem (1.14)

Given a vector function $\mathbf{F} = \mathrm{xy}\hat{\mathbf{u}}_\mathrm{x} + \left(3\mathrm{x} - \mathrm{y}^2\right)\hat{\mathbf{u}}_\mathrm{y}$ evaluate the line integral from P_1 to P_2 along

a) the direct path (1).

b) the indirect path $P_1 \rightarrow A \rightarrow P_2$ (path (2)).

Answer (1.14)

The line P_1P_2 can be written $\mathbf{s} = (3\hat{\mathbf{u}}_\mathrm{x} + 3\hat{\mathbf{u}}_\mathrm{y}) + (3\hat{\mathbf{u}}_\mathrm{x} + 2\hat{\mathbf{u}}_\mathrm{y})\,\mathrm{L}$ where L varies from L = 0 to L = 1. L= 0 corresponds to $\mathrm{P}_1\ (3\hat{\mathbf{u}}_\mathrm{x} + 3\hat{\mathbf{u}}_\mathrm{y})$ whereas L=1 corresponds to $\mathrm{P}_2\quad (6\hat{\mathbf{u}}_\mathrm{x} + 5\hat{\mathbf{u}}_\mathrm{y})$.

So $\mathbf{ds} = (3\hat{\mathbf{u}}_\mathrm{x} + 2\hat{\mathbf{u}}_\mathrm{y})\,\mathrm{dL}$ or dx= 3dL and dy= 2dL.

(a) Now $\mathbf{F}\cdot\mathbf{ds}$ = 3xy dL + 2(3x - y^2) dL

$\therefore\quad \int_{\mathrm{P}_2}^{\mathrm{R}} \mathbf{F}\cdot\mathbf{ds} = \int_{\mathrm{P}_2(\mathrm{L}=0)}^{\mathrm{P}_1(\mathrm{L}=1)} \left[3\mathrm{xy} + 6\mathrm{x} - 2\mathrm{y}^2\right] \mathrm{dL}$

But x = (3 + 3L) y = 3 + 2L along the line (components of **S**)

$\therefore \int_{\mathrm{P}_2}^{\mathrm{P}_1} \mathbf{F}\cdot\mathbf{ds} = \int_0^1 \mathrm{dL}\{3(3+3\mathrm{L})(3+2\mathrm{L}) + 6(3+3\mathrm{L})\,2(3+2L)^2\}$

$$\therefore\quad \int_{\mathrm{P}_2}^{\mathrm{P}} \mathbf{F}\cdot\mathbf{ds} = \int_0^1 \mathrm{dL}\left[27 + 39\mathrm{L} + 10\mathrm{L}^2\right]$$
$$= 27 + (137/6) = \frac{299}{6}\ .$$

(b) Along path (2)

$$\int_2 \mathbf{F}\cdot\mathbf{ds} = \int_3^6 F_{\mathbf{X}}(y=3)\mathrm{dx} + \int_3^5 \mathrm{F_Y}(\mathrm{x}=6)\mathrm{dy}$$
$$= 3\int_3^6 \mathrm{xdx} + \int_3^5 \left(18 - \mathrm{y}^2\right)\mathrm{dy}$$
$$= \left(\frac{3}{2}\right)(27) + 36 - \frac{98}{3} = 36 + \left(\frac{47}{6}\right) = \frac{263}{6}$$

The line integral is different for the two paths.

Therefore **F** is not a conservative field. Indeed, curl $\mathbf{F}=\left|\begin{array}{c}0\\0\\3\mathrm{x}\end{array}\right|$ and therefore curl **F** does not vanish everywhere.

Problem (1.15)

Given the vector function $\mathbf{E}=\mathrm{y}\hat{\mathbf{u}}_{\mathrm{x}}+\mathrm{x}\hat{\mathbf{u}}_{\mathrm{y}}$. Evaluate the line integral $\int_1^2 \mathbf{E}\cdot\mathbf{dL}$ from P_1 (2,1,-1) to P_2 (8,2,-1)

a) along the parabola x = $2y^2$,

b) along the straight line joining the two points.

c) Is **E** a conservative vector field?

Answer (1.15)

$$\operatorname{curl}\mathbf{E}=\left|\begin{array}{ccc}\hat{\mathbf{u}}_{\mathbf{x}} & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & x & 0\end{array}\right|=\left|\begin{array}{c}0\\0\\(1-1)=0\end{array}\right|\equiv 0\ .$$

Therefore **E** is a conservative vector field.

(a)
$$\begin{aligned}\int_1^2 \mathbf{E}\cdot\mathbf{dL} &= \int_1^2 E_{\mathbf{X}}\,\mathrm{dx}+\int_1^2 E_Y\,\mathrm{dy}\\ &=\int_2^8 \mathrm{ydx}+\int_1^2 \mathrm{xdy}\end{aligned}$$

But $y=\sqrt{x/2}\quad x=2y^2$ along the parabola

$$\therefore\int_1^2\mathbf{E}\cdot\mathbf{dL}=\int_2^8\frac{\mathrm{x}^{1/2}\mathrm{dx}}{\sqrt{2}}+2\left[\int_1^2 \mathrm{y\hat{}\{2\}dy}=\left.\frac{\sqrt{2\mathrm{x}^3}}{3}\right|_2^8+\left.\frac{2\mathrm{y}^3}{3}\right|_1^2\right.$$
$$=\frac{1}{3}\left[2^5-2^2+14\right]=42/3=14$$

(b) Since curl **E** ≡ 0 the line integral along the second path must also be equal to 14.

Check

Let $\mathbf{r}_1=2\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_{\mathrm{y}}-\hat{\mathbf{u}}_{\mathrm{z}}$ (the vector to P_1)

Let $\mathbf{r}_2=8\hat{\mathbf{u}}_{\mathrm{x}}+2\hat{\mathbf{u}}_{\mathrm{y}}-\hat{\mathbf{u}}_{\mathrm{z}}$ (the vector to P2).

Then any point on the straight line from P_1 to P_2 can be specified by $\mathbf{L}=\mathbf{r}_1+L\left(\mathbf{r}_2-\mathbf{r}_1\right)$ where L runs from L = 0 (P_1) to L = 1 (P_2)

$\therefore\mathbf{L}=(2\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_{\mathrm{y}}-\hat{\mathbf{u}}_{\mathrm{z}})+(6\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_{\mathrm{y}})$

$\mathbf{dL}=(6\hat{\mathbf{u}}_{\mathbf{x}}+\hat{\mathbf{u}}_{\mathbf{y}})\,\mathrm{dL}$

$\therefore\quad \mathbf{E}\cdot\mathbf{dL}=6\mathrm{ydL}+\mathrm{xdL}$

However, along the st. line **L** x = 2 + 6L y = 1 + L

$\therefore\quad \mathbf{E}\cdot\mathbf{dL}=6(1+L)\quad dL+(2+6L)\quad dL=(8+12L)\quad dL$

$\therefore\quad \int_{\mathrm{P}_1}^{\mathrm{P}_2}\mathbf{E}\cdot\mathbf{dL}=\int_0^1 \mathrm{dL}\quad(8+12\mathrm{L})=8+6=14\qquad$ Q.E.D.

13.2: Chapter 2

Problem (2.1)

Given an electric field of the form $\mathbf{E}=100\mathrm{x}\ \hat{\mathbf{u}}_{\mathrm{x}} \mathrm{V}/\mathrm{m}$ find the total charge contained in the following volumes:

1) A cubical volume 1 cm on a side centered on the origin. The cube edges are parallel with the x, y, and z axes.

2) A cylindrical volume having a radius of 1 cm and a height of 2 cm centered at the origin. The axis of the cylinder is parallel with the z-axis.

Answer (2.1)

Ex = 100x

Ey = 0

Ez = 0

$\therefore \operatorname{div} \mathbf{E}=100=\rho_{\mathrm{f}}/\varepsilon_{\mathrm{o}}$

This electric field corresponds to a uniform charge distribution $\rho_f=100\varepsilon_0$ Coulombs/m^3

∴ The total charge in

(1) The cube

$$\begin{aligned} Q &= (100\varepsilon_0)\left(10^{-6}\right)=10^{-4}\varepsilon_0 \\ &= \frac{10^{-13}}{36\pi}=8.84\times 10^{-16} \text{ Coulombs.} \end{aligned}$$

(2) The cylinder

$$\begin{aligned} Q &= (100\varepsilon_0)\left(2\pi\times 10^{-6}\right)=2\pi\varepsilon_0\times 10^{-4} \\ &= \frac{10^{-13}}{18}=5.56\times 10^{-15}\text{Coulombs} \end{aligned}$$

since $\varepsilon_0=\frac{1}{\mu_0 C^2}=\frac{10^{-9}}{36\pi}$

Note that the above charge distribution though uniform must have planar symmetry (because Ey = Ez = 0).

Problem (2.2)

A free charge distribution is given by ρ_f = ar Coulombs/m^3 for 0 ≤ r ≤ R and ρ_f = 0 for r > R. (The electric polarization P is everywhere zero).

a) Calculate the components of the electric field in and around this charge distribution. The problem has spherical symmetry so one can use Gauss' theorem (the divergence theorem).

b) Calculate the potential function corresponding to the electric field of part (a). Choose the arbitrary constants so that (1) V → 0 as r → ∞.

(2) V is continuous at r= R.

In this way show that the potential at r = 0 is given by $V(0)=\frac{aR^3}{3\varepsilon_0}$ Volts.

Answer (2.2)

(a) $\operatorname{div}\mathbf{E}=\frac{\rho_{\mathrm{f}}}{\varepsilon_{\mathrm{o}}}$ since P ≡ 0

and therefore div P = 0.

So $\int_{\mathrm{V}} \operatorname{div}\mathbf{E} d\tau=\int_{\mathrm{s}} \mathbf{E}\cdot\mathbf{ds}=\frac{Q(\mathrm{r})}{\varepsilon_0}$

where Q(r) is the charge contained within a sphere of radius r.

$Q(r)=a\int_0^r(r)4\pi r^2 dr=\pi a r^4$ Coulombs for r ≤ R.

But $\mathbf{E}$ has only a radial component by symmetry. Therefore for a spherical surface of radius r , $\int_S \mathbf{E}\cdot\mathbf{ds}=4\pi r^2 E_r$

So for r≤R $4\pi r^2 E_r = \frac{\pi a r^4}{\varepsilon_0}$

or $\mathrm{E_r} = \frac{\mathrm{ar^2}}{4\varepsilon_O}\mathrm{Volts/m}$.

for r>R the charge is independent of r: Q = πaR^4

$$\therefore \quad E_r = \frac{\pi a R^4}{4\pi\varepsilon_O r^2}\ \text{Volts/m (Coulomb's Law)}\ .$$

(b) Since **E** has only a radial component, the potential function will depend only upon r:

$$E_r = -\left(\frac{\partial V}{\partial r}\right)$$

∴ in the region r ≤ R $V = \frac{-a r^3}{12\varepsilon_0} + V_0$

in the region r > R $V = \frac{\pi a R^4}{4\pi\varepsilon_0 r}$

(The constant is zero so that V → 0 as r → ∞).

At r = R we require V to be continuous. Therefore

$$V_0 - \frac{aR^3}{12\varepsilon_0} = \frac{aR^3}{4\varepsilon_0} = \frac{3aR^3}{12\varepsilon_0}$$

So $V_0 = \frac{4aR^3}{12\varepsilon_0} = aR^3/3\varepsilon_0$

The potential at the center of the charge distribution is therefore ($aR^3/3\ \varepsilon_o$) Volts

Problem (2.3)

A cube of side length L m is centered on the origin and its edges are parallel with the x, y, and z axes. The electric dipole vector per unit volume, **P**, is given by $\mathbf{P} = \mathrm{P_0}\left(\mathrm{x}\hat{\mathbf{u}}_\mathrm{x} + \mathrm{y}\hat{\mathbf{u}}_\mathrm{y} + \mathrm{z}\hat{\mathbf{u}}_\mathrm{z}\right)\ \ \mathrm{Coulombs}\ /\mathrm{m}^2$

a) Calculate the bound charge density ρ_b = - div **P**.

b) Calculate the surface bound charge density on each face of the cube.

c) Show that the total bound charge on the cube is zero.

Answer (2.3)

(a) $\mathbf{P} = \ \mathrm{Po}\ \left(\mathrm{x}\hat{\mathbf{u}}_\mathbf{x} + \mathrm{y}\hat{\mathbf{u}}_\mathrm{y} + \mathrm{z}\hat{\mathbf{u}}_\mathrm{z}\right)\quad \text{inside cube}$

P ≡ 0 Outside cube

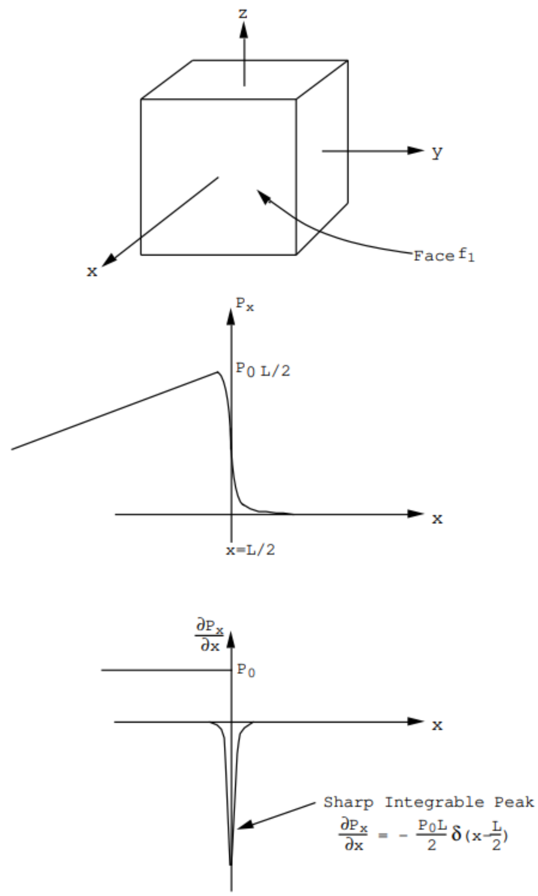
∴ ρ_b = - div **P** = - 3P_o inside the cube

= 0 outside the cube.

The total bound charge inside the cube is therefore Q_v = - 3$P_o L^3$ Coulombs

(b) The discontinuity in the normal component of **P** gives an effective surface charge density on each face of the cube.

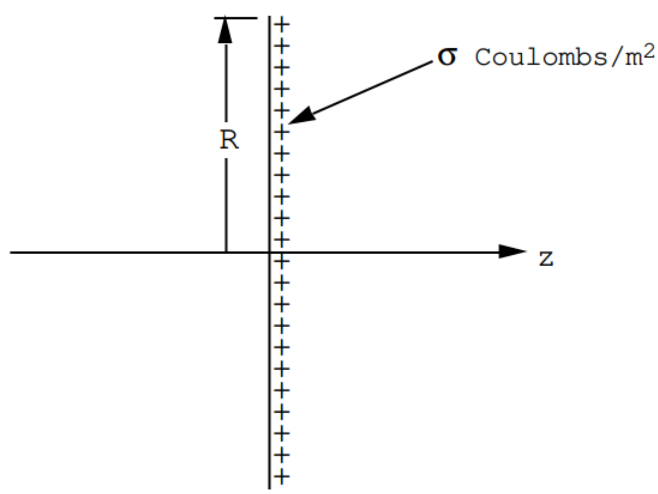
For example, on the face f_1 there is a discontinuity in P_x which is illustrated in the sketch below.



∴ The surface bound charge density is $\sigma_b = +\frac{P_0 L}{2}$ Coulombs/m^2 The total surface charge on f_1 is $Q_s = \sigma_b L^2 = P_o L^3/2$ Coulombs. There is a similar charge on each of the other faces. Therefore the total surface charge = 6Qs = $3P_o L^3$. The same as the volume charge.

Problem (2.4)

A disc of charge whose diameter is R meters is centered on the origin with its plane normal to the z-axis as shown in the sketch.



(a) Calculate the potential function V(z) on the axis of the disc. Sketch V(z).

(b) Make a sketch of $E_z(z)$. Show that as z → 0 $\mathbf{E}_z = \frac{\sigma}{2\varepsilon_0}$ for z > 0 and $E_z = -\frac{\sigma}{2\varepsilon_0}$ for z < 0.

(This problem is not as trivial as it looks. Remember V(z) must be an even function of z. It must also go to zero as | z | → ∞, and it must be continuous at z = 0. The answer is $V(z) = \frac{\sigma}{2\varepsilon_0}[\sqrt{R^2+z^2} - |z|]$.

Answer (2.4)

$$V(z) = \frac{1}{4\pi\varepsilon_0} \quad \int_0^R \frac{2\pi r dr\sigma}{\sqrt{r^2+z^2}} = \frac{\sigma}{4\varepsilon_0}\int_{z^2}^{R+z^2}\frac{du}{\sqrt{u}}$$

where u = r^2 + z^2 du = 2r dr

$$\therefore V(z) = \frac{\sigma}{2\varepsilon_0}[\sqrt{R^2+z^2} - \sqrt{z^2}]$$

(There is a temptation to write $\sqrt{z^2} = z$ but this would be wrong because one must use the +'ve root of z^2 even when z is negative. Hence $\sqrt{z^2} = |z|$.)

For z > 0 but z small $V \rightarrow -\frac{\sigma_z}{2\varepsilon_0} + \frac{\sigma_R}{2\varepsilon_0}$

For z < 0 but z small $V \rightarrow \frac{\sigma_z}{2\varepsilon_0} + \frac{\sigma R}{2\varepsilon_0}$

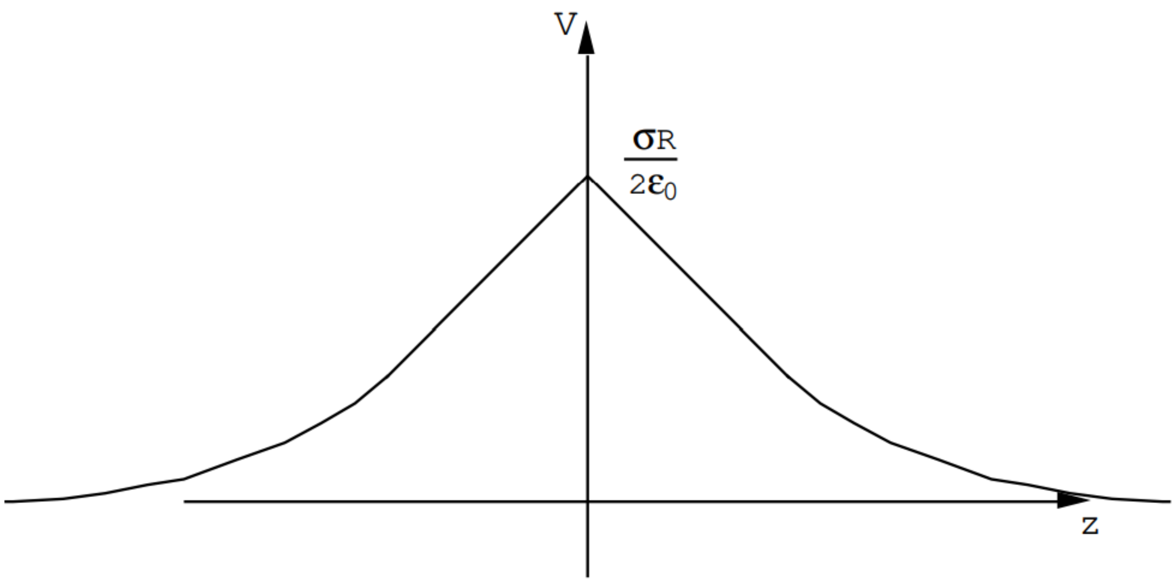
Therefore for z > 0 $\lim_{z\to 0} -\frac{\partial V}{\partial z} = E_z = \frac{\sigma}{2\varepsilon_0}$.

for z < 0 $\lim_{z\to 0} -\frac{\partial V}{\partial z} = E_z = -\frac{\sigma}{2\varepsilon_0}$.

for z > 0 but |z| >> R $V \rightarrow \frac{\pi\sigma R^2}{4\pi\varepsilon_0}\left(\frac{1}{z}\right)$,

for z < 0 but |z| >> R $\mathrm{V} \rightarrow -\frac{\pi\sigma \mathrm{R}^2}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{z}}\right)$,

ie. the potential looks like a point charge q = $\pi R^2\sigma$ Coulombs.



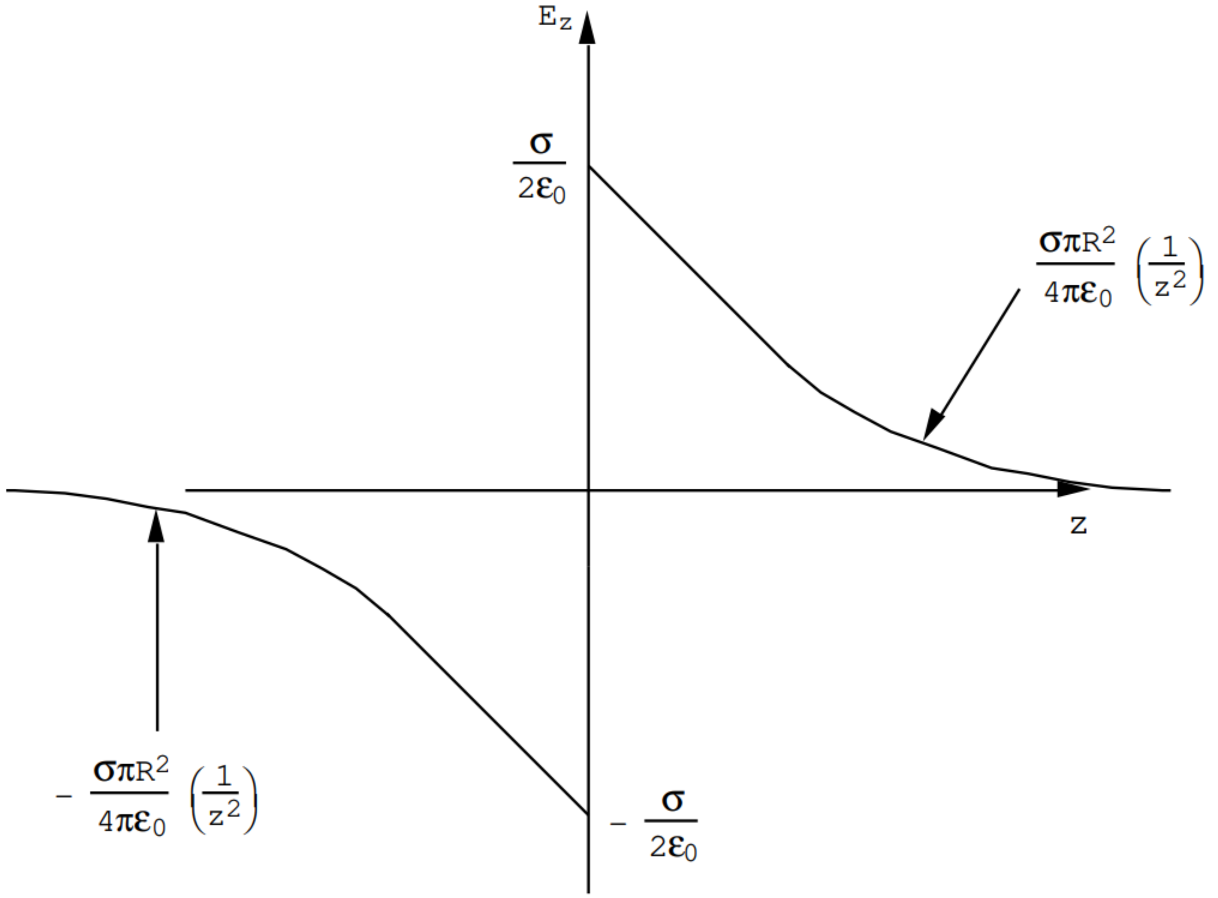
(b) $E_z = -\frac{\partial V}{\partial z}$

for z > 0 $E_z = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$

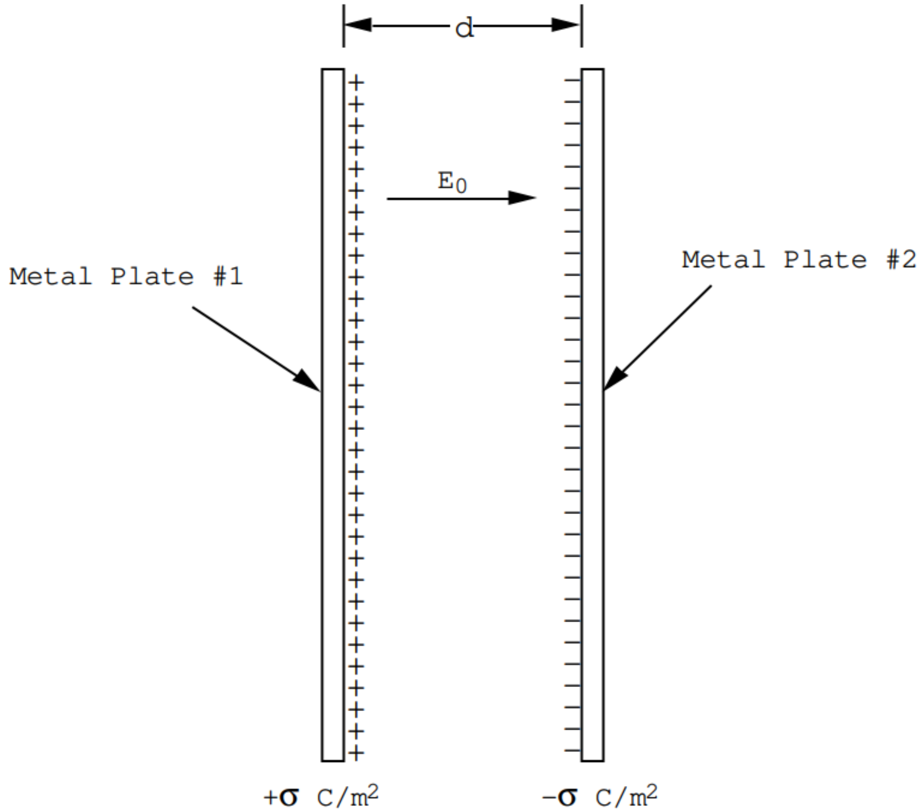
∴ @ z = 0 $E_z = +\frac{\sigma}{2\varepsilon_0}$.

for z < 0 $\mathrm{E_Z} = \frac{\sigma}{2\varepsilon_0}\left(-1 + \frac{\mathrm{z}}{\sqrt{\mathrm{R}^2+\mathrm{z}^2}}\right)$

$\therefore$ @ z = 0 $E_z = -\frac{\sigma}{2\varepsilon_0}$



Problem (2.5)



Two thin infinite plane metal plates are parallel and separated by a gap d meters as shown in the sketch. Plate #1 carries a surface charge density of +**σ** Coulombs/m^2. Plate #2 carries a surface charge of -**σ** Coulombs/m^2. In the metal **E** = 0, otherwise the charges in the metal would move and one would not have an electrostatic problem. Let the direction normal to the plates be the z direction.

(a) Use Gauss' theorem to calculate the electric field strength in the gap between the plates. Let this value be E_o.

(b) What is the value of D_z between the plates?

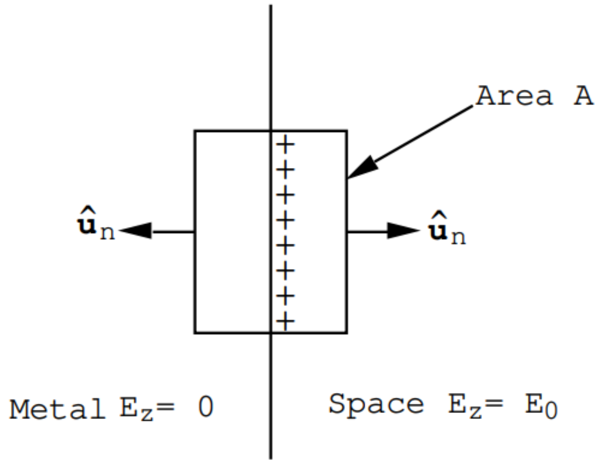
(c) What is the potential difference between the two metal plates?

(d) Suppose that a slab of matter whose thickness was (d/2) meters was slipped between the two metal plates. Suppose further that this slab were polarized such that $P_z = P_o$. What would now be the potential difference between the two plates?

(e) Show that D_z is continuous across the faces of the polarized slab.

Answer (2.5)

(a) Since we have two infinite sheets of charge the electric field is uniform and parallel with z (normal to the plates). Use a pill box which penetrates the metal surface on the left



$$\int_S (\mathbf{E}\cdot\mathbf{n})\mathrm{d}S = Q/\varepsilon_0 \backslash) \qquad \backslash(\therefore \mathrm{E_o A} = \frac{\sigma_A}{\varepsilon_o}$$

or $E_o = \sigma/\varepsilon_o$ Volts/meter.

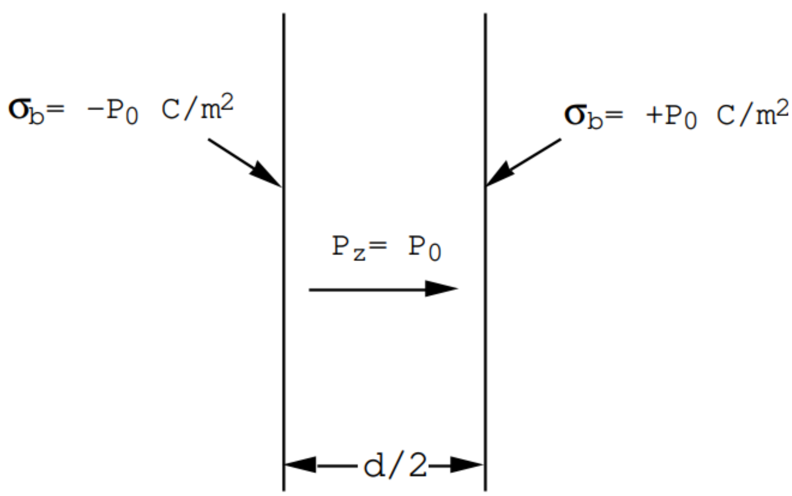
(b) $D_z = \varepsilon_0 E_z + P_z$ Here $P_z = 0$

$\therefore D_z = \varepsilon_0 E_o = \sigma$ Coulombs/m^2.

Notice that for $\boldsymbol{\sigma}$ = 1 Coulomb/m^2 the electric field would be 1.13 x 10^{11} Volts/meter. This is huge: air breaks down in a field of ~ 3 x 10^6 Volts/meter. Therefore 1 Coulomb/m^2 is a huge charge density.

(c) $\Delta V = E_o d = \sigma d/\varepsilon_0$ Volts.

(d)



Outside the slab $E_z = 0$

Inside the slab $E_z = -\frac{P_0}{\varepsilon_0}$

When placed between the two metal plates the field distributions add. Therefore in the gap $E_z = E_o$ but in the slab $E_z = E_O - \frac{P_O}{\varepsilon_0}$.

The total potential drop between the plates will be

$$\Delta V = E_O\left(\frac{d}{2}\right) + \left[E_O - \frac{P_O}{\varepsilon_0}\right]\left(\frac{d}{2}\right)$$

$$\therefore \Delta V = E_O d - \frac{P_o d}{2\varepsilon_0} = \frac{\sigma d}{\varepsilon_0} - \frac{P_O d}{2\varepsilon_0}$$

The potential drop is decreased by the presence of the slab.

(e) In the gap between the slab and the plates $D_z = \varepsilon_0 E_o = \sigma$ Coulombs/m^2.

In the slab

$$\begin{aligned} D_z &= \varepsilon_0 E_z + P_0 \\ &= \varepsilon_0 E_0 - P_0 + P_0 \\ &= \varepsilon_0 \mathrm{E}_0 = \sigma \text{ Coulombs}/\mathrm{m}^2 \end{aligned}$$

D_z is continuous across the slab faces!

Problem (2.6)

An ellipsoid of revolution has the shape of a cigar with its axis oriented along z. The length of the cigar is 1 cm and its diameter is $\frac{1}{2}$ cm. The cigar is uniformly polarized: The polarization is given by

$$\mathbf{P} = P_X\hat{\mathbf{u}}_X + P_Y\hat{\mathbf{u}}_Y + P_Z\hat{\mathbf{u}}_Z$$

where

$P_x = 0.1$ Coulombs/m^2

$P_y = 0.2$ Coulombs/m^2

$P_z = 0.3$ Coulombs/m^2.

Calculate the electric field components in the ellipsoid. (They turn out to be huge ~ 10^{10} V/m. Air breaks down in a field of ~ 10^6 V/m).

For the cigar whose length is 2d and whose diameter is 2R the depolarizing coefficient is given by (where $\frac{R}{d} < 1$)

$N_z = \left(\frac{1-\varepsilon^2}{\varepsilon^3}\right)\left(\frac{1}{2}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right) - \varepsilon\right)$ where $\varepsilon = \sqrt{1-\left(\frac{R}{d}\right)^2}$.

Answer (2.6)

For this problem the ratio $\frac{R}{d} = \frac{1}{2}$ and therefore $\varepsilon = \sqrt{3/4}$.

according to my calculations, $N_z = 0.1736$.

But the sum rule states that $N_x + N_y + N_z = 1$ and therefore the sum $N_x + N_y = 0.826$. By symmetry $N_x = N_y$, therefore $N_x = N_y = 0.413$.

$\mathbf{E} = -\frac{N_X P_X}{\varepsilon_0}\hat{\mathbf{u}}_x - \frac{N_Y P_Y}{\varepsilon_0}\hat{\mathbf{u}}_Y - \frac{N_Z P_z}{\varepsilon_0}\hat{\mathbf{u}}_z$, where $\varepsilon_0 = 8.85 \times 10^{-12}$MKS ,

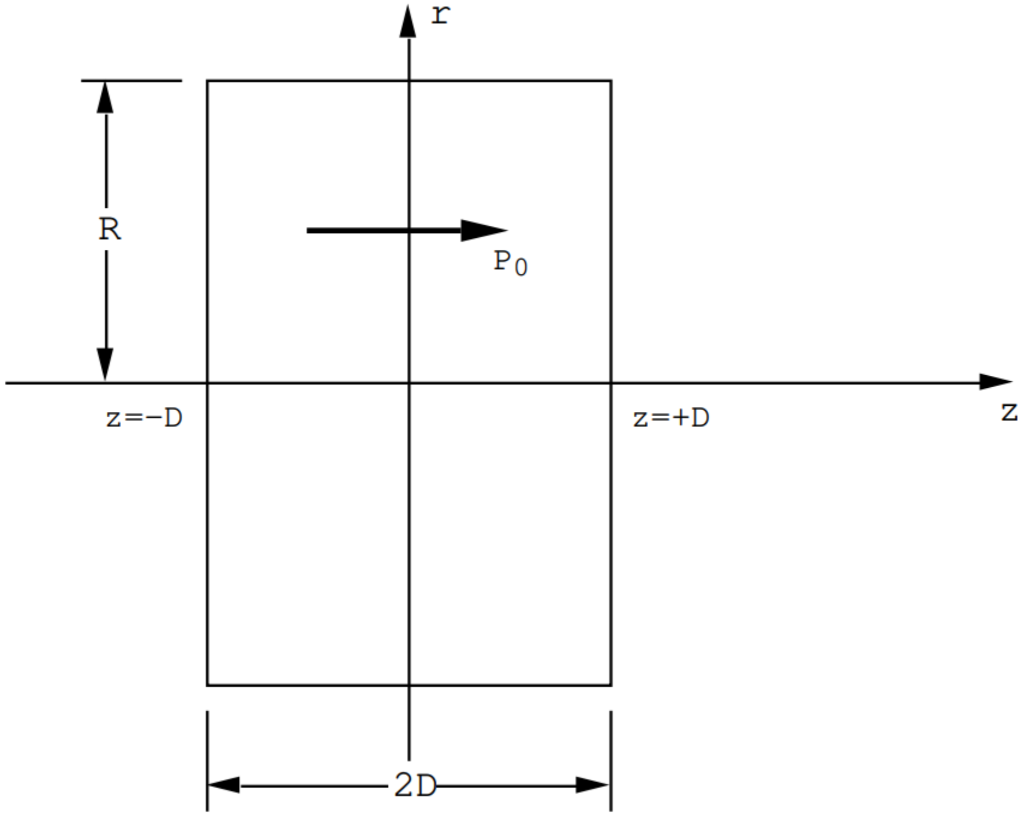
therefore $E_x = -0.47\times10^{10}$ Volts/meter,

$E_y = -0.94\times10^{10}$ Volts/meter

$E_z = -0.59\times10^{10}$ Volts/meter.

Problem (2.7)

An uncharged uniformly polarized disc of radius R meters and thickness 2D meters is shown in the figure. The polarization, P_0 Coulombs/m^2, is directed along the axis of the disc.



(a) Calculate the effective bound charge density, $\boldsymbol{\rho}_b$= - div **P**, everywhere.

(b) Use the bound charge density of part (a) to calculate the potential function along the axis of the disc.

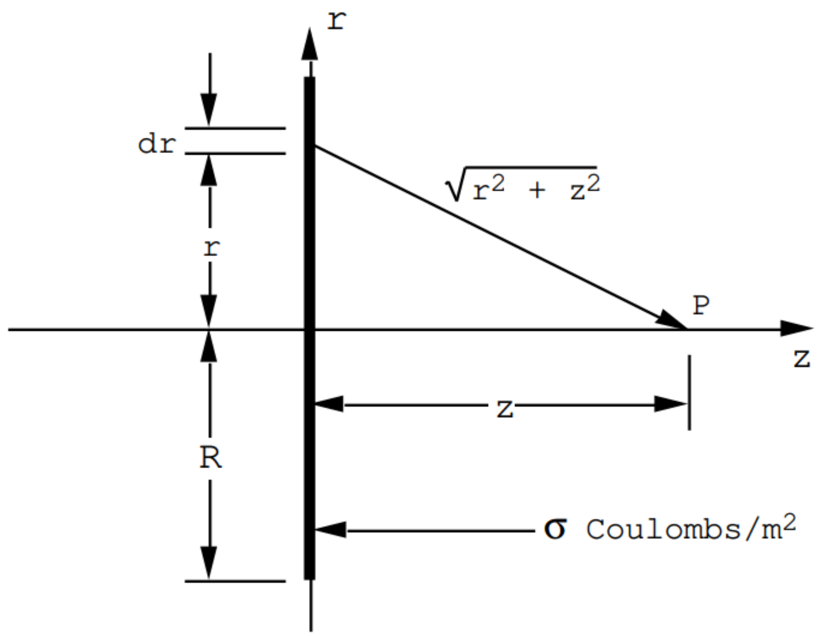
(c) Calculate the electric field along the axis of the disc. Check your answer by looking at three limits: (1) the limit (D/R)<<1; (2) the limit z>0 and (z/R)>>1; and (3) the limit z<0 and (|z|/R)>>1.

(d) Calculate the displacement vector **D**= ε_0**E** + **P** for all points along the axis of the disc.

Answer(2.7)

(a) **P**= 0 everywhere outside the disc and therefore $\boldsymbol{\rho}_b$= - div**P** = 0 everywhere outside the disc. Everywhere inside the disc **P** is constant and so its divergence is zero; $\boldsymbol{\rho}_b$= 0 inside the disc. The discontinuity in the normal component of **P** on the surfaces at z=-D and at z=+D produces surface bound charge densities. The surface charge density carried by the surface at z= -D is $\boldsymbol{\sigma}_b$= -P_0 Coulombs/m^2; the surface charge density carried by the surface at z= +D is $\boldsymbol{\sigma}_b$= +P_0 Coulombs/m^2.

(b) In order to calculate the potential function along the axis of the disc that is generated by the two surface charge distributions, it is useful to begin by considering just one plane surface charge distribution; see the sketch below.



$$\mathrm{dV_p} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi \mathrm{rdr}}{\sqrt{\mathrm{r}^2+\mathrm{z}^2}},$$

and

$$V_p = \frac{\sigma}{4\varepsilon_0}\int_0^R \frac{2rdr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2} - \sqrt{z^2}).$$

Notice that the potential function must be symmetric in z. There is a temptation to replace $\sqrt{z^2}$ in Equation (1) by z but that would be quite wrong because z is an odd function. One must replace $\sqrt{z^2}$ by |z|.

For z>0, but (z/R)<<1, $V \rightarrow \frac{\sigma_R}{2\varepsilon_0} - \frac{\sigma_z}{2\varepsilon_0}$.

For z<0, but (|z|/R)<<1, $V \rightarrow \frac{\sigma_R}{2\varepsilon_0} + \frac{\sigma_z}{2\varepsilon_0}$.

Therefore near the charged disc the electric field has the value E_z= +σ/2ε_0 on the right, and E_z= -σ/2ε_0 on the left; this is the expected result based upon an analysis of an infinite uniformly charged plane. Far from the charged disc, $\frac{R}{|z|} << 1$, one finds

For z>0, $V \rightarrow \frac{\pi R^2 \sigma}{4\pi\varepsilon_0}\left(\frac{1}{z}\right)$.

For z<0, $V \rightarrow -\frac{\pi R^2 \sigma}{4\pi\varepsilon_0}\left(\frac{1}{z}\right)$.

From a great distance the disc of charge looks like a point charge, where Q= $\pi R^2 \sigma$ Coulombs.

Returning to the problem of the polarized disc, the potential function along the axis of the disc can be written by inspection using Equation (1).

For z≥ D:

$$V(z) = \frac{P_0}{2\varepsilon_0} \quad (\sqrt{R^2+(z-D)^2} - \sqrt{R^2+(z+D)^2} + 2D).$$

For -D≤ z ≤ +D:

$$V(z) = \frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2} - \sqrt{R^2+(z+D)^2} + 2z).$$

For z≤ -D:

$$V(z)=\frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2}-\sqrt{R^2+(z+D)^2}-2D).$$

(c) The electric field along the axis is given by $E_z=-\frac{\partial V}{\partial z}$.

For z≥ +D:

$$E_z(z)=\frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right).$$

For -D ≤ z ≤ +D:

$$E_z(z)=\frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}-2\right).$$

For z ≤ -D:

$$E_z(z)=\frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right).$$

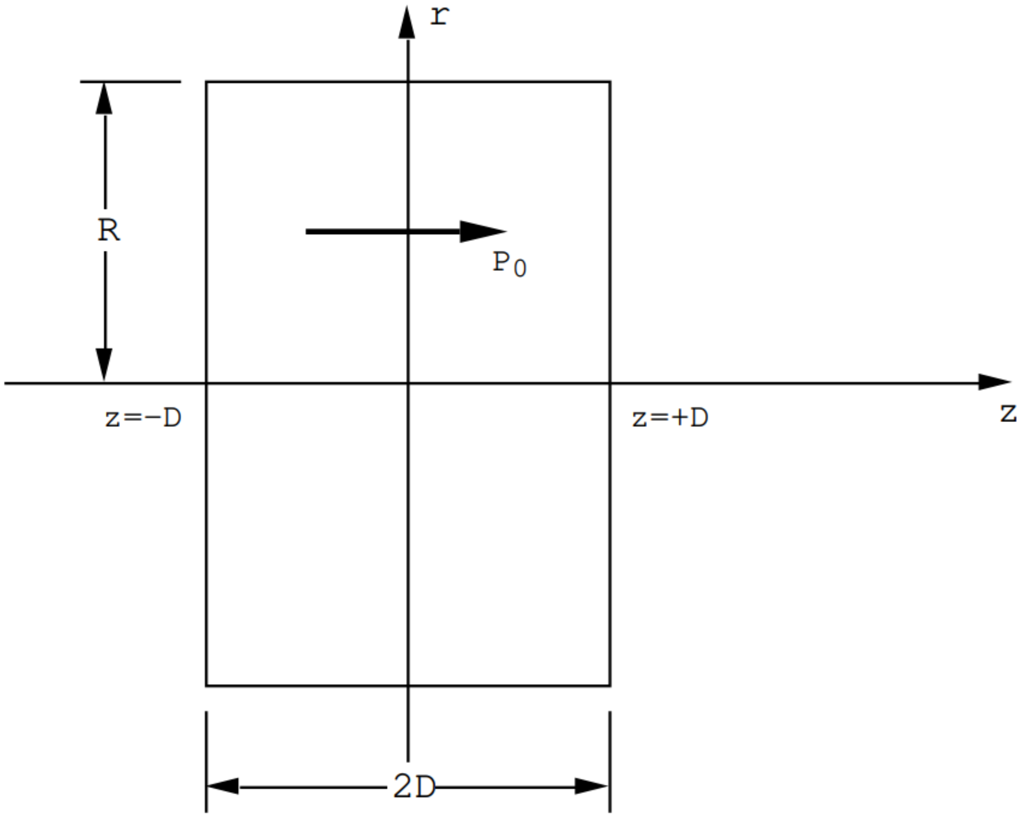
In the limit as (D/R)→0 the field outside the disc goes to zero like (2D/R); notice that the electric field is symmetric in z. The field inside the disc approaches the value $E_z=\frac{2D}{R}-\frac{P_0}{\varepsilon_0}$; i.e. the field approaches the value expected for an infinite pair of oppositely charged planes in the limit D→0.

In the limit |z|/R→∞, the electric field approaches the limit $E_z=\frac{1}{4\pi\varepsilon_0}4\pi R^2DP_0\frac{1}{|z|^3}$; i.e. the field due to a point dipole of moment p= $2D\pi R^2P_0$.

(d) By definition **D**= ε_0**E**+**P**. Outside the slab **D**= ε_0**E** since **P**=0. Inside the slab the term P_0 just cancels the constant term in E_z. The displacement vector is continuous through the surfaces of the slab. It is given for all points along the axis by $D_z(z)=\frac{P_0}{2}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right)$.

Problem (2.8)

An uncharged uniformly polarized disc of radius R meters and thickness 2D meters is shown in the figure. The polarization, P_0 Coulombs/m^2, is directed along the axis of the disc.



Calculate the potential function along the axis of the disc for z≥ D by summing the potential contributions from a collection of point dipoles. Show that

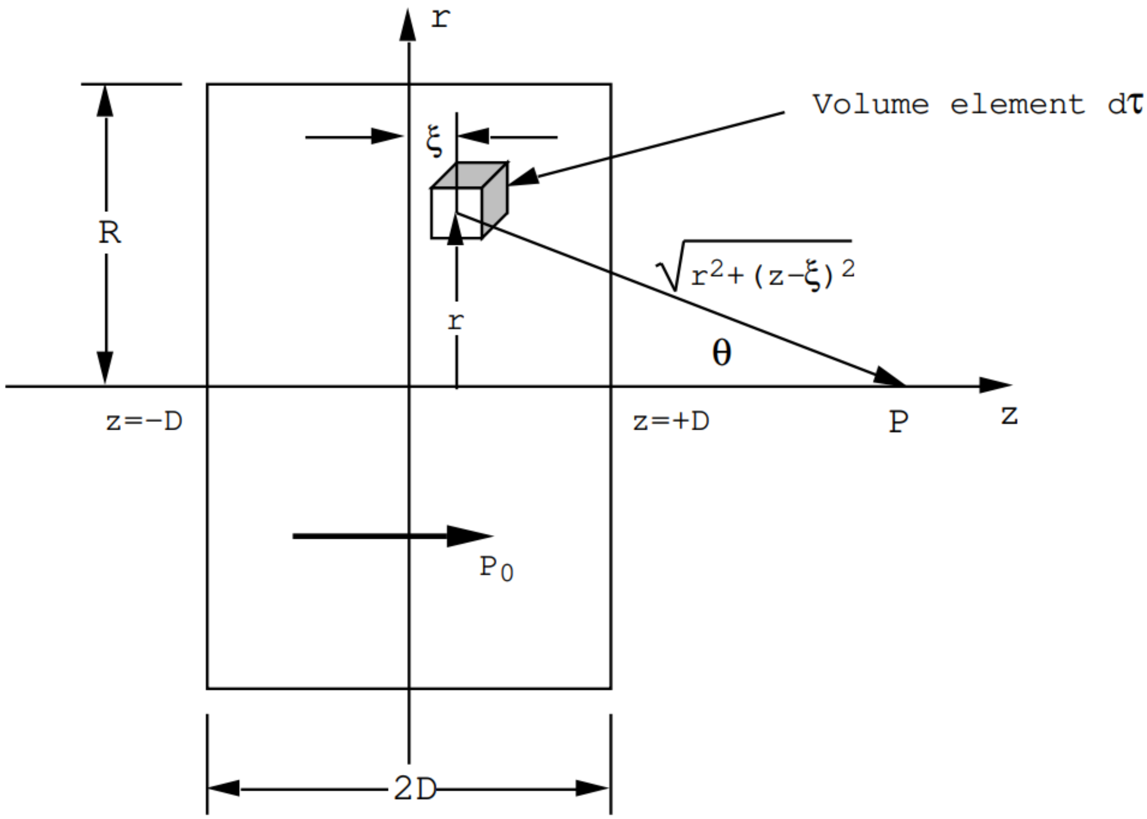
$$V(z)=\frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2}-\sqrt{R^2+(z+D)^2}+2D),$$

and therefore that the field for z≥D is given by

$$E_z(z)=\frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right).$$

(Compare with the results of Problem(2.7)).

Answer (2.8)



The contribution from the illustrated volume element to the potential at P can be written

$$\mathrm{dV} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{P}_0 \mathrm{d}\tau \cos\theta}{(r^2+(z-\xi)^2)}.$$

The total potential at z is given by

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int_0^R \int_{-D}^{D} \frac{P_0(z-\xi)d\tau}{(r^2+(z-\xi)^2)^{3/2}},$$

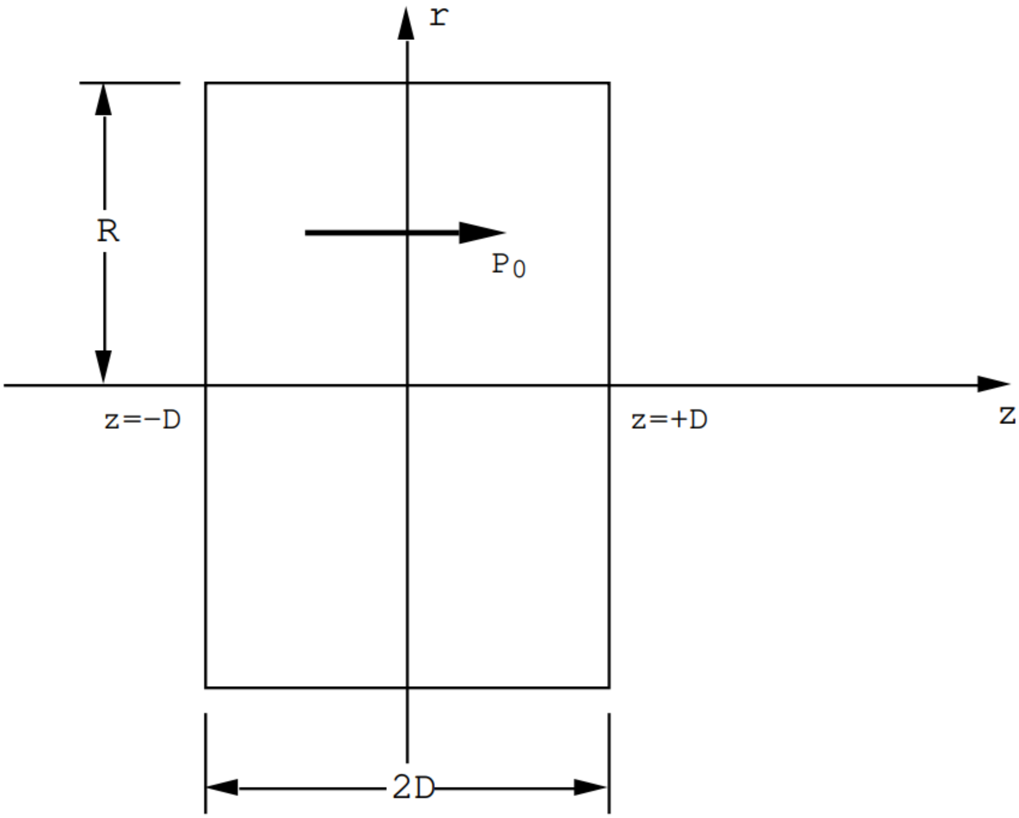
where $d\tau = 2\pi r dr d\xi$. The integrations can be readily carried out. The result is

$$V(z) = \frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2} - \sqrt{R^2+(z+D)^2} + 2D) \text{ Volts.}.$$

$$E_z(z) = -\frac{\partial V}{\partial z} = \frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}} - \frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right).$$

Problem (2.9)

An uncharged uniformly polarized disc of radius R meters and thickness 2D meters is shown in the figure. The polarization, P_0 Coulombs/m^2, is directed along the axis of the disc.



The electric field at the center of the disc is, by direct calculation (see Problem(2.7)),

$$E_Z(0) = -\frac{P_0}{\varepsilon_0}\left(1 - \frac{D}{\sqrt{R^2 + D^2}}\right).$$

Compare this value for the electric field with that obtained using the depolarizing coefficient for an ellipsoid of revolution having the same ratio of (D/R) as the disc. Carry out the calculation for (a) $(D/R) = \frac{1}{10}$, and for (b) $(D/R) = \frac{1}{100}$.

Answer (2.9)

The appropriate depolarizing coefficient for a pancake shaped ellipsoid is stated in the E&M notes, Figure (2.15):

$$N_z = \frac{R^2 D}{\left(R^2 - D^2\right)^{3/2}}\left(\frac{\sqrt{R^2 - D^2}}{D} - \mathrm{Tan}^{-1}\left(\frac{\sqrt{R^2 - D^2}}{D}\right)\right) \text{ for } (D/R) < 1.$$

If (D/R)<<1, $N_z \cong 1 - \frac{\pi}{2}\left(\frac{D}{R}\right)$. Using the exact expression for N_z

(a) For R= 10D one finds N_z= 0.861, and this gives

$$E_z(0) = -0.861\left(\frac{P_0}{\varepsilon_0}\right).$$

By direct calculation, the exact value is $-0.8995\left(\frac{P_0}{\varepsilon_0}\right)$.

(b) For R= 100D one finds N_z= 0.9845, and therefore

$$E_z(0) = -0.9845\left(\frac{P_0}{\varepsilon_0}\right).$$

By direct calculation, the value is $-0.9900\left(\frac{P_0}{\varepsilon_0}\right)$.

Cylindrical discs are often approximated as ellipsoids of revolution, especially in magnetic problems, for purposes of estimating the first order correction to the field at the center of a disc having an infinite radius, $E_z(0) = -\frac{P_0}{\varepsilon_0}$.

Problem (2.10)

An uncharged sphere of radius R is polarized in such a way that the polarization vector **P** is radial, and its magnitude is given by $\mathrm{P_r(r) = P_0\left(\frac{r}{R}\right)^2}$.

(a) Calculate the electric field at all points inside the sphere.

(b) Calculate the electric field at all points outside the sphere.

Answer (2.10)

The polarization vector possesses only a radial component, therefore $\operatorname{div}\mathbf{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right)$. The bound charge density is $\rho_b = -\operatorname{div}\mathbf{P} = -\left(\frac{4P_0}{R^2}\right) r$.

(a) The electric field inside the sphere can be calculated from Gauss' theorem because the field must be radial by symmetry. Thus

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^r \rho_b 4\pi r^2 dr = -\frac{4\pi P_0 r^4}{\varepsilon_0 R^2}$$

or

$$\mathrm{E_r} = -\frac{\mathrm{P_0}}{\varepsilon_0}\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2.$$

(b) There is a surface charge density on the sphere, $\rho_s = P_0$ Coulombs/m^2 because of the discontinuity in the normal component of the polarization vector. The total charge contained within a sphere whose radius is slightly larger than the radius R is zero. Therefore the electric field is zero everywhere outside the sphere.

Problem (2.11)

Consider an uncharged sphere having a very large radius R which is uniformly polarized along the z direction. The polarization is $P_z = P_0$.

(a) What is the direction and strength of the electric field inside the sphere? How does this field depend upon the radius of the sphere?

(b) A tiny spherical cavity of radius b, b/R<<1, is cut out of the sphere at some point not too far from its center. The polarization in the remainder of the big sphere remains unchanged. Use the principle of superposition to calculate the electric field strength inside the small cavity of radius b.

Answer (2.11)

(a) The depolarizing factors obey the sum rule $N_x + N_y + N_z = 1$. But for a sphere $N_x = N_y = N_z$, therefore each is equal to (1/3). Inside the sphere $E_z = -\frac{P_0}{3\varepsilon_0}$. This fieldstrength does not depend at all on the radius of the sphere.

(b) The field inside the tiny sphere of polarized material which has been cut out of the big sphere is $\mathrm{E_z} = -\frac{\mathrm{P_0}}{3\varepsilon_0}$. When this is added to the field in the cavity of radius b it must give a total field equal to the field strength before the tiny sphere was removed. It can therefore be concluded that the field in the cavity is zero!

Problem (2.12)

Consider an uncharged cylinder of radius R and length L. The axis of the cylinder lies along the zaxis. Let both R and L become infinitely large, but in such a way that the ratio (R/L)→0.

(a) Let the material of the cylinder be polarized along its length, i.e. $P_z = P_0$. What is the direction and strength of the electric field inside the cylinder?

(b) Let the material of the cylinder be polarized transverse to its axis, along the x-axis say; i.e. $P_x = P_0$. What is the direction and magnitude of the electric field inside the cylinder?

Answer (2.12)

(a) The depolarizing coefficient for a very long needle in the direction of its length is zero. Therefore, when the cylinder is polarized along its axis there is no electric field inside it.

(b) By symmetry the transverse depolarizing coefficients must be equal: $N_x=N_y$. But from the sum rule $N_x+N_y= 1$, since $N_z=0$. It follows that $N_x=N_y=(1/2)$. The electric field inside the cylinder is given by $E_x=-\frac{P_0}{2\varepsilon_0}$.

This problem demonstrates that the dipole field has such a long range that the electric field inside an infinitely large body depends upon its shape.

Problem (2.13)

Consider two charges q1=Q and q2=- αQ. The charge q1 is located at (-b,0,0); the charge q2 is located at (b,0,0).

(a) Let α=2. Show that the equipotential V=0 is a sphere of radius $R=\frac{4b}{3}$ centered at $x_0=-\frac{5b}{3}$.

(b) Let α= 1/2. Show that the equipotential V=0 is again a sphere of radius $R=\frac{4b}{3}$ but centered at $x_0=+\frac{5b}{3}$.

The equipotential V=0 can be replaced by a grounded metal sphere without disturbing the potential distribution outside the sphere. This construction therefore provides the solution of the problem of a point charge brought up to a grounded conducting sphere.

Answer (2.13)

$V_p=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right)$, where $q_1=Q$ and $q_2=-\alpha_Q$

$r_1=\sqrt{(x+b)^2+y^2+z^2}$, and $r_2=\sqrt{(x-b)^2+y^2+z^2}$. Therefore V=0 when $r_2=\alpha_{r_1}$, or $(r_2)^2=\alpha^2(r_1)^2$. A bit of algebra gives

$$x^2+y^2+z^2+2bx\left(\frac{\alpha^2+1}{\alpha^2-1}\right)+b^2=0 \qquad (1)$$

By adding $b^2\left(\frac{\alpha^2+1}{\alpha^2-1}\right)^2$ to both sides of (1) this equation can be written

$$\left(x+b\left(\frac{\alpha^2+1}{\alpha^2-1}\right)\right)^2+y^2+z^2=\left(\frac{2\alpha b}{|\alpha^2-1|}\right)^2 \qquad (2)$$

This is the equation of a sphere centered at $x_0=-b\frac{\alpha^2+1}{\alpha^2-1}$, and having a radius $R=\frac{2\alpha b}{|\alpha^2-1|}$.

Problem (2.14)

Consider two charges q_1=Q and q_2= -Q. The charge q_1 is located at (-b,0,0); the charge q_2 is located at (b,0,0).

(a) Show that V=0 on the plane x=0. The region to the right of x=0 can be replaced by a conducting metal without disturbing the potential in the region x<0. This construction provides the solution of the problem of a point charge brought up to a grounded conducting plane.

(b) Show that the charge q_1=Q is attracted to a grounded metal plane with a force

$$F_x=\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{4b^2}\ \text{Newtons}.$$

Answer (2.14)

The electric field at x=-b is just that due to a point charge -Q located at x=+b. Therefore $E_x=\frac{Q}{4\pi\varepsilon_0}\frac{1}{4b^2}$, and $E_y= E_z= 0$. Thus the force on the charge pulling it towards the metal surface is just

$$F_x=\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{4b^2}\ \text{Newtons}..$$

13.3: Chapter 3

Problem (3.1)

It is desired to construct a 100 pF capacitor using a mylar spacer 10^{-4} m. thick, ε_r= 3.0, placed between two metal plates. How large an area is required for the metal electrodes?

Answer (3.1)

$C=\frac{AE_r\varepsilon_0}{D}=\frac{3A}{10^{-4}}\left(8.84\times 10^{-12}\right)\text{ Farads }=10^{-10}\text{ Farads;}$

$A=3.77\times 10^{-4}\text{ m}^2=3.8\text{ cm}^2$,

i.e. one requires electrodes approx. 2x2 cm square.

Problem (3.2)

A small drop of oil is characterized by a relative dielectric constant ε_r= 1.5 and a density of 800 kg/m^3; its radius is R= 10^{-4} m. It is placed between condenser plates which are parallel and which are separated by 1 cm. The oil drop is uncharged. A potential difference of 100 Volts is placed across the capacitor plates. The relative dielectric constant of air may be taken to be ε_r= 1.00.

(a) Estimate the dipole moment induced on the drop by the electric field.

(b) How large a field gradient would be required to suspend the drop in the gravitational field?

Answer (3.2)

(a) The potential function outside the drop has the form

$$V_2=-E_0r\cos\theta+\frac{b\cos\theta}{r^2};$$

The potential function inside the drop has the form

$$\mathrm{V}_1=-\mathrm{a\,r}\cos\theta.$$

At r= R these potentials must satisfy the two boundary conditions

(1) The potential function must be continuous;

and (2) The normal component of **D** must be continuous.

These boundary conditions require

$$a+\frac{b}{R^3}=E_0$$

$$a-\frac{2}{\varepsilon_r}\frac{b}{R^3}=\frac{E_0}{\varepsilon_r}.$$

These equations have the solutions

$a=\left(\frac{3E_0}{\varepsilon_r+2}\right)\qquad b=\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)R^3E_0$

The dipole moment on the sphere is therefore given by

$$\mathrm{p_z}=4\pi\varepsilon_0\mathrm{R}^3\left(\frac{\varepsilon_\mathrm{r}-1}{\varepsilon_\mathrm{r}+2}\right)\mathrm{E}_0$$

For the present case

$$p_z=\left(\frac{10^{-12}}{9\times 10^9}\right)\left(\frac{0.5}{3.5}\right)\left(\frac{100}{10^{-2}}\right)=1.59\times 10^{-19}\text{ Coulomb-meters.}$$

(b) The gravitational force on the drop is

$$F_g=mg=\frac{4\pi R^3}{3}(800)(9.8)=3.28\times 10^{-8}\text{Newtons.}$$

In order to suspend the drop one would require

$$p_z \frac{dE_z}{dz} = 3.28 \times 10^{-8}\mathrm{N}.$$

This implies a field gradient of $\frac{\mathbf{dE_z}}{\mathbf{dz}} = \mathbf{2.07 \times 10^{11}\ Volts/meter.}$ This is an enormous field gradient!

Problem (3.3)

This problem concerns the calculation of the dielectric constant for a material composed of a lattice of atoms each of which carries a permanent electric dipole moment which is free to rotate. The calculation follows the article by L.Onsager, J.Amer.Chem.Soc.58,1486-1493(1936). According to the Onsager model, each electric dipole, of strength p, is located at the center of a sphere of radius R: inside the sphere the relative dielectric constant is 1, outside the sphere the relative dielectric constant is ε_r. The spherical hole is supposed to represent the volume of the atom which carries the dipole. The average electric field in the material far from the dipole under examination is uniform, it has the value E, and it is directed along z.

(a) Calculate the field in the cavity in the absence of the dipole moment; let this field be E_c.

(b) Calculate the field inside the cavity for the case when the dipole is present in the cavity but the average applied field strength is zero, i.e. E=0. Let the field in the cavity due to the presence of the dipole be the reaction field **R**. Notice that the reaction field is always parallel with the direction of the dipole. The field outside the cavity is a dipole field; what is the corresponding effective dipole moment?

(c) The total field in the cavity due both to the presence of the dipole and due to the applied field E can be obtained by superposition. The result is the vector sum of the cavity field, $\mathbf{E}_c$, and the reaction field, **R**. However, **R** exerts no torque on the dipole because it is parallel with it. The potential energy of the electric dipole because of the presence of the cavity field is given by

$$\mathrm{U_d} = -\mathbf{p} \cdot \mathbf{E}_\mathrm{C} = -\mathrm{pE_C}\,\mathrm{Cos}\,\theta$$

If pE_c is small compared with kT it can be shown, using standard statistical mechanics, that the average value of Cosθ due to thermal agitation is

$$< \cos\theta >= \left(\frac{\mathrm{pE_C}}{3\mathrm{kT}}\right)$$

Use this result to calculate the mean value of the polarization per unit volume. Let the number density of dipoles be N per m^3.

(d) Use the results of part (c) to show that the dielectric constant of the medium is related to the individual atomic dipole moment, p, through the expression

$$\varepsilon_\mathrm{r} - 1 = \left(\frac{\varepsilon_\mathrm{r}}{1 + 2\varepsilon_\mathrm{r}}\right)\left(\frac{\mathrm{Np}^2}{\varepsilon_0 \mathrm{kT}}\right)$$

This relation can be used to deduce the dipole moment of polar molecules from the measured values of the static dielectric constant.

(e) A certain material contains a density of molecules $\mathrm{N} = \frac{1}{3} \times 10^{29}$ per meter^3, and each molecule carries an electric dipole moment $p = \frac{1}{2} x 10^{-29}$ Coulomb-meters. Calculate the relative dielectric constant, ε_r, at 300K.

Answer (3.3)

(a) The Cavity Field.

Outside the cavity $\mathrm{V}_2 = -\mathrm{Er}\cos\theta + \frac{\mathrm{b}\cos\theta}{\mathrm{r}^2}$

Inside the cavity $V_1 = -E_C r\cos\theta$

At r=R: $V_1 = V_2$

$$\varepsilon_0 \frac{\partial V_1}{\partial r} = \varepsilon \frac{\partial V_2}{\partial r},$$

from which $\mathbf{E}_\mathrm{c} = \frac{3\varepsilon_\mathrm{r}\mathrm{E}}{(2\varepsilon_\mathrm{r}+1)}$.

(b) The Reaction Field.

Outside the cavity the potential function is that of a dipole; for this part of the problem there is no external field, so that $V_2 = \frac{b\cos\theta}{r^2}$.

Inside the cavity the potential function, V_1, must include the singular dipole field due to the point dipole plus a reaction field due to the polarization of the medium:

$$V_1 = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} + \mathrm{arcos}\,\theta$$

At r=R: $V_1 = V_2$

$$\varepsilon_0\frac{\partial V_1}{\partial r} = \varepsilon\frac{\partial V_2}{\partial r}$$

from which

$$\mathrm{V}_1 = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{p}\cos\theta}{\mathrm{r}^2} - \frac{2\mathrm{p}}{4\pi\varepsilon_0\mathrm{R}^3}\left(\frac{\varepsilon_{\mathrm{r}}-1}{2\varepsilon_{\mathrm{r}}+1}\right)\mathrm{r}\cos\theta$$

and

$$\mathrm{V}_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\varepsilon_{\mathrm{rp}}}{2\varepsilon_{\mathrm{r}}+1}\right)\frac{\cos\theta}{\mathrm{r}^2}$$

From these expressions one obtains

$$|\mathbf{R}| = \frac{2p}{4\pi\varepsilon_0 R^3}\left(\frac{\varepsilon_{\mathbf{r}}-1}{2\varepsilon_{\mathbf{r}}+1}\right)$$

and the effective dipole moment for the region external to the cavity is given by

$$p^* = \frac{3\varepsilon_r^2 p}{(2\varepsilon_r+1)},$$

where $\mathrm{V}_2 = \frac{\mathrm{p}^*}{4\pi\varepsilon_{\mathrm{r}}\varepsilon_0}\frac{\cos\theta}{\mathrm{r}^2}$.

(c) The mean polarization per unit volume is parallel with the field and is given by

$$P = Np < \cos\theta >$$

Consequently,

$$P = \frac{Np^2}{3kT}\left(\frac{3\varepsilon_r}{2\varepsilon_r+1}\right)E = \frac{Np^2}{kT}\left(\frac{\varepsilon_r}{2\varepsilon_x+1}\right)E \text{ coulombs }/m^2$$

(d) D = εE = ε0E + P,

or $(\varepsilon_r-1)\,E = P/\varepsilon_0$.

Therefore

$$(\varepsilon_{\mathrm{r}}-1) = \frac{\mathrm{Np}^2}{\varepsilon_0\mathrm{kT}}\left(\frac{\varepsilon_{\mathrm{r}}}{2\varepsilon_{\mathrm{r}}+1}\right),$$

or

$$\frac{\mathrm{Np}^2}{\varepsilon_0\mathrm{kT}} = \frac{(\varepsilon_{\mathrm{r}}-1)(2\varepsilon_{\mathrm{r}}+1)}{\varepsilon_{\mathrm{r}}}$$

(e)

$$\frac{\mathrm{Np}^2}{\varepsilon_0\mathrm{kT}} = \frac{(1/3)\left(10^{29}\right)(1/4)\left(10^{-29}\right)\left(10^{-29}\right)}{\left(8.84\times10^{-12}\right)\left(1.38\times10^{-23}\right)(300)} = 22.8$$

From this $\varepsilon_{\mathrm{r}}^2 - 11.9\varepsilon_{\mathrm{r}} - (1/2) = 0$

and

$\varepsilon_r = 11.93$

Problem (3.4)

The radius of the sun is 6.98×10^5 km. and its surface temperature is 6000°C, corresponding to an energy kT= 0.34 electron Volts. Treat the sun as a conducting sphere isolated in space and calculate the net positive charge required to produce a potential of 0.34 Volts relative to zero potential at infinity.

Answer (3.4)

At the surface of a conducting sphere the potential is given by

$\mathrm{V} = \frac{\mathrm{Q}}{4\pi\varepsilon_0}\frac{1}{\mathrm{R}}$

Therefore, Q= $(4\pi\varepsilon_0)(0.34)(6.98 \times 10^8) = 2.64 \times 10^{-2}$ Coulombs. This is a surprisingly small amount of charge. It corresponds to a deficit of 1.65×10^{17} electrons.

Problem (3.5)

A capacitor is constructed of two concentric metal cylinders. The relevant radius of the inner electrode is a, the relevant radius of the outer electrode is b. The space between the electrodes is filled with air for which ε_r= 1.00.

(a) What is the capacitance per unit length of this device?

(b) The above capacitor, whose length is L= 10 cm, is placed upright in a dish of oil, ε_r= 3.00, so that the space between the cylindrical electrodes is filled with oil to a depth of 5 cm. What is the capacitance of this configuration if the radii are a= 5 cm and b= 6 cm?

(c) The capacitor of part (b) is charged to a potential difference of 1000 Volts. How high will the oil rise between the capacitor electrodes if the density of the oil is 800 kg/m^3?

Answer (3.5)

(a) Let the charge on the inner electrode be Q Coulombs/meter, that on the outer electrode -Q Coulombs/meter. The field is radial, so from Gauss' law

$2\pi r\, E_r = Q/\varepsilon_0$,

and

$E_r = \frac{Q}{2\pi\varepsilon_0 r}$.

The potential difference between the electrodes is

$$\Delta \mathrm{V} = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{E_r\,dr} = \frac{\mathrm{Q}}{2\pi\varepsilon_0}\ln(\mathrm{b}/\mathrm{a})$$

But Q= CΔV, therefore

$$c = \frac{2\pi\varepsilon_0}{\ln(b/a)}\ \text{Farads/meter.}$$

(b) If oil is placed betweeen the electrodes the capacitance per unit length becomes

$C_{oil} = \varepsilon_r C$ Farads/meter.

For a system having a length of L= 5cm= 0.05 meters the capacitance is

$$c = \frac{(2\pi\varepsilon_0)(0.05)}{\ln(6/5)} = 15.2 \times 10^{-12}\ \mathrm{F} = 15.2\ \mathrm{pF}$$

The oil filled part has a capacitance which is 3 times this value: C_{oil}= 45.7 pF. The total capacitance is the sum of the above figures:

$$\mathrm{C_{tot}} = \mathrm{C_{oil}} + \mathrm{C} = 60.9\mathrm{pF}$$

(c) The electrostatic field energy per unit length of capacitor is given by

$$\mathrm{U_E} = \int_{\mathrm{a}}^{\mathrm{b}} (2\pi \mathrm{rdr}) \varepsilon_{\mathrm{r}} \varepsilon_0 \frac{\mathrm{E}^2}{2},$$

where $E = \frac{Q}{2\pi\varepsilon_0\varepsilon_r}$ and $\Delta \mathrm{V} = \frac{\mathrm{Q}}{2\pi\varepsilon_0\varepsilon_\mathrm{r}} \ln(\mathrm{b/a})$.

That is $E = \frac{\Delta V}{\ln(b/a)} \frac{1}{r}$, so that

$$\mathrm{U_E} = \pi\varepsilon_0\varepsilon_\mathrm{r} \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{rdr} \frac{(\Delta \mathrm{V})^2}{(\ln(\mathrm{b/a}))^2} \frac{1}{\mathrm{r}^2} = \frac{\pi\varepsilon_0\varepsilon_\mathrm{r}(\Delta\mathrm{V})^2}{\ln(\mathrm{b/a})}$$

If the oil level between the capacitor electrodes rises by dz, the increase in electrostatic field energy will be given by

$$\mathrm{dU_E} = \frac{\pi\varepsilon_0 (\varepsilon_\mathrm{r} - 1)(\Delta\mathrm{V})^2}{\ln(\mathrm{b/a})} \mathrm{dz}$$

since a slice dz thick of air (ε_r=1) is replaced by oil (ε_r= 3.0). But this change in energy must be equal to the work done by the electrostatic forces: Fdz= dU_E, so that

$$\mathrm{F} = \frac{\pi\varepsilon_0 (\varepsilon_\mathrm{r} - 1)(\Delta\mathrm{V})^2}{\ln(\mathrm{b/a})} \text{ Newtons}.$$

This force will support a column of oil whose height is H meters, where

$$\mathrm{F} = \pi\left(\mathrm{b}^2 - \mathrm{a}^2\right) \mathrm{H}\rho\mathrm{g}$$

For our problem F = π(1.1x10^{-3})(800)(9.8)H,

or F = 27.09H Newtons

Therefore $\mathbf{H = 1.12 \times 10^{-5} meters\ = 11.2\mu m}$.

Problem (3.6)

An electron is located a distance d in front of the plane interface with a material characterized by a relative dielectric constant ε_r= 3.00.

(a) Calculate the force on the electron.

(b) How much work must be done on the electron to bring it from infinity to a distance a= 10^{-10} m from the surface?

Answer (3.6)

(a) In the vacuum at the interface

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q'}{r}\right)$$

and

$\mathrm{D_n} \sim \left(\mathrm{Q} - \mathrm{Q'}\right)$.

In the slab at the interface

$$V = \frac{1}{4\pi\varepsilon_0} \frac{1}{\varepsilon_r} \frac{Q''}{r}$$

and

$$\mathrm{D_n} \sim Q''.$$

Therefore $Q - Q' = Q''$

$$Q + Q' = \frac{Q''}{\varepsilon_r},$$

so that

$$Q'' = \frac{2\varepsilon_r Q}{(\varepsilon_r + 1)}$$

and

$$Q' = Q\left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right).$$

For the present problem Q"= 3Q/2 and Q'= -Q/2.

The force on the charge Q is given by (z is measured towards the interface)

$$F_z = Q\left(\frac{Q/2}{4\pi\varepsilon_0(2d)^2}\right) = \frac{Q^2}{32\pi\varepsilon_0 d^2}.$$

This is an attractive force.

(b) The electron will be attracted to the interface, consequently the work done to bring it up from infinity is negative. The binding energy is given by

$$U = -\int_{-\infty}^{-a} F_z dz = \frac{Q^2}{32\pi\varepsilon_0 a},$$

or

$$U = \frac{\left((1.6)^2\times 10^{-38}\right)\left(9\times 10^9\right)}{\left(8\times 10^{-10}\right)} = \mathbf{2.88\times 10^{-19} Joules\ = 1.8eV}$$

Problem (3.7)

(W.Shockley, J.Appl.Phys.9 ,635(1938)).

A capacitor is made using conducting concentric cylinders with a vacuum in the space between the electrodes. The radii of the relevant surfaces are a,b where b>a. Place a charge q at position r between the electrodes. What is the charge induced on each of the electrodes?

The solution of this problem is related to the calculation of the noise spectrum in vacuum tubes. The current through such a tube is carried by discrete charges, electrons, and as each electron leaves one electrode it induces a characteristic current spike in an external circuit. The time variation of the current pulse depends upon the electron transit time. The Fourier transform of the time variation of the current pulse gives the noise spectrum.

Hint for the solution.

(1) Use the linearity between charge and Voltage to write three equations involving generalized capacitance coefficients (see Equations (10.111)). One can think of the test charge q as being located on a very tiny spherical electrode.

(2) Construct two thought experiments:

(a) Put a charge Q on the inner electrode(#1), a charge -Q on the outer electrode(#2) which is grounded, and put no charge on the tiny sphere (electrode #3); i.e. Q_1= Q, Q_2= -Q, and Q_3= 0. The corresponding potentials are V_1, which can be calculated, V_2= 0 (grounded electrode), and V_3 which can also be calculated assuming that electrode 3 is so small that it makes a negligible perturbation of the field between the electrodes.

(b) Put V_1= V_2= 0 and let the charge on electrode #3 be q.

The results of these two experiments enables one to deduce that the induced charge on the inner electrode is given by

$$Q_1 = -q\frac{\ln(b/r)}{\ln(b/a)}.$$

Similarly, one can show that $Q_2 = -q\frac{\ln(r/a)}{\ln(b/a)}$. Thus Q_1 + Q_2 = -q corresponding to charge conservation.

Answer (3.7)

Put a charge Q on the inner electrode and ground the outer electrode so that V_2=0. In the space between the electrodes the potential is given by

$$V(r)=\frac{Q}{2\pi\varepsilon_0}\ln(b/r),$$

corresponding to the electric field $E_r=\frac{Q}{2\pi\varepsilon_0}$. The potential at the position of the uncharged electrode #3 is just V(r). One has

$$Q_1=Q,\quad V_1=\frac{Q}{2\pi\varepsilon_0}\ln(b/a);$$

$$Q_2=-Q,\quad V_2=0;$$

$$Q_3=0,\quad V_3=\frac{Q}{2\pi\varepsilon_0}\ln(b/r).$$

But

$$\begin{aligned}Q_1&=C_{11}V_1+C_{12}V_2+C_{13}V_3\\ Q_2&=C_{12}V_1+C_{22}V_2+C_{23}V_3\\ Q_3&=C_{13}V_1+C_{23}V_2+C_{33}V_3\end{aligned}$$

Therefore

$$Q=C_{11}\frac{Q}{2\pi\varepsilon_0}\ln(b/a)+C_{13}\frac{Q}{2\pi\varepsilon_0}\ln(b/r)\qquad(1)$$

$$-Q=C_{12}\frac{Q}{2\pi\varepsilon_0}\ln(b/a)+C_{23}\frac{Q}{2\pi\varepsilon_0}\ln(b/r)\qquad(2)$$

$$0=C_{13}\frac{Q}{2\pi\varepsilon_0}\ln(\mathrm{b/a})+\mathrm{C}_{33}\frac{\mathrm{Q}}{2\pi\varepsilon_0}\ln(\mathrm{b/r})\qquad(3).$$

From (3) $\frac{C_{13}}{C_{33}}=-\frac{\ln(b/r)}{\ln(b/a)}$.

Now let V_1=0, V_2=0, and Q_3=q. Then

q = $C_{33}V_3$ and Q_1= $C_{13}V_3$,

from which

$$\mathbf{Q_1}=\left(\frac{\mathbf{c_{13}}}{\mathbf{c_{33}}}\right)\mathbf{q}=-\mathbf{q}\frac{\ln(\mathbf{b/r})}{\ln(\mathbf{b/a})}.$$

When r=b Q_1=0 as it should; no charge is induced on the inner electrode, but there is a charge -q induced on the outer electrode. When r=a the full charge -q is induced on the inner electrode. The induced charge -q is transferred from the outer to the inner electrode through the external circuit during the time required for the charge q to move from one electrode to the other.

Problem (3.8)

Let an air-filled capacitor (ε_r= 1.00) be constructed of two square shaped metal plates of length L on a side separated by a space D. The edges of the two plates are parallel. Now let one of the plates be rotated slightly around one of its edges so that the two electrodes make an angle θ with respect to one another; along one edge the spacing is D and along the other edge the spacing is D+Lθ. Estimate the capacitance of this wedged capacitor. This can be done by equating $\frac{\mathrm{CV}^2}{2}$ with the electrostatic field energy $\frac{\varepsilon_0}{2}\int E^2 d\tau$, and by making a plausible assumption about the electric field distribution between the wedged conductors. The electrostatic field energy is an extremum (a minimum) for the correct field distribution and therefore its value is insensitive to small deviations of the field from its correct distribution.

I assumed that

$$E_\theta = \frac{V}{(R+x)\theta},$$

where Rθ= D, and where V is the potential difference between the electrodes. This assumption makes E perpendicular to the electrode surfaces and it preserves a constant potential difference between the plates as x goes from 0, corresponding to one edge of the plates, to x=L corresponding to the other edge. Unfortunately, div **E** is not zero so that the its potential function does not satisfy LaPlace's equation. Nevertheless, this calculation will give an upper bound for the change in capacitance with wedge angle.

Answer (3.8)

Let x be the distance from the narrow edge of the wedge between the two conductors. At any point x one can use the volume element

$$d\tau = L(R+x)\theta dx;$$

this expression is based upon a cylindrical co-ordinate system in which the z-axis lies at the apex of the wedge. If

$$E_\theta = \frac{V}{(R+x)\theta},$$

then the field energy is given by (neglecting edge effects)

$$U_E = L\int_0^L \left(\frac{\varepsilon_0 E^2}{2}\right)(R+x)\theta dx = \frac{\varepsilon_0 L V^2}{2\theta}\int_0^L \frac{dx}{(R+x)},$$

or

$$U_E = \frac{\varepsilon_0 L V^2}{2\theta}\int_R^{R+L}\frac{du}{u} = \frac{\varepsilon_0 L V^2}{2\theta}\ln\left(1+\frac{L}{R}\right),$$

from which $\mathbf{c} \cong \frac{\varepsilon_0 \mathbf{L}}{\theta}\ln\left(\mathbf{1}+\frac{\mathbf{L}}{\mathbf{R}}\right)$.

If $\frac{L}{R} << 1$, (small wedge angle), this gives

$$C = C_0 = \frac{\varepsilon_0 L^2}{R\theta} = \frac{\varepsilon_0 L^2}{D},$$

the correct expression for a parallel plate capacitor. From the expansion $\ln(1+x) = x - \frac{x^2}{2} + \ldots$ the correction to the parallel plate value C_0 is given by

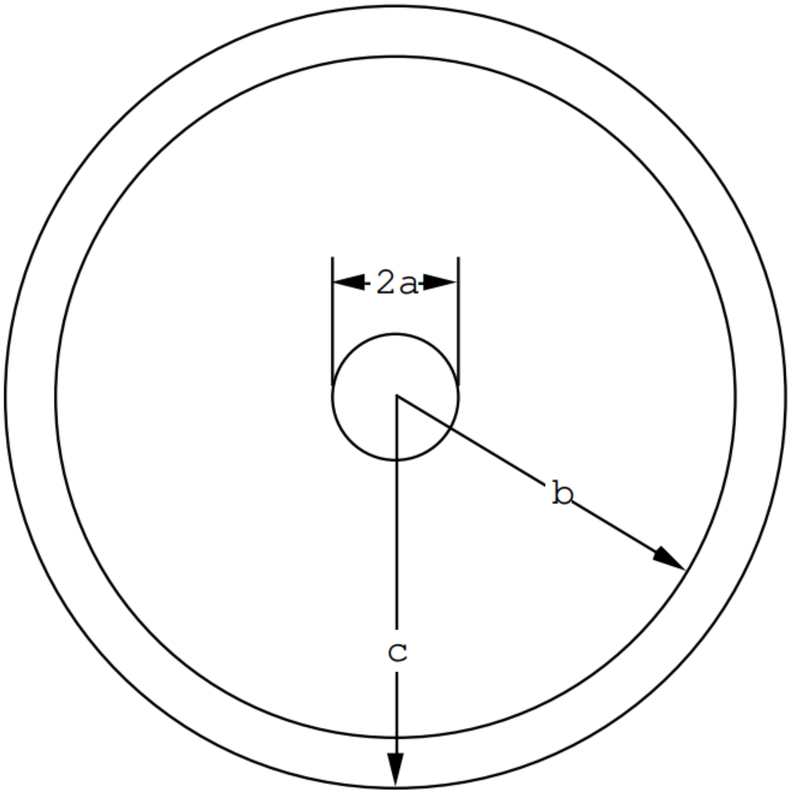
$$C \cong C_0\left(1-\frac{L}{2R}\right) = C_0\left(1-\frac{L\theta}{2D}\right).$$

The effect of tilting the plates is to reduce the capacitance by an amount corresponding to the average increase in spacing between the plates.

13.4: Chapter 4

Problem (4.1)

A current I amps flows in the inner conductor of an infinitely long co-axial line and returns via the outer conductor. The radius of the inner conductor is a, and b and c are the inner and outer radii of the outer conductor (see the sketch). The current density is uniform in the two conductors. Calculate the magnetic flux density in all regions. The magnetization density can be set equal to zero everywhere.



Answer (4.1)

This problem exhibits cylindrical symmetry so that it is ideal for an application of Stokes' theorem. Let z be the direction along the cable. Then there is only a component Az of the vector potential $\left(\mathbf{A}=\frac{\mu_0 I}{4\pi}\oint_C\frac{d\mathbf{L}}{r}\right)$. Moreover, by symmetry A_z cannot depend upon the angle θ, nor can it depend upon z (infinite wire).

$\therefore A_z = A_z(r)$.

In cylindrical co-ordinates

$$\operatorname{curl}\mathbf{A}=\frac{1}{r}\begin{vmatrix}\hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & rA_\theta & A_z\end{vmatrix}, \therefore \text{B has only a } \theta \text{ component}$$

$$\mathrm{B}_\theta=-\frac{\partial \mathrm{A}_z}{\partial \mathrm{r}}.$$

But since there is no magnetization and no time dependence

Curl $\mathbf{B} = \mu_o \mathbf{J}_f$

$$\therefore \int_{\text{surface}}\operatorname{curl}\mathbf{B}\cdot\mathbf{ds}=\mu_0\int_{\text{surface}}\mathbf{J}_f\cdot\mathbf{ds}$$

or

$\oint_C \mathbf{B} \cdot d\mathbf{L} = \mu_0 \int_{\text{surface}} \mathbf{J}_f \cdot \mathbf{ds}$

Apply this to a circle of radius r:

Case (1) $r \leq a \quad J_f = \frac{I}{\pi a^2}$

$\therefore 2\pi r B_\theta = \mu_0 \left(\frac{I}{\pi a^2}\right)\left(\pi r^2\right) = \mu_0 I \left(\frac{r}{a}\right)^2$

$\therefore \quad B_\theta = \left(\frac{\mu_0 I}{2\pi a^2}\right) r$

So when r = 0 $B_\theta = 0$

when r = a $B_\theta = \frac{\mu_0 I}{2\pi a}$.

Case (2) a ≤ r ≤ b

In this case $\int_{\text{surface}} \mathbf{J}_f \cdot \mathbf{ds} \equiv I$

$\therefore 2\pi r B_\theta = \mu_o I$

$B_\theta = \frac{\mu_0 I}{2\pi r}$

When r=a $B_\theta = \frac{\mu_0 I}{2\pi a}$

When r=b $B_\theta = \frac{\mu_0 I}{2\pi b}$

Case (3) b ≤ R ≤ c

In the outer conductor

$|J_f| = \frac{I}{\pi\left(c^2 - b^2\right)}$

and the current flow is negative. Therefore this time one has

$2\pi r B_\theta = \mu_0 \left[I - \frac{I\pi\left(r^2 - b^2\right)}{\pi\left(c^2 - b^2\right)}\right]$

$B_\theta = \frac{\mu_0 I}{2\pi r}\left[1 - \left(\frac{r^2 - b^2}{c^2 - b^2}\right)\right]$

So when r = b $B_\theta = \frac{\mu_0 I}{2\pi \mathrm{b}}$

When r = c $B_\theta = 0$

Case (4) R ≥ C

Here $2\pi r B_\theta = \mu_o(I - I) \equiv 0 \therefore B_\theta = 0$.

There is no field outside this co-axial cable. Notice that the tangential component of **B** is continuous across the boundaries.

Problem (4.2)

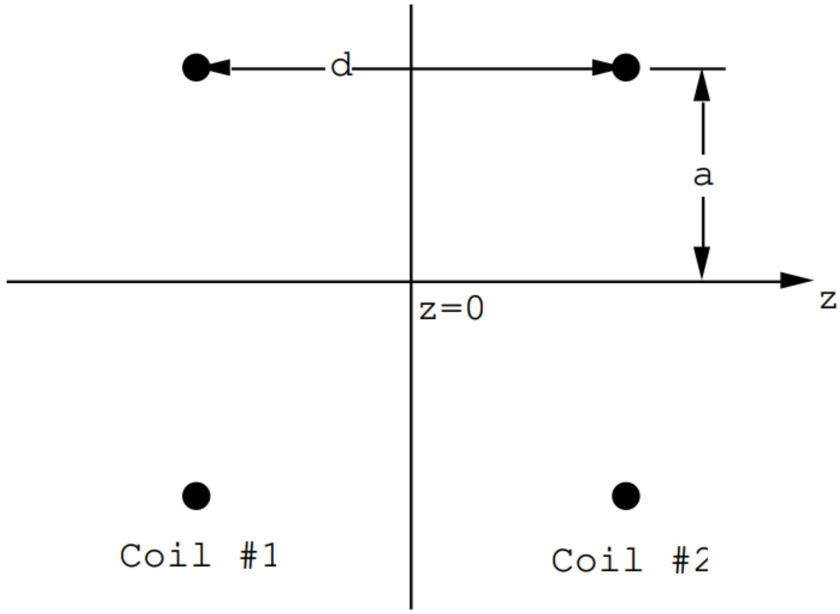
Two identical coaxial coils, each of N turns and radius a, are separated by a distance d as shown in the sketch. A current flows through each coil so that the fields of the two coils add at the origin.

(a) Calculate B_z at the origin

(b) Show that $\frac{dBz}{dz} = 0$ at z = 0.

(c) Find d such that $\frac{d^2 B_z}{dz^2} = 0$ at z = 0.

Such a configuration is the simplest system for generating a uniform magnetic field. It is known as a Helmholtz pair.



Answer (4.2)

The field of a single coil along its axis is

$$B_z = \frac{\mu_0 NI}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}$$

where z is measured from the center of the coil. For the above pair of coils

$$\mathrm{B_z} = \frac{\mu_\mathrm{o}\mathrm{NIa}^2}{2}\left\{\frac{1}{\left[\left(z-\frac{\mathrm{d}}{2}\right)^2+\mathrm{a}^2\right]^{3/2}}+\frac{1}{\left[\left(z+\frac{\mathrm{d}}{2}\right)^2+\mathrm{a}^2\right]^{3/2}}\right\}$$

(a) At z = 0 $\mathrm{B_z} = \mu_0 \mathrm{NI}\left(\frac{\mathrm{a}^2}{\left[\frac{\mathrm{d}^2}{4}+\mathrm{a}^2\right]3/2}\right)$

(b)

$$\frac{dB_z}{dz} \propto \frac{-3\left(z-\frac{d}{2}\right)}{\left[\left(z-\frac{d}{2}\right)^2+a^2\right]5/2}-\frac{3\left(z+\frac{d}{2}\right)}{\left[\left(z+\frac{d}{2}\right)^2+a^2\right]5/2}$$

Thus at z = 0 $\frac{dB_z}{dz} = 0$.

(c)

$$\frac{d^2B_z}{dz^2} \propto \frac{-3}{\left[\left(z-\frac{d}{2}\right)^2+a^2\right]^{5/2}}-\frac{3}{\left[\left(z+\frac{d}{2}\right)^2+a^2\right]5/2}+\frac{15\left(z-\frac{d}{2}\right)^2}{\left[\left(z-\frac{d}{2}\right)^2+a^2\right]^{7/2}}+\frac{15\left(z+\frac{d}{2}\right)^2}{\left[\left(z+\frac{d}{2}\right)^2+a^2\right]^{7/2}}$$

∴ at z = 0

$$\frac{d^2B_z}{dz^2} \propto \frac{-3\left(\frac{d^2}{4}+a^2\right)-3\left(\frac{d^2}{4}+a^2\right)+15\left(\frac{d^2}{4}\right)+15\left(\frac{d^2}{4}\right)}{\left[\frac{d^2}{4}+a^2\right]^{7/2}}$$
$$\propto 6d^2-6a^2 = 6\left(d^2-a^2\right)$$

So $\frac{d^2B_z}{dz^2} = 0$ if d = a.

Thus for a Helmholtz pair d = a.

The magnetic field strength at the center of the Helmholtz pair is given by

$B_z(0) = \frac{\mu_0 NI}{a}\left(\frac{4}{5}\right)^{3/2} = 0.716\frac{\mu_0 NI}{a}$.

Problem (4.3)

A solenoid is 1 meter long and it carries 10^4 turns of wire. The average radius of the coil is 0.1 meters. The coil carries a current of 10 Ampères.

(a) Calculate the field at the center of the solenoid.

(b) If the wire of the coil has a cross-sectional area of 10^{-6} meters^2 calculate the resistance of the coil. R = ρL/A and for copper ρ = 2 x 10^{-8} ohm meters.

(c) How much power is required to produce the magnetic field of part (a)?

This calculation explains why iron core magnets are used to generate fields of ~ 1 Tesla.

Answer (4.3)

$$\mathrm{B_z(z)} = \frac{\mu_o \mathrm{NI}}{2}\left\{\frac{(\mathrm{z}+\mathrm{L}/2)}{\sqrt{\mathrm{R}^2+\left(\mathrm{z}+\frac{\mathrm{L}}{2}\right)^2}} - \frac{(\mathrm{z}-\mathrm{L}/2)}{\sqrt{\mathrm{R}^2+\left(\mathrm{z}-\frac{\mathrm{L}}{2}\right)^2}}\right\}$$

N is the number of turns/m, L the length of the coil.

At z = 0 $\mathrm{B_z(o)} = \frac{\mu_o \mathrm{NI}}{2}\frac{\mathrm{L}}{\sqrt{\mathrm{R}^2+(\mathrm{L}/2)^2}}$

Here N = 10^4/m, I = 10 Amps, $\frac{L}{2} = \frac{1}{2}m$, and R = 0.1 m

(a) $\therefore B_z(0) = \frac{(2\pi\times10^{-2})}{\sqrt{.01+.25}} = 0.123$ Tesla i.e. ~ 10^3 x earth's field!

(b) L = (2πR)(10^4) = 6.283 x 10^3 m ∴ R = 125.7 Ohms

(c) For 10 Amps one would require 1257 Volts and a power = VI = 12,570 Watts!! = 12.57 kWatts!

Problem (4.4)

A square loop of wire 1 cm on a side carries a current of 2 Ampères.

(a) Estimate the magnitude of the magnetic field on the axis of the current loop and 1 meter from its center. The loop may be treated like a point dipole.

(b) Estimate the magnitude and direction of the magnetic field one meter from the center of the loop but at a point in the plane of the loop.

Answer (4.4)

The magnetic moment of the loop is M_o = IA = (2)(10^{-4}) Amp m^2.

Now $\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right]$

(a) On the axis of the dipole **m·r** = M_or

So $B_z = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{2M_0}{r^3}\right) = 4\times10^{-11}\mathrm{Tesla}$

(The earth's field is ~ 10^{-4} Tesla so this is very weak).

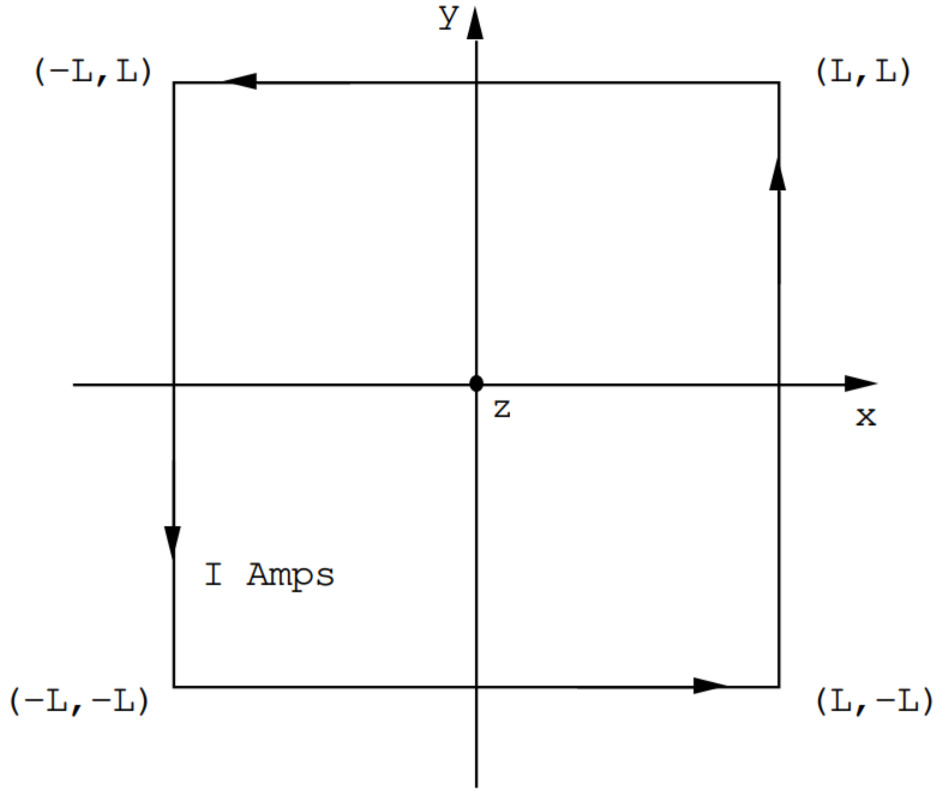
(b) On the equatorial plane **m·r** = 0

$\therefore B_z = -\frac{\mu_0}{4\pi}\frac{M_0}{r^3} = -2\times10^{-11}\mathrm{Tesla}$

Directed opposite to the dipole moment.

Problem (4.5).

Calculate the magnetic field along the z-axis of a square coil which carries a current of I Amps (see the sketch).



Each side of the square is 2L meters long.

Answer (4.5).

Along the axis of the coil there will be only a zcomponent of magnetic field by symmetry. In order to get the total field it is only necessary to calculate the z-component of the field generated by one side of the coil and then multiply by four. Consider the right hand side.

Let $d\mathbf{L} = dy\hat{\mathbf{u}}_y \quad \text{at} \quad (L, Y)$

The position of the element of length, d**L**, is specified by **r** where $\mathbf{r} = \mathrm{L}\hat{\mathbf{u}}_\mathrm{x} + \mathrm{y}\hat{\mathbf{u}}_\mathrm{y}$. The position of the point of observation along the z-axis is specified by $\mathbf{R} = z\hat{\mathbf{u}}_z$.

Therefore,

$(\mathbf{R} - \mathbf{r}) = -L\hat{\mathbf{u}}_\mathrm{x} - \mathrm{y}\hat{\mathbf{u}}_\mathrm{y} + \mathrm{z}\hat{\mathbf{u}}_\mathrm{z}$

and

$|\mathbf{R} - \mathbf{r}| = \sqrt{L^2 + y^2 + z^2}$.

From the law of Biot-Savard one obtains

$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{\mathrm{d}\mathbf{L} \times (\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^3};$$

from which

$$\mathrm{dB_z} = \frac{\mu_0}{4\pi} \mathrm{I} \frac{\mathrm{Ldy}}{\left(\mathrm{L}^2 + \mathrm{y}^2 + \mathrm{z}^2\right)^{3/2}},$$

and

$$B_z = \frac{\mu_0}{4\pi} I \frac{2L^2}{(z^2+L^2)\sqrt{z^2+2L^2}}.$$

This must be multiplied by 4x because the coil has four sides:

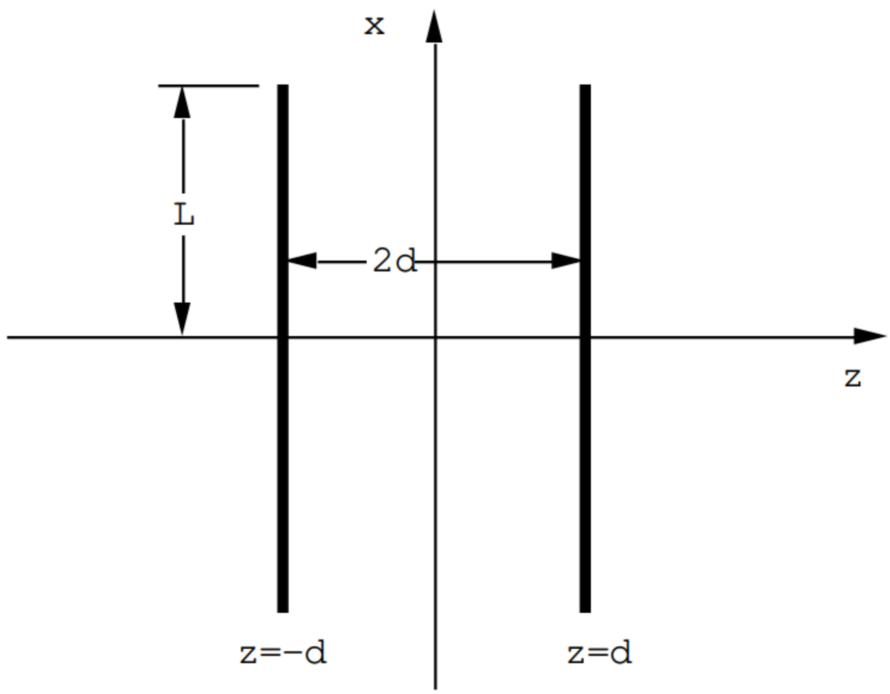
$$\mathrm{B_z(z)} = \frac{2\mu_0 \mathrm{I}}{\pi} \frac{\mathrm{L}^2}{\left(\mathrm{z}^2+\mathrm{L}^2\right)\sqrt{\mathrm{z}^2+2\mathrm{L}^2}}.$$

At z=0 $\mathrm{B_z}(0) = \frac{2\sqrt{2}}{\pi}\frac{\mu_0 I}{2\mathrm{L}} = 0.9003\frac{\mu_0 I}{2\mathrm{L}}$

This value can be compared with $B_z(0) = \frac{\mu_0 I}{2R}$ for a circular coil.

Problem (4.6).

(a) How far apart should two square coils be mounted in order to obtain as homogeneous as possible a magnetic field? See the sketch.



(One wants $\frac{d^2B_z}{dz^2} = 0$ at the center of the coil system. With a little thought one can convince oneself that at z=0 the quantity $\frac{d^2B_z}{dz^2}$ is exactly the same for each coil, so that the work of differentiation can be reduced by a factor two.)

(b) Over what distance along the z-axis will the field deviate by less than 1% from the field at the center of the coil system if L= 1 meter?

Such square coils are often more convenient to build than circular coils if the earth's magnetic field is to be cancelled over a large volume.

Answer (4.6)

(a) From the results of Problem (3.5) one can write

$\mathrm{B_z(z)} = \frac{2\mu_0 \mathrm{I}\,\mathrm{L}^2}{\pi}(\Psi_1+\Psi_2)$, where

$\Psi_1 = \frac{1}{\left((z+d)^2+L^2\right)\left((z+d)^2+2L^2\right)^{1/2}}$,

and $\psi_2 = \frac{1}{\left((z-d)^2+L^2\right)\left((z-d)^2+2L^2\right)^{1/2}}$.

$\frac{d\psi_1}{dz} = \frac{-(z+d)\left(3(z+d)^2+5L^2\right)}{\left((z+d)^2+L^2\right)^2\left((z+d)^2+2L^2\right)^{3/2}}$, and

$\frac{d\Psi_2}{dz} = \frac{-(z-d)\left(3(z-d)^2+5L^2\right)}{\left((z-d)^2+L^2\right)^2\left((z-d)^2+2L^2\right)^{3/2}}$.

Note that at z=0 $\frac{d}{dz}(\psi_1+\psi_2)\equiv 0$; the field gradient vanishes by symmetry.

$\frac{d^2\Psi_1}{dz^2} = \frac{N}{D}$, where

$$\begin{aligned} N = - & \left(9(z+d)2+5L^2\right)\left((z+d)^2+L^2\right)\left((z+d)^2+2L^2\right) \\ & +\left(12(z+d)^4+20L^2(z+d)^2\right)\left((z+d)^2+2L^2\right) \\ & +\left(9(z+d)^4+15L^2(z+d)^2\right)\left((z+d)^2+L^2\right) \end{aligned}$$

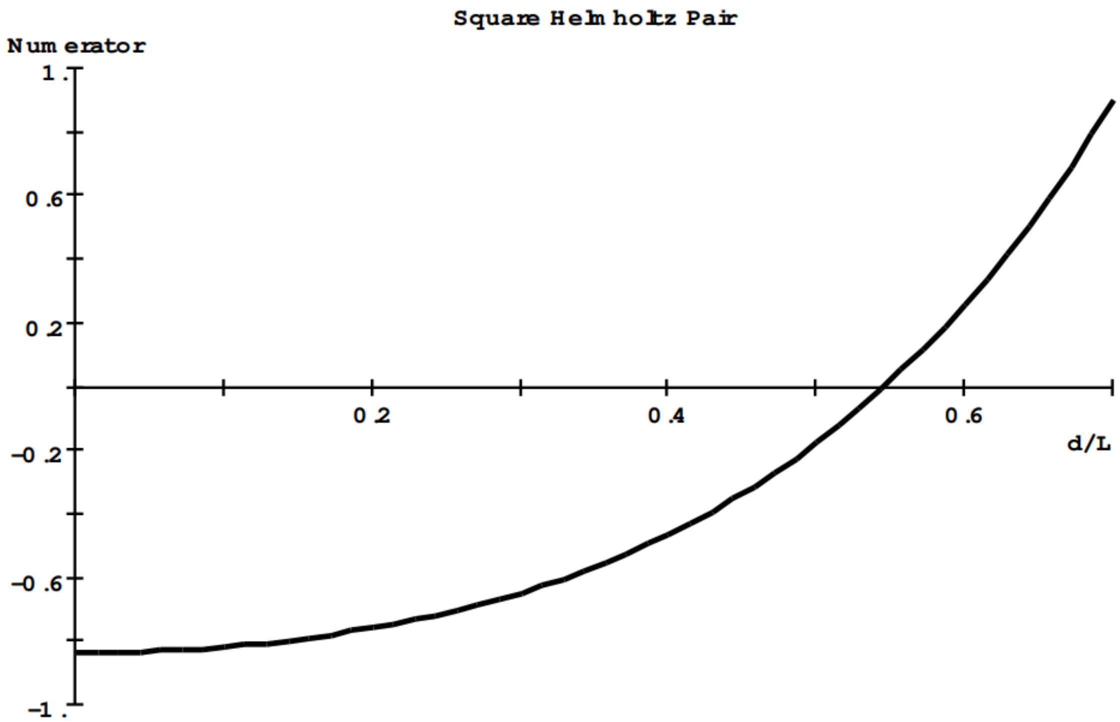
and

$$D = \left((z+d)^2+L^2\right)^3\left((z+d)^2+2L^2\right)^{5/2}.$$

At $z=0\frac{d^2\psi_1}{dz^2} = \frac{d^2\psi_2}{dz^2}$ so that for optimum uniformity We require the numerator in the second derivative to vanish at z=0. This condition gives

$\eta^6+3\eta^4+\left(\frac{11}{6}\right)\eta^2-\left(\frac{5}{6}\right)=0$ (1)

where $\eta=(d/L)$. The solution is $\frac{d}{L}=0.5445057$ (see the figure below). The coils should be placed 2d= 1.0890L apart.

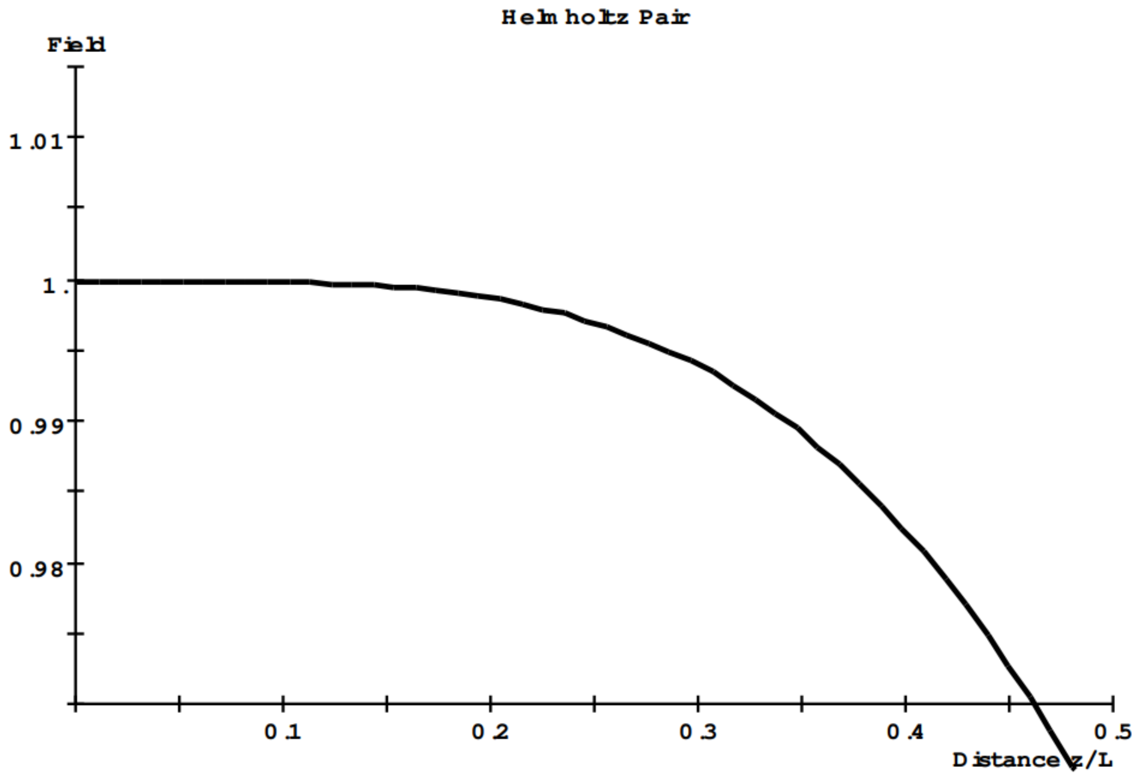


(b) The simplest way to examine the homogeneity is to plot the field function:

$$\mathrm{B}_z(\zeta) = \frac{2\mu_0 I}{\pi \mathrm{L}}\left(\frac{1}{((\zeta+\eta)^2+1)\sqrt{(\zeta+\eta)^2+2}}+\frac{1}{((\zeta-\eta)^2+1)\sqrt{(\zeta-\eta)^2+2}}\right),$$

where ζ= (z/L) and where η= (d/L)= 0.5445057. At the center of the coil system

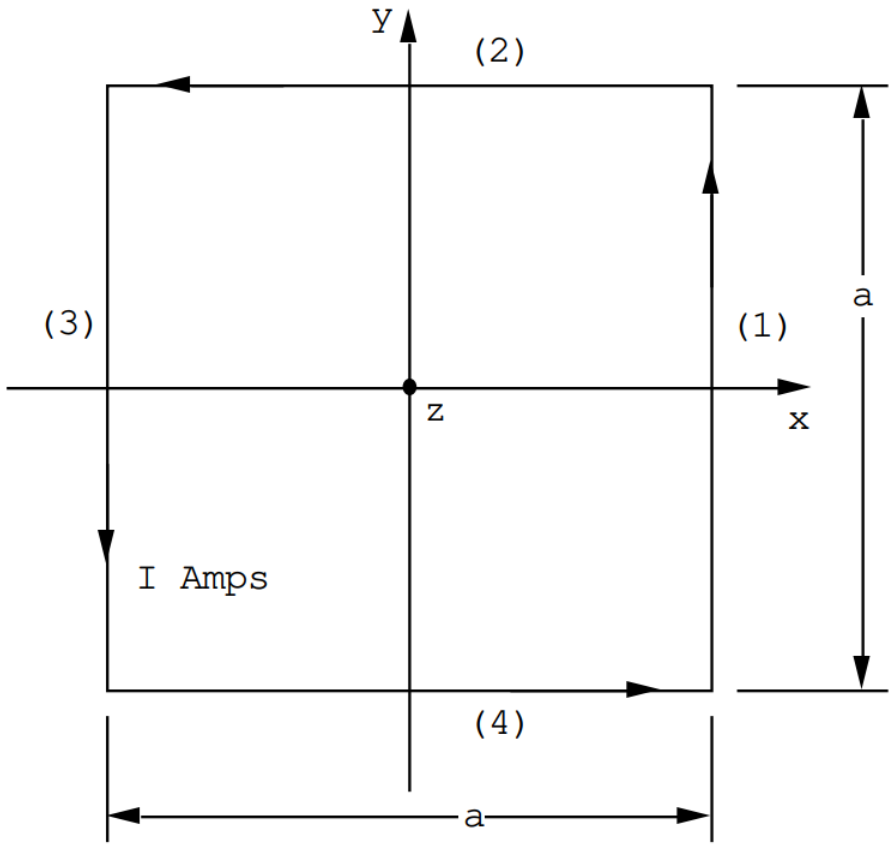
$\mathrm{B}_z(0) = \frac{2\mu_0 I}{\pi\mathrm{L}}(1.017958)$ Teslas.



From this graph one finds that the field has decreased by 1% when $\left(\frac{z}{L}\right)=0.344$. This means that the field varies by less than 1% over a central region whose length is 0.688L. It turns out that the field is homogeneous to within 1% within a **volume** whose diameter is 0.688L: i.e. within the sphere whose diameter is ~68 cm if L=1 meter.

Problem (4.7)

Consider a square loop of wire lying in the xy-plane as shown in the sketch. The loop carries a current of I amps and is centered on the origin.



(a) Show that the contribution to the vector potential at a point P(X,Y,Z) from side (1) has only a y component and that this component is given by

$$A_{Y1}=\frac{\mu_0 I}{4\pi}\ln\left(\frac{Y-(a/2)-\sqrt{(X-a/2)^2+(Y-a/2)^2+Z^2}}{Y+(a/2)-\sqrt{(X-a/2)^2+(Y+a/2)^2+Z^2}}\right).$$

(b) Show that the contribution to the vector potential at a point P(X,Y,Z) from side (3) has only a y component and that this component is given by

$$A_{Y3}=-\frac{\mu_0 I}{4\pi}\ln\left(\frac{Y-(a/2)-\sqrt{(X+a/2)^2+(Y-a/2)^2+Z^2}}{Y+(a/2)-\sqrt{(X+a/2)^2+(Y+a/2)^2+Z^2}}\right).$$

(c) Show that the contribution to the vector potential at a point P(X,Y,Z) from side (2) has only an x component and that this component is given by

$$A_{X2}=-\frac{\mu_0 I}{4\pi}\ln\left(\frac{X-(a/2)-\sqrt{(X-a/2)^2+(Y-a/2)^2+Z^2}}{X+(a/2)-\sqrt{(X+a/2)^2+(Y-a/2)^2+Z^2}}\right).$$

(d) Show that the contribution to the vector potential at a point P(X,Y,Z) from side (4) has only an x component and that this component is given by

$$A_{X4}=\frac{\mu_0 I}{4\pi}\ln\left(\frac{X-(a/2)-\sqrt{(X-a/2)^2+(Y+a/2)^2+Z^2}}{X+(a/2)-\sqrt{(X+a/2)^2+(Y+a/2)^2+Z^2}}\right).$$

(e) Now consider the point P(X,0,Z) which is specified by the vector $\mathbf{R}=X\hat{\mathbf{u}}_{\mathbf{X}}+Z\hat{\mathbf{u}}_{\mathbf{z}}$. Show that

$A_x = 0$,

$$A_y = \frac{\mu_0 I}{4\pi}\ln\left(\left(\frac{1+\frac{a}{2\sqrt{(x-a/2)^2+\frac{a^2}{4}+z^2}}}{1-\frac{a}{2\sqrt{(X-a/2)^2+\frac{a^2}{4}+Z^2}}}\right)\left(\frac{1-\frac{a}{2\sqrt{(x+a/2)^2+\frac{a^2}{4}+z^2}}}{1+\frac{a}{2\sqrt{(x+a/2)^2+\frac{a^2}{4}+z^2}}}\right)\right).$$

In the limit as a/R → 0, where $R = \sqrt{X^2+Z^2}$, the expression for A_y can be shown to go to the limit

$$A_y = \frac{\mu_0\, Ia^2}{4\pi}\left(\frac{X}{R^3}\right) = \frac{\mu_0}{4\pi}\left(\frac{m_z x}{R^3}\right),$$

where m_z= Ia^2 Amp-meters^2. This is just the x-component of the expression $\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{R}}{R^3}$, the dipole vector potential.

Answer (4.7)

We shall show the calculation for side (1). The procedure for the other three sides is very similar. For side (1) the element of length is given by

$d\mathbf{L} = dy\hat{\mathbf{u}}_y$.

This element is located at $\mathbf{r} = \frac{a}{2}\hat{\mathbf{u}}_x + y\hat{\mathbf{u}}_y$. The point of observation is located at $\mathbf{R} = X\hat{\mathbf{u}}_x + Y\hat{\mathbf{u}}_y + z\hat{\mathbf{u}}_z$, therefore

$$\mathbf{R}-\mathbf{r} = \left(X-\frac{a}{2}\right)\hat{\mathbf{u}}_x + (Y-y)\hat{\mathbf{u}}_y + Z\hat{\mathbf{u}}_z.$$

The length of this line is given by

$$|\mathbf{R}-\mathbf{r}| = \sqrt{\left(X-\frac{a}{2}\right)^2 + (Y-Y)^2 + Z^2}\,.$$

The contribution to the vector potential at P has only a y-component because the current element has only a y component:

$$dA_y = \frac{\mu_0 I}{4\pi}\frac{dy}{\sqrt{\left(X-\frac{a}{2}\right)^2+(Y-y)^2+Z^2}}, \quad \text{and}$$

$$A_y = \frac{\mu_0 I}{4\pi}\int_{-a/2}^{a/2}\frac{dy}{\sqrt{\left(X-\frac{a}{2}\right)^2+(Y-y)^2+Z^2}}.$$

This is a standard integral; it can be written

$$A_{Y1} = \frac{\mu_0 I}{4\pi}\ln\left(\frac{Y-(a/2)-\sqrt{(X-a/2)^2+(Y-a/2)^2+Z^2}}{Y+(a/2)-\sqrt{(X-a/2)^2+(Y+a/2)^2+Z^2}}\right).$$

(e) The expansion for A_y in the limit of (a/R) → 0 can be carried out as follows: (it is convenient to use the notation

$$\sqrt{(x-a/2)^2+\frac{a^2}{4}+z^2} = \sqrt{x^2+z^2-ax+\frac{a^2}{2}} = R_-,$$

and

$$\sqrt{(x+a/2)^2+\frac{a^2}{4}+z^2} = \sqrt{x^2+z^2+ax+\frac{a^2}{2}} = R_+.$$

$$\ln\left(\frac{\left(1+\frac{a}{2R_-}\right)\left(1-\frac{a}{2R_+}\right)}{\left(1-\frac{a}{2R_-}\right)\left(1+\frac{a}{2R_+}\right)}\right) = \ln\left(1+\frac{a}{2R_-}\right)+\ln\left(1-\frac{a}{2R_+}\right)-\ln\left(1-\frac{a}{2R_-}\right)-\ln\left(1+\frac{a}{2R_+}\right).$$

Expand to first order in small quantities:

$\cong \frac{a}{R_-} - \frac{a}{R_+}$,

since (a/R₋) and (a/R₊) are very small. One can finally write

$$(a/R_-) = \frac{a}{R}\left(1 - \frac{aX}{R^2} + \frac{a^2}{2R^2}\right)^{-1/2} \cong \frac{a}{R}\left(1 + \frac{aX}{2R^2} - \frac{a^2}{4R^2}\right),$$
$$(a/R_+) = \frac{a}{R}\left(1 + \frac{aX}{R^2} + \frac{a^2}{2R^2}\right)^{-1/2} \cong \frac{a}{R}\left(1 - \frac{aX}{2R^2} - \frac{a^2}{4R^2}\right),$$

where $\mathrm{R} = \sqrt{\mathrm{X}^2 + \mathrm{Z}^2}$.

It follows from this that to first order in small quantities

$$\mathrm{A_Y} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{a^2X}}{\mathrm{R^3}}.$$

Problem (4.8)

A short cylindrical solenoid has a radius of R= $5\mathrm{x}10^{-2}$ meters and a length of L= $5\mathrm{x}10^{-2}$ meters. It is wound with N= $8\mathrm{x}10^{4}$ turns/meter, and the windings carry a current of I= 10 Amps.

(a) What is the magnetic field at the center of the solenoid?

(b) What is the magnetic field strength on the axis of the solenoid but at the end face (z=L/2)?

Answer (4.8)

The magnetic field along the axis of a short solenoid is given by (z is measured from the solenoid center)

$$\mathrm{B_z(z)} = \frac{\mu_0\mathrm{NI}}{2}\left(\frac{(\mathrm{z}+\mathrm{L}/2)}{\sqrt{(\mathrm{z}+\mathrm{L}/2)^2+\mathrm{R}^2}} + \frac{(\mathrm{L}/2-\mathrm{z})}{\sqrt{(\mathrm{z}-\mathrm{L}/2)^2+\mathrm{R}^2}}\right).$$

(a) At z=0 $B_z(0) = \frac{\mu_0 NI}{2}\frac{L}{\sqrt{(L/2)^2+R^2}}$

For this problem $\frac{\mu_0\mathrm{NI}}{2} = 0.503\ \mathrm{Teslas.}$.

Therefore $\mathrm{B_z(0)}$= 0.450 Teslas.

(b) At z= L/2= $2.5\mathrm{x}10^{-2}$ meters:

$$\mathrm{B_z(L/2)} = \frac{\mu_0\mathrm{NI}}{2}\left(\frac{\mathrm{L}}{\sqrt{\mathrm{L}^2+\mathrm{R}^2}}\right) = 0.707\left(\frac{\mu_0\mathrm{NI}}{2}\right) = 0.356\ \mathrm{Teslas}\,.$$

Problem (4.9)

A short cylindrical disc has a radius of R= $5\mathrm{x}10^{-2}$ meters and a length of L= $5\mathrm{x}10^{-2}$ meters. It is uniformly magnetized; the magnetization density is parallel with the axis of the disc, the z-axis, and the magnetization has the value $\mathrm{M_0}$= $0.955\mathrm{x}10^{6}$ Amps/meter.

(a) What is the magnetic field at the center of the disc?

(b) What is the magnetic field strength on the axis of the disc but at the end face (z=L/2)?

Answer (4.9)

The magnetic field distribution generated by a uniformly magnetized disc is the same as that generated by the windings of a short solenoid. The magnetic field along the axis of a short solenoid is given by

$$\mathrm{B_z(z)} = \frac{\mu_0\mathrm{NI}}{2}\left(\frac{(\mathrm{z}+\mathrm{L}/2)}{\sqrt{(\mathrm{z}+\mathrm{L}/2)^2+\mathrm{R}^2}} + \frac{(\mathrm{L}/2-\mathrm{z})}{\sqrt{(\mathrm{z}-\mathrm{L}/2)^2+\mathrm{R}^2}}\right).$$

It is only necessary to replace the product NI in this formula by the magnetization $\mathrm{M_0}$.

(a) At z=0 $B_z(0)=\frac{\mu_0 M_0}{2}\frac{L}{\sqrt{(L/2)^2+R^2}}$.

For this problem $\frac{\mu_0 M_0}{2}=0.600$ Teslas.

Therefore $B_z(0)$= 0.537 Teslas.

(b) At z= L/2= 2.5x10^{-2} meters:

$$B_z(L/2)=\frac{\mu_0 M_0}{2}\left(\frac{L}{\sqrt{L^2+R^2}}\right)=0.707\left(\frac{\mu_0 M_0}{2}\right)=0.424\text{ Teslas}.$$

Problem (4.10).

Given a sphere which is uniformly polarized along the z direction i.e. $M_z = M_o$ Amps/meter.

(a) What is **H** inside the sphere?

(b) What is **B** inside the sphere?

(c) What is the value of B_z on the axis of the sphere but just outside the surface of the sphere?

(d) What is the value of **H** just outside the equator of the sphere?

(e) A neutron star is typically an object 10^4 meters in diameter having the density of nuclear matter (~ 10^{21} kg/m^3). The maximum magnetic field at its surface is estimated to be 10^8 Tesla. What is its average magnetization density, M_o?

(f) A neutron has a mass of 1.68 x 10^{-27} kg. From (e) what is the average magnetic moment of a neutron in a neutron star?

Answer (4.10).

(a) The demagnetizing factor for a sphere is 1/3. Therefore $H_z=-\frac{M_0}{3}$.

(b) $B_z=\mu_0\left(H_z+M_z\right)=\frac{2}{3}\mu_0 M_0$.

(c) From div **B** = 0 The normal component of **B** must be continuous $\therefore B_z=\frac{2}{3}\mu_0 M_0$.

(d) From curl **H**=0 (there are no currents) the tangential component of **H** must be continuous across the surface of the sphere. It follows that $H_z=-\frac{M_0}{3}$ Amps/meter at the equator just outside the sphere. From the fact that **M** has no component normal to the surface of the sphere at the equator it follows that the normal component of **H** must be continuous across the surface of the sphere at its equator and therefore **H** has only a z-component just outside the sphere on the equator. Also on the equator just outside the sphere $B_z=-\frac{\mu_0 M_0}{3}$. The tangential component of **B** is discontinuous.

(e) $\frac{2}{3}\mu_0 M_0=10^8$ Teslas.

$$\therefore M_0=1.19\times 10^{14} Amps/m(i.e. Large!!)$$

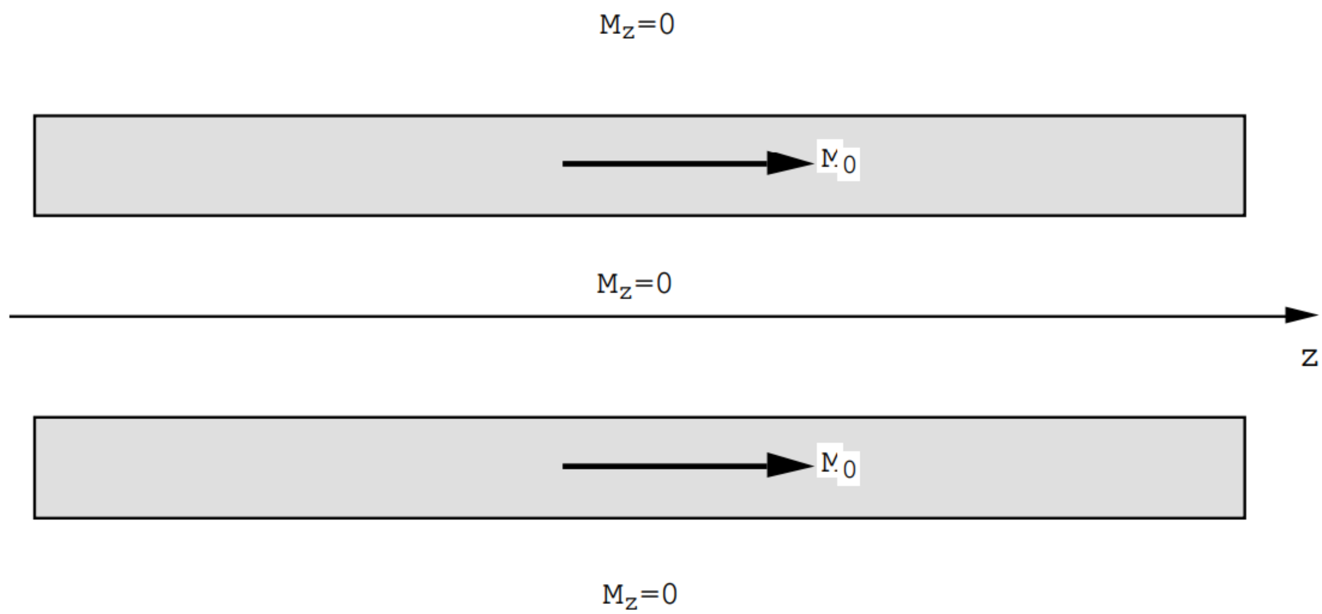
(f) The number of neutrons/m^3 = $=\frac{10^{21}}{1.68\times 10^{-27}}=5.95\quad \text{x}\quad 10^{47}$

$$\therefore\quad \langle\mu_N\rangle=\frac{11.9\times 1013}{5.95\times 10^{47}}=2.0\times 10^{-34}\quad \text{Amp m}^2\ .$$

The neutron magnetic moment is 9.7 x 10^{-27} Amp m^2 so that on average only 2 x 10^{-8} of a neutron is aligned.

Problem (4.11)

The material of a very long, hollow, rod is uniformly magnetized as shown in the sketch. (Although the rod is shown as having a finite length in the sketch, it is supposed to be infinitely long).



(a) What is the value of the magnetic field **B** outside the rod?

(b) What is the value of the magnetic fields **H**,**B** in the central hollow region where M_z=0?

(c) What are the values of **B**,**H** in the material of the rod where the magnetization is M_0?

Answer (4.11)

By superposition this problem can be reduced to the problem of nested solenoids. The outer surface discontinuity in the tangential component of M is equivalent to a solenoid for which NI= M_0. This current sheet produces a field $B_1 = \mu_0 M_0$. The inner surface discontinuity in the tangential component of M is equivalent to a solenoid for which NI= - M_0.

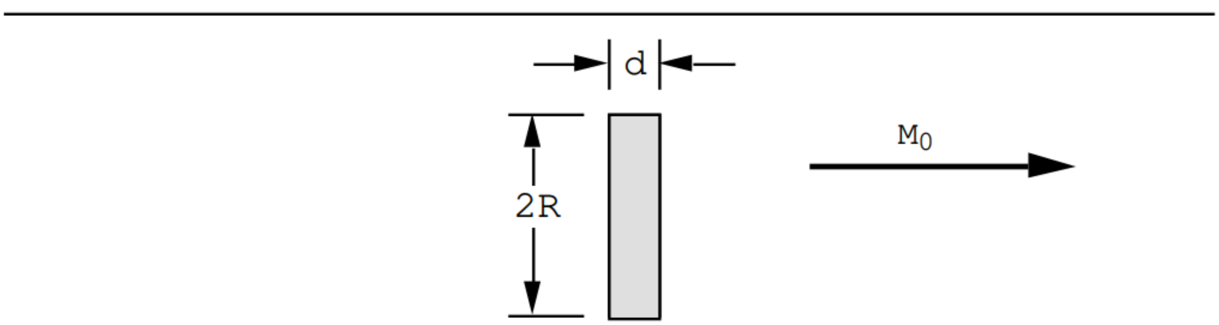
(a) Outside the rod the fields **B**,**H** are both zero.

(b) In the hollow region the fields due to the two current sheets cancel so that **B**= **H**= 0.

(c) In the region between the two current sheets the **B** field is that due to the outer current sheet; $B_z = \mu_0 M_0$. But by definition, $B_z = \mu_0(H_0 + M_0)$, and therefore H_z=0. Thus **H**= 0 <u>everywhere</u> because there are no real currents and no magnetic charge density to generate an **H**-field.

Problem (4.12)

An infinitely long rod is uniformly magnetized except for a disc shaped cavity shown shaded in the figure. Inside the cavity the magnetization is zero. What is the magnetic field strength at the center of the cavity?



Answer (4.12)

This problem can be worked as the superposition of a uniformly magnetized, infinitely long rod plus a uniformly magnetized disc, but for the disc M_z= - M_0. For the uniform rod $B_z = \mu_0 M_0$. Along the axis of the disc

$$B_z(z) = -\frac{\mu_0 M_0}{2}\left(\frac{(z+d/2)}{\sqrt{(z+d/2)^2+R^2}}+\frac{(d/2-z)}{\sqrt{(z-d/2)^2+R^2}}\right),$$

and at z=0

$$B_z(0) = -\frac{\mu_0 M_0}{2}\frac{d}{\sqrt{(d/2)^2+R^2}}.$$

The total field at the center of the disc will be

$$B_z(0) = \mu_0 M_0\left(1-\frac{d}{2\sqrt{(d/2)^2+R^2}}\right).$$

In the limit (d/R)→0 the field at the center of the cavity is just $B_0 = \mu_0 M_0$.

Problem (4.13)

A uniformly magnetized ellipsoid possesses magnetization components

$$M_X = 2\times 10^5 \text{ Amps/meter},$$
$$M_Y = 2\times 10^5 \text{ Amps/meter}$$
$$M_Z = 4\times 10^5 \text{ Amps/meter}$$

when referred to the principle axes of the ellipsoid. Demagnetizing coefficients for the ellipsoid are

$$N_X = 0.2,$$
$$N_Y = 0.3 \quad .$$

(a) Calculate the components of **H** inside the ellipsoid.

(b) Calculate the components of **B** inside the ellipsoid.

(c) Calculate the angle between **B** and **M**.

Answer (4.13)

The demagnetizing coefficients obey the sum rule

$$N_x + N_y + N_z = 1 \ .$$

For this problem

$$N_x = 0.20,$$
$$N_Y = 0.30,$$
$$N_z = 0.50.$$

(a)
$$H_X = -N_X \qquad M_X = -0.40\times 10^5 \text{ Amps/meter.}$$
$$H_Y = -N_Y \qquad M_Y = -0.60\times 10^5 \text{ Amps/meter.}$$
$$H_Z = -N_Z \qquad M_Z = -2.00\times 10^5 \text{ Amps/meter.}$$

(b) $\mathbf{B} = \boldsymbol{\mu}_0(\mathbf{H}+\mathbf{M})$, therefore

$$B_X = \mu_0\,(H_X+M_X) = 0.201 \text{ Teslas.}$$
$$B_Y = \mu_0\,(H_Y+M_Y) = 0.176 \text{ Teslas.}$$
$$B_z = \mu_0\,(H_Z+M_Z) = 0.251 \text{ Teslas.}$$

(c) $\mathbf{B}\cdot\mathbf{M} = |\mathbf{B}||\mathbf{M}| \quad \cos\theta;$

$$|\mathbf{M}| = 4.899\times 10^5 \text{ Amps/meter} \ ,$$

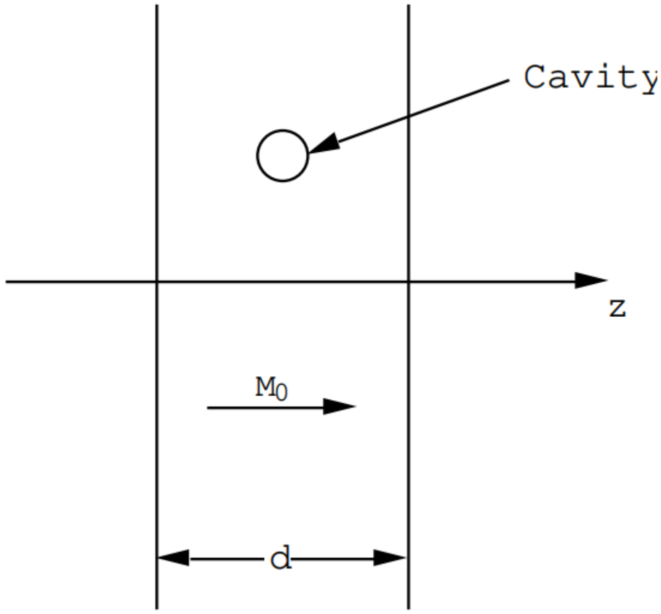
$$|\mathbf{B}| = 0.367\ Teslas,$$

$$\cos\theta = \frac{M_xB_x+M_yB_y+M_zB_z}{|\mathbf{M}||\mathbf{B}|} = \frac{1.758}{(4.899)(.3667)} = 0.9786\ .$$

So θ = 11.9°.

Problem (4.14)

A very large disc whose radius is infinite is magnetized along its normal as shown in the figure.



(a) What is **H** in the disc?

(b) What is **H** outside the disc?

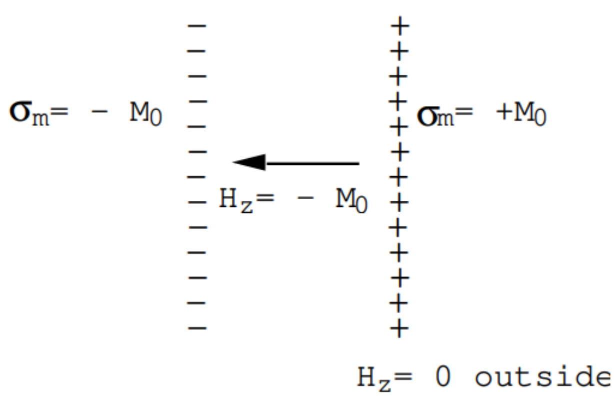
(c) What is **B** inside the slab?

(d) A spherical cavity is cut out of the material of the disc. Use the principle of superposition to calculate the magnetic field **B** in the cavity.

Answer (4.14)

(a) The demagnetizing factor for the direction along the disc normal is N_z = 1. Therefore $H_z = - M_o$.

(b) Outside the disc the field is zero by analogy with the equivalent electrostatic problem i.e. two infinite charge sheets



(c) $B_z = \mu_o (H_z + M_z) \equiv 0$.

(d) Inside a uniformly polarized sphere $B_z = \frac{2}{3}\mu_0 M_0$. Therefore in the cavity one must have $B_z = -\frac{2}{3}\mu_0 M_0$ so that the sum of the two fields gives zero when the sphere is put into the hole.

Problem (4.15)

A very long cylinder of magnetic material has a radius R. The axis of the cylinder lies along the z-axis. The magnetization depends upon the distance from the cylinder axis:

$$\mathrm{M_z = M_0(r/R) \quad Amps\ /\ meter.}$$

(a) Calculate the effective current density curl **M** both inside and outside the cylinder.

(b) Note that there is an effective surface current density on the surface of the cylinder due to the discontinuity in the tangential component of the magnetization. Calculate this surface current density, J_s.

(c) Calculate the radial dependence of the magnetic field in the cylinder.

Answer (4.15)

$$\operatorname{curl}\mathbf{M} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & 0 & 0 \\ 0 & 0 & \mathrm{M}_z \end{vmatrix} = \begin{vmatrix} 0 \\ -\frac{\partial M_z}{\partial r} \\ 0 \end{vmatrix},$$

(There is no angular or z dependence).

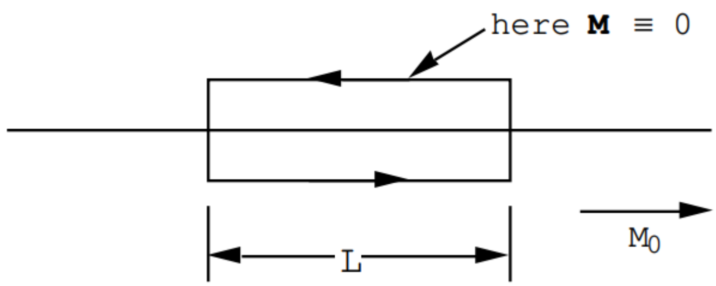
(a) $\therefore \mathrm{J}_\theta = -\frac{\partial \mathrm{M_z}}{\partial \mathrm{r}} = -\mathrm{M_0/R}$ ie. independent of position.

(b) At the outer surface there is a discontinuity in the tangential component of **M**. Use Stokes' theorem to obtain the effective surface current density:

$\mathbf{J}_b$ = curl **M**

$$\therefore \quad \int_{\text{surface}} \mathbf{J}_b \cdot \mathbf{ds} = \int_{\text{surface}} \operatorname{curl}\mathbf{M}\cdot\mathbf{ds} = \oint_C \mathbf{M}\cdot \mathrm{d}\mathbf{L}$$

Apply this to the loop shown below:



Current through the loop $\int \mathbf{J}_\mathrm{b}\cdot\mathbf{ds} = J_\mathrm{s}\mathrm{L}$

J_s is the effective surface current density.

$\oint_C \mathbf{M}\cdot \mathrm{d}\mathbf{L} = \mathrm{M_0 L}$

$\therefore \quad J_S L = M_0 L$

or $J_s = M_o$ Amps/m.

(c) Calculate the field along the axis of the cylinder. By symmetry there is only a z-component which is independent of z. The uniform effective current density, $-\frac{\mathrm{M_0}}{\mathrm{R}}$, can be treated like a nested solenoid problem in order to calculate the magnetic field along the cylinder axis.

The effective current sheet strength is $\mathrm{NI} = -\left(\frac{\mathrm{M_0}}{\mathrm{R}}\right)\mathrm{dr} \quad \mathrm{Amps/m}$.

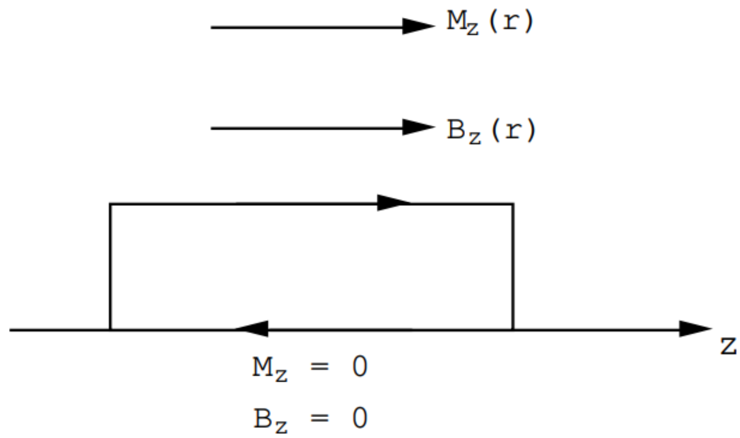
This produces the solenoid field contribution $\mathrm{dB_z} = -\mu_0\left(\frac{\mathrm{M_0}}{\mathrm{R}}\right)\mathrm{dr}$.

Integrate from r = 0 to r = R: $B_z = -\mu_o M_o$ Tesla. However, this is just cancelled by the surface current sheet which produces $B_z = \mu_o M_o$ Tesla.

$\therefore$ On the axis $B_z \equiv 0$.

Now use curl **B** = μ_o curl **M**

or $\oint_C \mathbf{B}\cdot d\mathbf{L} = \mu_0 \quad \int_C \mathbf{M}\cdot \mathbf{dL}$ and integrate around the loop shown in the figure:



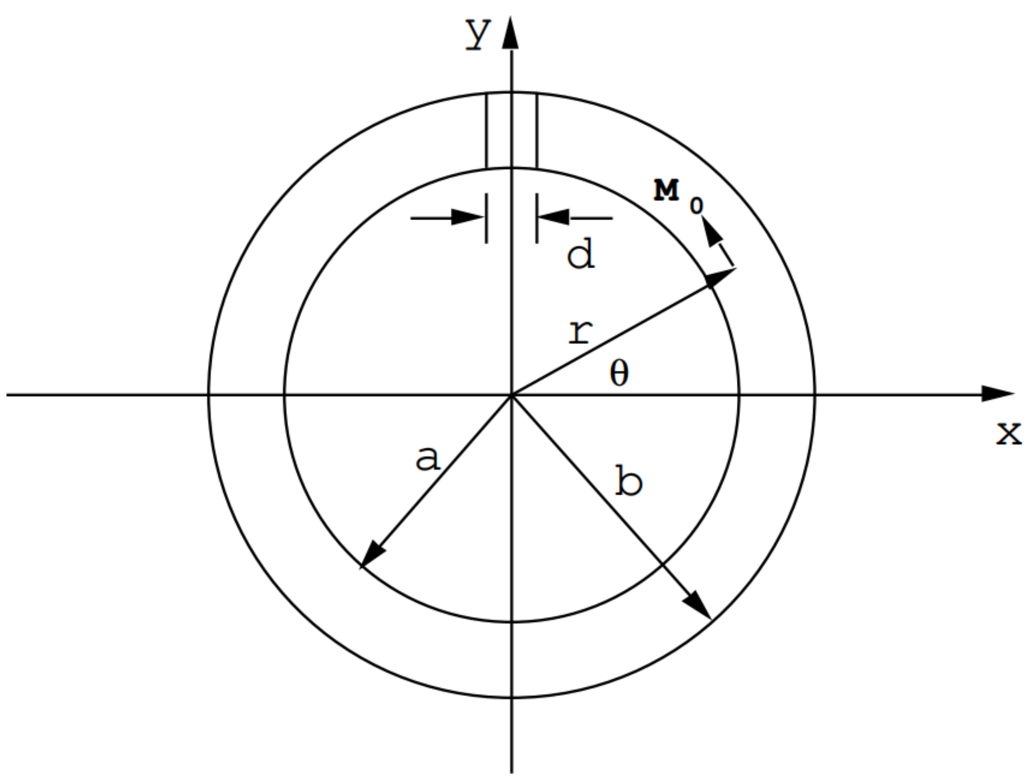
from which $B_z = \mu_o M_z$

$\therefore B_z(r) = \mu_o M_o(r/R)$

and $H_z \equiv 0$ everywhere.

Problem (4.16)

A permanent magnet is formed in the shape of a dough-nut having an inner radius a meters and an outer radius of b meters (see the figure). The magnetization density has the components M_r=0, M_θ= M_0, M_z=0 in cylindrical polar coordinates, where M_0 is constant.



(a) Calculate the field **H** everywhere.

(Answ: div **M**=0 everywhere, and there are no free currents. Therefore there are no sources for **H** and consequently **H**=0 everywhere.)

(b) Suppose that a gap d meters wide is opened in the ring as shown in the figure. Calculate the field **B** at the center of the gap.

(Answ: $\mathrm{B}_0=\mu_0 \mathrm{M}_0\left(1-\frac{\mathrm{d}}{\sqrt{4(\mathrm{b}-\mathrm{a})^2+\mathrm{d}^2}}\right)$ Teslas.)

Answer (4.16)

A uniform magnetic charge density will appear on the faces of the cut due to the discontinuity in **M**. The surface charge density on the left hand face is +M_0/m^2; the surface charge density on the right hand face is -M_0/m^2. These charge distributions produce a field at the gap center given by

$$\mathrm{H}_0=\frac{\mathrm{M}_0 \mathrm{d}}{4} \int_0^{\mathrm{R}} 2 \mathrm{rdr} \frac{1}{\left(r^2+(d / 2)^2\right)^{3 / 2}}$$

where R= (b-a)/2.

$$\mathrm{H}_0=\mathrm{M}_0 \quad\left(1-\frac{\mathrm{d}}{\sqrt{(\mathrm{b}-\mathrm{a})^2+\mathrm{d}^2}}\right),$$

B_0= $\mu_0 H_0$ directed along M_0, ie along -x in the above figure. This problem can also be solved by treating the magnetized plug removed from the gap as a short solenoid: for a short solenoid of radius R= (b-a)/2 and of length d the field at its center is given by

$$B_x=-\frac{\mu_0 M_0}{2} \frac{d}{\sqrt{\left(R^2+(d / 2)^2\right)}}.$$

This field plus the gap field, B_x^G, must equal the field in the gapless ring, - $\mu_0 M_0$, by superposition. Therefore

$$B_x^G=-\mu_0 M_0\left(1-\frac{d / 2}{\sqrt{(d / 2)^2+R^2}}\right)=-\mu_0 M_0\left(1-\frac{d}{\sqrt{d^2+(b-a)^2}}\right),$$

the same answer as above.

13.5: Chapter 5

Problem (5.1)

A very long solenoid is wound with N= 10^5 turns per meter. It is filled with a very permeable material, but one that becomes saturated at a critical value of B: i.e. for B< 0.30 teslas the relative permeability is μ_r= 10^3, but for B> 0.30 Teslas the relative permeability becomes very nearly equal to μ_r= 1.0.

(a) Make a sketch showing approximately how one would expect the **B**-field inside the material in the solenoid to vary with the dc current through the solenoid windings.

(b) Suppose that a secondary coil of radius R=2 cm and 1000 turns was wound on the above solenoid. Calculate the emf induced in the secondary coil if the current through the primary varies as

$$I(t)=I_0 \operatorname{Sin} \omega t,$$

where I_0= 1 mAmp (10^{-3} Amps), and ω= 2πF corresponds to 60 Hz.

(c) Calculate the emf induced in the secondary coil if a dc current of 10 mAmps flows through the solenoid windings in addition to the above ac current.

The controll of an output ac signal amplitude by means of a relatively small dc control current formed the basis for a device called a magnetic amplifier. In effect, the efficiency of a transformer could be altered by a dc current and therefore large amounts of ac power could be controlled by means of relatively small amounts of dc power. Magnetic amplifiers enjoyed a brief spell of popularity in the late 1950's and the early 1960's. They were superceded by the developement of transistors which could handle large amounts of power.

Answer (5.1).

(a) Inside the solenoid H= NI = 10^5I Amps/meter. When the **B** field is less than 0.30 Teslas the relative permeability is given by μ_r= 10^3, so that

$$\mathrm{B}=\mu \mathrm{H}=\mu_{\mathrm{r}} \mu_0 \mathrm{NI}=125.7 \text{ I Teslas.}$$

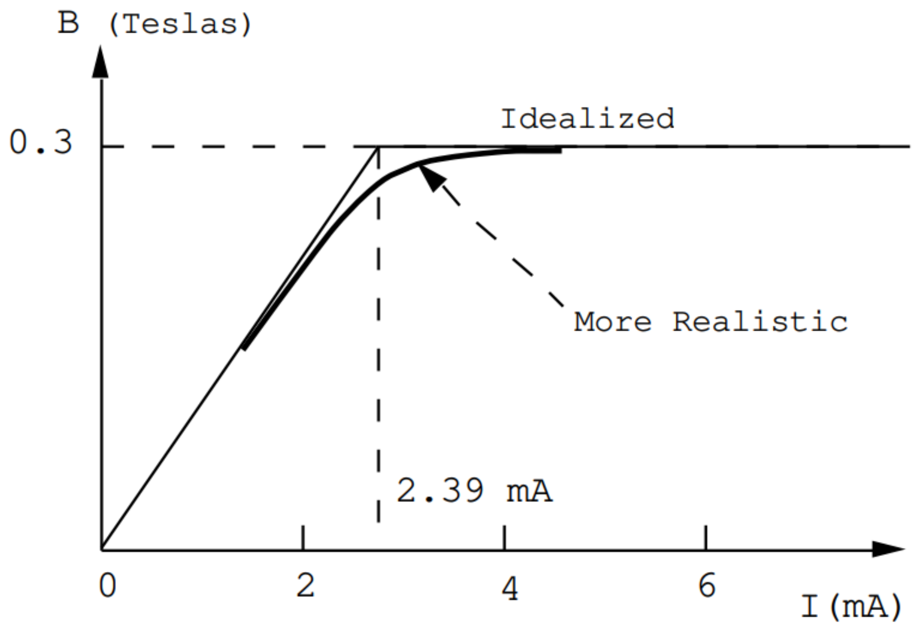
The current required to saturate the core is

$$I=\frac{0.30}{125.7}=\mathbf{2.39} \times \mathbf{10^{-3} Amps} \ \mathbf{= 2.39 \ mAmps}.$$

Upon saturation, the **B**-field increases only very slowly with the current because **M** remains fixed at the saturation value:

$$\mathrm{B}=\mu_0 \mathrm{M}+\mu_0 \mathrm{H} \cong 0.3 \text{ Teslas}$$

since μ_0H is relatively small. At I= 0.1 Amps (~40x the current required to saturate the core) μ_0H= 0.013 Teslas, an increase of only 3% in **B**.



(b) For a small current μ_r= 1000 and so for an amplitude of 1 mA the field varies as

$$B=0.126 \sin \omega t \text{ Teslas.}$$

The flux through the secondary coil is given by

$$\phi=\left(10^{3}\right)\left(\pi \mathrm{R}^{2}\right)(\mathrm{B})=0.158 \sin \omega t \text{ Webers.}$$

$$e=-\frac{d \phi}{d t}=59.6 \cos \omega t \text{ Volts,}$$

since ω= 2π(60) = 377 radians/sec.

(c) The dc current of 10 mAmps would bias the core of the solenoid into the region where the relative permeability is only μ_r= 1.0. The voltage induced in the secondary coil would decrease by a factor of 1000: the output signal would fall to ~60 mV from its initial value of ~60 Volts.

Problem (5.2)

A long straight thin wire carries a current of 5 Amps; it runs parallel with the interface between vacuum and a superconducting plane for which the relative permeability is μ_r= 0. Calculate the force on the wire due to its image if the wire is a distance z=1 cm from the plane. In a superconductor the field **B** is zero.

Answer (5.2)

The image current I' has the same magnitude as the driving current I, but is opposite in sign, and is located z from the interface, but in the superconductor;

I' = - I.

The current plus its image generate the magnetic field in the region outside the superconductor. The normal component of **B** is zero at the superconducting surface as is required by div **B**=0 plus the condition **B**=0 in the superconductor. The component of **B** or of **H** parallel with the interface does not matter since surface currents flow in the superconductor to shield its interior so that **H**=0.

The field generated by I' at the wire carrying the current I is given by

$$|\mathbf{H}|=\frac{\left|I^{\prime}\right|}{2 \pi(2 z)}=\frac{I}{4 \pi z}.$$

The force on the wire per unit length is given by

$$F=\frac{\mu_0 I^2}{4\pi z}=\frac{(25)\left(4\pi\times 10^{-7}\right)}{(4\pi)\left(10^{-2}\right)}=25\times 10^{-5}\text{Newtons / m}\,.$$

The direction of the force is such as to repell the wire from the interface. The above force is sufficient to lift a weight of approximately 25 miligrams per meter. This is pretty feeble; however, the force increases with the square of the current so that for 500 Amps the force would support ~0.25 kg/meter.

Problem (5.3)

A permanent magnetic dipole, **m**, is brought up to the plane interface between vacuum, μ_r=1, and a superconductor, μ_r=0. The dipole is located a distance z in front of the interface.

(a) Show that the image magnetic charge induced in the superconductor by the magnetic charge q_m a distance z in front of the interface is equal to q_m and is located a distance z behind the interface. The image charge is required in order to satisfy the condition div**B**=0 and also the condition **B**=0 in the superconductor.

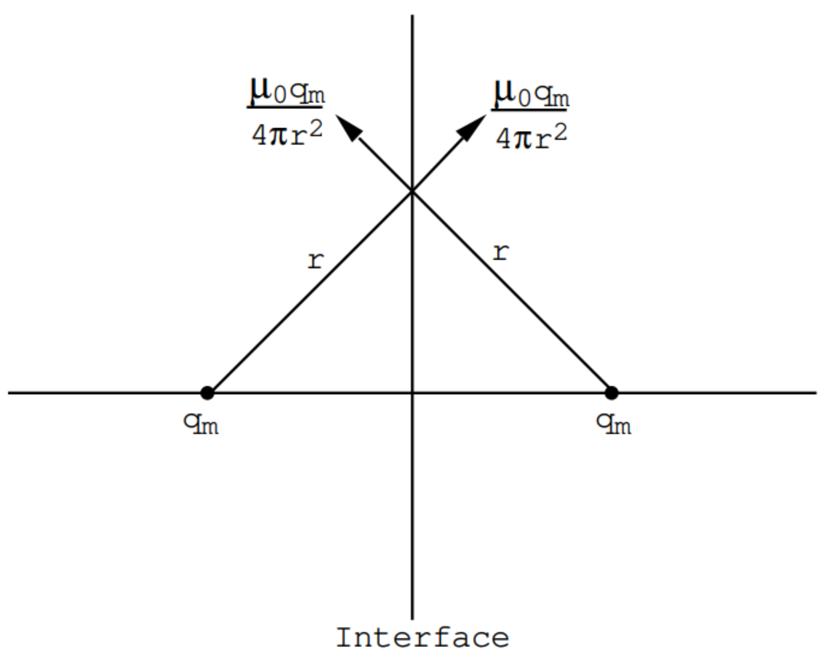
(b) Use the results of part (a) in order to calculate the force exerted on a magnetic dipole by its image when the dipole is oriented parallel with the interface.

(c) Calculate the force on the dipole when it is oriented normal to the interface.

(d) Given 1 cc of permanently magnetized material, estimate the heighth at which it would float above a superconducting plane. Let the density of the material be 4.5 gm/cc, and let its magnetization density be M= 1.59x10^5 Amps/meter (these parameters are appropriate for Barium ferrite BaO.6Fe_2O_3 - this is a common ferromagnetic insulator called Ferroxdure).

Answer (5.3)

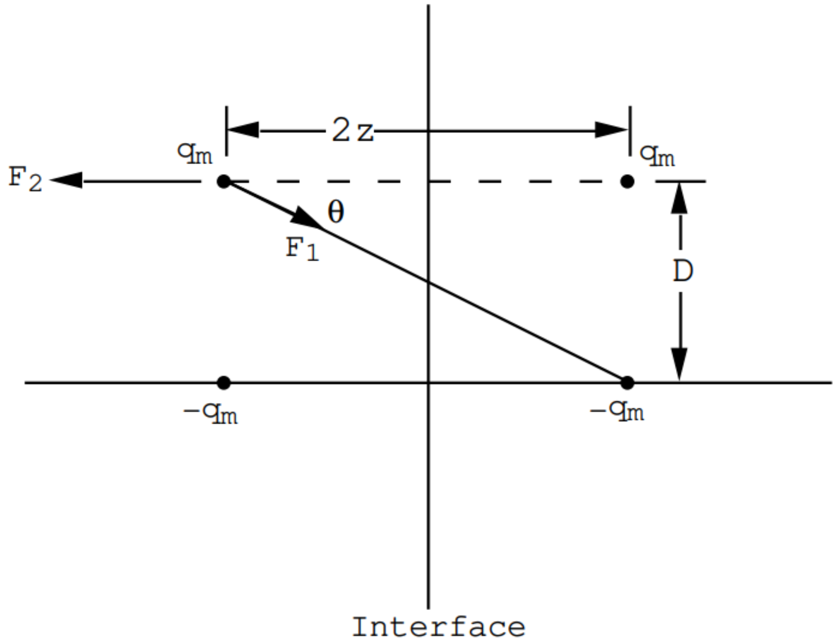
The field in the vacuum which is generated by a point magnetic charge and its image must be such that the normal component of **B** vanishes on the superconducting surface (see the figure).



In this way the conditions div**B**=0 and **B**=0 can both be satisfied. The tangential component of **B** need not be zero; surface currents flow in a very thin surface layer (~ 10^{-8} meters thick) which shield the interior of the superconductor.

(b) Dipole moment parallel with the Interface.

The component of force normal to the surface is the same for each charge on the dipole, therefore there is no torque acting on the dipole. The two forces acting on a given charge are:



an attractive force

$$F_1 = \frac{\mu_0 q_m^2}{4\pi} \frac{\cos\theta}{\left(4z^2 + D^2\right)},$$

and a repulsive force

$$F_2 = \frac{\mu_0 q_m^2}{4\pi} \frac{1}{4z^2}.$$

The net repulsive force on the dipole is given by

$$F = \frac{2\mu_0 q_m^2}{4\pi} \left(\frac{1}{4z^2} - \frac{2z}{\left(4z^2 + D^2\right)^{3/2}} \right),$$

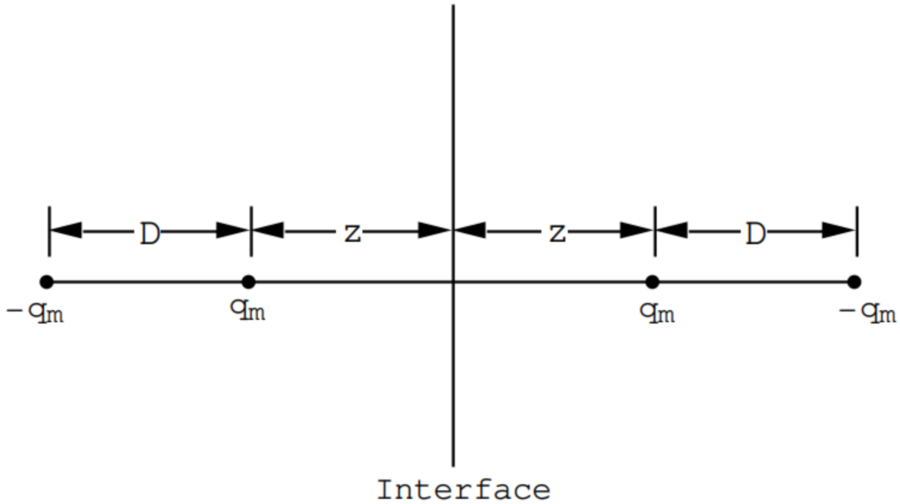
or

$$F = \frac{2\mu_0 q_m^2}{4\pi} \left(\frac{3D^2}{32z^4} \right).$$

But m= q_mD so that the repulsive force can be written

$$\mathbf{F} = \frac{\mu_0}{4\pi} \frac{\mathbf{3m^2}}{\mathbf{16z^4}}.$$

(c) Dipole moment normal to the Interface.



The net repulsive force on the dipole is given by

$$F=\frac{\mu_0 q_m^2}{4\pi}\left(\frac{1}{4z^2}+\frac{1}{4z^2(1+D/z)^2}-\frac{2}{4z^2(1+D/2z)^2}\right).$$

Now $\left(1+\frac{D}{z}\right)^{-2}=1-\frac{2D}{z}+3\left(\frac{D}{z}\right)^2+\ldots$

and $\left(1+\frac{D}{2z}\right)^{-2}=1-\frac{D}{z}+\frac{3}{4}\left(\frac{D}{z}\right)^2+\ldots$

so that

$$F=\frac{\mu_0}{4\pi}\frac{q_m^2}{4z^2}\left(\frac{3}{2}\frac{D^2}{z^2}\right)=\frac{\mu_0}{4\pi}\frac{\mathbf{3m^2}}{\mathbf{8z^4}}$$

This repulsive force is twice as large as the repulsive force when the dipole is oriented so that it is parallel with the interface.

(d) The weight of 1 cc of Ferroxdure is 4.5 gm or 4.5x10^{-3} kg. The gravitational force is F_g= 4.41x10^{-2} Newtons. The total magnetic moment for this piece of material is

$$\mathrm{m}=\mathrm{MV}=\left(1.59\times10^5\right)\left(10^{-6}\right)=0.159\mathrm{Amp}\cdot\mathrm{m}^2.$$

The repulsive force when the moment is parallel with the plane (the stable configuration) will be

$$F=\left(10^{-7}\right)\left(\frac{3}{16}\right)\frac{\left(2.53\times10^{-2}\right)}{z^4}=4.41\times10^{-2}\mathrm{N}.$$

or

$$\mathbf{z=1.02\times10^{-2}meters.}$$

The magnet would float approximately 1 cm above the superconducting plane.

Problem (5.4)

A short solenoid is constructed of 100 turns wound evenly on a cylindrical form. The length of the windings is L= 10 cm, and the mean radius of the coil is R= 5 cm. Find an expansion in Legendre polynomials for the magnetic potential in the interior of the solenoid, and estimate the radius of the region around the solenoid center within which the field is uniform to better than 1%. In the expansion of the field along the solenoid axis you may discard terms of order z^4 and higher powers.

Answer (5.4)

The field along the axis of a short solenoid is given by

$$B_z = \frac{\mu_0 NI}{2}\left(\frac{(L/2+z)}{\sqrt{R^2+\left(\frac{L}{2}+z\right)^2}}+\frac{(L/2-z)}{\sqrt{R^2+\left(\frac{L}{2}-z\right)^2}}\right), \quad \text{eqn } (3.2.10).$$

Note that the term in the brackets is a dimensionless number so that one can use z,R,L measured in cm rather than meters. For our case N= 1000 turns/meter, so that for a current of 1 Amp

$B_0 = \frac{\mu_0 NI}{2} = 6.283\times 10^{-4}\text{Teslas}$,

and

$$B_z(z) = B_0\left(\frac{(5+z)}{\sqrt{(z+5)^2+25}}+\frac{(5-z)}{\sqrt{(z-5)^2+25}}\right),$$

where z is measured in cm. The idea is to expand this function in powers of z.

At z=0 $B_z(0) = B_0\sqrt{2}$

$\left.\frac{dB_z}{dz}\right|_0 = 0$

$\left.\frac{d^2B_z}{dz^2}\right|_0 = -\frac{3B_0\sqrt{2}}{100} = -0.04243B_0$

$\left.\frac{d^3B_z}{dz^3}\right|_0 = 0$

$\left.\frac{d^4B_z}{dz^4}\right|_0 = -\frac{3B_0}{1000\sqrt{2}} = -0.0021213B_0$

etc.

But

$$B_z(z) = B_0\sqrt{2}+\frac{1}{2}\left.\frac{d^2B_2}{dz^2}\right|_0 z^2+\frac{1}{24}\left.\frac{d^4B_2}{dz^4}\right|_0 z^4+\cdots$$

Thus

$$B_z(z) = B_0\sqrt{2}\left(1-\frac{3z^2}{200}+\frac{z^4}{16000}\cdot+\ldots O\left(z^6\right)\right)$$

This field can be obtained from a potential function

$$B_z = -\frac{\partial V}{\partial z},$$

where

$$V(z) = -B_0\sqrt{2}\left(z-\frac{z^3}{200}+O\left(z^5\right)\right). \qquad (1)$$

V(z) must satisfy $\nabla^2 V=0$, therefore

$$V(z) = \sum_{n=1}^{\infty} a_n r^n P_n(\cos\theta);$$

the terms in $\frac{1}{r^{n+1}}$ must be omitted because they blow up at r=0.

$$V(r,\theta) = a_1 r\cos\theta+\frac{a_2r^2}{4}\quad(1+3\cos 2\theta)+\frac{a_3r^3}{8}(3\cos\theta+5\cos 3\theta)+\frac{a_4r^4}{64}(9+20\cos 2\theta+35\cos 4\theta)+$$

At θ=0 the radius r becomes equal to the cylindrical co-ordinate z, and since Cosθ=1 this series becomes

$$V(z) = a_1z+a_2z^2+a_3z^3+a_4z^4+\ldots \qquad (2)$$

from which by comparison with eqn.(1) one finds

$$a_1 = -B_0\sqrt{2}$$
$$a_2 = 0$$
$$a_3 = \frac{B_0\sqrt{2}}{200}$$
$$a_4 = 0,$$

and so on. Thus to terms of order z^5 one has

$$\mathbf{V}(\mathbf{r},\theta) = -\mathbf{B}_0\sqrt{\mathbf{2}}\left(\mathrm{r}\cos\theta - \frac{\mathbf{r}^3}{1600}(3\cos\theta + 5\cos 3\theta)\right.$$

$$B_r = -\frac{\partial V}{\partial r} = B_0\sqrt{2}\left(\cos\theta - \frac{3r^2}{1600}(3\cos\theta + 5\cos 3\theta)\right),$$

$$B_\theta = -\frac{1\partial V}{r\partial\theta} = B_0\sqrt{2}\left(-\sin\theta + \frac{3r^2}{1600}\quad(\sin\theta + 5\sin 3\theta)\right).$$

The first two terms, i.e. $B_r = B_0\sqrt{2}\cos\theta$ and $B_\theta = -B_0\sqrt{2}\sin\theta$, correspond to a uniform field $B_z = B_0\sqrt{2}$ Teslas. The correction to the axial field component, B_z, is given by

$$\frac{\mathbf{\Delta B_z}}{\mathbf{B_0\sqrt{2}}} = -\frac{\mathbf{3r^2}}{\mathbf{1600}}\left(\mathbf{1} + \mathbf{2}\cos^2\theta + \mathbf{5}\cos\theta\cos\mathbf{3}\theta + \mathbf{5}\sin\theta\sin\mathbf{3}\theta\right)$$

since $B_z = B_r \mathrm{Cos}\theta - B_\theta \mathrm{Sin}\,\theta$. The term in the brackets varies between -4 and +8.

The correction to the transverse magnetic field component, B_ρ, where $B_p = B_r \mathrm{Sin}\,\theta + B_\theta\cos\theta$ is given by

$$\frac{\mathbf{B}_\rho}{\mathbf{B_0\sqrt{2}}} = -\frac{\mathbf{3r^2}}{\mathbf{1600}}(\mathbf{2}\sin\theta\cos\theta + \mathbf{5}\sin\theta\cos\mathbf{3}\theta - \mathbf{5}\cos\theta\sin\mathbf{3}\theta).$$

The term in the brackets varies from -4 to +4. It is clear from these expressions that the deviations ΔB_z,Bρ will be less than $B_0\sqrt{2}\left(\frac{3r^2}{200}\right)$ for all angles. The field will be uniform to better than 1% within a sphere of radius r= 0.816 cm, and uniform to better than 10% within a sphere of radius r= 2.582 cm around the center of the solenoid.

Problem (5.5).

A magnetic shield is made of a permeable material in the form of a long cylinder having an inner radius R_1 and an outer radius R_2. The relative permeability of the cylinder material is μ_r. If this shield is placed in a uniform magnetic field, B_0, that is directed transverse to the cylinder axis what will be the field inside the cylinder? You may treat the cylinder as if it were infinitely long. Inside and outside the cylinder the relative permeability is μ_r=1.

Answer (5.5)

There are clearly three regions involved in this problem:

(1) the region inside the cylinder, μ_r=1;

(2) the region inside the cylinder walls, μ_r;

(3) the region outside the cylinder, μ_r=1.

In each of these three regions ∇^2V=0, where **H**= - gradV. Therefore, in each region the potential can be expanded in a series of the form

$$V(r,\theta) = \sum_{n=1}^{\infty}\left(a_n r^n + \frac{b_n}{r^n}\right)\cos n\theta.$$

It proves not to be necessary to use terms for n>1. Inside the cylinder: V_1 = arCosθ

In the cylinder walls: $V_2 = a_0 r\cos\theta + \frac{b_0\cos\theta}{r}$,

one must use both terms because neither term becomes singular in the cylinder walls;

Outside the cylinder: \(V_{3}=-H_{0} r \cos \theta+\frac{b C o s \theta}{r}).

Boundary Conditions.

At $r=R_1$ (the inner wall)

$$V_1 = V_2$$

$$\frac{dV_1}{dr} = \mu_r \frac{\partial V_2}{\partial r}$$

or

$$a = a_0 + \frac{b_0}{R_1^2} \qquad (1)$$

$$\mathrm{a} = \mu_\mathrm{r} \mathrm{a}_0 - \frac{\mu_\mathrm{r} \mathrm{b}_0}{\mathrm{R}_1^2} \qquad (2)$$

At $r=R_2$ (the outer wall)

$$\mathrm{V}_2 = \mathrm{V}_3$$

$$\mu_r \frac{\partial V_2}{dr} = \frac{\partial V_3}{dr}$$

or

$$a_0 + \frac{b_0}{R_2^2} = -H_0 + \frac{b}{R_2^2} \qquad (3)$$

$$\mu_r a_0 - \frac{\mu_r b_0}{R_2^2} = -H_0 - \frac{b}{R_2^2} \qquad (4)$$

These 4 equations can be solved for the 4 unknowns a, a_0, b_0, and b. The result is

$$a = \frac{-4\mu_r H_0}{\left((\mu_r + 1)^2 - (\mu_r - 1)^2 \left(\frac{R_1}{R_2} \right)^2 \right)}.$$

The ratio of the field inside the cylinder to the field outside the cylinder is given by

$$\frac{B_{in}}{B_{out}} = \frac{4\mu_r}{(\mu_r + 1)^2 - (\mu_r - 1)^2 \left(\frac{R_1}{R_2} \right)^2)},$$

or

$$\mathbf{\frac{B_{in}}{B_{out}} \cong \frac{4}{\mu_r \left(1 - \left(\frac{R_1}{R_2} \right)^2 \right)}}.$$

The relative permeability for Supermalloy is $\mu_r \sim 10^5$ for B< 0.7 Teslas. In a typical application for shielding a photomultiplier tube one would have R_1= 2.5 cm and R_2= 2.6 cm or $\frac{R_1}{R_2} = 0.962$. For such a case

$$\mathbf{\frac{B_{in}}{B_{0ut}} = \frac{4 \times 10^{-5}}{0.0754} = 5.3 \times 10^{-4}}.$$

Problem (5.6)

A magnetic shield is constructed of a permeable material in the form of a long cylinder of length L and having an inner radius R_1 and an outer radius R_2. The relative permeability of the shield material is μ_r. Let this shield be placed in a field B_0 parallel with the cylinder axis. Estimate the field at the center of the shield if L is much greater than the radii R_1 and R_2.

This problem cannot be easily solved in closed form; however, one can argue as follows:

(1) Most of the field inside the cylinder will be sucked into the permeable material of the cylinder walls. One can estimate the strength of B inside the cylinder wall from conservation of flux.

(2) Assuming that B is constant within the cylinder walls then the magnetization M will also be uniform. The discontinuities in M at the cylinder ends will act as field sources. These sources can be used to estimate the field at the center of the cylinder. As a crude first approximation one can assume that all of the magnetic charges on the cylinder ends are the same distance from its center because L>>R.

Answer (5.6)

The region inside the cylinder originally contained the flux $\phi = \mathrm{B_0}\pi\mathrm{R_1^2}$. This flux becomes concentrated in the cylinder wall. The resulting B-field in the wall must be such that

$$\mathrm{B_W}\pi\left(\mathrm{R_2^2}-\mathrm{R_1^2}\right)=\mathrm{B_0}\pi\mathrm{R_1^2}.$$

Therefore $\mathrm{B_W}=\frac{\mathrm{B_0}}{\left(\left(\frac{\mathrm{R_2}}{\mathrm{R_1}}\right)^2-1\right)}$.

But $\mathrm{B_W}=\mu_0(\mathrm{H}+\mathrm{M})\cong\mu_0\mathrm{M}$

since for a very permeable material $\mathrm{H}=\frac{\mathrm{B}}{\mu_0\mu_r}\sim 0$.

Consequently, $\mathrm{M}=\frac{\mathrm{B_W}}{\mu_0}=\frac{\mathrm{B_0}/\mu_0}{\left(\left(\frac{\mathrm{R_2}}{\mathrm{R_1}}\right)^2-1\right)}$, Amps/m.

The discontinuities in M give two rings of magnetic charge, each of average radius R= $(R_1+R_2)/2$, and of total strength

$$Q=M\pi\left(R_2^2-R_1^2\right)=M\pi R_1^2\left(\left(\frac{R_2}{R_1}\right)^2-1\right).$$

If the length of the cylinder is much longer than its radii, then crudely speaking, the field at the center of the cylinder must be given approximately by two point charges, +Q and -Q, located a distance L/2 from the center of the cylinder: thus

$$\mathrm{H}\sim\frac{2\mathrm{Q}}{4\pi(\mathrm{L}/2)^2}=\frac{2\mathrm{B_0R_1^2}}{\mu_0\mathrm{L^2}},$$

or

$$\boldsymbol{\frac{B}{B_0}\cong 2\left(\frac{R_1}{L}\right)^2}. \tag{13.5.1}$$

Notice that this expression is independent of μ_r, but μ_r must be large enough so that H inside the shield material can be neglected when compared with M. In order to obtain effective shielding the field B_w in the cylinder walls must be less than the saturation field. For an iron based shielding material the field B at saturation is typically ~ 1 Tesla. If the driving field B_0 is the earth's magnetic field, ~10^{-4} Tesla, the ratio of the inner radius R_1 to the shield thickness d must be less than 5000. This condition is easily met since for typical values R_1= 2.5 cm and R_2= 2.6 cm the ratio $\frac{R_1}{d}=25$.

Problem (5.7)

A solenoid is constructed of N=100 turns of wire. The mean diameter of the windings is D=5 cm and the length of the windings is L= 10 cm. This coil is to be used to generate a field of 10 Tesla in vacuum. In the following calculations the coil may be approximated as an infinitely long solenoid.

(a) What current would be required to generate a field of 10 Teslas?

(b) Estimate the magnetic force acting to change the length of the solenoid windings. Do these forces tend to lengthen or to shorten the windings?

(c) Estimate the tension in the wire of which the coil is wound.

Answer (5.7)

(a) For a long solenoid in vacuum $\mathrm{B}=\frac{\mu_0 \mathrm{NI}}{\mathrm{L}}$, where N= 100 is the total number of turns, and L is the length. For the present example, L= 0.10 m and

$$\mathrm{B}=\frac{\left(4 \pi \times 10^{-7}\right)\left(10^2\right) I}{0.1}=4 \pi \times 10^{-4} \quad \mathrm{I} \quad \text{Teslas.}$$

In order to generate 10 Teslas the current required is

$$\mathbf{I}=\mathbf{7.96} \times \mathbf{10^3} \textbf{ Amps.}$$

(b) The energy stored in the solenoid is approximately given by

$$\mathrm{U_B}=\mathrm{V} \frac{\mathrm{B}^2}{2 \mu_0}=\left(\frac{\pi \mathrm{D}^2}{8}\right) \frac{\mu_0 \mathrm{N}^2 \mathrm{I}^2}{\mathrm{L}}, \text{ Joules} .$$

Therefore $\frac{\partial \mathrm{U_B}}{\partial \mathrm{L}}=-\left(\frac{\pi \mathrm{D}^2}{8}\right) \frac{\mu_0 \mathrm{N}^2 \mathrm{I}^2}{\mathrm{L}^2}=-\frac{\pi \mathrm{D}^2}{8 \mu_0} \mathrm{B}^2$.

The solenoid will tend to contract along its length. The force on the windings is given by

$$F=\left|\frac{\partial U_B}{\partial L}\right|=\frac{(\pi)(25)\left(10^{-4}\right)\left(10^2\right)}{\left(32 \pi \times 10^{-7}\right)}=7.8 \times 10^4 \text{ Newtons} .$$

This force would suspend a weight of 8000 kg! The turns of the solenoid must be very securely held in place.

(c) The field B is independent of the solenoid diameter. One can write

$$\mathrm{U_B}=\frac{\pi \mathrm{D}^2 \mathrm{L}}{8 \mu_0} \mathrm{B}^2,$$

so that

$$\frac{\partial \mathrm{U_B}}{\partial \mathrm{D}}=\frac{\pi \mathrm{DL}}{4 \mu_0} \mathrm{B}^2 .$$

If the mean diameter increases by dD the length of the solenoid wire increases by dS= Nπ dD, therefore

$$\frac{\partial \mathrm{U_B}}{\partial \mathrm{S}}=\frac{1}{\mathrm{N} \pi} \frac{\partial \mathrm{U_B}}{\partial \mathrm{D}}=\frac{\mathrm{DLB}^2}{4 \mathrm{N} \mu_0} .$$

The tension on the wire will be given by

$$F=\frac{\left(50 \times 10^{-4}\right)\left(10^2\right)}{(4)(4 \pi)\left(10^2\right)\left(10^{-7}\right)}=0.995 \times 10^3 \text{ Newtons} .$$

This force is approximately the equivalent of a 100 kg weight.

Problem (5.8)

A rigid loop of wire has the form of a triangle, The base of the triangle is 5 cm long and the height of the triangle is 5 cm. This object is placed in a uniform magnetic field of B= 1 Tesla such that its area embraces no flux. What will be the torque on the triangle if it carries a current of 1 Amp?

Answer (5.8).

The magnetic energy of the system contains three terms:

$$\mathrm{U_B}=\frac{1}{2} \mathrm{L}_{11} \mathrm{I}_1^2+\frac{1}{2} \mathrm{L}_{22} \mathrm{I}_2^2+\mathrm{L}_{12} \mathrm{I}_1 \mathrm{I}_2 .$$

The first two terms are the self-energy of the sources of the uniform field and the self-energy of the triangle: these terms do not change when the triangle is rotated. The last term is dependent on the angle between the plane of the triangle and the applied magnetic field. The flux through the triangle is given by

$$\phi_2=\mathrm{BASin} \theta=(\mathrm{KASin} \theta) \quad I_1$$

where I_1 is the current associated with the source field B. This expression gives the mutual inductance coefficient L_{12}:

$$L_{12} = KASin\theta,$$

and

$$\frac{\partial U_B}{\partial\theta} = I_1 I_2 \quad KACos\theta = (I_2 BA)\,Cos\,\theta.$$

The torque on the triangle is just

$$\tau = \frac{\partial U_B}{\partial\theta} = \mathbf{I_2 BA}\cos\theta = \mathbf{25 \times 10^{-4}\ Newton\ meters\ at\ } \theta = \mathbf{0.}$$

Problem (5.9)

A charged particle moves in a uniform magnetic field B which changes slowly with time. (Slowly here means that the rate of change is slow compared with the cyclotron frequency).

(a) Show that the radius R of the particle orbit must change in such a way that

$$BdR = -\frac{RdB}{2}.$$

This change in radius is the consequence of the changing magnetic field that creates an electric field that exerts a force on the particle.

(b) Show that the change in radius of part (a) corresponds to a change in the orbit area in such a way as to keep the flux through the orbit constant.

(c) Show that the orbital magnetic moment associated with the particle motion remains constant as the field changes.

Answer (5.9)

(a) The force on a charged particle in a magnetic field is given by qvB where v is the transverse component of velocity. One can ignore any motion along the magnetic field for this problem. From mechanics, and for a particle of mass m,

$$\frac{mv^2}{R} = qvB,$$

so that $v = \frac{q}{m}BR$. (1)

If B changes with time there is induced an electric field since curl $\mathbf{E} = -\frac{d\mathbf{B}}{\partial t}$. In cylindrical polar co-ordinates there will be only a component E_θ because the field is uniform and has only a z-component:

$$\frac{1}{r}\frac{d}{dr}(rE\theta) = -\frac{\partial B_z}{\partial t}.$$

But since B_z is independent of position $E_\theta = -\frac{R}{2}\left(\frac{\partial B_z}{\partial t}\right)$ along the particle orbit. A moments thought will reveal that the direction of E_θ is such as to cause the particle velocity to increase, therefore

$$m\frac{dv}{dt} = q\,|E_\theta| = \frac{qR}{2}\frac{dB}{dt},$$

or

$$dv = \left(\frac{qR}{2m}\right)dB. \qquad (2)$$

However, from (1) $dv = \left(\frac{q}{m}\right) \quad (BdR + RdB)$

so that from (2) $\frac{qR}{2m}dB = \frac{qB}{m}dR + \frac{qR}{m}dB$

and so $\frac{qB}{m}dR = -\frac{qR}{2m}dB$

or $BdR = -\frac{\mathbf{RdB}}{\mathbf{2}}$ (3)

(b) The flux through the particle orbit is

$$\phi=\pi \mathrm{R}^2 \mathrm{B}.$$

$$\mathrm{d}\phi=\pi \quad \left(2 \mathrm{RBdR}+\mathrm{R}^2 \mathrm{dB}\right).$$

But from (3) above

$$\mathrm{d}\phi=\pi\left(-\mathrm{R}^2 \mathrm{dB}+\mathrm{R}^2 \mathrm{dB}\right)=0.$$

In other words, the flux through the orbit is conserved.

(c) The magnetic moment associated with the orbit is given by

$$\mathrm{m}_{\mathrm{z}}=\left(\pi \mathrm{R}^2\right) \mathrm{I},$$

where the current I is given by $I=\frac{qV}{2\pi R}$. Thus

$$\mathrm{m}_{\mathrm{z}}=\left(\pi \mathrm{R}^2\right)\left(\frac{\mathrm{qv}}{2 \pi \mathrm{R}}\right)=\frac{\mathrm{qvR}}{2}.$$

Using eqn.(1) this can be written $\boldsymbol{m_z=\left(\frac{qR}{2}\right)\left(\frac{q}{m}\right)(BR)=\frac{q^2}{2\pi m}\phi}$.

Since the flux is conserved so is the magnetic moment.

Problem (5.10)

An electron in an atomic n=1 state can be described by the wave function

$$\psi=\frac{2}{\sqrt{4\pi}}\left(\frac{z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

where $a_0=\frac{h^2}{4\pi^2 me^2}=0.53\times 10^{-10}\text{meters}$;

Z is the nuclear charge. The above wave function corresponds to an s-state which possesses zero angular momentum. The electron, however, carries a spin magnetic moment of 1 Bohr magneton μ_B= 9.27x10^{-24} Joules/Tesla oriented along the z direction. If this magnetic moment is smeared out over the above charge distribution it corresponds to a magnetization density

$$\mathrm{M}_{\mathrm{z}}(\mathrm{r})=\mu_{\mathrm{B}} \psi \psi^{\star}.$$

(a) Calculate the effective current density $\mathbf{J}_M$= curl**M** caused by the spatial variation of the above magnetization density. Use spherical co-ordinates. (Hint: do not be in a rush to evaluate M_z(r) in terms of ψψ*).

(b) Show that the magnetic field at the nucleus, i.e. at r=0, due to the spatial variation of the above magnetization density is given by

$$B_z(0)=\left(\frac{2\mu_0}{3}\right) M_z(0).$$

This field at the nucleus is responsible for the hyperfine coupling between the nuclear spin and the electron spin.

(c) Evaluate the hyperfine field at the nucleus of a hydrogen atom, Z=1.

Answer (5.10)

(a) The magnetization density in spherical polar coordinates is given by

$$\begin{aligned}\mathrm{M}_{\mathrm{r}} &=\mathrm{M}_{\mathrm{z}} \cos\theta \\ \mathrm{M}_\theta &=-\mathrm{M}_{\mathrm{z}} \sin\theta\end{aligned}.$$

Neither of these components depends upon the angle φ:

$$\begin{aligned}&\operatorname{cur}\ln)_r=0\\ &\operatorname{cur}\ln)\theta=0\\ &\operatorname{cur}\ln)\phi=\frac{1}{r}\left(\frac{\partial rM\theta}{\partial r}-\frac{\partial M_r}{\partial\theta}\right)=J_\phi\end{aligned}$$

or

$$J_{\phi}=\frac{1}{r}M_{\theta}+\frac{\partial M_{\theta}}{\partial r}-\frac{1}{r}\frac{\partial M_{r}}{\partial\theta}.$$

But

$$\begin{gathered}
\mathrm{M}_{\theta}=-\mathrm{M}_{\mathrm{z}}\sin\theta \\
\frac{\partial\mathrm{M}_{\theta}}{\partial\mathrm{r}}=-\sin\theta\left(\frac{\partial\mathrm{M}_{\mathrm{z}}}{\partial\mathrm{r}}\right) \\
\mathrm{M}_{\mathrm{r}}=\mathrm{M}_{\mathrm{Z}}\cos\theta \\
\frac{\partial\mathrm{M}_{\mathrm{r}}}{\partial\theta}=-\mathrm{M}_{\mathrm{z}}\sin\theta,
\end{gathered}$$

and so

$$\mathrm{J}_{\phi}=-\sin\theta\left(\frac{\partial\mathrm{M}_{z}}{\partial r}\right).$$

From the law of Biot-Savard one has:

$$\mathrm{dB}_{\mathrm{z}}=\frac{\mu_{0}}{4\pi}\frac{\mathrm{J}_{\phi}\mathrm{d}\tau}{\mathrm{r}^{2}}\sin\theta$$

where $\mathrm{d}\tau=\left(r^{2}\mathrm{dr}\right)(\mathrm{Sin}\,\theta\mathrm{d}\theta)(\mathrm{d}\phi)$.

Inserting the expression for J_{φ} one obtains

$$\mathrm{dB}_{\mathrm{z}}=-\left(\frac{\mu_{0}}{4\pi}\right)\left(\frac{\partial\mathrm{M}_{\mathrm{z}}}{\partial\mathrm{r}}\right)\mathrm{dr}\sin^{3}\theta\mathrm{d}\theta\mathrm{d}\phi.$$

The integrals over θ,φ give $\frac{8\pi}{3}$, and the integral over r simply gives the value of the magnetization density at r=0:

$$B_{z}(0)=\frac{2\mu_{0}}{3}M_{z}(0).$$

The field at the nucleus is given by

$$\mathbf{B_{z}(0)=\frac{2\mu_{0}\mu_{B}}{3}\frac{1}{\pi}\left(\frac{z}{a_{0}}\right)^{3}}.$$

(b) When the expression for $B_z(0)$ is evaluated for Z=1 the result is

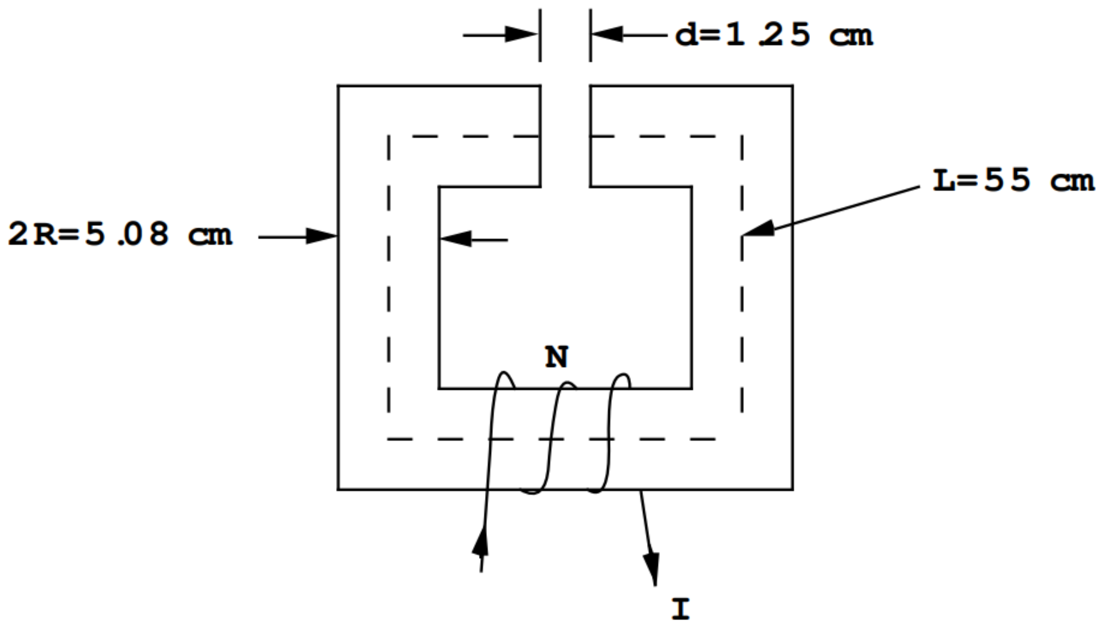
$$\mathbf{B_{z}(0)=\frac{(8)\left(10^{-7}\right)\left(9.27\times10^{-24}\right)}{(3)(0.53)^{3}\left(10^{-30}\right)}=16.6\ Teslas.}.$$

This is a very large magnetic field: a typical iron core laboratory magnet produces approximately 1 Tesla.

13.6: Chapter 6

Problem (6.1)

An electromagnet is constructed of a soft iron yoke, see the diagram. The yoke radius is R= 2.54 cm, and the gap is d= 1.25 cm. The distance from pole face to pole face along the dotted line is L= 55 cm. The number of turns on the coil is 1000 windings. Estimate the current required to generate a field of 1.0 Teslas at the center of the magnet gap. A field of 1.0 T in the iron yoke corresponds to a field H of 130 Amps/m.



Answer (6.1)

$\oint_C \mathbf{H}\cdot d\mathbf{L} = NI$

Therefore, 130L + d/μ_0 = NI.

or

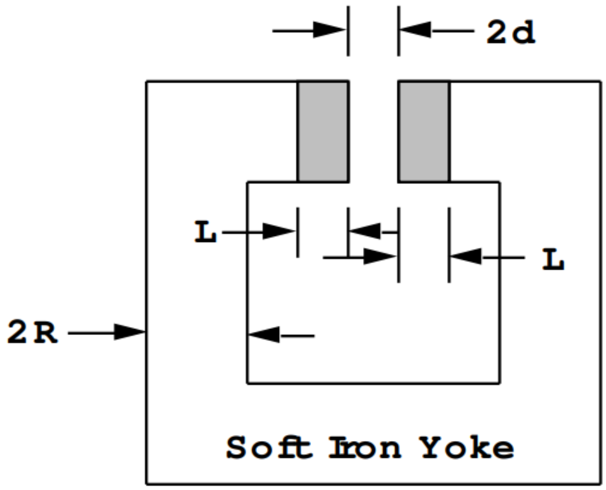
$(13)\,(0.55) + (0.0125)/\left(4\pi\times10^{-7}\right) = 10^3 I$,

and

I= 10.0 Amps.

Problem (6.2)

Modern permanent magnet materials such as FeNdB can be used to generate substantial magnetic fields. Consider the configuration shown in the diagram where the cross-hatched regions represent FeNdB permanent magnets.



The saturation magnetization density in each magnet is M_0= $0.8x10^6$ Amps/m., ie. B= $\mu_0 M_0$ = 1.01 Teslas. Let R=L= 1.0 cm., and let d= 1/2 cm.

Calculate the field B at the midpoint of the gap. The approximate effect of the iron yoke is to make each permanent magnet appear to be infinitely long due to the magnetization induced in the soft iron. In soft iron having a very large permeability the magnetization must be continuous at the iron-magnet interface because a discontinuity in **M** would produce an **H**- field which would produce a large **M** in the iron and as a result **B** would not be continuous.

Answer (6.2)

The field generated at the gap center can be approximated using a superposition argument. If there were no gap the field would be that due to an infinitely long solenoid having NI= M_0, ie. B= $\mu_0 M_0$.

The field in the gap,B_G, plus the field at the center due to a magnetized section 2d long must equal $\mu_0 M_0$. A section 2d long possessing a magnetization density M_0 produces a field at its center given by the short solenoid formula $B_s = \frac{\mu_0 M_0 d}{\sqrt{d^2+R^2}}$. Therefore

$$\mathrm{B_G + B_S} = \mu_0 \mathrm{M_0},$$

and

$$\mathrm{B_G} = \mu_0 \mathrm{M_0} \left(1 - \frac{\mathrm{d}}{\sqrt{\mathrm{d}^2 + \mathrm{R}^2}}\right).$$

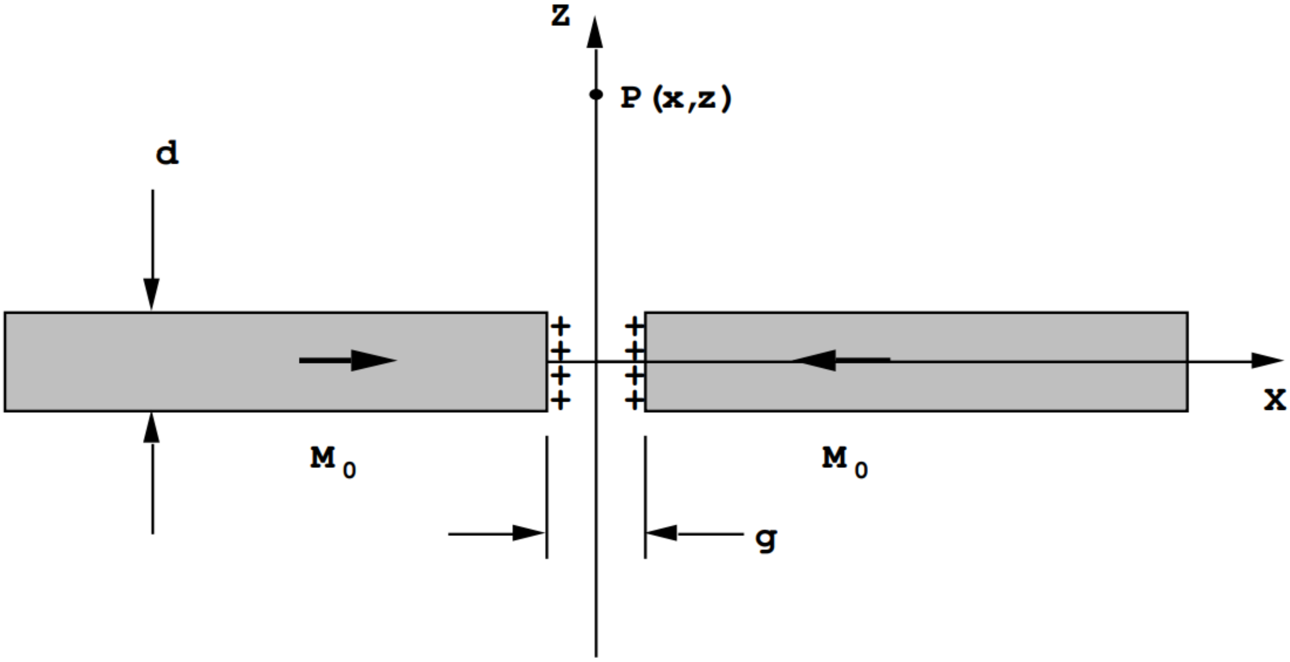
In this case d=1/2 cm and R=1 cm, so B_G= 0.558 Teslas.

R. Oldenbourg and W.C. Phillips,

(Rev.Sci.Instrum.57,2362(1986) and 57,3139(1986)), used 2d=1.9 cm and soft iron pole tips tapered to 0.35 cm in diameter to produce a field of 2 Teslas in a 0.2 cm gap.

Problem (6.3)

This problem has to do with the fields produced in the region between two magnetized bits on a hard disc, see the figure. The gap is g= 10^{-8} meters, the thickness is d= 10^{-8} meters, and the width of each magnetized region is w= $0.4x10^{-6}$ meters. the magnetization is M_0= $6.4x10^5$ Amps/m.



The end of each magnetized region bears a surface charge of M_0 per square meter due to the discontinuity in the magnetization density.

Calculate the field at P(x,o,z) on the centerline of the gap between the two magnetized regions due to the magnetic surface charges.

Answer (6.3)

On the centerline **H** has only a z-component by symmetry.

Let $\mathbf{r} = (g/2)\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z}$,

and $\mathbf{R} = Z\mathbf{u_z}$.

The vector from the element of magnetic charge $dq = dydzM_0$ to the point of observation is $\rho = \mathbf{R} - \mathbf{r} = -(g/2)\mathbf{u_X} - y\mathbf{u_Y} + (Z-z)\mathbf{u_z}$, and $|\rho| = \sqrt{(g/2)^2 + y^2 + (Z-z)^2}$.

For one of the end faces

$$H_z = \frac{M_0}{4\pi}\int_{-d/2}^{d/2} dz(z-z) \quad \int_{-W/2}^{W/2} dy \frac{1}{(y^2 + (g/2)^2 + (Z-z)^2)\, 3/2}.$$

Now

$$\int_{-w/2}^{w/2} \frac{dy}{(y^2+a^2)^{3/2}} = \frac{w}{a^2\sqrt{(w/2)^2+a^2}},$$

therefore

$$H_z = \frac{M_0 w}{4\pi}\int_{-d/2}^{d/2} dz \frac{(Z-z)}{((g/2)^2 + (Z-z)^2)\sqrt{(Z-z)^2 + (g/2)^2 + (w/2)^2}}$$

Let $v = (Z-z)^2 + (g/2)^2$, then the integral becomes

$$H_z = \frac{M_0 w}{8\pi}\int_{(Z-d/2)^2+(g/2)^2}^{(Z+d/2)^2+(g/2)^2} dv \quad \frac{1}{v\sqrt{v+(w/2)^2}}.$$

This is a standard integral:

$$H_z = \frac{M_0}{2\pi}\left\{\tanh^{-1}\left[\frac{2}{w}\sqrt{(w/2)^2+(g/2)^2+(Z-d/2)^2}\right] - \tanh^{-1}\left[\frac{2}{w}\sqrt{(w/2)^2+(g/2)^2+(z+d/2)^2}\right)\right]\right\}.$$

(This is for one face of the gap magnetic charge distribution- it must be multiplied by 2 to obtain the total field).

For $(w/2)=20\times10^{-8}$, $(g/2)=(1/2)\times10^{-8}$, and $(d/2)= (g/2)$, and if $Z= 1.0\times10^{-8}$ meters $B_z= \mu_0 H_z= 0.206$ Teslas. For the above parameters B_z is a maximum for $Z=0.71\times10^{-8}$ meters. The maximum value of $B_z= 0.224$ T. At $Z=2\times10^{-8}$m the field has dropped to $B_z= 0.121$ T.

13.7: Chapter 7

Problem (7.1)

In his original experiments on radio waves Hertz used two spheres approximately 0.5 meters in diameter and separated by approximately 0.5 meters. These spheres were charged to a potential difference of 2x10^5 Volts; as a result one sphere carried a charge of Q_1= +Q= 5.56x10^{-6} Coulombs, and the other sphere carried Q_2= -Q Coulombs. The two spheres were suddenly connected together electrically by means of a spark gap (ionized air is an excellent conductor), and the charge oscillated forth and back between the spheres at a frequency which was determined by the geometry but which was of the order of 100 MHz. You may model this system as a point electric dipole oscillating at 100 MHz, where the dipole amplitude is P_0= 2.78x10^{-6} Coulomb-meters.

a. Calculate and compare the terms in the expressions for the electric and magnetic fields generated by an oscillating electric dipole as measured at a point in the equatorial plane 1 meter from the dipole (θ= π/2).
b. Calculate and compare the terms in the expressions for the electric and magnetic fields generated by an oscillating electric dipole as measured at a point in the equatorial plane 1 km from the dipole.

Answer (7.1)

$$\mathrm{E_T}=\frac{1}{4\pi\varepsilon_0}2\cos\theta\left(\frac{\mathrm{P_z}}{\mathrm{R^3}}+\frac{\dot{\mathrm{P}}_\mathrm{z}}{\mathrm{CR^2}}\right),$$

$$\mathrm{E}_\theta=\frac{1}{4\pi\varepsilon_0}\sin\theta\left(\frac{\mathrm{P_z}}{\mathrm{R^3}}+\frac{\dot{\mathrm{P}}_\mathrm{z}}{\mathrm{CR^2}}+\frac{\ddot{\mathrm{P}}_\mathrm{z}}{\mathrm{c^2R}}\right),$$

$$\mathrm{E}_\phi=0,$$

$$\mathrm{CB}_\phi=\frac{1}{4\pi\varepsilon_0}\sin\theta\left(\frac{\dot{\mathrm{P}}_\mathrm{z}}{\mathrm{CR^2}}+\frac{\ddot{\mathrm{P}}_\mathrm{z}}{\mathrm{c^2R}}\right).$$

$$\mathrm{B}_\theta=\mathrm{B_r}=0,$$

At $\theta=\frac{\pi}{2}$ E_r= 0. For R= 1 meter, f= 10^8 Hz, ω= 6.28x10^8 radians/sec

(1) $\frac{P_z}{4\pi\varepsilon_0 R^3}=2.5\times10^4$ Volts/meter.

(2) $\frac{\dot{P}_z}{4\pi\varepsilon_0 \quad cR^2}=5.2i\times10^4 \quad$ Volts/meter

(3) $\frac{\ddot{P}_z}{4\pi\varepsilon_0 C^2R}=-1.1\times10^5$ Volts/meter

Even at R= 1 meter the field is dominated by the radiation term.

B_ϕ = - (3.7 - 1.7i) x 10^{-4} Teslas, i.e. approximately four times the earth's magnetic field.

(b) R= 1 km = 10^3 meters.

(1) $\frac{P_z}{4\pi\varepsilon_0 R^3}=2.5\times10^{-5}$ Volts/meter

(2) $\frac{\dot{\mathrm{P}}_\mathrm{z}}{4\pi\varepsilon_0\mathrm{cR^2}}=5.2\mathrm{ix}10^{-2}$ Volts/meter.

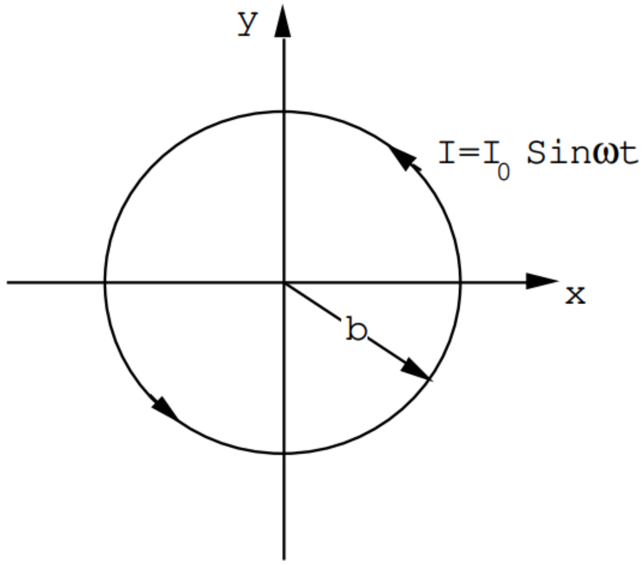
(3) $\frac{\ddot{P}_z}{4\pi\varepsilon_0 \quad c^2R}=-110 \quad$ Volts/meter

B_ϕ = - 3.67 x 10^{-7} Teslas.

Notice that the radiation field is now much larger than the near field components.

Problem (7.2)

Consider a small current loop of radius b. It carries a current I(t) = I_o sinωt. Calculate the electric and magnetic fields observed at a point P located at **R** relative to the center of the current loop. There is no net charge density anywhere on the loop i.e. $\rho_f \equiv 0$. Calculate **A** for the observer at **R**(X,Y,Z,t) and keep only the terms to lowest order in (b/R) both in the distance from an element d**L** on the current loop and in the retarded time $t_R=t-\frac{R}{C}$.



Show that to first order in (b/R) the components of the vector potential are given by

$$A_X=-\frac{\mu_0}{4\pi}\left(\pi b^2 I_0\right)\left(\frac{Y}{R}\right)\left(\frac{\operatorname{Sin}\omega(t-R/c)}{R^2}+\frac{\omega \operatorname{Cos}\omega(t-R/c)}{CR}\right)$$

$$A_y=\frac{\mu_0}{4\pi}\left(\pi b^2 I_0\right)\left(\frac{X}{R}\right)\left(\frac{\sin\omega(t-R/c)}{R^2}+\frac{\omega\cos\omega(t-R/c)}{cR}\right)$$

or

$$A_\phi=\frac{\mu_0}{4\pi}\left(\pi b^2 I_0\right)\sin\theta\left(\frac{\sin\omega(t-R/c)}{R^2}+\frac{\omega\cos\omega(t-R/c)}{CR}\right)$$

and

$A_\theta = A_R = 0$. Also V = 0 because div**A** = 0.

Show that for very large R the fields are given by

$$B_\theta=\left(\frac{\mu_0}{4\pi}\right)\left(\pi b^2\right)\frac{\sin\theta}{c^2R}\left(\frac{d^2I}{dt^2}\right)_{t_R}$$

$$E_\phi=-\left(\frac{\mu_0}{4\pi}\right)\left(\pi b^2\right)\frac{\sin\theta}{cR}\left(\frac{d^2I}{dt^2}\right)_{t_R}$$

where $t_R=\left(t-\frac{R}{c}\right)$.

Answer (7.2)

Start from the general expression for the vector potential:

$$\mathbf{A}(\mathbf{R})=\frac{\mu_0}{4\pi}\int\frac{J(\mathbf{r})d\tau}{|\mathbf{R}-\mathbf{r}|}$$

In carrying out the integral the integrand vanishes except on the wire.

$$\therefore\quad \mathbf{A}(\mathbf{R})=\frac{\mu_0}{4\pi}\oint\frac{I\left(t_R\right)\mathbf{dl}}{|\mathbf{R}-\mathbf{r}|}$$

Now $\mathbf{dl}=$ bd $\phi\left[-\sin\phi\hat{\mathbf{u}}_x+\cos\phi\hat{\mathbf{u}}_y\right]$

and $|\mathbf{R}-\mathbf{r}|^2=(X-b\cos\phi)^2+(Y-b\sin\phi)^2+Z^2=X^2+Y^2+Z^2-2bX\cos\phi-2b\sin\phi Y+b^2$

$\therefore |\mathbf{R}-\mathbf{r}| \cong R\left[1-\frac{bX\cos\phi}{R^2}-\frac{bY\sin\phi}{R^2}\right]$

Therefore

$$\mathrm{A_X}=\frac{\mu_0 I_0}{4\pi\mathrm{R}}\int_0^{2\pi}-\mathrm{b}\sin\phi d\phi\left[1+\frac{\mathrm{bX}\cos\phi}{\mathrm{R}^2}+\frac{\mathrm{bY}\sin\phi}{\mathrm{R}^2}\right]\times\sin\omega\left(t-\frac{R}{c}+\frac{bX\cos\phi}{cR}+\frac{bY\sin\phi}{cR}\right)$$

But $\sin\left[\omega\left(t-\frac{R}{c}\right)+\omega\delta\right]=\cos\omega\delta\sin\omega\left(t-\frac{R}{c}\right)+\sin\omega\delta\cos\omega\left(t-\frac{R}{c}\right)$

Here $\delta=\frac{bX\cos\phi}{cR}+\frac{bY\sin\phi}{cR}$

and ωδ << 1 i.e. of order $\frac{V}{c}<<1$

thus $\cos\omega\delta\cong 1$

$\sin\omega\delta\simeq\omega\delta$

$\therefore\sin\left[\omega\left(t-\frac{R}{c}\right)+\omega\delta\right]=\sin\omega\left(t-\frac{R}{c}\right)+\omega\delta\cos\omega\left(t-\frac{R}{c}\right)$

So

$$\begin{gathered}\mathrm{A_X}=-\frac{\mu_0}{4\pi}\frac{\mathrm{I_0 b}}{\mathrm{R}}\int_0^{2\pi}\mathrm{d}\phi\sin\phi\left\{\left(1+\frac{\mathrm{bX}\cos\phi}{\mathrm{R}^2}+\frac{\mathrm{bY}\sin\phi}{\mathrm{R}^2}\right)\right.\\ \left.\times\left[\sin\omega\left(t-\frac{R}{c}\right)+\cos\omega\left(t-\frac{R}{c}\right)\left(\frac{\omega b\times\cos\phi}{cR}+\frac{\omega bY\sin\phi}{cR}\right)\right]\right\}\end{gathered}$$

The only terms which survive the integration over the angles are

$\mathrm{A_x}=-\frac{\mu_o}{4\pi}\left(\mathrm{I_o}\pi\mathrm{b}^2\right)\left(\frac{\mathrm{Y}}{\mathrm{R}}\right)\left[\frac{\sin\omega\left(\mathrm{t}-\frac{\mathrm{R}}{\mathrm{c}}\right)}{\mathrm{R}^2}+\frac{\omega}{\mathrm{cR}}\cos\omega\left(\mathrm{t}-\frac{\mathrm{R}}{\mathrm{c}}\right)\right]=-\alpha(\mathrm{Y}/\mathrm{R})$

Similarly

$$\mathrm{A_Y}=\frac{\mu_0}{4\pi}\frac{\mathrm{I_0 b}}{\mathrm{R}}\int_0^{2\pi}\mathrm{d}\phi\cos\phi\left[1+\frac{\mathrm{bX}\cos\phi}{\mathrm{R}^2}+\frac{\mathrm{bY}\sin\phi}{\mathrm{R}^2}\right]\times\sin\omega\left(t-\frac{R}{c}+\frac{bx\cos\phi}{cR}+\frac{bY\sin\phi}{cR}\right)$$

But $\sin\omega\left(t-\frac{R}{c}+\delta\right)=\cos\omega\delta\sin\omega\left(t-\frac{R}{c}\right)+\sin\omega\delta\cos\omega\left(t-\frac{R}{c}\right)$

$$\begin{aligned}&\cong\left(\frac{\omega b\times\cos\phi}{cR}+\frac{\omega bY\sin\phi}{cR}\right)\quad\cos\omega\left(t-\frac{R}{c}\right)+\sin\omega\left(t-\frac{R}{c}\right)\\ &\therefore A_Y=\left(\frac{\mu_0}{4\pi}\right)\left(\frac{I_0 b}{R}\right)\int_0^{2\pi}d\phi\cos\phi\left\{\left[1+\frac{bX\cos\phi}{R^2}+\frac{bY\sin\phi}{R^2}\right]\sin\omega\left(t-\frac{R}{c}\right)\right.\\ &\qquad+\left(\frac{\omega bX\cos\phi}{cR}+\frac{\omega bY\sin\phi}{cR}\right)\cos\omega\left(t-\frac{R}{c}\right)\\ &\qquad+\frac{\mathrm{bX}\cos\phi}{\mathrm{R}^2}\left(\frac{\omega\mathrm{bX}\cos\phi}{\mathrm{cR}}+\frac{\omega\mathrm{bY}\sin\phi}{\mathrm{cR}}\right)\cos\omega\left(t-\frac{\mathrm{R}}{\mathrm{c}}\right)\\ &\qquad\left.+\frac{\mathrm{bY}\sin\phi}{\mathrm{R}^2}\left(\frac{\omega\mathrm{bX}\cos\phi}{\mathrm{cR}}+\frac{\omega\mathrm{bY}\sin\phi}{\mathrm{cR}}\right)\cos\omega\left(\mathrm{t}-\frac{\mathrm{R}}{\mathrm{c}}\right)\right\}\end{aligned}$$

The only terms which survive the integration are

$\mathrm{A_y}=\left(\frac{\mu_0}{4\pi}\right)\left(\mathrm{I_O}\pi\mathrm{b}^2\right)\left(\frac{\mathrm{X}}{\mathrm{R}}\right)\left[\frac{\sin\omega(\mathrm{t}-\mathrm{R}/\mathrm{c})}{\mathrm{R}^2}+\frac{\omega}{\mathrm{cR}}\cos\omega\left(\mathrm{t}-\frac{\mathrm{R}}{\mathrm{c}}\right)\right]=\alpha(\mathrm{X}/\mathrm{R})$

But

$$\begin{aligned}\frac{X}{R}&=\sin\theta\cos\phi\\ \frac{Y}{R}&=\sin\theta\sin\phi\end{aligned}$$

and

$$\begin{aligned}\mathrm{A_X}&=-\alpha\sin\theta\sin\phi\\ \mathrm{A_Y}&=\alpha\sin\theta\cos\phi\\ \mathrm{A_z}&=0.\end{aligned}$$

Since

$$\hat{\mathbf{u}}_{\mathrm{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\hat{\mathbf{u}}_{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$\hat{\mathbf{u}}_{\phi} = (-\sin\phi, \cos\phi, 0),$$

then

$$\mathbf{A}\cdot\hat{\mathbf{u}}_{\mathrm{r}} = 0, \quad \mathbf{A}\cdot\hat{\mathbf{u}}_{\theta} = 0, \quad \text{and} \quad \mathbf{A}\cdot\hat{\mathbf{u}}_{\phi} = \alpha\sin\theta.$$

By comparison with the above one finds

$$\mathrm{A}_{\phi} = \left(\frac{\mu_0}{4\pi}\right)\left(I_0\pi\mathrm{b}^2\right)\sin\theta\left[\frac{\sin\omega\left(t-\frac{R}{c}\right)}{R^2} + \frac{\omega}{cR}\cos\omega\left(t-\frac{R}{c}\right)\right]$$

and $A_r = A_\theta = 0$.

Also V = 0. Let $m_o = I_o\pi b^2$

$$\operatorname{curl}\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & r\sin\theta\hat{\mathbf{u}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\mathrm{A}_\phi \end{vmatrix}$$
$$= \begin{vmatrix} \frac{1}{\mathrm{r}\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} \\ -\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} \\ 0 \end{vmatrix}$$

$$\therefore B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \quad A_\phi)$$
$$= \left(\frac{\mu_0}{4\pi}\right)\frac{2m_0\cos\theta}{R}\left[\frac{\sin\omega\left(t-\frac{R}{c}\right)}{R^2} + \frac{\omega}{cR}\cos\omega\left(t-\frac{R}{c}\right)\right]$$

$$\mathrm{B}_\theta = \left(\frac{\mu_0}{4\pi}\right)\mathrm{m}_0\sin\theta\left[\frac{\sin\omega\left(t-\frac{\mathrm{R}}{\mathrm{c}}\right)}{\mathrm{R}^3} + \frac{\omega}{\mathrm{cR}^2}\cos\omega\left(t-\frac{\mathrm{R}}{\mathrm{c}}\right) - \frac{\omega^2}{c^2R}\sin\omega\left(t-\frac{R}{c}\right)\right]$$

$$\mathrm{B}_\phi = 0$$

$$E_\phi = -\frac{\partial A_\phi}{\partial t} = -\left(\frac{\mu_0}{4\pi}\right)m_0\sin\theta \quad \left[\frac{\omega}{R^2}\cos\omega\left(t-\frac{R}{c}\right) - \frac{\omega^2}{cR}\sin\omega\left(t-\frac{R}{c}\right)\right]$$

and $E_r = E_\theta = 0$.

For large R only the terms in 1/R are important.

$$\mathrm{B}_\theta = \left(\frac{\mu_0}{4\pi}\right)\pi\mathrm{b}^2\sin\theta\left(-\frac{\omega^2}{\mathrm{c}^2\mathrm{R}}\right)\mathrm{I}_0\sin\omega\left(\mathrm{t}-\frac{\mathrm{R}}{\mathrm{c}}\right)$$

$$B_\theta = \left(\frac{\mu_0}{4\pi}\right)\frac{\pi b^2\sin\theta}{c^2R}\left(\frac{d^2I}{dt^2}\right)_{t_R}$$

Similarly

$$E_\phi = -\left(\frac{\mu_0}{4\pi}\right)\frac{\pi b^2\sin\theta}{\mathrm{cR}}\left(\frac{d^2I}{dt^2}\right)_{t_R} \qquad \text{ie } \cdot E_\phi = -cB_\theta$$

Problem (7.3).

A magnetic dipole transmitter consists of 10 turns of wire wound on a form whose radius is 10 cm. An alternating current whose amplitude is 100 Amps and whose frequency is 100 MHz is passed through the coil.

(a) What is the maximum magnetic moment of the above coil?

(b) Assuming that the above coil can be approximated by a point magnetic dipole, calculate and compare the terms in the expressions for the electric and magnetic fields generated by an oscillating magnetic dipole as measured at a point in the equatorial plane 1 meter from the dipole (θ= π/2).

(c) Calculate and compare the terms in the expressions for the electric and magnetic fields generated by an oscillating magnetic dipole as measured at a point in the equatorial plane 1 km from the dipole.

Answer (7.3)

(a) The maximum magnetic moment is m_z= IA , or in this case m_z= (1000)(.01π)= 31.4 Amp.m^2.

(b) The fields generated by an oscillating magnetic dipole are given by

$$B_r = \frac{\mu_0}{4\pi} 2\cos\theta\left(\frac{m_z}{R^3} + \frac{\dot{m}_z}{cR^2}\right)$$

$$B_\theta = \frac{\mu_0}{4\pi}\sin\theta\left(\frac{m_z}{R^3} + \frac{\dot{m}_z}{cR^2} + \frac{\ddot{m}_z}{c^2R}\right)$$

$$B_\phi = 0$$

$$E_\phi = -c\left(\frac{\mu_0}{4\pi}\sin\theta\left(\frac{\dot{m}_z}{cR^2} + \frac{\ddot{m}_z}{c^2R}\right)\right)$$

$$E_r = E_\theta = 0$$

For this problem θ= π/2, Cosθ=0 and Sinθ=1. Also R= 1 meter and ω= 2πf = 6.28x10^8 radians/sec.

(1) $\frac{\mu_0}{4\pi}\frac{m_z}{R^3} = 3.14 \times 10^{-6}\text{Teslas}$

(2) $\frac{\mu_0}{4\pi}\frac{\dot{m}_z}{cR^2} = 6.58\text{ix}10^{-6}\text{Tes1as}$

since $\frac{\dot{m}_z}{c} = \frac{i\omega m_z}{c} = 65.80i$

(3) $\frac{\mu_0}{4\pi}\frac{\ddot{m}_z}{c^2R} = -13.78 \times 10^{-6}\text{Teslas}$

since $\ddot{m}_z = -\omega^2 m_z = -1.24 \times 10^{19}$, and $E_\phi = (4134 - 1970i)\text{Volts/meter.}$

(c) For R= 1 km = 10^3 meters

(1) $\frac{\mu_0}{4\pi}\frac{m_z}{R^3} = 3.14 \times 10^{-15}\text{Teslas}$.

(2) $\frac{\mu_0}{4\pi}\frac{\dot{m}_z}{cR^2} = 6.58\text{ix}10^{-12}\text{Tes1as.}$

(3) $\frac{\mu_0}{4\pi}\frac{\ddot{m}_z}{c^2R} = -13.78 \times 10^{-9}\text{Teslas}$, and $E_\phi = \left(4.13 - 1.97\text{ix}10^{-3}\right)\text{ Volts/meter.}$

Problem (7.4).

An electron is at rest at the origin of co-ordinates. It is suddenly given an acceleration of a = 1.76 x 10^{17} m/sec^2 for 10^{-14} seconds after which it continues with a uniform velocity. This acceleration, which is directed along the z axis, was produced by a 1000 V pulse applied across a gap of 1 mm at t = 0. An observer is located at X = 1 meter, Y= Z= 0 m.

(a) Make a sketch showing how the x-component of the electric field measured by the observer varies with time (observer's time--his clock is synchronized with that at the origin).

(b) Ditto showing how E_z varies with time.

Answer (7.4).

Think of putting both a stationary charge of +1.6x10^{-19} C. and a stationary charge of - 1.6x10^{-19} C. at the origin: the net charge is zero so that these together add nothing to the fields. However, the + charge and the moving electron together form a dipole p_z= - qz(t) where q= 1.6x10^{-19} C. The time varying dipole generates the fields

$$E_r = \frac{1}{4\pi\varepsilon_0} 2\cos\theta\left(\frac{p_z}{R^3} + \frac{\dot{p}_z}{cR^2}\right)$$

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\sin\theta\quad\left(\frac{p_2}{R^3} + \frac{\dot{p}_2}{cR^2} + \frac{\ddot{p}_2}{c^2R}\right).$$

The left over static charge at the origin generates the static field

$$E_X = \frac{1}{4\pi\varepsilon_0}\frac{-q}{R^2}$$

at **R**= (1,0,0).

The time scale T_0 for this problem is the time required for light to travel 1 m; T_0= 3.33x10^{-9} seconds. The velocity of the electron after the acceleration is V= 1.76x10^3 m/sec. Therefore on the time scale of interest here the electron has moved only VT_0= 5.9x10^{-6} meters, thus the change in position is negligible compared with the observer distance of 1 m. over the time scales of interest here (~10^{-8} secs., z= 1.76x10^{-5} m), one finds

$$\mathrm{p_z} = -\left(1.6\times 10^{-19}\right)\left(1.76\times 10^{-5}\right) = -2.82\times 10^{-24}\mathrm{Cm.}$$

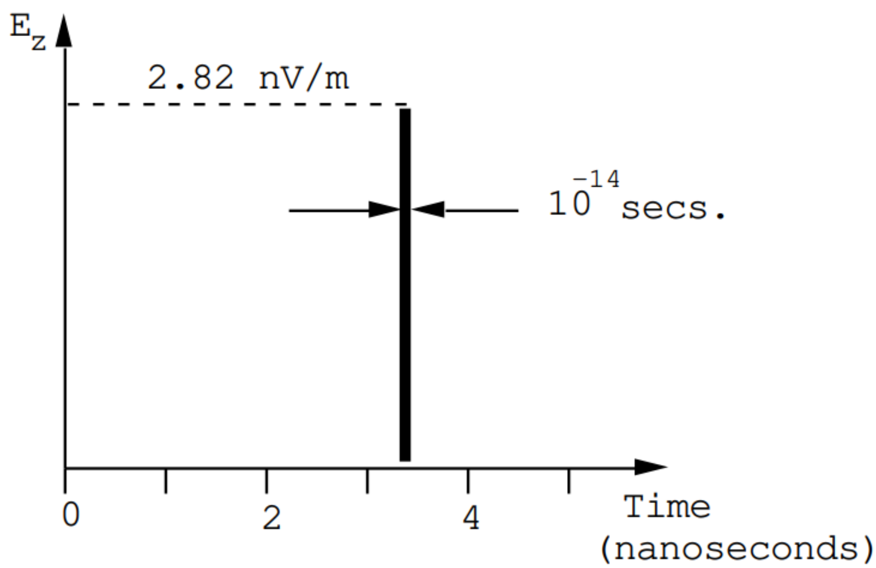
$$\frac{\dot{p}_z}{c} = -0.94\times 10^{-24}\mathrm{C.}$$

$$\frac{\dot{p}_z}{c^2} = -3.13\times 10^{-19}\mathrm{C/m,}\ \text{during the acceleration.}$$

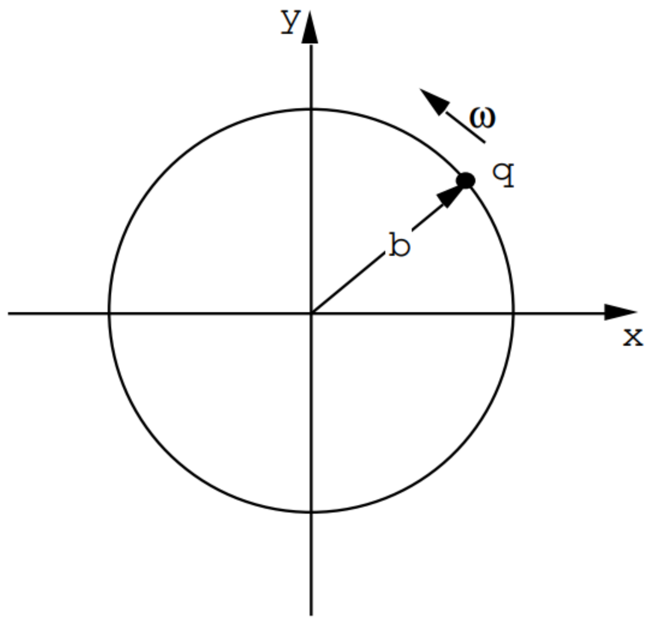
These fields are directed along +z for an observer at **R**= (1,0,0). It is clear therefore that the acceleration spike in E_z will be very large compared with the other two terms in E_θ.

The observer at (1,0,0) will see a steady field of E_x= - 14.4x10^{-10} V/m. An electric field spike will be observed beginning at T_0= 3.33x10^{-9} secs after the impulse: this spike E_z= 28.2x10^{-10} V/m will last for 10^{-14} secs. After the spike has passed the component E_z will remain at the level of 8.46x10^{-15} V/m. over the time scale of interest here. It is clear that this residual component is very small compared with the radiation spike.

In summary: the acceleration field of 282 x 10^{-9} V/m which lasts 10^{-14} seconds is directed along $-\hat{\mathbf{u}}_\theta$ because the charge is negative. Therefore for an observer at P(1,0,0) the electric field is directed along z. The acceleration begins at t=0. However, the time required for the field to reach the observer is $\frac{1}{c} = 3.33\times 10^{-9}$ seconds (a distance of 1 meter at the velocity of light). Therefore at t = 3.33 x 10^{-9} seconds the observer will see a transverse pulse which lasts 10^{-14} secs. This is superposed on a steady electric field of E_x = -14.4x10^{-10} V/m. (Steady on the time-scale of interest here.)



Problem (7.5).



A particle carrying a charge q revolves in a circle at a constant speed v = bω. This motion can be decomposed into two coupled motions

$$x = b\cos\omega t$$
$$y = b\sin\omega t$$

Let b be very small compared with the distance to the observer so that this radiation source can be treated like two orthogonal point dipoles qx and qy.

(a) Consider an observer at P = (R,0,0). Show that this observer will see a radiation field polarized along y and given by

$$E_y = \frac{\mathrm{qb}\omega^2 \sin\omega(\mathrm{t}-\mathrm{R}/\mathrm{c})}{4\pi\varepsilon_o \mathrm{c}^2\mathrm{R}} \quad \text{Volts/m.}$$

(b) Consider an observer at P = (0,R,0). Show that this observer will see a radiation field polarized along x and given by

$$E_x = \frac{\mathrm{qb}\omega^2 \cos\omega(\mathrm{t}-\mathrm{R}/\mathrm{C})}{4\pi\varepsilon_o \mathrm{c}^2\mathrm{R}} \quad \text{Volts/m.}$$

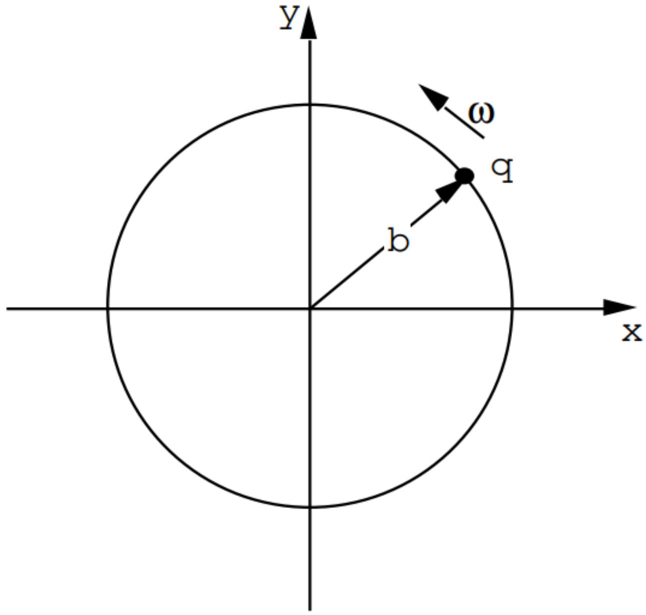
(c) Consider an observer at P = (0,0,R). Show that this observer will see circularly polarized light whose electric field components are given by

$$\mathrm{E_X} = \frac{\mathrm{qb}\omega^2 \cos\omega(\mathrm{t}-\mathrm{R}/\mathrm{c})}{4\pi\varepsilon_{\mathrm{O}} \mathrm{c}^2\mathrm{R}} \quad \text{Volts / m.}$$

$$\mathrm{E_y} = \frac{\mathrm{qb}\omega^2 \sin\omega(\mathrm{t}-\mathrm{R}/\mathrm{c})}{4\pi\varepsilon_o \mathrm{c}^2\mathrm{R}} \quad \text{Volts /m.}$$

Answer (7.5).

This motion can be regarded as a superposition of two linear motions. The electric field (radiation field) amplitude produced by an accelerated charge is given by



$E_\theta = \frac{1}{4\pi\varepsilon_o}\frac{qa\sin\theta}{c^2 r} = \frac{1}{4\pi\varepsilon_0}\frac{\ddot{p}\sin\theta}{c^2 r}$, a evaluated at t_R = t - R/c For each of the above cases θ = π/2

$\therefore E_\theta = \frac{qa}{4\pi\varepsilon_o c^2 r}$

The above results follow immediately since $a_x = -b\omega^2 \cos\omega t$ and $a_y = -b\omega^2 \sin\omega t$. The observer at (R,0,0) will see radiation only due to p_y. The observer at (0,R,0) will see radiation only due to p_x. The observer at (0,0,R) will see radiation from both p_x and p_y. The observer located along the z-axis will see circularly polarized radiation because if

$$E_X = E_0 \cos\omega\left(t - \frac{R}{c}\right),$$

and

$$E_y = E_0 \sin\omega\left(t - \frac{R}{c}\right),$$

these two fields together form a vector of fixed magnitude E_0 rotating at the angular frequency ω.

13.8: Chapter 8

Problem (8.1)

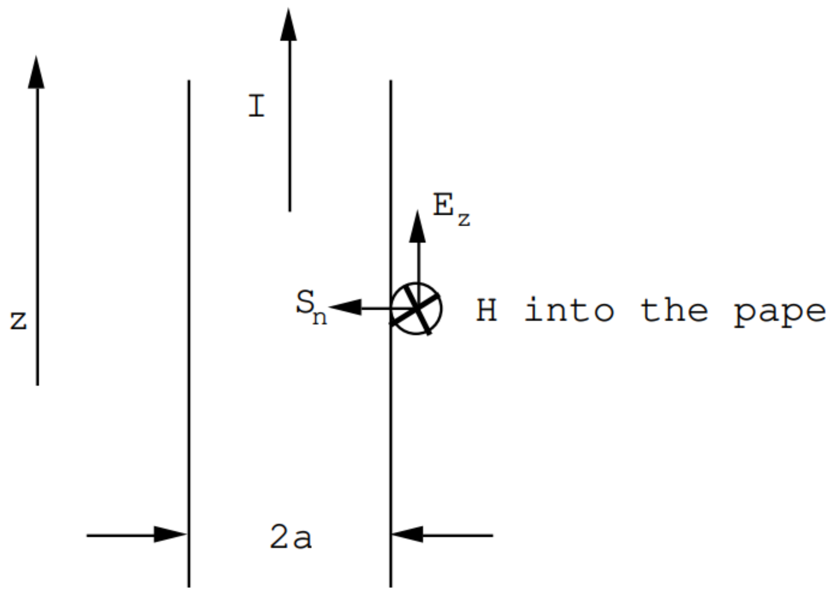
A long straight non-magnetic wire carries a steady current of I Amps. The resistance of the wire is R Ohms/meter. Use Poynting's theorem to show that the energy flow into the wire is I^2R/meter.

Answer (8.1)

At the surface of the wire the magnetic field is tangential. From Curl**B** = μ_o**J**$_f$ (No **M**, no time variation) one has, using Stokes' theorem,

$$2\pi \mathrm{a} \mathrm{B}_\theta = \mu_0 I$$

So $\mathrm{B}_\theta = \frac{\mu_0 I}{2\pi_2}$ or $\mathrm{H}_\theta = \frac{\mathrm{I}}{2\pi \mathrm{a}}$.



The electric field is also tangential E_z = IR Volts/m.

∴ **S** = **E** x **H** is normal to the wire surface and $\mathrm{S_n} = \frac{\mathrm{I^2R}}{2\pi \mathrm{a}}$. Energy flow/m = ($S_n$)(2$\pi$a) = **$I^2$R Watts/m.**

Problem (8.2).

One meter of wire is bent into a circular form to make a magnetic dipole antenna. The wire carries a current I(t) = I_o sin ωt where I_o = 2 Amps, ω = 2πf, and f = 50 MHz. At what rate does this loop radiate energy?

Answer (8.2).

For a magnetic dipole at the origin, m_z, one has

$B_r = \left(\frac{\mu_0}{4\pi}\right) 2\cos\theta \left[\frac{m_z}{R^3} + \frac{\dot{m}_z}{cR^2}\right]$

$\mathrm{B}_\theta = \frac{\mu_0}{4\pi} \sin\theta \left[\frac{\mathrm{m_z}}{\mathrm{R^3}} + \frac{\dot{\mathrm{m}}_\mathrm{z}}{\mathrm{CR^2}} + \frac{\ddot{\mathrm{m}}_\mathrm{z}}{\mathrm{c^2R}}\right]$

$E_\phi = -cB_\theta = -\left(\frac{\mu_0}{4\pi}\right) \mathrm{c}\sin\theta \left[\frac{\dot{m}_z}{cR^2} + \frac{\ddot{m}_z}{c^2R}\right]$

(part of B_θ that is proportional to the time derivatives).

Very far from the magnetic dipole one has only the radiation field terms ~ 1/R:

$E_\phi = -\left(\frac{\mu_0}{4\pi}\right) \quad \sin\theta \frac{\ddot{m}_z}{CR}$

$B_\theta = \left(\frac{\mu_0}{4\pi}\right)\sin\theta\frac{\ddot{m}_z}{c^2R}$

or $H_\theta = \frac{1}{4\pi}\sin\theta\frac{\ddot{m}_z}{c^2R}$

S = **E** x **H** has only a radial component

$S_r = \frac{\mu_0}{(4\pi)^2}\sin^2\theta\frac{(\ddot{m}_z)^2}{c^3R^2}$.

If $I = I_o \sin\omega t$ then $m_z = (\pi a2)\, I_o \sin\omega t$

$$\ddot{m}_z = -\omega^2 \quad (\pi a^2 I_0) \quad \sin\omega t.$$

Let $m_o = \pi a^2 Io$. In this problem $2\pi a = 1$ m so $m_o = 0.159$ Amp m^2.

At R and at the angle θ the time averaged Poynting vector is given by

$\langle S_r\rangle = \frac{\mu_0}{(4\pi)^2}\sin^2\theta\frac{\omega^4 m_0^2}{2c^3R^2} = 0.0363\frac{\sin^2\theta}{R^2} \quad \text{Watts}\ /m^2$

Integrate this over a sphere of radius R. The element of surface area is $dA = 2\pi R^2 \sin\theta\, d\theta$

∴ radiated power $P = (0.0363)(2\pi)\int_0^\pi \sin^3\theta d\theta \quad \text{Watts}$

but $\int_0^\pi \sin^3\theta d\theta = 4/3$

∴ P = **0.305 Watts**

Problem (8.3)

An electric dipole whose strength is $p_0 = 10^{-7}$ Coulomb-meters oscillates at a frequency f= 50 MHz. Let the dipole be oriented along z.

(a) Estimate the electric field amplitude measured by an observer on the x-axis at a mean distance of 5 meters from the dipole. Compare the near field terms with the far-field, or radiation term. Note that the $\dot{p}$ term is in quadrature with the other two terms so that it contributes only about 2% to the electric field amplitude.

(b) How big is the phase shift between the time variation of the dipole and the electric field measured by the observer?

(c) What intensity would the observer measure at the coordinates (5,0,0); i.e. what is the value of $<S_x>$)?

(d) What would be the intensity of the radiation measured by an observer on the z-axis 5 meters from the oscillating dipole?

(e) At what total rate does this dipole radiate energy?

(f) This dipole can be modelled by two spheres each having a radius of 0.1 meters and separated by 0.5 meters center to center. One sphere carries an initial charge of $Q= +2\times10^{-7}$ Coulombs, the other sphere carries an initial charge of -2×10^{-7} Coulombs. The two spheres are suddenly connected by a conducting wire and the two charges oscillate back and forth. Estimate how long is required for this system to radiate away e^{-1} of its initial energy. (The stored energy is proportional to Q^2; the rate at which energy is radiated away is proportional to Q^2 because $p_z= Qd$. It follows that the energy stored on the two spheres will decay exponentially in time).

Answer (8.3).

(a) $p_z = p_0 e^{-i\omega t}$

where $\omega = 2\pi f = 3.14\times10^8$ radians/sec, and $\frac{\omega}{c} = 1.047\mathrm{m}^{-1}$.

$$E_r = \frac{1}{4\pi\varepsilon_0}2\cos\theta\left(\frac{p_z}{R^3} + \frac{\dot{p}_z}{cR^2}\right)$$

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\sin\theta\left(\frac{p_z}{R^3} + \frac{\dot{p}_z}{cR^2} + \frac{\ddot{p}_z}{c^2R}\right)$$

$$E_\phi = 0$$

$$cB_\phi = \frac{1}{4\pi\varepsilon_0}\sin\theta\left(\frac{\dot{p}_z}{cR^2} + \frac{\ddot{p}_z}{c^2R}\right)$$

$$B_r = B_\theta = 0.$$

On the x-axis θ= π/2, and Cosθ=0, Sinθ=1. Consequently, one has only to worry about the electric field component, E_θ. Taking out the common factors one has

first term: $\frac{1}{R^3} = \frac{1}{125} = 0.008$

second term: $-\frac{i\omega}{cR^2} = -0.042i$

third term: $\frac{-\omega^2}{c^2 R} = -0.219$

The total field is proportional to -0.211 - 0.042i. The quadrature term makes only a ~ 2% correction to the field. For an observer along x and 5 meters from the dipole the electric field is polarized along z: it is given by

$$|\mathrm{E_z}| = 193 \text{ Volts/meter.}$$

(b) The phase shift between the time variation of the dipole and the electric field at the observer is

$$\Delta\phi = \frac{\omega R}{c} = 5.235 \text{ radians } = 300^\circ$$

(c) For an observer at (5,0,0)

$$\mathrm{E}_\theta = -90(2.11 + 0.419\mathrm{i})$$

$$\mathrm{H}_\phi = -0.2387(2.193 + 0.419\mathrm{i})$$

Therefore

$$\langle S_X\rangle = \frac{1}{2}\text{Real}\left(E_\theta H_\phi^\star\right) = 10.74(4.627 + 0.176 + 0.035i) \quad \text{Watts } /m^2,$$

$$\langle S_X\rangle = 51.6 \text{ Watts } /m^2$$

(d) For an observer at (0,0,5) the angle θ is zero, and consequently E_θ=0 and B_ϕ= H_ϕ= 0; thus $\langle S_z \rangle$= 0.

(e) $\langle \mathrm{P}\rangle = \frac{1}{3}\frac{\mathrm{C}}{4\pi\varepsilon_0}\mathrm{p}_0^2\left(\frac{\omega}{\mathrm{c}}\right)^4 = 10.8\text{kWatts}$

Problem (8.4)

A 10 turn circular coil of wire is centered on the origin and the plane of the coil lies parallel with the xy plane. The coil has a mean radius of 5 cm and it carries a current I(t)= I_0Sinωt where I_0= 100 Amps, and ω= 2πf where f= 20 MHz. An observer in the xy plane, and 1 km distant from the coil, measures the emf induced in a piece of straight wire 1 meter long due to the radiation field produced by the oscillating current in the coil.

(a) In what direction should the observer orient the wire in order to obtain the maximum signal?

(b) What maximum receiver power would you expect the observer to measure using a matched receiver? The radiation resistance of a short wire of length L meters (L/λ<<1) is given by $\mathrm{R} = 80\pi^2\left(\frac{\mathrm{L}}{\lambda}\right)$ Ohms.

(c) Calculate the total average rate at which energy is radiated by the oscillating magnetic dipole formed by the coil.

Answer (8.4).

(a) The wire should be oriented parallel with the xy plane and perpendicular to the line joining the observer to the coil. The electric field has only an E_ϕ component.

(b) For this problem one can ignore the near field terms and calculate only the radiation field terms. These are

$$\mathrm{B}_\theta = \frac{\mu_0}{4\pi}\sin\theta\frac{\ddot{m}_z}{c^2 R} \quad \text{Teslas },$$

$$\mathrm{E}_\phi = -\mathrm{cB}_\theta \text{ Volts/meter.}$$

In this problem m_z= $m_0 e^{-i\omega t}$ where m_0= 7.85 Amp-meters^2, ω= 1.257x10^8 radians/sec., $\frac{\omega}{c} = 0.419\mathrm{m}^{-1}$ and λ=15 m. For an observer in the x-y plane the angle θ= π/2 so that

$$\mathrm{B}_\theta = -\frac{\mu_0}{4\pi}\frac{\omega^2}{\mathrm{c}^2}\frac{\mathrm{m}_0}{\mathrm{R}} = 1.378\times 10^{-10}\text{Teslas}$$

.

The electric field strength at the observer will be $E_\phi = -cB\theta = 41.3 \times 10^{-3}$ Volts/m . The current induced in the wire will have the spatial variation $I(z) = I_0 \sin\left(\frac{2\pi}{\lambda}\left[\frac{L}{2} - z\right]\right)$ for z>0 with a similar variation for z<0. The average power delivered to the antenna will be

$< P_i >= \frac{1}{2}(2)E_0 I_0 \int_0^L dz \sin\left(\frac{2\pi}{\lambda}\left[\frac{L}{2} - z\right]\right) = E_0 I_0 \frac{\lambda}{2\pi}\left(1 - \cos\left(\frac{\pi L}{\lambda}\right)\right)$,

or for small L/λ $< P_i > \cong E_0\, I_0 \left(\frac{\pi}{4}\right)\left(\frac{L^2}{\lambda}\right)$. Half this power is delivered to the load:

$E_0 I_0 \left(\frac{\pi}{4}\right)\left(\frac{L^2}{\lambda}\right) = I_0^2 R$, where $R = 20\pi^2\left(\frac{L}{\lambda}\right)^2 = 0.877$ Ohms.

I_0 = 2.468x10^{-3} Amps. The power delivered to the matched load is $< P_L >= \frac{R}{2} I_0^2 = 2.67 \times 10^{-6}$ Watts .

(c) The total rate at which a magnetic dipole radiates energy is given by

$$P_M = \frac{c}{3}\frac{\mu_0}{4\pi}\left(\frac{\omega}{c}\right)^4 m_0^2 \text{ Watts } = 19.0 \text{ Watts.}$$

This contributes an amount Z Ohms to the coil resistance where $\frac{I_0^2 z}{2} = 19.0$ Watts . This gives Z= 3.8x10^{-3} Ohms since the current amplitude was assumed to be 100 Amps. This means that very heavy wire should be used for the oscillator tank coil if one wishes most of the input power to be radiated as electromagnetic energy rather than dissipated in the coil as heat.

Problem (8.5).

Two identical electric dipoles are driven by the same oscillator at a frequency of 20 MHz but there is a phase shift of β radians between them. The dipoles are both oriented along the z-axis, but one dipole is located at (5,0,0), the other is located at (-5,0,0). Describe the angular variation of the maximum radiation field intensity produced by these two dipoles as measured by an observer confined to the x-y plane and located a constant distance of 1 km from the origin; i.e. make a plot of the time-averaged Poynting vector as a function of the angle ϕ measured from the x-axis for (a) β=0 radians, and (b) $\beta = \frac{\pi}{2}$ radians.

Answer (8.5).

One has only to worry about the far field terms generated by the oscillating dipoles, however the phase at the observer is very important in this problem.

Dipole #1:

$$E_z^1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 \frac{p_0}{R} e^{-i\omega\left(t - \left[\frac{R - d\cos\phi}{c}\right]\right)} \exp(i\beta)$$

Dipole #2:

$$E_z^1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 \frac{p_0}{R} e^{-i\omega\left(t - \left[\frac{R + d\cos\phi}{c}\right]\right)}.$$

In writing these expressions explicit account has been taken of the fact that the distance from the observer to each dipole is slightly different. But slight as it may be compared with R, the difference in distance is a large fraction of a wavelength (λ= 15 meters). The total electric field measured by the observer in the x-y plane is given by

$$E_z = \frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 \frac{p_0}{R} e^{i\omega(t - R/c)} e^{i\beta/2}\left(e^{i\frac{\omega}{c}dCos\phi + i\beta/2} + e^{-i\frac{\omega}{c}} d\cos\phi - i\beta/2\right),$$

or

$$E_S = 2E_0 e^{i\omega(t-R/c)} e^{i\beta/2} \cos\left(\frac{\omega}{c} d\cos\phi + \beta/2\right),$$

where $E_0 = \frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 \frac{p_0}{R}$.

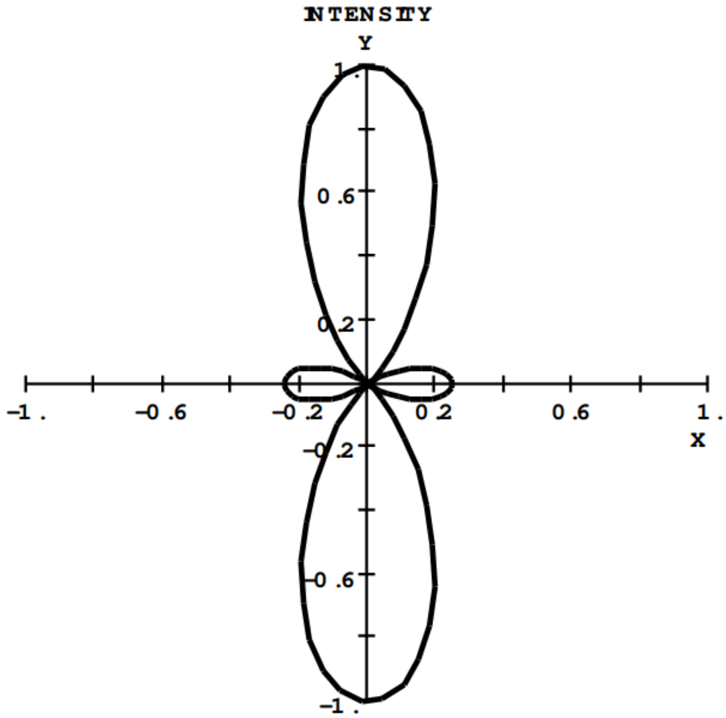
The time averaged Poynting vector is proportional to the square of the electric field strength; for one oscillator $< S_0 >= \frac{E_0^2}{2Z_0}$, where Z_0= 377 Ohms. We may therefore write

$$\langle S\rangle = 4 < S_0 > \mathrm{Cos}^2\left(\frac{\omega}{c} d\,\mathrm{Cos}\,\phi + \beta/2\right).$$

(a) β=0, and d= 5.0 m. (ω/c)= 0.419 cm^{-1} so

$$\langle S\rangle = 4 < S_0 > \cos^2(2.094\cos\phi).$$

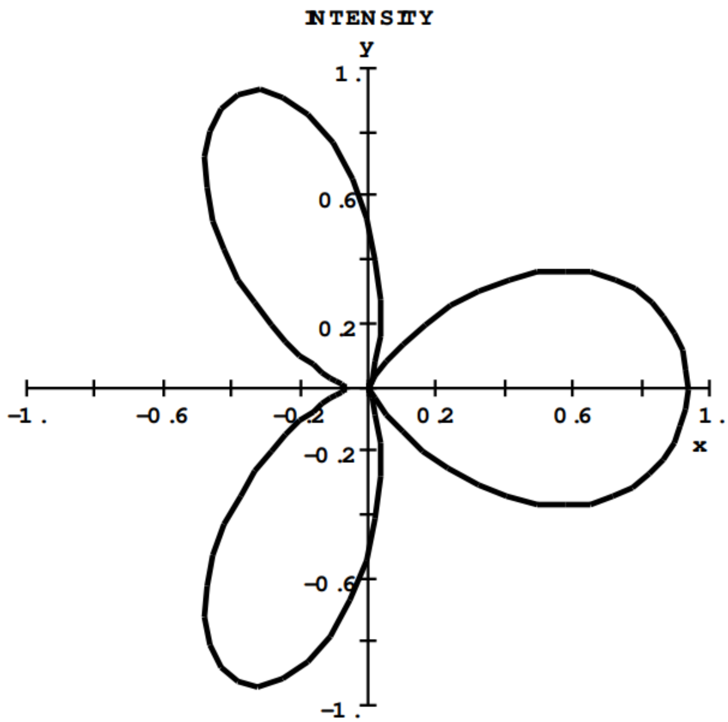
A polar plot of $\mathrm{Cos}^2(2.094\mathrm{Cos}\phi)$ is shown below.



(b) For a phase shift of $\pi/2$ between the oscillators one finds

$$\langle S\rangle = 4S_0\,\mathrm{Cos}^2\left(\frac{\omega}{c} d\cos\phi + \frac{\pi}{4}\right) = 4S_0\cos^2\left(2.094\cos\phi + \frac{\pi}{4}\right).$$

A plot of the $\cos^2\left(2.094\cos\phi + \frac{\pi}{4}\right)$ pattern is shown below.



Problem (8.6).

100 Watts/m^2 of monochromatic laser light having a free space wavelength of λ= 0.5145 μm is used to illuminate a stationary hydrogen atom. For this problem atomic hydrogen can be modelled by an oscillator having a single resonant frequency given by the n=1 to n=2 transition at 10.18 eV. the oscillator strength may be taken to be unity

(a) Estimate the total power removed from the incident beam by the hydrogen atom.

(b) How much energy would be scattered per second into a photomultiplier tube having an aperature of 3 cm and located 10 cm from the atom? The axis of the photomultiplier tube is oriented perpendicular to the incident beam in such a direction as to intercept the maximum scattered power.

Answer (8.6).

(a) Calculate the resonant frequency associated with the n=1 to n=2 transition in the hydrogen atom:

$$\begin{gathered} 10.18\mathrm{eV} = 16.31 \times 10^{-19}\mathrm{Joules.} \\ \mathrm{hf}_0 = 16.31 \times 10^{-19}, \quad \mathrm{so} \\ \mathrm{f}_0 = 2.461 \times 10^{15}\mathrm{Hz\ and\ } \lambda = 0.1219\mu\mathrm{m.} \\ \omega_0 = 2\pi\mathrm{f}_0 = 1.546 \times 10^{16}\ \mathrm{radians/sec.} \end{gathered}$$

The equation of motion for the bound electron in the hydrogen atom can be written

$$m\frac{d^2 z}{dt^2} + kz = -eE_0 e^{-i\omega t},$$

where $\mathrm{k/m} = \omega_0^2$. Under the influence of a driving field at circular frequency ω the electron amplitude is given by

$$z_0 = -\frac{(e/m)E_0}{\left[\omega_0^2 - \omega^2\right]}.$$

Now calculate the electric field amplitude in the incident beam. The frequency of the incident light is f= c/λ= 5.831x10^{14} Hz, and ω= 3.664x10^{15} radians/sec. The power in the incident beam is

$$< S >= \frac{E_0 H_0}{2} = \frac{E_0^2}{2Z_0} = 10^2 Watts/m^2,$$

where Z_0= 377 Ohms. From this expression E_0= 275 Volts/m. For this field amplitude the electron amplitude is

z_0 = -2.14×10^{-19} meters, corresponding to

an induced dipole moment amplitude $p_0 = - ez_0 = 3.43\times10^{-38}$ Coulomb-meters. The averaged power radiated by the oscillating atomic electron is

$$P_E = \frac{c}{3}\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^4 p_0^2 = 23.53\times 10^{-30}\text{Watts}$$

A photon at 0.5145 μm carries an energy of 38.64×10^{-20} Joules. The hydrogen atom scatters the equivalent of 6.1×10^{-11} photons per second. One would have to wait approximately 3200 years in order to get only 6 photons! (1 year= 3.15×10^7 seconds).

(b) The area of the photomultiplier tube is πr^2= 7.07 cm^2, therefore $\frac{\text{Area}}{\text{R}^2} = \text{d}\Omega = \frac{7.07}{100} = 0.0707$ steradians. The power intercepted by the tube will be given by

$$P_T = \frac{1}{8\pi}\left(\frac{c}{4\pi\varepsilon_0}\right)\left(\frac{\omega}{c}\right)^4 p_0^2 d\Omega,\ \text{since}\ \sin\theta = 1$$

Therefore P_T= 1.99×10^{-31} Watts. In order to get one count per second one would need N= 19.5×10^{11} hydrogen atoms. This corresponds to 7.2×10^{-11} liters at NTP or 7.2×10^{-8} cc of gas at NTP. Such an experiment would be feasible using 1 mm cubed of gas at NTP (10^{-3} cc); the dark count for a good tube is of order 1.0 counts per second at room temperatures.

Problem (8.7).

A very thin plane, uniform, sheet of dipoles is located in the y-z plane. The dipole moments are aligned along the z-axis and they vary in time like $e^{-i\omega t}$. The polarization density for such a sheet can be written

$$\mathrm{P_z(x,y,z,t) = P_0\delta(x)e^{-i\omega t}}.$$

(a) Show that this time-varying polarization density generates a vector potential which can be written in the form

X>0: $\mathrm{A_z(X,Y,Z,t) = \frac{\mu_0 P_0 C}{2} e^{i(kX-\omega t)}}$;

X<0: $\mathrm{A_Z(X,Y,Z,t) = \frac{\mu_0 P_0 C}{2}} e^{\mathrm{-i(kX+\omega t)}}$.

(Hint: one can use the particular solution for the vector potential in terms of the current density. Note also that **A** cannot depend upon either Y or Z from the symmetry of the problem; this means that one can carry out the calculation for Y=Z=0. You will run into an indeterminate constant upon carrying out the integral; this does not matter because any constant, no matter how large, can be added to the potential without altering the fields calculated from it.)

(b) Use the above vector potential to calculate the scalar potential from the Lorentz condition $\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$. Use the potentials to calculate the corresponding electric and magnetic fields.

(c) What is the rate at which energy is radiated away from this sheet of dipoles?

Answer (8.7).

The vector potential is generated by the total current density $\boldsymbol{J}_{\text{tot}} = \mathbf{J}_{\text{f}} + \text{curl}\,\mathbf{M} + \frac{\partial \mathbf{p}}{\partial t}$. For this problem the current density is entirely due to the term $\frac{\partial \mathbf{p}}{\partial t}$; this term has only a zcomponent, so the vector potential which it generates will have only a z-component.

$$J_z = \frac{\partial P_z}{\partial t} = -i\omega P_0\delta(x)e^{-i\omega t},$$

therefore

$$\mathrm{A_z(X,Y,Z,t)} = \frac{\mu_0}{4\pi}\int_{\mathrm{Space}} \frac{\mathrm{dxdydz}\,(-\mathrm{i}\omega \mathrm{P_0})\,\delta(\mathrm{x}) e^{-\mathrm{i}\omega(\mathrm{t}-\mathrm{R/c})}}{\mathrm{R}}$$

where

$$R = \left((X-x)^2 + (Y-y)^2 + (Z-z)^2\right)^{1/2}.$$

From the plane symmetry the vector potential A_z cannot depend upon the co-ordinates Y,Z; it is convenient to take Y=Z= 0. One can use polar co-ordinates in the plane and write

$$R = \sqrt{r^2 + X^2},$$

and dydz= 2πrdr. The integral over x can be carried out with no effort because of the δ-function. One obtains

$$\mathrm{A_z(X,t)} = \frac{\mu_0}{4\pi}(-\mathrm{i}\omega \mathrm{P_0})\int_0^\infty \frac{2\pi \mathrm{rdr} e^{-\mathrm{i}\omega \mathrm{t}} e^{\mathrm{i}\frac{\omega\sqrt{\mathrm{r^2+X^2}}}{\mathrm{c}}}}{\sqrt{\mathrm{r^2+X^2}}}.$$

The integral can be carried out with the help of the substitutions

$u = \sqrt{r^2+X^2}$ and udu = rdr.

$$\mathrm{A_z(X,t)} = -\frac{\mathrm{i}\omega\mu_0 \mathrm{P_0}}{2} \mathrm{e}^{-\mathrm{i}\omega \mathrm{t}} \int_{|\mathrm{X}|}^\infty \mathrm{du} e^{\mathrm{i}\omega \mathrm{u/c}}.$$

The constant is ill-defined; ignore it because a constant has no effect on the fields calculated from the vector potential.

$\mathrm{A_z(X,t)} = \left(\frac{\mu_0 \mathrm{cP_0}}{2}\right) \mathrm{e}^{\mathrm{i(k|X|}-\omega \mathrm{t})}$, where k= ω/c.

For X>0 this represents a wave propagating to the right.

For X<0 this represents a wave propagating to the left.

(b) The scalar potential can be calculated from the Lorentz condition

$$\mathrm{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0.$$

For this example, div**A**=0 so that the scalar potential is also zero.

The electric field has only a z-component which is given by $\mathrm{E_z} = -\frac{\partial \mathrm{A_z}}{\partial t}$.

The magnetic field has only a y-component which is given by $\mathrm{B_Y} = -\frac{\partial \mathrm{A_z}}{\partial \mathrm{X}}$.

For X>0: $\mathrm{E_z} = \frac{\mathrm{i}\omega\mu_0 \mathrm{cP_0}}{2} e^{\mathrm{i(kX}-\omega \mathrm{t})} \quad \mathrm{Volts/m}$

$$B_Y = -\frac{ik\mu_0 P_0 c}{2} e^{i(kX-\omega t)} = -\frac{i\omega\mu_0 P_0}{2} e^{i(kX-\omega t)} \qquad \text{Teslas}.$$

For X<0: $\mathrm{E_z} = \frac{\mathrm{i}\omega\mu_0 \mathrm{cP_0}}{2} e^{-\mathrm{i(kX}+\omega \mathrm{t})} \quad \mathrm{Volts/m},$

$$B_Y = \frac{ik\mu_0 P_0 c}{2} e^{-i(kX+\omega t)} = \frac{i\omega\mu_0 P_0}{2} e^{-i(kX+\omega t)} \qquad \text{Teslas}.$$

(c) The plane sheet of polarization radiates a plane wave in each direction which is shifted in phase by 90° with respect to the polarization sheet. The average rate at which energy is radiated in either direction is

$$<|S_X|> = \frac{C}{8}\mu_0 \quad \omega^2 P_0^2 \quad \text{Watts}/m^2.$$

Problem (8.8).

A very thin plane, uniform, sheet of dipoles is located in the y-z plane. The dipole moments are aligned along the x-axis and they vary in time like $e^{-i\omega t}$. The polarization density for such a sheet can be written

$P_X(X,Y,z,t) = P_0\delta(x)e^{-i\omega t}$.

This problem differs from Prob.(8.7) in the orientation of the dipoles. It is still true that **A** cannot depend upon either Y or Z because of the planar symmetry.

(a) Show that this time-varying polarization density generates a vector potential which can be written in the form

X>0: $A_X(X,Y,Z,t)=\frac{\mu_0 P_0 c}{2}e^{i(kX-\omega t)}$,

X<0: $A_X(X,Y,Z,t)=\frac{\mu_0 P_0 c}{2}e^{-i(kX+\omega t)}$.

(Hint: one can use the particular solution for the vector potential in terms of the current density. You will run into an indeterminate constant upon carrying out the integral; this does not matter because any constant, no matter how large, can be added to the potential without altering the fields calculated from it.)

(b) Show that the electric and magnetic fields corresponding to the above potentials are zero.

Answer (8.8).

This problem can be solved by direct integration after the manner of Problem(8.7). A more elegant and usefull method starts from the differential equation for the vector potential:

$$\nabla^2 A_x-\frac{1}{c^2}\frac{\partial A_x}{\partial t^2}=-\mu_0 J_x=i\omega\mu_0\, P_0 e^{-i\omega t}\delta(x)$$

The free space solutions of this differential equation are:

(i) on the right,x>0; $A_X=A_0 e^{i(kx-\omega t)}$,

(ii) on the left,x<0; $A_X=A_0 e^{-i(kx+\omega t)}$,

where k= ω/c. These plane waves satisfy the homogeneous wave equation

$$\nabla^2 A_X-\frac{1}{c^2}\frac{\partial A_X}{\partial t^2}=0.$$

The amplitudes of the two waves must be the same by symmetry. Near x=0 one requires

$$\frac{\partial^2 A_x}{\partial x^2}+\left(\frac{\omega}{c}\right)^2 A_x=i\omega\mu_0 P_0\delta(x).$$

Integrate this equation over a vanishingly small interval around x=0, from x=-ε to x=ε. The result is

$$\left.\frac{\partial A_x}{\partial x}\right|_{\varepsilon}-\left.\frac{\partial A_x}{\partial x}\right|_{-\varepsilon}+\left(\frac{\omega}{c}\right)^2 O(\varepsilon)=i\omega\mu_0 P_0.$$

$$\lim_{\varepsilon\to 0}\left.\frac{\partial A_x}{\partial x}\right|_{\varepsilon}=ikA_0$$

$$\lim_{\varepsilon\to 0}\left.\frac{\partial A_X}{\partial x}\right|_{-\varepsilon}=-ikA_0,$$

and the term of order ε goes to zero in the limit as ε→0. Therefore,

$$2ikA_0=i\omega\mu_0 P_0$$

and

$$A_0=\frac{\mu_0 c P_0}{2}.$$

(b) The scalar potential can be calculated from the Lorentz condition

$$\operatorname{div}\mathbf{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}=0.$$

Consider the potentials for x>0; the potentials for x<0 can be handled in the same way.

$$\operatorname{div}\mathbf{A}=\frac{i\omega\mu_0 P_0}{2}e^{i(kx-\omega t)},$$

consequently

$$V=\frac{c^2\mu_0 P_0}{2}e^{i(kx-\omega t)}=\frac{P_0}{2\varepsilon_0}e^{i(kx-\omega t)}.$$

$$\mathbf{E}=-\operatorname{grad}V-\frac{\partial\mathbf{A}}{\partial t},$$

so that

$$E_X=-\frac{ikP_0}{2\varepsilon_0}e^{i(kx-\omega t)}+\frac{i\omega\mu_0 P_0 c}{2}e^{i(kx-\omega t)}.$$

But $\frac{i\omega\mu_0 P_0 c}{2}=\frac{i\omega\mu_0 P_0 c^2}{2c}=\frac{ikP_0}{2\varepsilon_0}$

so that $E_x \equiv 0$

Furthermore, curl**A** ≡ 0 so that there is also no magnetic field component.

Problem (8.9).

very thin plane, uniform, sheet of dipoles is located in the y-z plane. The dipole moments are aligned along the x-axis and their time variation can be described by

$$e^{i(qy-\omega t)}.$$

The polarization density for such a sheet can be written

$$P_X(x,y,z,t)=P_0\delta(x)e^{i(qy-\omega t)}.$$

Calculate the electric and magnetic fields generated by this space and time varying polarization density.

Hint: It is very difficult to calculate the vector potential by direct integration of the expression

$$\mathbf{A}(\mathbf{R},t)\frac{\mu_0}{4\pi}\int\frac{d\tau\left.\frac{\partial\mathbf{p}}{\partial t}\right|_{t_R}}{|\mathbf{R}-\mathbf{r}|}.$$

It is better to work directly with the differential equation

$$\nabla^2 A_x-\frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2}=\mu_0 i\omega P_0\delta(x)e^{i(qy-\omega t)}.$$

For x>0 let $A_X=A_0e^{i(kx+qy-\omega t)}$

For x<0 let $A_x=A_0e^{i(-kx+qy-\omega t)}$.

These functions satisfy the homogeneous equation and represent plane waves propagating away from the dipole plane. The left and right propagating plane waves have the same amplitude by symmetry; this can be deduced directly from the integral for **A**(**R**,t). Near x=0 one finds

$$\frac{\partial^2 A_x}{\partial x^2}+\left[\left(\frac{\omega}{c}\right)^2-q^2\right]A_x=i\omega\mu_0 P_0\delta(x),$$

where the factor $e^{i(qy-\omega t)}$ has been cancelled out on both sides. Integrate this equation from -ε to +ε where ε is allowed to go to zero. This gives

$$\left.\frac{\partial A_x}{\partial x}\right|_{\varepsilon}-\left.\frac{\partial A_x}{\partial x}\right|_{-\varepsilon}+O(A_0\varepsilon)\left[\left(\frac{\omega}{c}\right)^2-q^2\right]=i\omega\mu_0 P_0,$$

which may be used to find $A_0=\frac{\mu_0 P_0}{2}\left(\frac{\omega}{k}\right)$, where $\left(\frac{\omega}{c}\right)^2=k^2+q^2$ (the term proportional to ε goes to zero with ε).

Answer (8.9).

Following the procedure outlined in the problem, one obtains for x>0: $A_X = \left(\frac{\omega\mu_0 P_0}{2k}\right) e^{i(kx+qy-\omega t)}$

for x<0: $A_X = \left(\frac{\omega\mu_0 P_0}{2k}\right) e^{i(-kx+qy-\omega t)}$.

For x>0 one finds div $\mathbf{A} = \frac{\partial A_x}{\partial x} = \left(\frac{i\omega\mu_0 P_0}{2}\right) e^{i(kx+qy-\omega t)}$;

from div $\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$ this gives

for x>0; $V = \frac{P_0}{2\varepsilon_0} e^{i(kx+qy-\omega t)}$.

For x<0 a similar calculation gives

$$V = -\frac{P_0}{2\varepsilon_0} e^{i(-kx+qy-\omega t)}.$$

$\mathbf{E} = -\operatorname{grad} V - \frac{\partial \mathbf{A}}{\partial t}$, from which

for x>0 $E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_X}{\partial t} = \frac{iP_0}{2\varepsilon_0}\left(\frac{q^2}{k}\right) e^{i(kx+qy-\omega t)}$,

$E_Y = -\frac{\partial V}{\partial y} = -\frac{iqP_0}{2\varepsilon_0} e^{i(kx+qy-\omega t)}$,

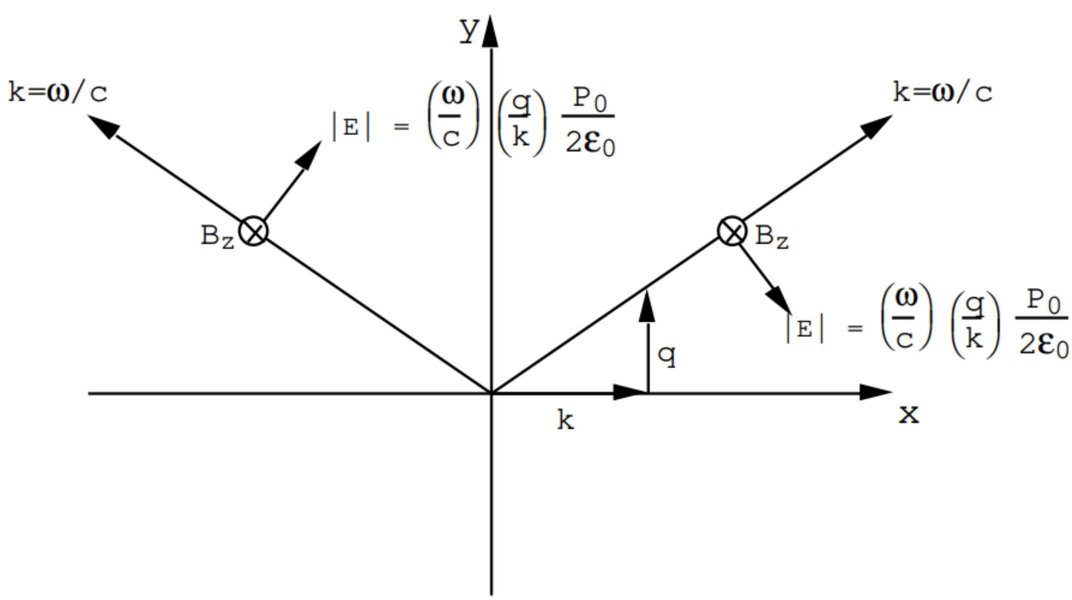
and **B** = curl**A**, so that

$$B_z = -\frac{\partial A_X}{\partial y} = -\frac{i\omega\mu_0 P_0}{2}\left(\frac{q}{k}\right) e^{i(kx+qy-\omega t)}.$$

The amplitude of the electric field is

$$|E| = \sqrt{E_x^2 + E_Y^2} = \left(\frac{\omega}{c}\right)\left(\frac{q}{k}\right)\frac{P_0}{2\varepsilon_0},$$

therefore cB_z= |E| which is the correct ratio for a plane wave propagating through empty space. A similar calculation shows that a plane wave propagates to the left; its **B** field is the same as the wave propagating to the right. See the diagram below.



Problem (8.10).

The electron on a hydrogen atom is characterized by the resonant frequency f_o = 15 x 10^{15} Hz. The dipole moment induced on each hydrogen atom by an electric field can be written p = αE where α is the polarizability.

(a) Estimate the polarizability of a hydrogen atom for an electric field oscillating at a frequency of 10^{18} Hz.

(b) Consider a hydrogen atom at the origin. A plane wave is incident on the atom where $E_z = E_o\, e^{i(kx-wt)}$ where E_o = 1 Volt/meter and $\omega = 2\pi \times 10^{18}$ rad/sec. How large are the electric field components measured by an observer 1 meter distant and located in the x-y plane?

Answer (8.10).

(a) The equation of motion of the electron on the H atom can be written

$$m\frac{d^2z}{dt^2} + kz = |e|E_0 e^{-iwt}$$

or

$$\frac{d^2z}{dt^2} + \frac{k}{m}z = -\frac{|e|}{m}E_0 e^{-iwt}.$$

The resonant frequency is $\frac{3}{2} \times 10^{16}\text{Hz}$ therefore $\frac{\mathrm{k}}{\mathrm{m}} = \left(3\pi \times 10^{16}\right)^2 (\mathrm{rad/sec})^2 = \Omega^2$.

If $z = z_0 e^{-i\omega t}$ then $z_0 = \frac{-(|e|/m)E_0}{(\Omega^2-\omega^2)}$.

The dipole moment is $P_z = -|e|z = \frac{(e^2/m)}{(\Omega^2-\omega^2)}E_0 e^{-i\omega t}$,

$\therefore \alpha = \frac{e^2/m}{(\Omega^2-\omega^2)} \simeq -\frac{e^2}{m\omega^2}$ since $\omega^2 >> \Omega^2$.

At f = 10^{18} Hz $\omega = 6.28 \times 10^{18}$ and $\alpha = -0.714 \times 10^{-45}$ Coulomb meters.

The induced dipole moment is $P_z = \alpha E_z$. For an observer in the x-y plane

$$\theta = \frac{\pi}{2} \quad \therefore \quad \cos\theta = 0$$
$$\sin\theta = 1$$

There is no radial component of electric field!

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_z}{R^3} + \frac{\dot{P}_z}{cR^2} + \frac{\ddot{P}_z}{c^2R}\right]$$

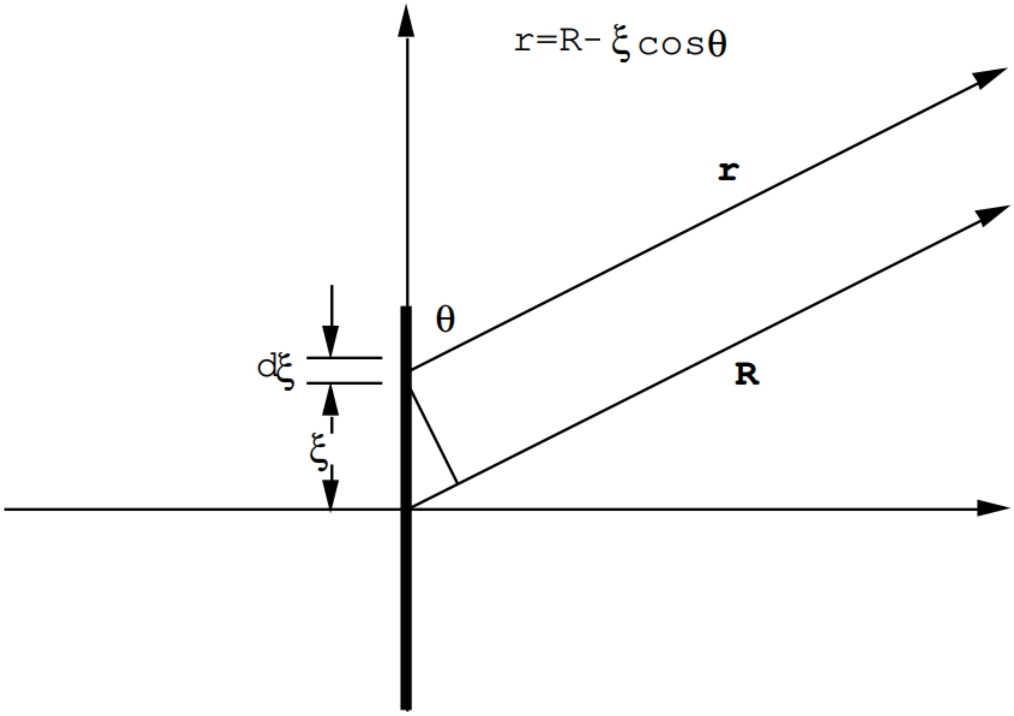
Now $\frac{\omega}{c} = 2.09 \times 10^{10}$ $\left(\frac{\omega}{c}\right)^2 = 4.39 \times 10^{20}$

So the quantities 1 & $\frac{\omega}{\mathrm{c}}$ are negligible c.f. $\left(\frac{\omega}{c}\right)^2$

$\mathrm{E}_\theta = -\frac{(0.71)(10^{-45})(1)}{4\pi\varepsilon_0}\left(4.39 \times 10^{20}\right) = \mathbf{-2.82 \times 10^{-15}}$ **Volts/meter.**

Problem (8.11).

A short thin-wire center-fed dipole antenna of length L meters is oriented along the z-axis with its center at the origin as shown in the sketch.



The current distribution on the antenna is given by:

$$z>0: \qquad I(\xi)=I_0\left(1-\frac{2\xi}{L}\right)e^{-i\omega t}$$

$$z<0: \qquad I(\xi)=I_0\left(1+\frac{2\xi}{L}\right)e^{-i\omega t}.$$

Show that the radiation resistance of the antenna is given by

$$\mathrm{R}=20\pi^2\left(\frac{\mathrm{L}}{\lambda}\right)^2 \text{ Ohms}\,.$$

Hint: The distance from the element dξ to the observer is given approximately by

$$r=R-\xi\cos\theta=R\left(1-\frac{\xi\cos\theta}{R}\right);$$

here ξ/R is a very small quantity. Expand all relevant terms as a power series in (ξ/R) and keep only terms proportional to (ξ/R). Also use the approximation that $e^{-i\omega\xi\cos\theta/c}$ can be set equal to $\left(1-i\frac{\omega\xi}{c}\cos\theta\right)$.

Answer (8.11).

$$\mathrm{dA_z}=\frac{\mu_0}{4\pi}\frac{\mathrm{d}\xi\mathrm{I}(\xi)}{(\mathrm{R}-\xi\cos\theta)}\mathrm{e}^{-\mathrm{i}\omega\left(\mathrm{t}-\frac{\mathrm{R}}{\mathrm{c}}+\frac{\xi\cos\theta}{\mathrm{c}}\right)},$$

$$\mathrm{A_z}\cong\left\{\frac{\mu_0}{4\pi\mathrm{R}}\int_{-\mathrm{L}/2}^{\mathrm{L}/2}\mathrm{d}\xi\left(1+\frac{\xi_{\cos}\theta}{\mathrm{R}}\right)\mathrm{I}(\xi)e^{\frac{-\mathrm{i}\omega\xi}{\mathrm{c}}}\cos\theta_{\}}\mathrm{e}^{-\mathrm{i}\omega(\mathrm{t}-\mathrm{R}/\mathrm{c})}\right..$$

Drop the term $e^{-i\omega(t-R/c)}$; it is just a factor throughout.

Let $e^{-i\omega\xi\cos\theta/c}=(1-i\omega\xi\cos\theta/c)$. Then setting k=ω/c, one finds

$$A_z \cong \frac{\mu_0 I_0}{4\pi R} \quad \int_0^{L/2} d\xi \quad \left(1 - \frac{2\xi}{L}\right)\left(1 + \frac{\xi\cos\theta}{R}\right)(1 - ik\xi\cos\theta) + \frac{\mu_0 I_0}{4\pi R}$$
$$\int_{-L/2} d\xi \left(1 + \frac{2\xi}{L}\right)\left(1 + \frac{\xi\cos\theta}{R}\right)(1 - ik\xi\cos\theta)$$

where $\frac{\xi}{R} << 1$ and it is assumed that kξ<<1. With these approximations

$$A_z = \frac{\mu_0 I_0}{4\pi R}\left(\frac{L}{2}\right) e^{-i\omega[t-R/c]}.$$

In spherical polar co-ordinates

$$A_R = \frac{\mu_0 I_0}{4\pi}\left(\frac{L}{2}\right)\frac{\cos\theta}{R} e^{-i\omega[t-R/c]},$$
$$A_\theta = -\frac{\mu_0 I_0}{4\pi}\left(\frac{L}{2}\right)\frac{\sin\theta}{R} e^{-i\omega[t-R/c]}.$$

B= curl**A**. In this case **B** has only the component B_ϕ. The radiation component of this field is proportional to 1/R and is

$$B_\phi = -i\frac{\mu_0 I_0}{4\pi}\left(\frac{L}{2}\right)\left(\frac{\omega}{c}\right)\frac{\sin\theta}{R} e^{-i\omega[t-R/c]}.$$

The electric field is

$E_\theta = cB_\phi = -i\frac{\mu_0 I_0}{4\pi}\left(\frac{L}{2}\right)(\omega)\frac{\sin\theta}{R} e^{-i\omega[t-R/c]}$.

$H_\phi = B_\phi/\mu_0 = -i\frac{I_0}{4\pi}\left(\frac{L}{2}\right)\left(\frac{\omega}{c}\right)\frac{\sin\theta}{R} e^{-i\omega[t-R/c]}$.

The time average of the Poynting vector, $<S_R>$ is

$\langle S_R\rangle = \frac{1}{2}\text{Real}\left(E_\theta H_\phi^*\right) = \frac{1}{2}\frac{\mu_0 I_0^2}{16\pi^2}\frac{L^2}{4}\frac{\omega^2}{c}\frac{\sin^2\theta}{R^2}$.

Integrate this expression over the sphere of radius R to get the total radiated power:

$$\langle P\rangle = \left(\frac{\pi}{12}\right)(CH_0)\left(\frac{I_0^2 L^2}{\lambda^2}\right) \text{ Watts} .$$

But $c\mu_0$= 120π, and by definition the radiation resistance R_R is such that $\langle P\rangle = \frac{I_0^2 R_R}{2}$, therefore

$$R_R = 20\pi^2\left(\frac{L}{\lambda}\right)^2 \text{ Ohms} .$$

13.9: Chapter 9

Problem (9.1).

A plane wave is polarized with its electric vector along z. The wave propagates along the y-axis. The electric field is given by

$$\mathrm{E_2(y,t) = E_0 e^{i(ky-\omega t)}} \quad \text{Volts/meter.}$$

This wave is propagating in vacuum; its amplitude is E_o = 5V/m and its wavelength is 0.10 meters.

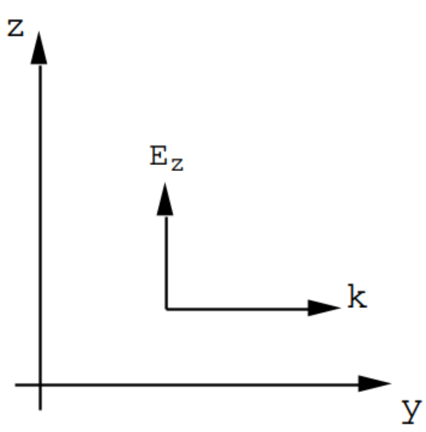
a. What is the frequency of the wave?
b. How large is the magnetic field associated with this wave and in what direction is it oriented?
c. What is the average rate at which energy is transported by this wave (per square meter)?
d. This wave encounters an electron. At what rate does the electron remove energy from the wave?
e. The wave propagates through an electron gas whose density is 10^{15} per cubic centimeter. If each electron acts like an independent scattering center estimate the distance the wave will travel before its amplitude has been reduced to (1/e) of its initial value.

Answer (9.1).

a) ω = ck in free space

$\therefore f = \frac{c}{\lambda} = 3 \times 10^9 \mathrm{Hz} = 3\mathrm{GHz}$

b) **B** is along x:



$\mathrm{B_x = \frac{E_O}{c} e^{i(ky-\omega t)}}$

$\therefore \mathrm{H_X = \frac{E_O}{c\mu_O}} e^{\mathrm{i(ky-\omega t)}} = \mathrm{H_O e^{i(ky-\omega t)}}$

$\therefore \mathrm{H_0 = \frac{E_0}{c\mu_0}} = \frac{5}{(3\times 10^8)(4\pi\times 10^{-7})} = \frac{5}{120\pi} = \frac{5}{377}$

$\mathbf{H_0 = 13.26 \times 10^{-3}\,Amps/meter\,.}$

(c) $S_Y = E_z H_X = (5)\left(13.3\times 10^{-3}\right)\cos^2(ky-\omega t)$

$\langle S_Y\rangle = \left(\frac{1}{2}\right)(5)(13.3)\times 10^{-3}\text{Watts }/m^2 = \mathbf{33.16 \times 10^{-3}}$ **Watts/m^2.**

d) For the electron $\mathrm{ma} = -|e|\mathrm{E_0 e^{-i\omega t}}$

$$\therefore a = -\frac{|e|}{m} E_0 e^{-i\omega t}$$

The energy scattered by an accelerated charge and integrated over all angles is given by

$\frac{dW}{dt} = \left(\frac{2}{3}\right)\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{|e|^2}{c^3} a^2$

Time Average: $\left\langle \frac{dW}{dt} \right\rangle = \left(\frac{1}{3}\right)\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^4}{m^2c^3}E_0^2$

or $\left\langle \frac{dW}{dt} \right\rangle = 0.88\times 10^{-31}E_0^2$ since $\frac{e^4}{m^2c^4} = 9.80\times 10^{-50}$.

E_o = 5 V/m initially so in this case

$$\left\langle \frac{\mathbf{dW}}{\mathbf{dt}} \right\rangle = \mathbf{22.05\times 10^{-31}\,Watts.}$$

e) Now $\frac{dW}{dt} = \alpha E_0^2$ for 1 electron.

There are 10^{15} electrons/cc = 10^{21} electrons/m^3 = N. Consider a section of the wave having an area of 1 m^2 and look at the energy change in traversing a distance dy:

The energy change in dy is

$$\Delta \mathrm{W} = -\left(\alpha \mathrm{E}^2\right)(\mathrm{Ndy}) = -\mathrm{N}\alpha \mathrm{dyE}^2$$

But the Poynting vector is given by (time average)

$< S >= \frac{E^2}{2c\mu} = \frac{E^2}{240\pi}$

$\therefore \quad < S > \quad (y+dy) - \langle S(y)\rangle = -N\alpha E^2 \quad dy$

or $\mathrm{dy}\frac{<\mathrm{dS}>}{\mathrm{dy}} = -\mathrm{N}\alpha\mathrm{E}^2 \quad \mathrm{dy}$

or $\frac{<dS>}{dy} = -N\alpha E^2$.

But $\langle S\rangle = \frac{E^2}{240\pi}$

$\frac{d<S>}{dy} = \frac{E}{120\pi}\frac{dE}{dy}$

$\therefore \frac{E}{120\pi}\frac{dE}{dy} = -N\alpha E^2$

or $\frac{dE}{dy} = -N\alpha(120\pi)E = -377\mathrm{N}\alpha E$

where $\alpha = 0.88\times 10^{-31}\frac{\mathrm{Watts}}{V^2}$ and $\mathrm{N} = 10^{21}$ electrons $/\mathrm{m}^3$.

$\therefore \mathrm{E} = \mathrm{E_O e^{-y/L}}$

where $1/L = (377)(N) \quad \alpha = 0.33\times 10^{-7}\mathrm{m}^{-1}$

or **L = 3.01 x 10^7 meters = 3.01 x 10^4 km.**

So the wave can travel ~ 30,000 km before its amplitude has dropped to e^{-1} of its initial value.

Problem (9.2).

A plane wave is propagating thru empty space with a wavevector given by

$$\mathbf{k} = 6.283\frac{(\hat{\mathbf{u}}_\mathrm{x} + \hat{\mathbf{u}}_\mathrm{y})}{\sqrt{2}} \qquad \text{per meter.}$$

The electric vector has a strength of $\frac{1}{10}$ Volts/meter.

a. Calculate the frequency and wavelength of this radiation.
b. How large is the magnetic field B associated with this wave.
c. At what average rate is energy being transported by this wave (Watts/meter^2).
d. What is the average stored electrical energy in the wave? (Joules/m^3)
e. What is the average stored magnetic energy in the wave? (Joules/m^3).

Answer (9.2).

Assume that the wave is polarized with **E** along z - this will ensure that **k·E** = 0. Then

$$\mathrm{E_z = E_O e^{i(k_x x + k_y y)}\cdot e^{-i\omega t}}$$

where $k_x = \frac{2\pi}{\sqrt{2}}$ and $k_Y = \frac{2\pi}{\sqrt{2}}$ meters^{-1}.

But curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = i\omega \mathbf{B}$

$$\text{curl } \mathbf{E} = \begin{vmatrix} \hat{\mathbf{u}}_x & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} = \begin{vmatrix} \frac{\partial E_z}{\partial y} \\ -\frac{\partial E_z}{\partial x} \\ 0 \end{vmatrix}$$

$\therefore i\omega B_x = ik_y E_z$ or $B_X = \left(\frac{k_Y}{\omega}\right) E_Z$

$i\omega B_y = -ik_X E_z \; B_Y = -\left(\frac{k_X}{\omega}\right) \quad E_z$

Now $\frac{\omega}{c} = 2\pi$ ∴ f = 3 x 10^8 Hz i.e. 300 MHz

and λ = 1 meter.

$H_x = \frac{B_X}{\mu_O} = \frac{E_O}{C\mu_O\sqrt{2}} e^{i(k_X x + k_Y Y - \omega t)}$

$H_Y = \frac{B_Y}{\mu_O} = -\frac{E_O}{C\mu_O\sqrt{2}} e^{i(k_X x + k_Y Y - \omega t)}$

where $c\mu_o$ = (3 x 10^8)(4π x 10^{-7}) = 120 π = 377 Ohms. The Poynting Vector is **S** = **E** x **H** and it is directed along k. Since **E** and **H** are perpendicular

$$|\mathbf{E} \times \mathbf{H}| = \frac{E_0^2}{C\mu_0} \cos^2(k_X \mathbf{x} + k_{YY} - \omega t)$$

Time Average $< S > = \frac{E_0^2}{2c\mu_0} = \frac{.01}{(2)(377)} = \mathbf{1.326 \times 10^{-5} Watts / m^2}$

Amplitude $|\mathbf{B}| = \sqrt{B_x^2 + B_y^2} = \frac{E_0}{C} = \frac{10^{-1}}{3\times 10^8} = \mathbf{3.333 \times 10^{-10} Tesla}$

Amplitude $|\mathbf{H}| = \frac{|\mathbf{B}|}{\mu_0} = \frac{10^{-1}}{377} = \mathbf{2.652 \times 10^{-4} Amps/m}$.

(d) Energy density stored in the electric field is given by

$$w_E = \frac{\varepsilon_0 E^2}{2}$$

$$w_E = \frac{\varepsilon_o E_O^2}{2} \cos^2(k_x x + k_y Y - \omega t)$$

$$\therefore \text{Avg} < w_E > = \frac{\varepsilon_0 E_0^2}{4} = \frac{E_0^2}{(4)(36\pi)\left(10^9\right)}$$

$$= \mathbf{2.21 \times 10^{-14} Joules / m^3}.$$

(e) The average energy stored in the magnetic field is given by

$$< w_B > = \frac{|B_0|^2}{4\mu_0} = \frac{\varepsilon_0}{4} E_0^2 = \mathbf{2.21 \times 10^{-14} Joules / m^3}$$

Problem (9.3).

The electric field of an electromagnetic wave

$$\mathbf{E} = E_0 \hat{\mathbf{u}}_x \cos\left[10^8 \pi \left(t - \frac{z}{c}\right) + \theta\right] \quad V/m$$

is the sum of

$$\mathbf{E}_1 = 0.03 \hat{\mathbf{u}}_x \sin 10^8 \pi \left(t - \frac{z}{c}\right) \quad V/m$$

and $\mathbf{E}_2 = 0.04 \hat{\mathbf{u}}_x \cos\left[10^8 \pi \left(t - \frac{z}{c}\right) - \frac{\pi}{3}\right] \quad V/m$

Find E_o and θ.

Answer (9.3).

The electric field can be written

$E_x = E_0 \cos\theta \cos 10^8\pi\left(t-\frac{z}{c}\right) - E_0 \sin\theta \sin 10^8\pi\left(t-\frac{z}{c}\right)$

Also $\mathrm{E}_{1x} = 0.03\left[\sin 10^8\pi\left(\mathrm{t}-\frac{z}{c}\right)\right]$

and $E_{2x} = 0.04\left[\cos\left(\frac{\pi}{3}\right)\cos 10^8\pi\left(t-\frac{z}{c}\right) + \sin\left(\frac{\pi}{3}\right)\sin 10^8\pi\left(t-\frac{z}{c}\right)\right)$

$\therefore E_{1x} + E_{2x} = 0.02\cos 10^8\pi\left(t-\frac{z}{c}\right) + 0.0646\sin 10^8\pi\left(t-\frac{z}{c}\right)$

∴ compare with above. $E_o \cos\theta = 0.02$

$-E_o \sin\theta = 0.06464$

$\therefore \tan\theta = -\frac{0.06464}{0.02} = -3.232 \;\therefore \theta = \mathbf{-72.81^\circ = -1.271}\pi$.

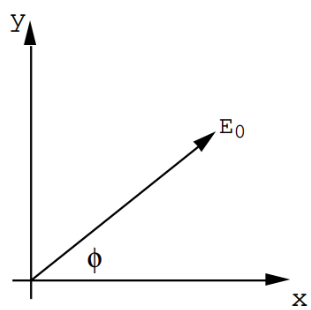
and $E_0 = \frac{0.02}{\cos\theta} = \mathbf{0.06766 \quad v/m}$.

Problem (9.4).

An optical device called a λ/2-plate (half-wave plate) is characterized by two axes which can be labeled x and y. The velocity of a plane wave polarized along y is different from the velocity of a plane wave polarized along x. The plate thickness is such that a phase shift of π is introduced between waves polarized along x and along y. Consider an incident plane polarized beam of light such that the electric vector makes an angle ϕ with the x-axis. Show that the plane of polarization of the exit beam will be rotated through 2ϕ. This mechanism is used in experiment to make fine adjustments to the plane of polarization.

(Hint: Decompose the electric field vector of the incident plane wave into the sum of two plane waves; one having the electric vector polarized along x, $E_x = E_0\cos\phi$, the other having the electric vector polarized along y, $E_y = E_0\sin\phi$).

Answer (9.4).



This can be written as the sum of two plane waves:

$E_x = E_o \cos(kz-wt)\cos\phi = E_o\cos\phi\cos\omega t$ (at z=0)

$E_y = E_o \cos(kz-wt)\sin\phi = E_o\sin\phi\cos\omega t$ (at z=0).

If the y-axis is slow then the exit waves will have the form

$$E_x = E_0\cos\phi\cos(k_x d - \omega t)$$

$$\mathrm{E_y = E_O}\sin\phi\cos(\mathrm{k_y d} - \omega t).$$

However, $v_x = \frac{c}{n_x}$ and $V_Y = \frac{c}{n_Y}$, where n_x,n_y are the indices of refraction for propagation of light along the x and y axes. One has $\omega = v_x k_x$ and $\omega = v_y k_y$ so that if y is a slow axis $k_y > k_x$.

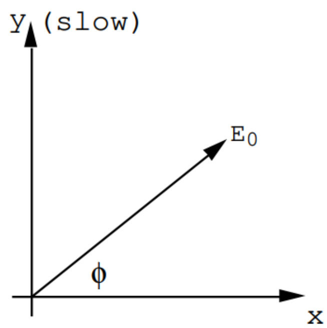
therefore $E_y = E_0 \sin\phi \cos(k_x d-\omega t+\pi)$,

or $E_y = - E_o \sin\phi\cos(k_x d-\omega t)$.

But $E_x = E_0\cos\phi \cos(k_x d-\omega t)$, so these electric field components correspond to a plane wave in which the direction of polarization has been rotated through 2ϕ (clockwise). A similar argument also gives 2ϕ clockwise if x is the slow axis: the phase of the resulting wave is just shifted by 180°.

Problem (9.5).

A quarter wave plate is similar to the half-wave plate of problem (9.4) except that the thickness is adjusted so that in its passage through the plate light polarized parallel with one principle axis is shifted by $\pi/2$ in phase relative to light polarized with its electric vector parallel with the other axis. (See the sketch).



Let light be incident on the $\frac{\lambda}{4}$ - plate which is polarized so that its electric vector makes an angle ϕ with the fast axis. Show that the transmitted light will be elliptically polarized. For what angle ϕ will the transmitted light be circularly polarized?

Answer (9.5).

At z = 0 $E_x = E_o \cos\phi \cos\omega t$

$E_y = E_o \sin\phi \cos\omega t$,

Plane polarized incident light.

At exit where z = d

$$\begin{aligned} \mathrm{E_X} &= \mathrm{E_O} \cos\phi \cos(\mathrm{k_X d} - \omega \mathrm{t}) \\ \mathrm{E_Y} &= \mathrm{E_O} \sin\phi \cos(\mathrm{k_Y d} - \omega \mathrm{t}) \\ &= \mathrm{E_0} \sin\phi \cos\left(\mathrm{k_X d} - \omega \mathrm{t} + \frac{\pi}{2}\right) \end{aligned}$$

since y is the slow axis for which $k_y > k_x$. This follows from the relations $\omega = v_X k_X = \left(\frac{c}{n_X}\right) k_X$ and $\omega = v_y k_y = \left(\frac{c}{n_Y}\right) k_Y$, ie. $k_Y = \left(\frac{n_Y}{n_X}\right)$.

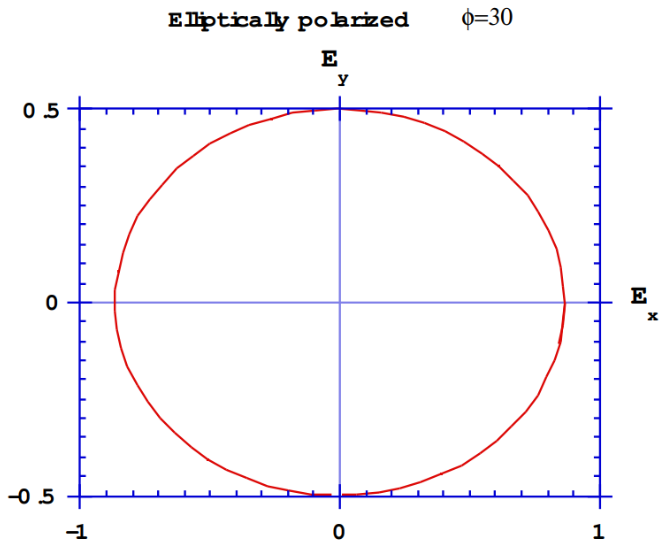
Thus $\mathrm{E_Y} = -\mathrm{E_O} \sin\phi \sin(\mathrm{k_x d} - \omega \mathrm{t})$.

These components can be written

$$\begin{aligned} \mathrm{E_X} &= \mathrm{E_0} \cos\phi \cos(\omega \mathrm{t} - \mathrm{k_X d}) \\ \mathrm{E_Y} &= \mathrm{E_0} \sin\phi \sin(\omega \mathrm{t} - \mathrm{k_X d}) \end{aligned} .$$

$(\omega t - k_x d)$	E_x	E_Y
0	$E_0 \cos\phi$	0
$\pi/2$	0	$+E_0 \sin\phi$
$\pi/4$	$\frac{E_0 \cos\phi}{\sqrt{2}}$	$+\frac{E_0}{\sqrt{2}} \sin\phi$
π	$-E_0 \cos\phi$	0
$\frac{3\pi}{2}$	0	$-E_0 \sin\phi$
$-\pi/4$	$\frac{E_0 \cos\phi}{\sqrt{2}}$	$-\frac{E_0}{\sqrt{2}} \sin\phi$

The light will become circularly polarized for $\phi = \pi/4$.



Problem (9.6).

A charged particle moves in a circular orbit of radius b meters centered on the origin and lying in the x-y plane. The co-ordinates of the particle can be described by the relations

$$x = b\cos\omega t$$

$$y = b\sin\omega t$$

where ω= 2πf = 3x10^{15} radians/second. The motion is equivalent to the superposition of two point dipoles

$$p_x = p_0\cos\omega t = Qb\cos\omega t$$

$$\mathrm{p_Y = p_0\sin\omega t = Qb\sin\omega t}$$

where p_0= 10^{-30} Coulomb-meters.

a. An observer is located at x=0, y=0, z=1 meter. How will the electric field at the observer vary in time? What intensity of radiation will be observed?
b. An observer is located at x=0, y=0.707, z=0.707 meters. How will the electric field at the observer vary in time? What intensity of radiation will be observed?
c. An observer is located at x=0, y=1, z=0 meters. How will the electric field at the observer vary in time? What intensity of radiation will be observed?

Answer (9.6).

(a) The observer is at right angles to both dipoles. The radiation fields are given by (R=1 meter, Sinθ=1)

$$\mathrm{E_X = \frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\cos(\omega t - R/c)}$$

$$\mathrm{E_Y = \frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\sin(\omega t - R/c)}$$

The electric field is right hand circularly polarized. The intensity of the radiation will just be given by

$$S_z = \frac{E_0^2}{Z_0} = \frac{1}{Z_0}\left(\frac{p_0}{4\pi\varepsilon_0}\right)^2\left(\frac{\omega}{c}\right)^4,$$

independent of time because $\cos^2(\omega[t - R/c]) + \sin^2(\omega[t - R/c]) = 1$. Z_0= 377 Ohms, thus S_z = 2.149x10^{-15} Watts/m^2. Notice that the intensity does not fluctuate with time for a circularly polarized wave.

(b) For the observer at (0,0.707,0.707) the electric fields will be given by

$$E_X=\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\cos(\omega[t-R/c])$$

$$E_\theta=-\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\frac{\sin(\omega[t-R/c])}{\sqrt{2}}$$

Therefore

$$E_Y=\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\frac{\sin(\omega[t-R/c])}{2}$$

$$E_Z=-\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\frac{\sin(\omega[t-R/c])}{2}.$$

This electric field corresponds to right hand elliptically polarized radiation.

In a co-ordinate system rotated so that the new Z-axis is pointed along the line joining the observer to the origin one has

$$E_X=\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\cos(\omega[t-R/c])$$

and

$$E_\eta=\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\frac{\sin(\omega[t-R/c])}{\sqrt{2}}.$$

The time averaged intensity is given by

$$<S>=\frac{E_X^2}{2Z_0}+\frac{E_\eta^2}{2Z_0},$$

where Z_0= 377 Ohms, and E_X,E_η are the electric field amplitudes. In this case $E_\eta=E_X/\sqrt{2}$, so that

$$<S>=\left(\frac{3}{4}\right)\left(2.15\times 10^{-15}\right)=1.611\times 10^{-15}\text{Watts}/\text{m}^2.$$

(c) An observer at (0,1,0) sees a radiation field due entirely to the dipole oriented along the x-axis. The electric field will be linearly polarized and

$$E_x=\frac{1}{4\pi\varepsilon_0}\left(\frac{\omega}{c}\right)^2 p_0\cos(\omega t-R/c).$$

The corresponding intensity will be just half the intensity measured by the observer on the z-axis:

$$\langle S\rangle=1.074\times 10^{-15}\text{Watts}/\text{m}^2.$$

Problem (9.7).

Consider the sum of 5 phasors:

$$S=e^{i\phi}+e^{2i\phi}+e^{3i\phi}+e^{4i\phi}+e^{5i\phi}.$$

This is the sum of 5 waves: the phase shift between each pair of waves is ϕ.

a. Calculate the sum for ϕ = 0
b. Calculate the sum for $\phi=\frac{\pi}{10}$
c. Calculate the sum for ϕ = π/5
d. Calculate the sum for $\phi=\frac{2\pi}{5}$
e. Make a sketch of S as a function of ϕ.

A graphical construction is useful for summing phasors. Notice that one has to do with a geometrical series.

Answer (9.7).

(a) S = 5.0

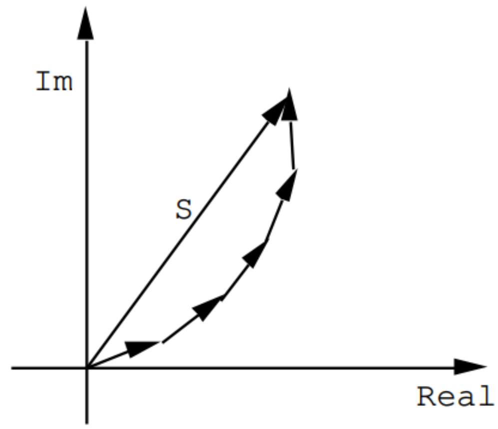
(b) $S = e^{i\phi}\left(1 + e^{i\phi} + e^{2i\phi} + e^{3i\phi} + e^{4i\phi}\right)$

$= e^{i\phi \frac{\left(e^{5i\phi}-1\right)}{\left(e^{i\phi}-1\right)}}$

S = 2.657 + 3.657i

|S| = 4.520

Angle = 54.00°

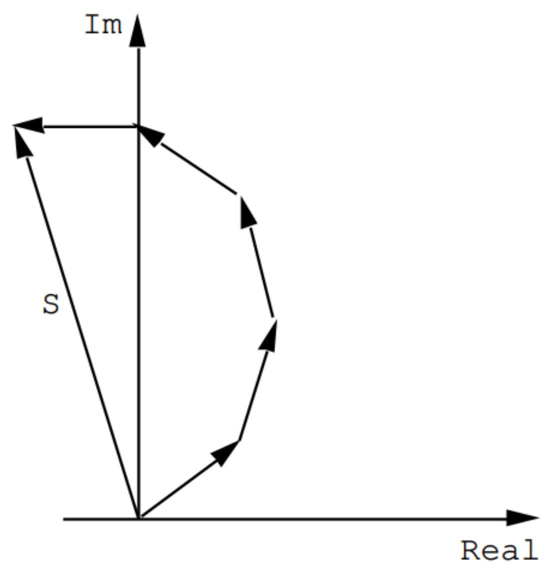


(c) $\phi = \frac{\pi}{5} = 36^{\circ}$

S = -1.00 + 3.078i

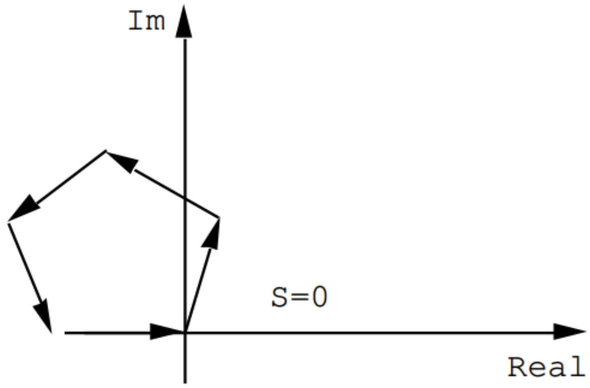
Angle = 108.00°

|S| = 3.236



(d) $\phi = \frac{2\pi}{5} = 72^{\circ}$

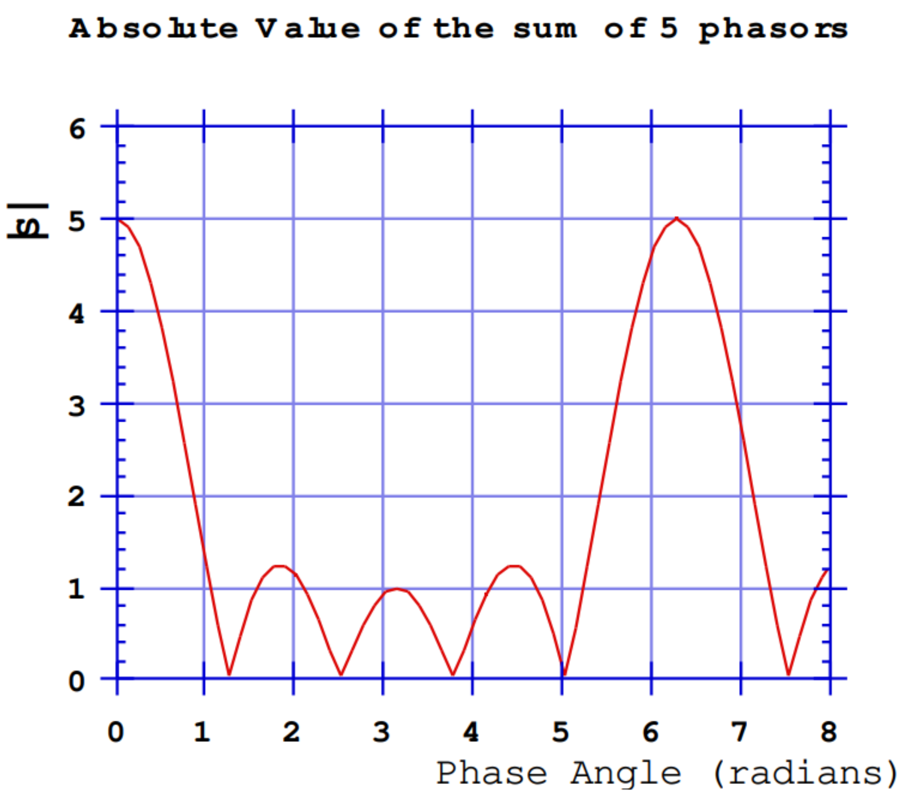
S = 0



(e) (i) When ϕ is a multiple of 2π S = 5.0

(ii) The sum is a geometric progression

$$S = e^{i\phi}\left(1 + e^{i\phi} + e^{2i\phi} + e^{3i\phi} + e^{4i\phi}\right)$$
$$= e^{i\phi\frac{\left(e^{5i\phi}-1\right)}{\left(e^{i\phi}-1\right)}}$$
$$|S|^2 = SS^\star = \frac{1-\cos 5\phi}{1-\cos\phi}$$



Notice that for N phasors

$$|S|^2 = \frac{1-\cos N\phi}{1-\cos\phi}$$

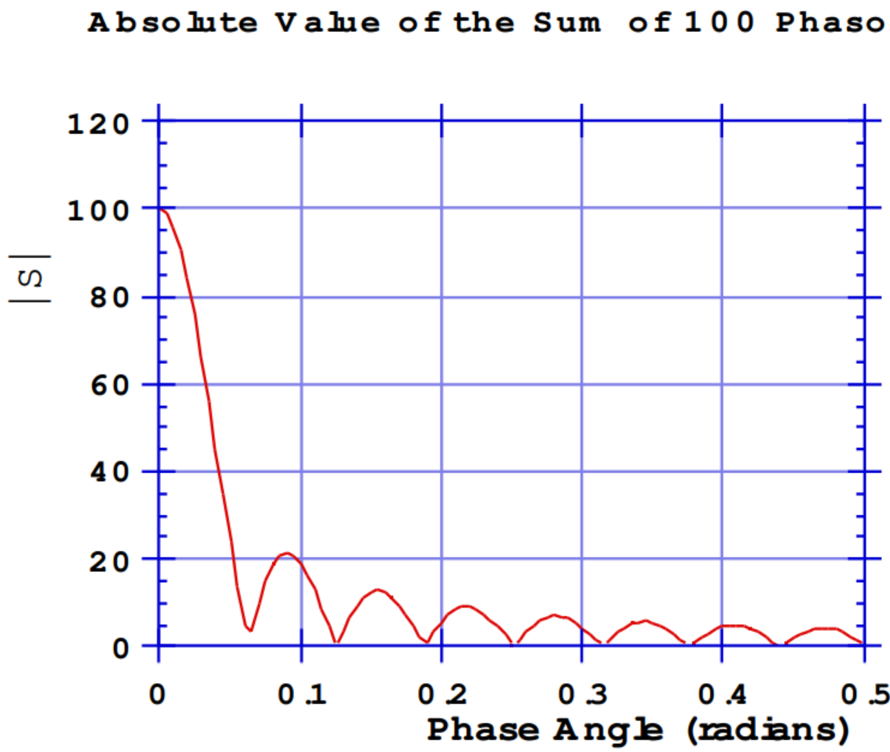
So when ϕ = 0 or 2π $|S|^2 = N^2$.

These high peaks drop to zero when Nϕ = 2π or $\phi = \frac{2\pi}{N}$ and $|S|^2$ = 0 for multiples of $\frac{2\pi}{N}$.

There are peaks at $\phi=(\text{ odd integer } \geq 3) \times \frac{\pi}{N}=(g)\left(\frac{\pi}{N}\right)$.

However $\cos \left(\frac{g \pi}{N}\right) \simeq 1-\frac{g^2 \pi^2}{2 N^2}$

$\therefore|\mathrm{S}|^2 \cong \frac{4 \mathrm{~N}^2}{\mathrm{~g}^2 \pi^2} \sim\left(\frac{.41}{\mathrm{~g}^2}\right) \mathrm{N}^2$, so these subsidiary peaks drop off as .045, .0162, etc. and $|S|^2$ drops off rapidly with phase angle.



Problem (9.8).

Eight atoms are located on the corners of a cube whose sides are a long. One corner of the cube is located at the origin of co-ordinates, and the sides of the cube are parallel with the co-ordinate axes. The polarizability of each atom is α, i.e. in the presence of an electric field the atom developes a dipole moment given by **p=** $\boldsymbol{\alpha}$**E**. Let an incident free space plane wave of the form

$$\mathrm{E}_{\mathrm{z}}=\mathrm{E}_0 \mathrm{e}^{\mathrm{i}(\mathrm{kx}-\omega \mathrm{t})}$$

fall on the group of 8 atoms, where k= 2π/a.

(a) Write an expression for the electric field which would be measured by an observer whose spherical polar co-ordinates are (R,θ, ϕ). Your answer should be in the form of the electric field amplitude generated by an atom at the origin multiplied by the structure factor, S.

(b) Explicitly evaluate the structure factor for this problem for an observer confined to the x-y plane (θ= π/2). Make a plot of the absolute square of the structure factor as a function of the angle ϕ (a quantity proportional to the intensity of the scattered radiation).

Answer (9.8).

The electric field component E_θ at the position of the observer due to the atom at $\mathbf{r}_m$ is given by

$$\mathrm{E}_{\mathrm{m}}=\frac{-\sin \theta}{4 \pi \varepsilon_0}\left(\frac{\omega}{\mathrm{c}}\right)^2 \frac{\alpha \mathrm{E}_0}{\mathrm{R}} e^{-\mathrm{i} \omega(\mathrm{t}-\mathrm{R} / \mathrm{c})} e^{\mathrm{i}\left[\left(\mathbf{k}_{\mathrm{i}}-\mathbf{k}_{\mathrm{f}}\right) \cdot \mathbf{r}_{\mathrm{m}}\right]}$$

where

$$\mathbf{k}_{\mathrm{i}}=\frac{\omega}{\mathrm{c}}\hat{\mathbf{u}}_{\mathrm{x}}=\frac{2\pi}{\mathrm{a}}\hat{\mathbf{u}}_{\mathrm{X}},$$

and

$$\mathbf{k}_{\mathrm{f}}=\frac{\omega}{\mathrm{c}}\hat{\mathbf{u}}_{\mathrm{T}}.$$

But $\hat{\mathbf{u}}_{r}=\sin\theta\cos\phi\hat{\mathbf{u}}_{\mathrm{x}}+\sin\theta\sin\phi\hat{\mathbf{u}}_{\mathrm{y}}+\cos\theta\hat{\mathbf{u}}_{z}$,

so that

$$(\mathbf{k}_{\mathrm{i}}-\mathbf{k}_{\mathrm{f}})=\frac{\omega}{\mathrm{c}}((1-\sin\theta\cos\phi)\hat{\mathbf{u}}_{\mathrm{x}}-\sin\theta\sin\phi\hat{\mathbf{u}}_{\mathrm{y}}-\cos\theta\hat{\mathbf{u}}_{\mathrm{z}})$$

or

$(\mathbf{k}_{\mathrm{i}}-\mathbf{k}_{\mathrm{f}})=\frac{2\pi}{\mathrm{a}}((1-\mathrm{Sin}\,\theta\cos\phi)\hat{\mathbf{u}}_{\mathrm{x}}-\sin\theta\sin\phi\hat{\mathbf{u}}_{\mathrm{y}}-\cos\theta\hat{\mathbf{u}}_{\mathrm{z}})$.

The total electric field amplitude measured by the observer is the sum of the fields scattered by each atom; it will be proportional to the structure factor

$$S=\sum_{m=1}^{8}e^{i(\mathbf{k}_{1}-\mathbf{k}_{f})\cdot\mathbf{r}_{m}}$$

where

$$\begin{gathered}\mathbf{r}_{1}=0\\ \mathbf{r}_{2}=\mathrm{a}\hat{\mathbf{u}}_{\mathrm{x}}\\ \mathbf{r}_{3}=\mathrm{a}\left(\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_{\mathrm{y}}\right)\\ \mathbf{r}_{4}=\mathrm{a}\hat{\mathbf{u}}_{\mathrm{y}}\\ \mathbf{r}_{5}=\mathrm{a}\left(\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_{\mathrm{z}}\right)\\ \mathbf{r}_{6}=\mathrm{a}\left(\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_{\mathrm{y}}+\hat{\mathbf{u}}_{z}\right)\\ \mathbf{r}_{7}=\mathrm{a}\left(\hat{\mathbf{u}}_{\mathrm{y}}+\hat{\mathbf{u}}_{z}\right)\\ \mathbf{r}_{8}=\mathrm{a}\left(\hat{\mathbf{u}}_{\mathrm{y}}+\hat{\mathbf{u}}_{\mathrm{z}}\right).\end{gathered}$$

$$\begin{aligned}S=1&+e^{i\frac{a\omega}{c}(1-\sin\theta\cos\phi)}+e^{i\frac{a\omega}{c}}(1-\sin\theta\cos\phi-\sin\theta\sin\phi)+e^{-i\frac{a\omega}{c}(\sin\theta\sin\phi)}+e^{i\frac{a\omega}{c}}(1-\sin\theta\cos\phi-\cos\theta)\\ &+e^{i\frac{a\omega}{c}}(1-\sin\theta\cos\phi-\sin\theta\sin\phi-\cos\theta)+e^{-i\frac{a\omega}{c}}(\sin\theta\sin\phi+\cos\theta)+e^{-i\frac{a\omega}{c}\cos\theta}.\end{aligned}$$

For the case θ= π/2 the structure factor becomes

$S=2\left(1+e^{-i\frac{\omega a\sin\phi}{C}}+e^{i\frac{\omega a(1-\mathrm{Cos}\,\phi)}{C}}+e^{i\frac{\omega a(1-\cos\phi-\sin\phi)}{c}}\right)$.

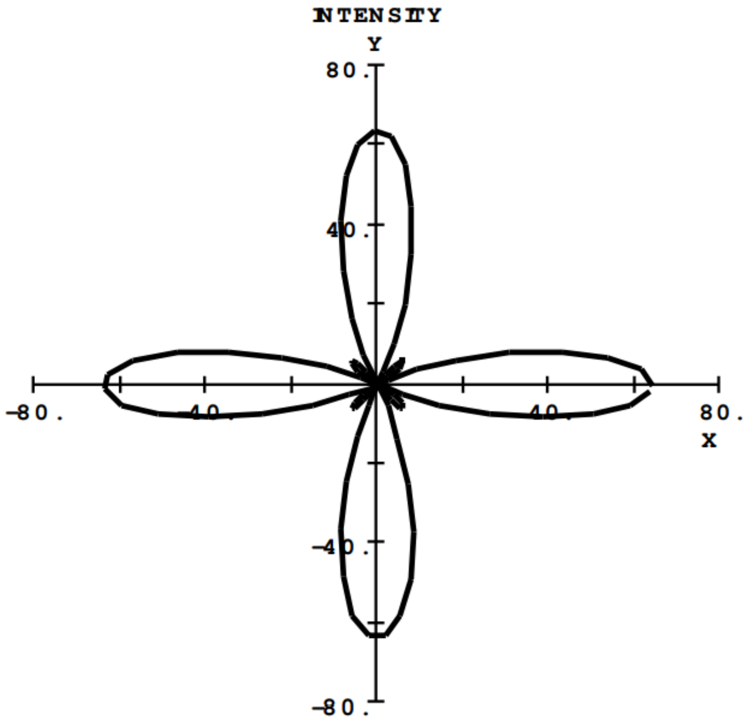
But $k=\frac{\omega}{c}=\frac{2\pi}{a}$; therefore $\frac{\omega a}{c}=2\pi$, and

$S=2\left(1+e^{-2\pi i\,\mathrm{Sin}\,\phi}+e^{2\pi i(1-\cos\phi)}+e^{2\pi i(1-\cos\phi-\sin\phi)}\right)$.

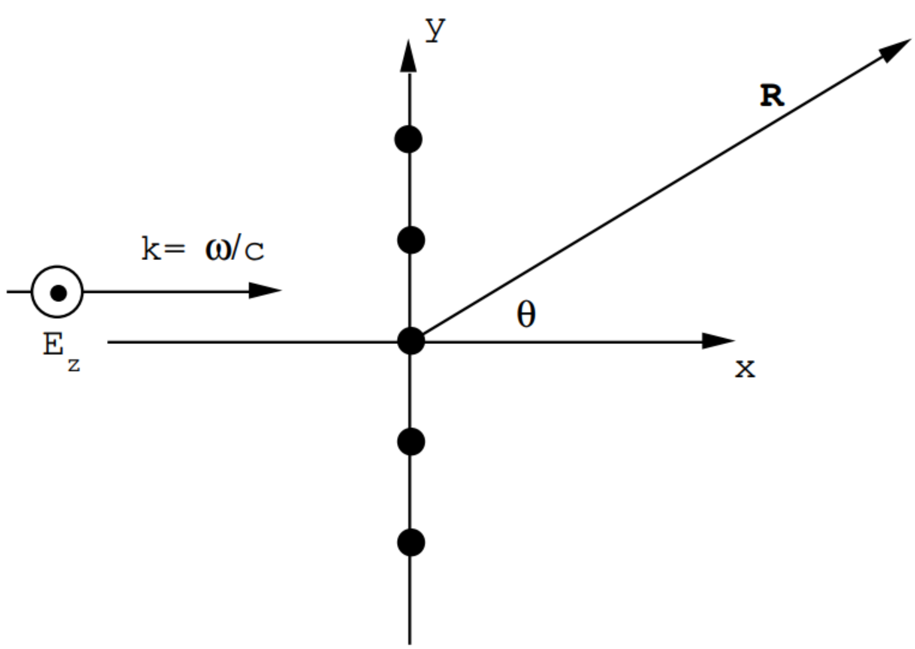
A little tedious algebra (no gain without pain!) can be used to put the absolute square of S into the form

$\mathrm{SS}^{\star}=16(1+\cos(2\pi\sin\phi)+\cos(2\pi\cos\phi)+\cos(2\pi\sin\phi)\cos(2\pi\cos\phi))$.

A polar plot of the quantity SS* shows the way in which the intensity varies with angle for an observer in the x-y plane. In this plot SS*Sinϕ is plotted against SS*Cosϕ.



Problem (9.9).



A plane wave whose electric field is given by

$$E_z = E_O e^{i(kx - \omega t)}$$

is incident upon 5 hydrogen atoms which are spaced a distance a = 1.5 x 10^{-10} meters along the y-axis as shown in the sketch. The frequency associated with the electric field is 10^{18} Hz.

An observer in the x-y plane is located very far away in a position specified by the angle θ shown in the sketch.

(a) Calculate the structure factor for the scattered radiation.

(b) Make a sketch of the angular variation of the intensity measured by P as θ ranges from 0 to $\pi/2$.

Answer (9.9).

The electric field amplitude at R due to a single atom is independent of θ and the electric field is polarized along z. However, the fields from the 5 atoms interfere because for fixed time of observation, the phase of each wave is shifted. The structure factor is given by

$$S = 1 + e_1^{-iq\cdot r} + e_2^{-iq\cdot r} + - - - + e^{-iq\cdot r_5}$$

where $\mathbf{q} = (\mathbf{k}_\mathrm{f} - \mathbf{k}_\mathrm{i})$.

In this problem

$\mathbf{k}_\mathrm{i} = \frac{\omega}{\mathrm{c}}\hat{\mathbf{u}}_\mathrm{x}$
$\mathbf{k}_\mathrm{f} = \frac{\omega}{\mathrm{c}}[\cos\theta\hat{\mathbf{u}}_\mathrm{x} + \sin\theta\hat{\mathbf{u}}_\mathrm{y}]$
$\therefore \mathrm{q} = \frac{\omega}{\mathrm{c}}[(\cos\theta - 1)\hat{\mathrm{u}}_\mathrm{x} + \sin\theta\hat{\mathrm{u}}_\mathrm{y}]$.

The atomic positions are given by

$$\mathbf{r}_n = n\,\mathrm{a}\,\hat{\mathbf{u}}_y$$

$$\therefore \quad \mathbf{q}\cdot\mathbf{r}_n = n\frac{a\omega}{c}\sin\theta = n\phi.$$

Thus $S = 1 + e^{-i\phi} + e^{-2i\phi} + e^{i\phi} + e^{2i\phi}$

(the first term corresponds to n = 0),

or $S = 1 + 2\cos\phi + 2\cos 2\phi$,

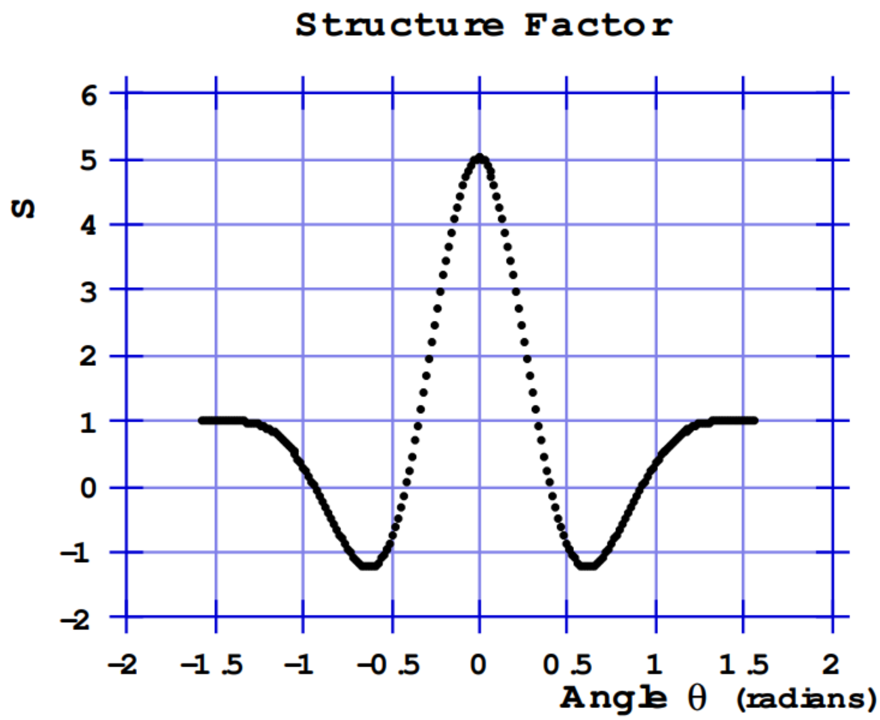
or $S = 1 + 2\cos\left(\frac{a\omega}{c}\sin\theta\right) + 2\cos\left(\frac{2a\omega}{c}\sin\theta\right)$

In this problem $\frac{\omega}{c} = \frac{2\pi}{3} \times 10^{10}$ and $a = \frac{3}{2} \times 10^{-10}\mathrm{m}$.

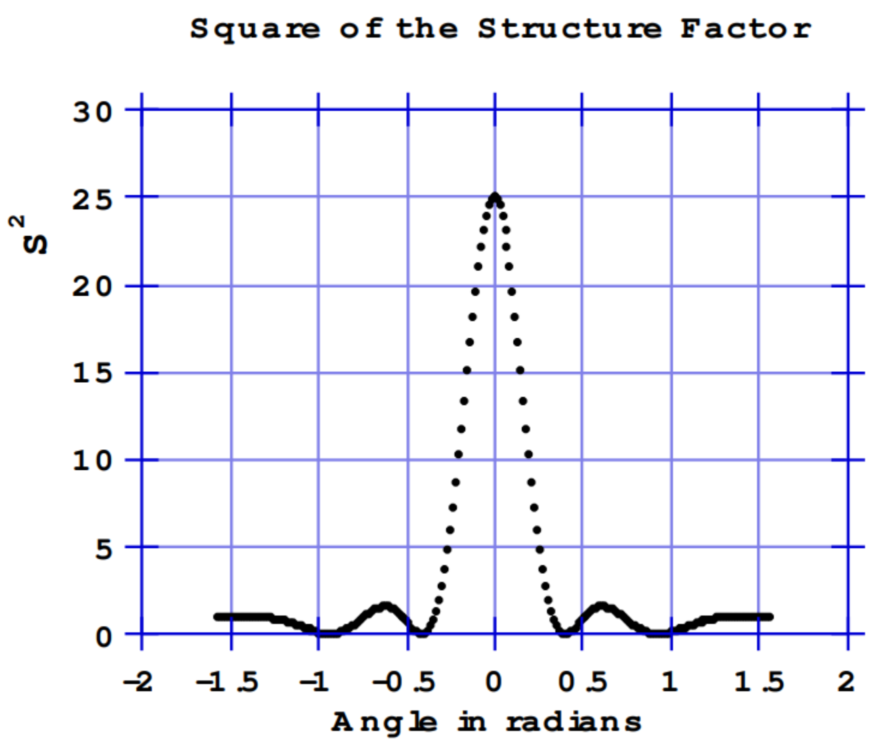
$\therefore \frac{a\omega}{c} = \pi$ and

$\boldsymbol{S = 1 + 2\cos(\pi\sin\theta) + 2\cos(2\pi\sin\theta)}$

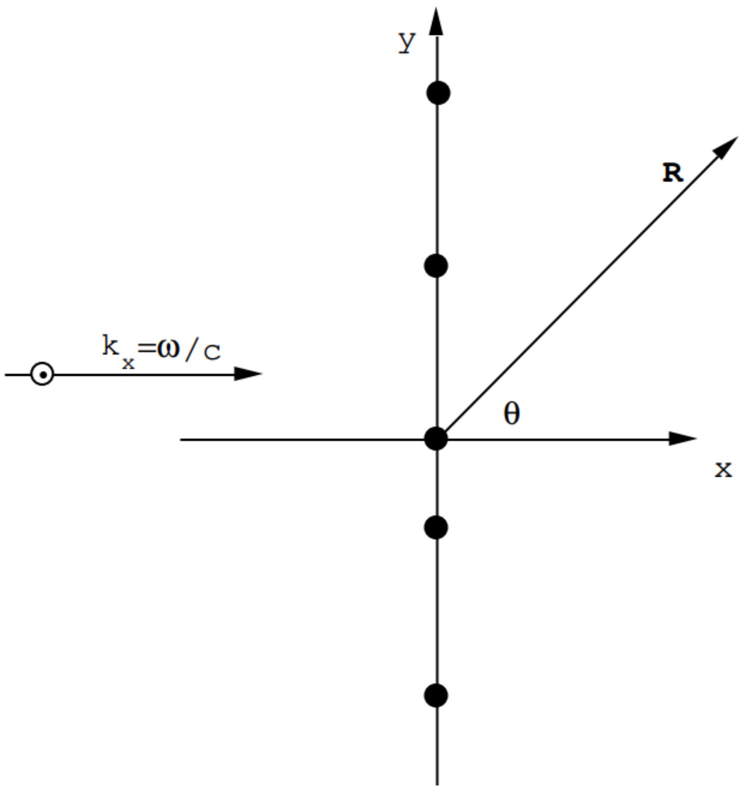
(See the figure below).



The intensity measured at P is proportional to $|S|^2$, or to S^2 in this case since S is real. Note the strong forward scattering pattern (see the figure below).



Problem (9.10).



This is a repeat of the previous problem but the scattering centers are not equally spaced. A plane wave is incident from the left $\mathrm{E_z = E_O e^{i(kx-\omega t)}}$

The atoms are at:

$$\mathbf{r}_1 = 0$$
$$\mathbf{r}_2 = \frac{8a}{7}\hat{\mathbf{u}}_\mathrm{y} \qquad \mathbf{r}_3 = \left(\frac{16\mathrm{a}}{7}\right)\hat{\mathbf{u}}_\mathrm{y}$$
$$\mathbf{r}_4 = -\left(\frac{4\mathrm{a}}{7}\right)\hat{\mathbf{u}}_\mathrm{y} \quad \mathbf{r}_5 = -\left(\frac{12\mathrm{a}}{7}\right)\hat{\mathbf{u}}_\mathrm{y}.$$

This is a more or less random spacing which preserves an average spacing of a.

Calculate the dependence upon the angle θ of the intensity of the scattered radiation which would be observed at a distant point P. a = 1.5 x 10^{-10} m and ω = 2π x 10^{18} rad./sec.

Answer (9.10).

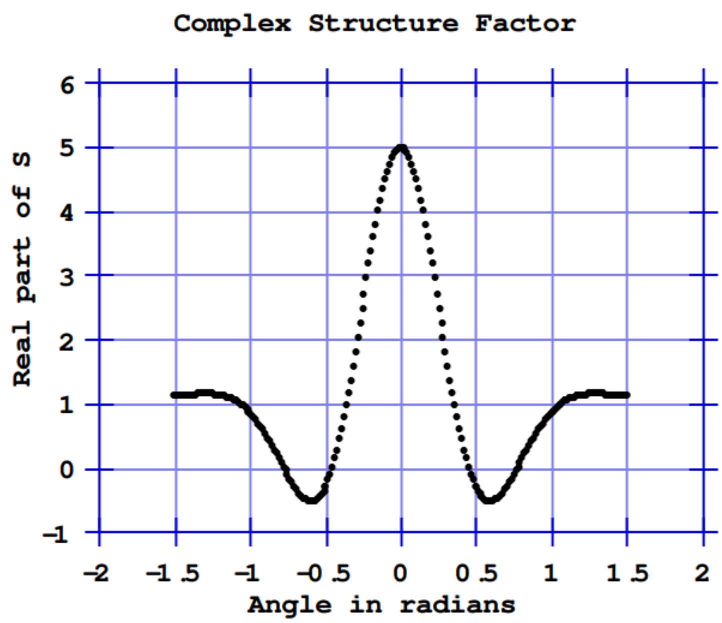
$$\mathbf{k}_i = \frac{\omega}{\mathrm{c}}\hat{\mathbf{u}}_\mathrm{x}$$
$$\mathbf{k}_\mathrm{f} = \frac{\omega}{\mathrm{c}}[\cos\theta\hat{\mathbf{u}}_\mathrm{x} + \sin\theta\hat{\mathbf{u}}_\mathrm{Y}]$$
$$\therefore \mathbf{q} = (\mathbf{k}_\mathrm{f} - \mathbf{k}_\mathrm{i}) = \frac{\omega}{\mathrm{C}}[(\cos\theta - 1)\hat{\mathbf{u}}_\mathrm{x} + \sin\theta\hat{\mathbf{u}}_\mathrm{y}]$$
$$\therefore \mathbf{q}\cdot\mathbf{r}_1 = 0 \qquad \mathbf{q}\cdot\mathbf{r}_2 = \frac{8}{7}\left(\frac{a\omega}{c}\right) \quad \sin\theta$$
$$\mathbf{q}\cdot\mathbf{r}_3 = \left(\frac{16}{7}\right)\left(\frac{a\omega}{c}\right)\sin\theta$$
$$\mathbf{q}\cdot\mathbf{r}_4 = -\left(\frac{4}{7}\right)\left(\frac{a\omega}{c}\right)\sin\theta$$
$$\mathbf{q}\cdot\mathbf{r}_5 = -\left(\frac{12}{7}\right)\left(\frac{\mathrm{a}\omega}{\mathrm{c}}\right)\sin\theta$$
$$\frac{\omega}{c} = \left(\frac{2\pi}{3}\right)\left(10^{10}\right) \qquad a = \frac{3}{2}\times 10^{-10} \qquad \therefore \frac{a\omega}{c} = \pi$$
$$S = 1 + e^{-i}8\pi\sin\theta/7 + e^{-i}16\pi\sin\theta/7 + e^{i}4\pi\sin\theta/7 + e^{i}12\pi\sin\theta/7$$
$$S = \left[1 + \cos\left[\left(\frac{8\pi}{7}\right)\sin\theta\right] + \cos\left[\left(\frac{16\pi}{7}\right)\sin\theta\right] + \cos\left[\left(\frac{4\pi}{7}\right)\sin\theta\right] + \cos\left[\left(\frac{12\pi}{7}\right)\sin\theta\right)\right]$$

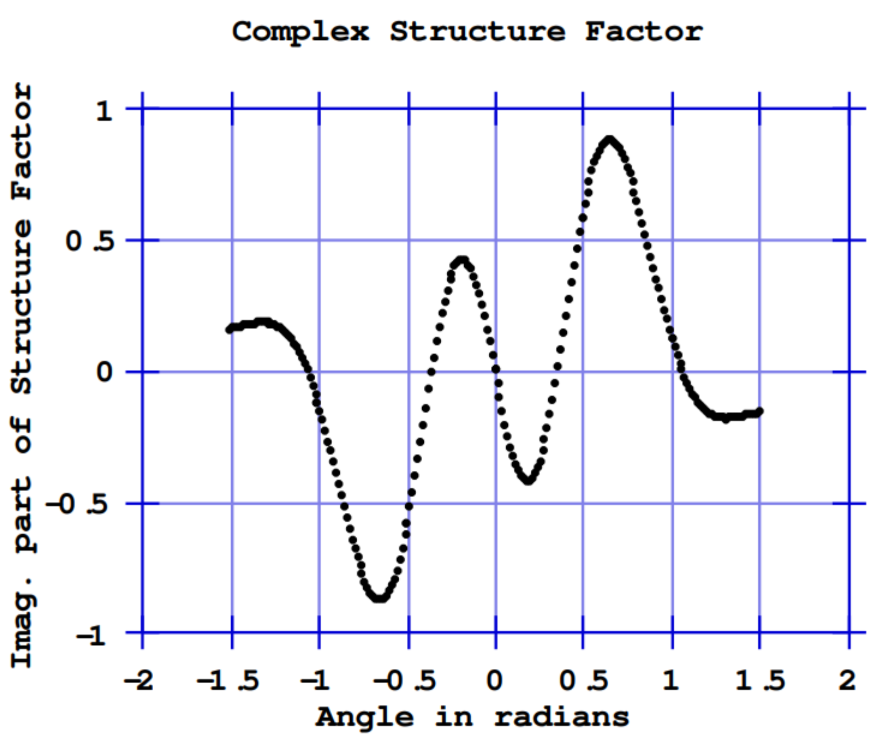
$$+i\left[-\sin\left[\left(\frac{8\pi}{7}\right)\sin\theta\right]-\sin\left[\left(\frac{16\pi}{7}\right)\sin\theta\right]+\sin\left[\frac{4\pi}{7}\sin\theta\right]+\sin\left[\left(\frac{12\pi}{7}\right)\sin\theta\right]\right]$$

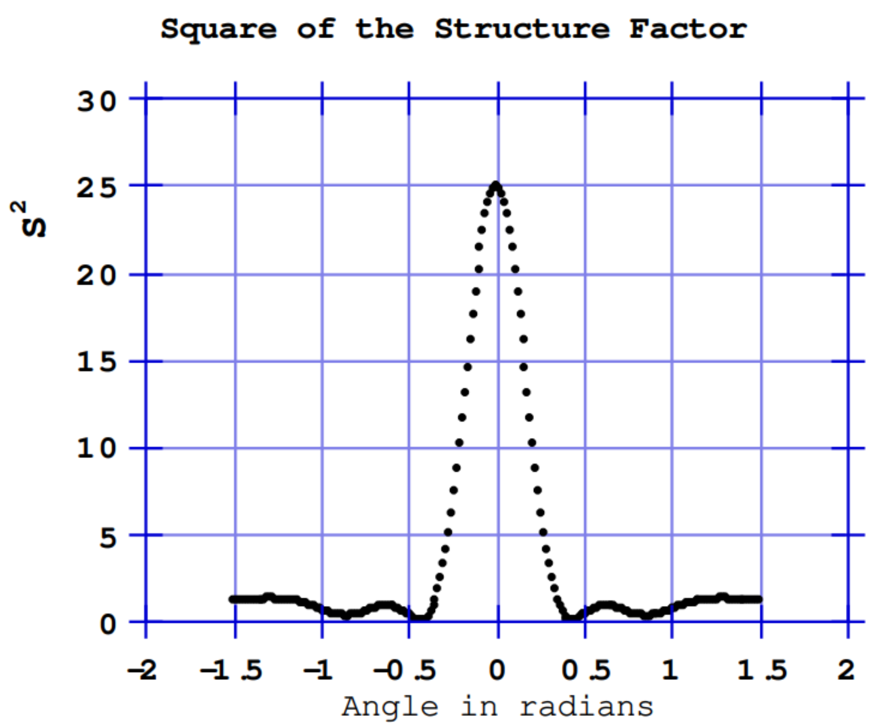
The structure factor can be written S = a + ib, then the intensity required is proportional to

$|S|^2 = SS^* = a^2 + b^2$

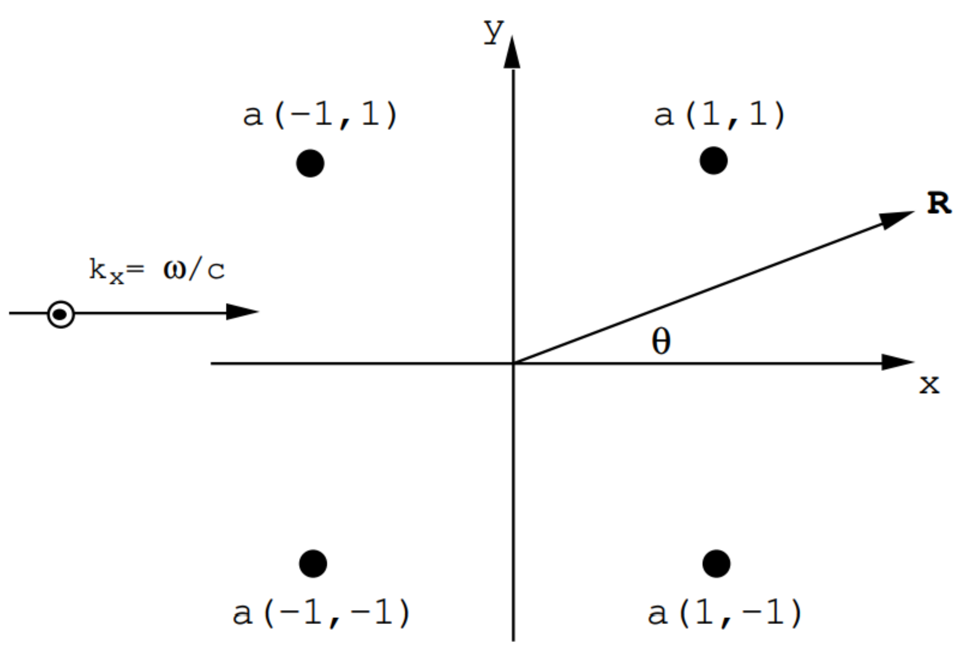
The result of the calculation is shown in the figures. The main peak at θ = 0 persists because all signals remain in phase no matter where the scatterers are located along the y axis. The main effect of the irregular spacing is to reduce the structure in the "wings" i.e. the oscillations at angles larger than 30˚.







Problem (9.11).



Four scattering centers are arranged on the grid shown above. A plane wave is incident from the left:

$$E_z = E_O e^{i(kx-\omega t)}$$

where ω = 2π x 10^{18} rad/sec. The parameter $a = \frac{3}{2} \times 10^{-10}\mathrm{m}$ and $\frac{a\omega}{c} = \pi$. Calculate the dependence of the scattered intensity on the angle of observation θ when the observer is very far away (R >> a).

Answer (9.11).

The structure factor is given by $S=\sum_n \exp(-i\mathbf{q}\cdot\mathbf{r}_n)$

where $\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i$

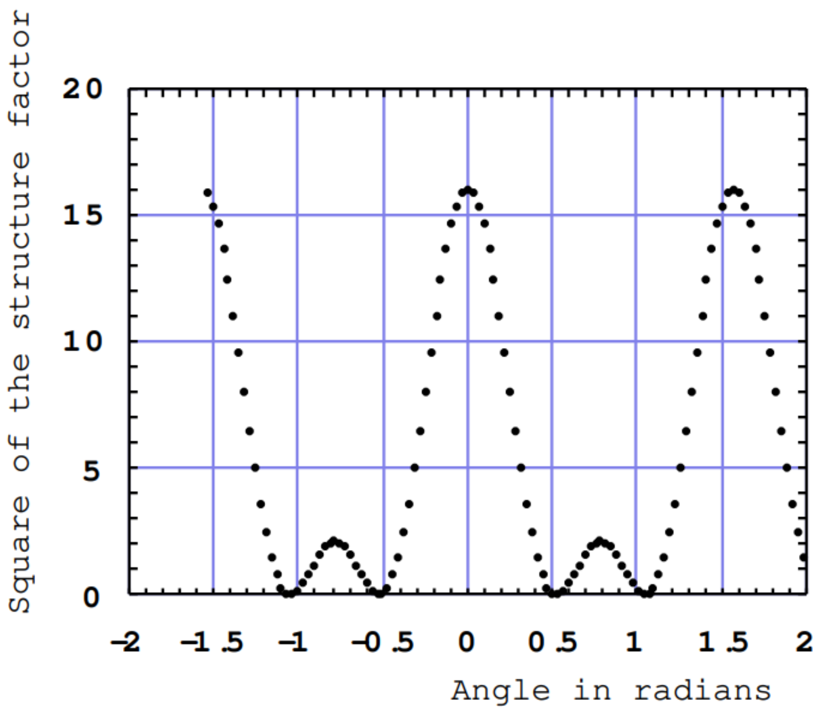
In this case $\mathbf{q}=\frac{\omega}{c}[(\cos\theta-1)\hat{\mathbf{u}}_x+\sin\theta\hat{\mathbf{u}}_y]$

$\therefore S=e^{-i\frac{\omega a}{C}}[(\cos\theta-1)+\sin\theta]+e^{-i\frac{\omega a}{c}}[(\cos\theta-1)-\sin\theta]+e^{-i\frac{\omega a}{a}}[-(\cos\theta-1)+\sin\theta]$
$+e^{+i\frac{\omega a}{c}}[(\cos\theta-1)+\sin\theta]$

or for $\frac{a\omega}{c}=\pi$

$S=2\cos\pi[\cos\theta+\sin\theta-1]+2\cos\pi[\cos\theta-\sin\theta-1]$.

This is real because of the symmetry around the origin. S^2 is plotted in the figure below.



13.10: Chapter- 10

Problem (10.1).

(a) Use Stokes' theorem to show that the Maxwell equation curl $\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$ can be written in the form

$$\oint_{C} \mathbf{E} \cdot d \mathbf{L}=-\frac{\partial}{\partial t} \quad \int_{\text{Surface S}} \mathbf{B} \cdot \mathbf{d S} \tag{1}$$

where the surface S is bounded by the closed curve c.

(b) Apply the above equation to a loop which straddles the boundary between two materials to show that the tangential component of **E** must be continuous across the boundary.

Answer (10.1).

(a) curl $\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$

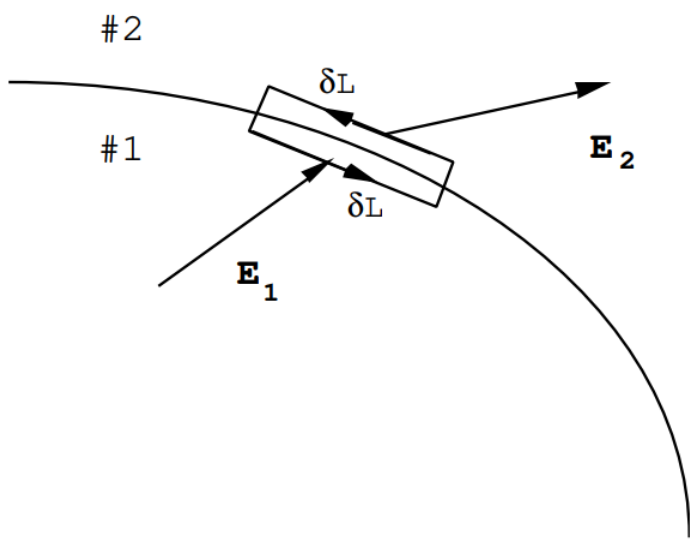
Integrate over a surface S bounded by a curve c:

$\int_{S} \text{curl} \quad \mathbf{E} \cdot d \mathbf{S}=-\frac{\partial}{\partial t} \int_{S} \mathbf{B} \cdot d \mathbf{S}$

But from Stokes' theorem

$\int_{S} \text{Curl}\, \mathbf{E} \cdot \mathrm{d} \mathbf{S}=\oint_{c} \mathbf{E} \cdot \mathrm{d} \mathbf{L}$, and the result follows.

(b) Apply the above to a loop δL long and of negligible width, δd.



Then $\oint_{c} \mathbf{E} \cdot \mathrm{d} \mathbf{L}=\left.\mathrm{E}_{2}\right)_{\text{tang}} \delta \mathrm{L}-\left.\mathrm{E}_{1}\right)_{\text{tang}} \delta \mathrm{L}$

$=-\frac{\partial}{\partial t}\left(\mathrm{B}_{\text{perp}} . \delta \mathrm{L} \delta \mathrm{d}\right) \Rightarrow 0$

therefore

$\left.\mathrm{E}_{2}\right)_{\text{tangential}}=\left.\mathrm{E}_{1}\right)_{\text{tangential}}$

Problem (10.2).

(a) Use Stokes' theorem to transform the Maxwell equation

$$\text{curl}\, \mathbf{H}=\mathbf{J}_{\mathrm{f}}+\frac{\partial \mathbf{D}}{\partial \mathrm{t}}$$

into

$$\oint_C \mathbf{H}\cdot d\mathbf{L} = \int_S \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S},$$

where the surface S is bounded by the closed curve, c.

(b) Use the above equation to show that at the surface of discontinuity between two materials the tangential component of **H** must be continuous.

Answer (10.2).

(a) $\text{Curl}\,\mathbf{H} \quad = \quad \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right)$

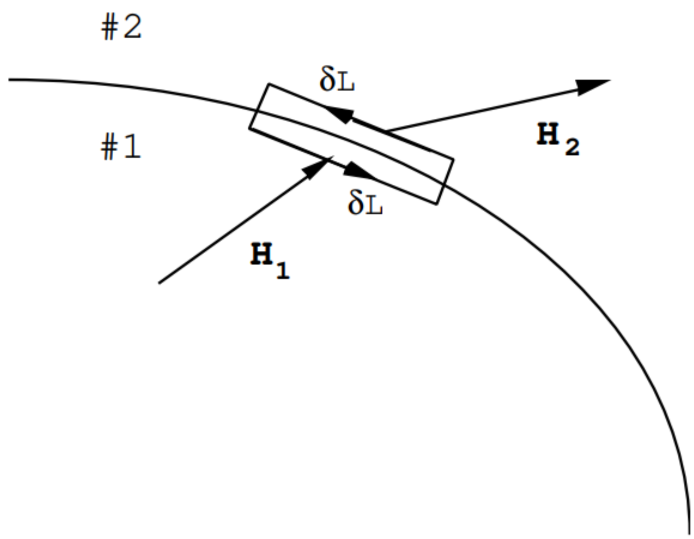
$\therefore \quad \int_S \text{Curl}\,\mathbf{H}\cdot d\mathbf{S} = \int_S \mathbf{J}_f\cdot d\mathbf{S} + \frac{\partial}{\partial t}\int_S \mathbf{D}\cdot d\mathbf{S}.$

But by Stokes' theorem:

$$\int_S \text{Curl}\,\mathbf{H}\cdot d\mathbf{S} = \oint_c \mathbf{H}\cdot d\mathbf{L}$$

from which the result follows.

(b) Apply the above theorem to a loop straddling the boundary. The loop is δL long and δd wide.



$\oint_c \delta\mathbf{H}\cdot d\mathbf{L} = \left.H_2\right)_{\text{tang}}\delta L - \left.H_1\right)_{\text{tang}}\delta L + \text{ terms 2nd order in } \delta d$

$\int_S \left(J_f + \frac{\partial D}{\partial t}\right)\cdot ds = \left(J_f + \frac{\partial D}{\partial t}\right)_{\text{normal}}\delta L\delta d \Rightarrow 0 \text{ as } \delta d \to 0$

$\therefore \quad \left.H_2\right)_{\text{tang}} = \left.H_1\right)_{\text{tang}}$

Problem (10.3).

a. From div **B** = 0 show that the normal component of **B** is continuous across the boundary between two different materials.
b. From div **D** = ρ_f show that there will be a surface charge density on the surface of discontinuity between two materials. Show that the magnitude of this surface charge density is given by

$$\rho_f = \left.D_2\right)_{\text{normal}} - \left.D_1\right)_{\text{normal}}$$

where $\left.D_2\right)_{\text{normal}}$ and $\left.D_1\right)_{\text{normal}}$ are the normal components of the vector **D**.

Answer (10.3).

(a) $\text{div}\,\mathbf{B} = 0$

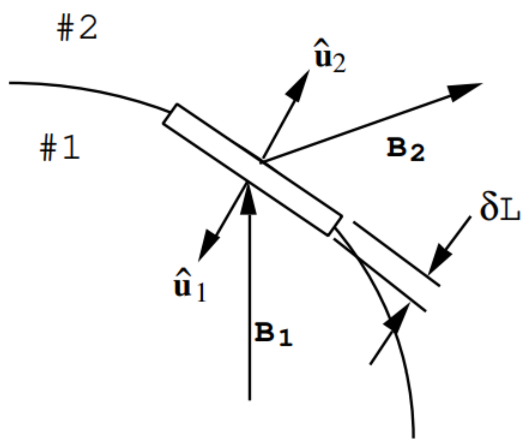
$\therefore \int_V \operatorname{div}\mathbf{B}\, d\tau = 0$

But by Gauss' theorem $\int_V \operatorname{div}\mathbf{B}\, d\tau = \int_S \mathbf{B}\cdot d\mathbf{S}$

where S is the surface bounding the closed volume V.

Therefore $\int_S \mathbf{B}\cdot d\mathbf{S} = 0$

Apply this to a pill box of area δA and thickness δL which straddles the boundary between material (1) and material (2)



$$\int_{\text{Pill Box}} \underset{\text{box}}{\mathbf{B}\cdot \mathrm{d}\mathbf{S}} = \left[\left.\mathrm{B}_2\right)_{\text{normal}} - \left.\mathrm{B}_1\right)_{\text{normal}}\right] \delta A + \text{ terms of 2nd order in } \delta L$$

(As shown, $\mathbf{B}_2 \cdot \hat{\mathbf{u}}_2$ makes a positive contribution and $\mathbf{B}_1 \cdot \hat{\mathbf{u}}_1$ makes a negative contribution).

Therefore $\left[\left.\mathrm{B}_2\right)_{\text{normal}} - \left.\mathrm{B}_1\right)_{\text{normal}}\right] \delta\mathrm{A} = 0$ for arbitrary δA and

$$\therefore \left.\mathrm{B}_2\right)_{\text{normal}} = \left.\mathrm{B}_1\right)_{\text{normal}}$$

(b) div **D** = ρ_f

∴ for any closed volume V bounded by a surface S

$\int_V \operatorname{div}\mathbf{D}\, d\tau = \int_V \rho_f d\tau$

But by Gauss' theorem:

$\int_V (div\mathbf{D}) d\tau = \int_S \mathbf{D}\cdot \mathbf{ds}$

Apply this to a pill-box which straddles material (1) and material (2):

Then $\int_S \mathrm{D}\cdot \mathrm{ds} = \left[\left.\mathrm{D}_2\right)_{\text{normal}} \delta\mathrm{A} - \left.\mathrm{D}_1\right)_{\text{normal}} \delta\mathrm{A}\right]$

+ higher order corrections of order δLδA.

$\therefore \left[\left.\mathrm{D}_2\right)_{\text{normal}} - \left.\mathrm{D}_1\right)_{\text{normal}}\right] \delta\mathrm{A} = \rho_\mathrm{f} \delta\mathrm{A}\delta\mathrm{L}$

So $\left[\left.\mathrm{D}_2\right)_{\text{normal}} - \left.\mathrm{D}_1\right)_{\text{normal}}\right] = \rho_\mathrm{f}\delta\mathrm{L} = \rho_\mathrm{s}$,

where (ρ_f δ_L) does not depend upon the length δ_L and therefore represents a surface charge ρ_s. A discontinuity in the normal component of **D** means that there exists a surface charge density.

Problem (10.4).

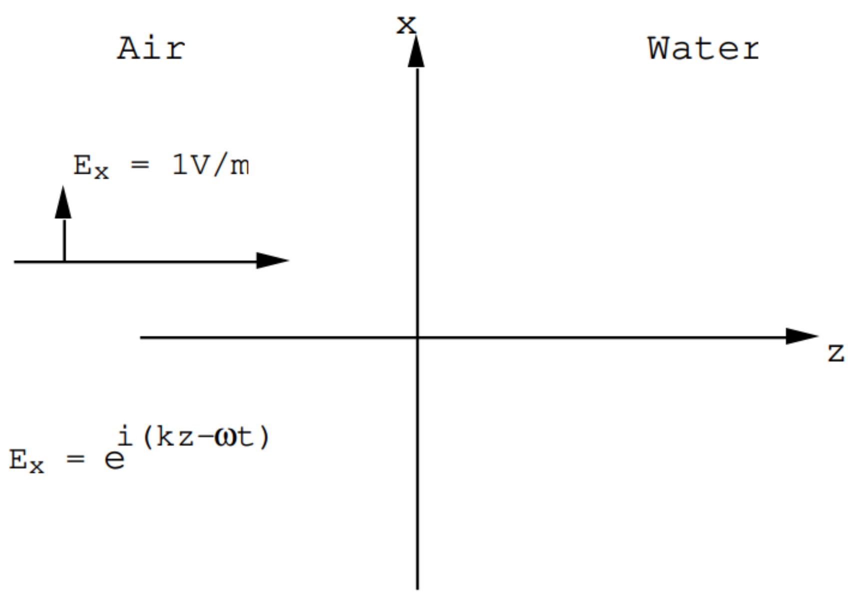
A plane wave falls at normal incidence on the plane surface of a large, deep, body of water. The real and imaginary parts of the index of refraction for water are n = 4/3 and κ = 10^{-8} corresponding to a time dependence ~ $e^{-i\omega t}$. The amplitude of the electric field in the incident wave is 1 V/m. Let the z-axis be directed into the water, and let the x, y axes lie in the surface of the water. Let the electric field be polarized along x. The index of refraction of air is n = 1, κ = 0.

a. Write an equation for the space and time variation of the electric field in the incident wave.

b. Write an equation for the space and time variations of **B**,**H** in the incident wave. What is the amplitude, H_o, of the **H** field?
c. Write expressions for the space and time variation of the reflected wave. Let the reflected electric field amplitude be E_R. Write the reflected magnetic field amplitude in terms of E_R.
d. Write expressions for the space and time variations of the electric and magnetic field waves (**H** field) transmitted into the water. Let the electric field amplitude at the water surface, at z = 0, be E_T. Write the magnetic field amplitude in terms of E_T.
e. State the boundary conditions which **E**, **H** must satisfy at the surface of the water.
f. Apply the boundary conditions of part (e) to obtain the reflected electric field amplitude, E_R, and the transmitted wave electric field amplitude, E_T.
g. What is the intensity of the incident wave? i.e. At what rate, in Watts/m^2, is energy transported to the water surface?
h. At what rate is energy absorbed by the water?
i. What will be the electric field amplitude at a depth of 2 m if the wavelength of the light is 1/2 micron?

Answer (10.4).

(a)



(b) $B_y = \frac{E_x}{c} = \frac{1}{c} e^{i(kz-\omega t)}$

$$\mathrm{H_y} = \frac{\mathrm{B_Y}}{\mu_\mathrm{O}} = \frac{1}{\mathrm{c}\mu_\mathrm{O}} \mathrm{e}^{\mathrm{i(kz-\omega t)}} = \frac{1}{120\pi} \mathrm{e}^{\mathrm{i(kz-\omega t)}} \quad \mathrm{Amps/m} \; .$$

$$\text{Amplitude } = \frac{1}{120\pi} = \frac{1}{377} \text{ Amps /m} \; .$$

c) Let the reflected electric field be

$$E_x = E_R e^{-i(kz+\omega t)}$$

(note change in sign of k).

Then $\mathrm{H_y} = -\frac{\mathrm{E_R}}{120\pi} \mathrm{e}^{-\mathrm{i(kz+\omega t)}}$

(d) In the water the propagation vector is given by $\mathrm{k} = \frac{\omega}{\mathrm{c}}(\mathrm{n} + \mathrm{i}\kappa)$

$$\therefore \mathrm{E_X} = \mathrm{E_T} e^{-\kappa \frac{\omega}{c} z} e^{i\left(\frac{n\omega z}{c} - \omega t\right)}$$

Now curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}} = \mathrm{i}\omega \mathbf{B} = \mathrm{i}\omega\mu_o \mathbf{H}$

$$\mathrm{i}\omega\mu_\mathrm{O} \mathbf{H} = \begin{vmatrix} \hat{\mathbf{u}}_\mathrm{x} & \hat{\mathbf{u}}_\mathrm{y} & \hat{\mathbf{u}}_\mathrm{z} \\ 0 & 0 & \frac{\partial}{\partial \mathrm{z}} \\ \mathrm{E_x} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} \\ 0 \end{vmatrix}$$

$$\therefore H_Y = \frac{1}{i\omega\mu_o}\frac{\partial E_x}{\partial z} = \frac{i\left(\frac{\omega}{c}\right)(n+i\kappa)E_X}{i\omega u_O} = \frac{(n+i\kappa)}{\mu_O c}E_X$$

and

$$H_y = \left(\frac{n+i\kappa}{\mu_O c}\right)E_T e^{-\kappa\frac{\omega z}{c}} e^{i\left(n\frac{\omega z}{c}-\omega t\right)}.$$

(e) At the interface the required boundary conditions are

(1) Tangential components of **E** must be continuous.

(2) Tangential components of **H** must be continuous.

(f) At z = 0

Incident Wave $E_x = (1)e^{-i\omega t}$

$$H_y = \frac{1}{c\mu_0}e^{-i\omega t}$$

Reflected Wave $E_x = E_R e^{-i\omega t}$

$$H_Y = -\frac{E_R}{c\mu_O}e^{-i\omega t}$$

Transmitted Wave $E_x = E_T e^{-i\omega t}$

$$H_Y = \frac{(n+i\kappa)}{c\mu_O}E_T e^{-i\omega t}$$

Continuity of E_x: $1 + E_R = E_T$ (1)

Continuity of H_y: $\frac{1}{c\mu_0} - \frac{E_R}{c\mu_0} = \frac{(n+i\kappa)}{c\mu_0}E_T$

or $1 - E_R = (n+i\kappa)E_T$ (2)

Solve eqns. (1) and (2) to obtain:

$$E_T = \frac{2}{(1+n)+i\kappa} = \frac{2[(n+1)-i\kappa]}{(n+1)^2+\kappa^2}$$

But $\kappa \simeq 0$ so $E_T = \frac{14/3}{(7/3)^2} = \frac{6}{7} = \underline{0.86\ Volts/m}.$

Also $E_T \simeq \left(\frac{2}{n+1}\right)$

and $E_R = E_T - 1 = \underline{-0.143\ Volts/m}.$

(NOTICE THE PHASE CHANGE IN THE ELECTRIC FIELD!!)

(g) Rate of transport of energy to the water surface is

$$S_z = E_x H_y$$

$$\langle S_z\rangle = \left(\frac{1}{2}\right)(1)\left(\frac{1}{c\mu_0}\right) = \frac{1}{754}\ \text{Watts}/\text{m}^2$$

$$= \underline{1.33\ mW/m^2}.$$

(h) The rate of energy reflected from the surface is

$$< S_z >_R = \frac{1}{2}(E_R)\frac{(E_R)}{c\mu_0} = \frac{(0.143)^2}{754} = \underline{0.027\ \text{mW}/\text{m}^2} = \underline{27\ \mu\text{W}/\text{m}^2}.$$

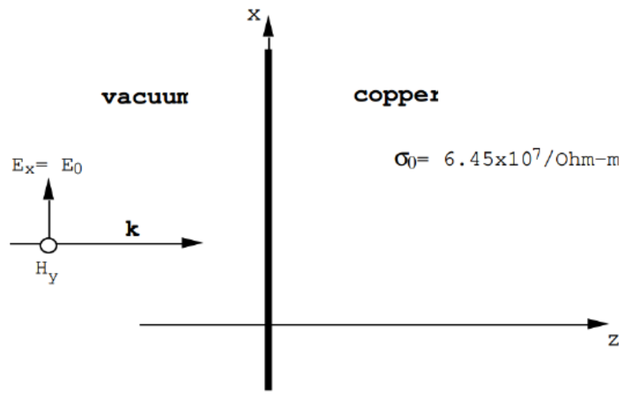
∴ Energy absorbed in H_2O = $\underline{1.30\ \text{mW/m}^2}$.

(i) At z = 2m

$$|E_X| = E_T e^{-\kappa 4\pi/\lambda} = E_T e^{-0.251} = 0.78\ E_T$$

∴ @ 2m the electric field strength = $\underline{0.67\ \text{V/m}}$.

Problem (10.5).



A wave having an electric field amplitude E_o = 1 V/m falls at normal incidence on a plane copper surface as shown in the above sketch. Its frequency is 10^6 Hz.

a. Write expressions for the electric and magnetic fields in the incident wave. How big is H_y?
b. Calculate the magnitude of the vacuum wave-vector.
c. Calculate the wave-vector in the metal (k_m) in the expressions:

$$E_x = E_T e^{i(k_m z - \omega t)}$$

$$H_Y = H_T e^{i(k_m z - \omega t)}$$

d. Calculate the amplitude of the electric field at the surface of the metal i.e. E_T.
e. Calculate the magnetic field amplitude at the surface of the metal i.e. H_T.
f. Calculate the time average Poynting vector for the incident wave i.e. $<S_o>$
g. Calculate the time average Poynting vector for the energy flow into the metal i.e. $<S_m>$
h. From (f) and (g) calculate the absorption coefficient $\alpha = <S_m>/<S_o>$.
i. Calculate the average rate of energy dissipation as Joule heat in the metal. Show that the integral of this quantity from z = o to ∞ is just equal to $<S_m>$ from (g) above.

Answer (10.5).

(a) $k = \frac{\omega}{c} = \frac{2\pi \times 10^6}{3 \times 10^8} = \mathbf{2.094 \times 10^{-2}\ m^{-1}}$

$E_x = E_0 e^{i(kz - \omega t)} = e^{i(kz - \omega t)}$ since E_o = 1 V/m.

$H_Y = \frac{E_0}{Z_0} e^{i(kz - \omega t)} = \left(\mathbf{2.653 \times 10^{-3}}\right) e^{i(kz - \omega t)}$

since Z_o = 377 Ohms.

(b) See above. k = 2.094 x 10^{-2} /meter.

(c) In the metal:

$$\text{curl}\,\mathbf{E} = \begin{vmatrix} \hat{\mathbf{u}}_x & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{\partial E_x}{\partial z} \\ 0 \end{vmatrix} = i\omega\mu_0 \mathbf{H},$$

$$\text{curl}\,\mathbf{H} = \begin{vmatrix} \hat{\mathbf{u}}_x & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = \begin{vmatrix} -\frac{\partial H_y}{\partial z} \\ 0 \\ 0 \end{vmatrix} = \sigma \mathbf{E}.$$

$$\therefore \frac{\partial E_x}{\partial z} = i\omega\mu_0 H_y = i k_m E_x$$

$$\text{or } \mathbf{H_y} = \left(\frac{\mathbf{k_m}}{\omega\mu_0}\right) \mathbf{E_x}$$

$$\text{and } \frac{\partial H_Y}{\partial z} = -\sigma E_X$$

$\therefore \mathrm{ik_m H_y} = -\sigma \mathrm{E_x}$

or $\mathbf{H_y} = \left(\frac{\mathbf{i}\sigma}{\mathbf{k_m}}\right)\mathbf{E_x}$.

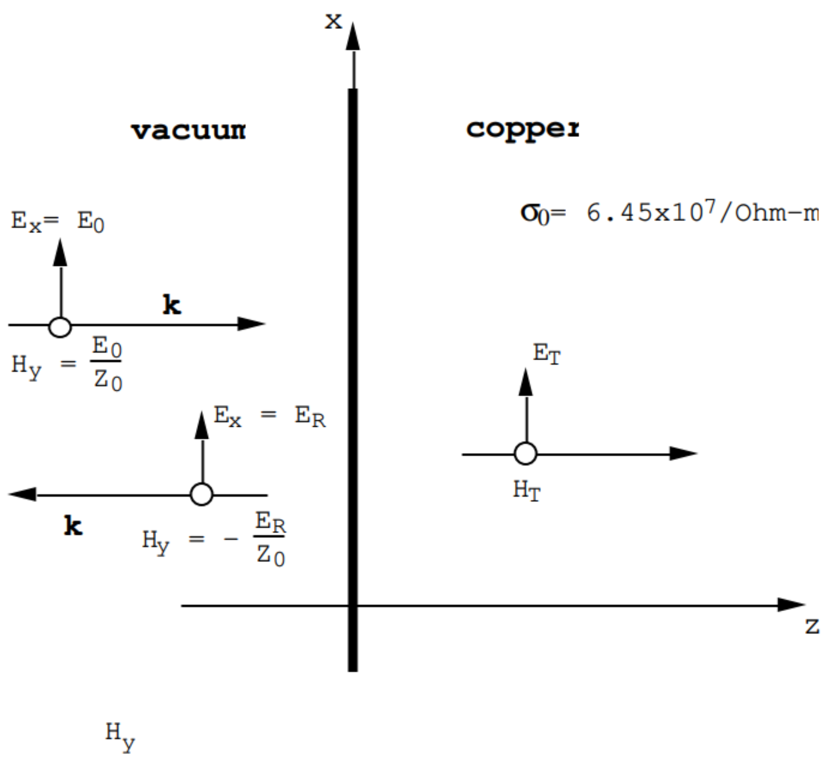
So $\frac{\mathrm{i}\sigma}{\mathrm{k_m}} = \frac{\mathrm{k_m}}{\omega\mu_o}$ or $k_m^2 = i\omega\mu_0\sigma$

$k_m^2 = i\left(2\pi \times 10^6\right)\left(4\pi \times 10^{-7}\right)\left(6.45 \times 10^7\right)$

= i (5.093 x 10^8)

$k_m = \left(\frac{1+i}{\sqrt{2}}\right)\left(2.257 \times 10^4\right) = \left(\mathbf{1.596} \times \mathbf{10^4}\right)(\mathbf{1}+\mathbf{i})$.

N.B. km is very large c.f. k = ω/c. Approx. 10^6 larger!!



At z = 0: a) Continuity of E_x: $\mathrm{E_O} + \mathrm{E_R} = \mathrm{E_T}$

b) Continuity of H_y: $\frac{E_0}{Z_0} - \frac{E_R}{Z_0} = H_T$

or $\mathrm{E_O} - \mathrm{E_R} = \mathrm{Z_O H_T}$

$\therefore 2\mathrm{E_O} = (\mathrm{E_T} + \mathrm{Z_O H_T}) = \left[1 + \frac{\mathrm{i}\sigma \mathrm{z_O}}{\mathrm{k_m}}\right]\mathrm{E_T}$

$2\mathrm{E_R} = (\mathrm{E_T} - \mathrm{Z_O H_T}) = \left[1 - \frac{\mathrm{i}\sigma \mathrm{Z_O}}{\mathrm{k_m}}\right]\mathrm{E_T}$

But $\mathrm{H_T} = \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\right)\mathrm{E_T}$

$\therefore \frac{E_R}{E_0} = \frac{1-\frac{i\sigma Z_0}{k_m}}{1+\frac{i\sigma Z_0}{k_m}} \quad \frac{E_T}{E_0} = \frac{2}{1+\frac{i\sigma Z_0}{k_m}}$.

$\frac{1}{k_m} = \left(\frac{10}{1.596}\right) \times 10^{-5}\left(\frac{1}{1+i}\right) = \left(\frac{5 \times 10^{-5}}{1.596}\right)(1-i)$

$= 3.133(1-i) \times 10^{-5}$

$\therefore \left(\frac{\mathrm{i}}{\mathrm{k_m}}\right) = \left(\mathbf{3.133} \times \mathbf{10^{-5}}\right)(\mathbf{1}+\mathbf{i})$.

So $\frac{i\sigma Z_0}{k_m} = (3.133)\left(10^{-5}\right)\left(6.45\times 10^7\right)(377)(1+i)$

$= \left(\mathbf{7.618\times 10^5}\right)\mathbf{(1+i)}$.

This is much larger than 1.

$\therefore \frac{E_T}{E_0} \cong \frac{2}{\frac{i\sigma Z_0}{k_m}} = \frac{-i2k_m}{\sigma Z_0} = \mathbf{1.313\times 10^{-6}(1-i).}$

$\frac{E_R}{E_0} = \frac{-\left[1+\frac{ik_m}{\sigma Z_0}\right]}{\left[1-\frac{ik_m}{\sigma Z_0}\right]} \simeq -\left[1+\frac{2ik_m}{\sigma Z_0}\right] \cong -1$

to approximately 1 part in 10^6!

e) From part (c) $\mathrm{H_Y} = \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\right)\mathrm{E_X}$ $\therefore \mathrm{H_T} = \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\right)$ $\mathrm{E_T}$

and $\mathrm{H_T} \cong \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\left(-\mathrm{i}\frac{2}{\sigma}\frac{\mathrm{k_m}}{\mathrm{Z_0}}\right)\right) = \frac{2}{\mathrm{z_0}}$

N.B. To first order in $\left(\frac{2}{\sigma}\frac{\mathrm{k_m}}{\mathrm{Z_0}}\right)$ the magnetic field amplitude in the metal is INDEPENDENT of σ,ω !!

The factor 2 comes from the sum $H_T = H_o + H_R$, where $\mathrm{H_0} = \frac{\mathrm{E_0}}{\mathrm{Z_0}}$ & $\mathrm{H_R} = \frac{|\mathrm{E_R}|}{\mathrm{Z_0}}$

But E_o = 1 V/m & E_R = -1 v/m (to 1 part in 10^6)

$\therefore \mathrm{H_T} = \frac{2}{\mathrm{Z_0}} = \mathbf{5.305\times 10^{-3}\ Amps/m.}$

(f) For the incident wave $\langle \mathrm{S_0}\rangle = \frac{1}{2}$ Real $\{\mathrm{E_X H_Y^*}\}$

$= \frac{\mathrm{E_0^2}}{2\mathrm{z_0}} = \frac{1}{2\mathrm{z_0}} = \mathbf{1.326\times 10^{-3}\ watts/m^2.}$

(g) At the metal surface (z = 0)

$$\begin{aligned}\langle S_m\rangle &= \frac{1}{2}\text{ Real }\left\{E_T H_T^*\right\}\\ &= \frac{1}{2}\text{Real}\left\{\left(1.313\times 10^{-6}\right)(1-i)\frac{(2)}{Z_0}\right\}\\ &= \left(\frac{1.313}{Z_0}\times 10^{-6}\right) = \mathbf{3.48\times 10^{-9}\ Watts/m^2.}\end{aligned}$$

(h) $\alpha = \langle S_m\rangle / \langle S_0\rangle = \frac{3.48}{1.326}\times 10^{-6} = \mathbf{2.627\times 10^{-6}.}$

(i) In the metal the current density is given by

$\mathrm{J_x} = \sigma\mathrm{E_x} = \sigma\mathrm{E_T}e^{\mathrm{i}(\mathrm{k_m z}-\omega\mathrm{t})}$

The Joule heat/volume (time averaged) is

$$\begin{aligned}\frac{dQ}{dt} &= \frac{1}{2}\text{Real}\{J_x E_x^*\}\\ &= \frac{1}{2}\text{Real}\left\{\sigma E_T e^{i(k_m z-\omega t)}\cdot E_T^* e^{-i(k_m^* z-\omega t)}\right\}\\ &= \frac{1}{2}\text{Real}\left\{\sigma|E_T|^2 e^{i(k_m-k_m^*)z}\right\}\end{aligned}$$

But $\mathrm{k_m} = \gamma(1+\mathrm{i})$ and $\mathrm{k_m^*} = \gamma(1-\mathrm{i})$ $\therefore \mathrm{k_m} - \mathrm{k_m^*} = 2\mathrm{i}\gamma$

and γ = 1.596 x 10^4 from part (c)

& $\mathrm{i}\left(\mathrm{k_m}-\mathrm{k_m^*}\right) = -2\gamma$

$\therefore \frac{dQ}{dt} = \frac{\sigma}{\mathbf{2}}|\mathbf{E_T}|^{\mathbf{2}}\mathbf{e^{-2\gamma z}.}$

Total rate of heat production $= \frac{\sigma|E_T|^2}{2}\int_0^\infty e^{-2\gamma z}dz = \frac{\sigma|E_T|^2}{4\gamma}.$

$$\therefore Q_{\text{Total}} = \frac{(6.45\times10^7)}{(4)(1.596\times10^4)}(1.313)^2\times10^{-12})(2) = \mathbf{3.48\times10^{-9}\ Watts/m^2.}$$

$$= < S_m > (\text{ from (g)}) \ .$$

Problem (10.6).

Light having a wavelength of 5145 Å (0.5145 μm) falls upon a plane copper surface at normal incidence. The intensity of the light is 10^5 Watts/m^2 (i.e. 100 mW in a laser beam 1x1 mm in cross-section). The complex index of refraction for copper at 5145 Å is $\sqrt{\varepsilon_r} = (1.19 + 2.60i)$ for a time dependence of $e^{-i\omega t}$.

(a) Calculate the amplitudes of the electric and magnetic fields in the incident wave.

(b) Calculate the amplitudes of the electric and magnetic fields in the reflected wave.

(c) Calculate the intensity of the reflected wave;i.e. calculate the time-averaged value of the Poynting vector.

(d) Calculate the wave-vector of the light in the copper. What is the phase velocity associated with the wave in the copper?

(e) Calculate the amplitudes of the electric and magnetic fields in the copper but near the surface at z=0.

(f) Calculate the time averaged value of the Poynting vector inside the copper but near the surface at z=0.

(g) How far into the copper does the light penetrate before its intensity has decreased to 1% of its intensity at the surface?

(h) Calculate the time averaged energy density, <W> , stored in the electric and magnetic fields in the copper but at the surface z=0. Show that $\langle S_z\rangle = \frac{c}{n} < W >$ Watts/m^2.

Answer (10.6).

(a) Incident wave:

$$E_x = E_0 e^{i(kz-\omega t)}$$

$$H_y = \frac{E_0}{Z_0} e^{i(kz-\omega t)},$$

where k= ω/c and $Z_0 = \mu_0 c$ = 377 Ohms.

$< S_z > = \frac{1}{2}\text{Real}\left(E_x H_y^*\right) = \frac{E_0^2}{2Z_0} = I_0 = 10^5 \text{ Watts }/m^2.$

Therefore, $E_0^2 = 75.4\times10^6$, and E_0= 8.683x10^3 Volts/m, and H_y= 23.03 Amps/m.

(b) From the boundary value problem

$$\frac{E_R}{E_0} = \frac{1-\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}} = \frac{(1-n)-i\kappa}{1+n+i\kappa} = r.$$

For this problem n=1.19 and κ=2.60;

r = -0.621- 0.45i, and therefore r= - $Re^{i\phi}$ where R=0.767, and Tanϕ= 0.725 so that ϕ= 35.93° = 0.627 radians. The minus sign means that the direction of the reflected wave amplitude is reversed relative to the amplitude in the incident wave.

$$|E_R| = R|E_0| = 6.66\times10^3 \text{ V/m},$$

and

$$|H_R| = R|H_y| = 17.66 \text{ Amps/m}.$$

(c) The intensity of the reflected wave is given by

$$I_R = R^2 I_0 = 0.588\times10^5 \text{ Watts }/m^2.$$

(d) In the copper $k_m^2 = \varepsilon_r\left(\frac{\omega}{c}\right)^2$

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} = 1.221\times10^7 m^{-1}.$$

$$k_m = (n+ik)\frac{\omega}{c} = (1.453 + i3.175)\times10^7 m^{-1}.$$

In the copper the fields are proportional to

$$e^{-\kappa\left(\frac{\omega}{c}\right)z}e^{i\left(n\frac{\omega}{c}z-\omega t\right)}.$$

The phase velocity is $\frac{c}{n}=2.52\times 10^{8}\ \mathrm{m/sec}$.

(e) $\frac{E_T}{E_0}=Te^{i\theta}=\frac{2}{1+\sqrt{\varepsilon}}=\frac{2}{(n+1)+i\kappa}$.

$\mathrm{Te^{i\theta}=(0.379-i0.450)}$,

and T= 0.588 and θ= - 49.9° = - 0.871 radians.

$$\mathrm{|E_T|=TE_0=5.11\times 10^{3}\ V/m.}$$

$$\mathrm{H_Y=\frac{(n+i\kappa)}{Z_0}E_T=(37.33+10.36i);}$$

$|H_y|=\frac{\sqrt{n^2+K^2}}{Z_0}E_T=\frac{2.859}{377}E_T=38.75\ \text{Amps}\ /\mathrm{m}.$

phase=0.271 rad= 15.51°.

(f) In the metal

$$E_x=E_Te^{-\kappa\left(\frac{\omega}{c}\right)z}e^{i\left(n\frac{\omega}{c}z-\omega t\right)}$$

$$\mathrm{H_y=\frac{(n+i\kappa)}{Z_0}E_T}e^{-\kappa\left(\frac{\omega}{c}\right)z}e^{\mathrm{i}\left(\mathrm{n}\frac{\omega}{c}\mathrm{z}-\omega \mathrm{t}\right)},$$

so at z=0 these become

$$E_x=E_Te^{-i\omega t}$$

and

$$\mathrm{H_y=\frac{(n+i\kappa)}{Z_0}E_T}e^{-\mathrm{i}\omega \mathrm{t}}.$$

$$<S_z>=\frac{1}{2}\mathrm{Real}(E_xH_y^*)=<S_z>=\frac{1}{2}\mathrm{Real}\left(E_T\frac{(n-i\kappa)}{Z_0}E_T^*\right),$$

$$\mathrm{<S_z>=\frac{nE_T^2}{2Z_0}=0.4119\times 10^{5}\ Watts/m^2.}$$

$$\mathrm{\langle S_z\rangle|_{Reflected}+\langle S_z\rangle|_{Transmitted}=1.0\times 10^{5}\ Watts\ /m^2.}$$

(g) The electric and magnetic field amplitudes are multiplied by $e^{-\kappa\left(\frac{\omega}{c}\right)z}$ and therefore the intensity is multiplied by

$$e^{-2\kappa\left(\frac{\omega}{c}\right)z}.$$

If $e^{-2\kappa\left(\frac{\omega}{c}\right)z}=0.01$ then $2\kappa\frac{\omega}{c}z=4.605$.

But ω/c= 1.221x10. $\mathrm{m^{-1}}$, therefore z= 0.725×10^{-7} meters, or z= 72.5 nm, or z= 0.0725 µm.

The free space wavelength of the light is 0.5145 µm, so that the light penetrates $\sim\left(\frac{\lambda}{7.1}\right)$, approximately 1/10 of a free space wavelength.

(h) At the surface of the copper the electric and magnetic field amplitudes are given by

$$E_x=E_Te^{-i\omega t},$$

$$\mathrm{H_y=\frac{(n+i\kappa)}{Z_0}E_T}e^{-\mathrm{i}\omega \mathrm{t}}.$$

$$\langle W_E\rangle=\frac{1}{4}\mathrm{Real}(E_xD_x)=\langle S_Z\rangle=\frac{1}{4}\mathrm{Real}\left(\varepsilon_0E_T^2\left(\left(n^2-\kappa^2\right)-2n\kappa\right)\right),$$

$$< W_E >= \frac{\varepsilon_0}{4}\left(n^2 - \kappa^2\right) E_T^2.$$

$$< W_B >= \frac{\mu_0}{4}\text{ Real }\left(H_y H_y^*\right) = \frac{\mu_0}{4}\frac{\left(n^2+\kappa^2\right)}{Z_0^2} E_T^2.$$

But $z_0^2 = \mu_0^2 c^2 = \frac{\mu_0}{\varepsilon_0}$, and

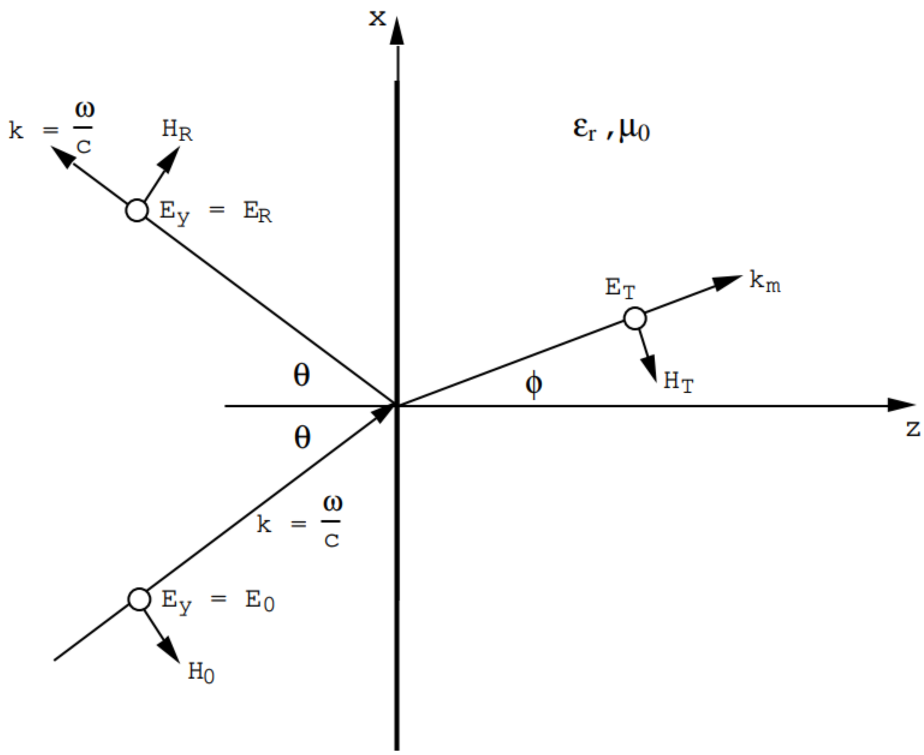
$$< W_B >= \frac{\varepsilon_0}{4}\left(n^2 + \kappa^2\right) E_T^2.$$

$$< \mathrm{W} >=< \mathrm{W_E} > + < \mathrm{W_B} >= \frac{\varepsilon_0}{2}\mathrm{n}^2\mathrm{E_T^2} = 1.63\times 10^{-4}\text{ Joules }/\mathrm{m}^3.$$

$$\langle \mathrm{S_z}\rangle = \frac{\mathrm{n}}{2\mu_0 \mathrm{c}}\mathrm{E_T^2} = \frac{\mathrm{c}\varepsilon_0\mathrm{n}}{2}\mathrm{E_T^2} = \left(\frac{\mathrm{c}}{\mathrm{n}}\right)\frac{\mathrm{n}^2\varepsilon_0}{2}\mathrm{E_T^2} = \left(\frac{\mathrm{c}}{\mathrm{n}}\right) < \mathrm{W} >,$$

where for this case c/n= $2.52\mathrm{x}10^8$ m/sec.

Problem (10.7).



An s-polarized electromagnetic wave is incident on a plane interface at the angle θ (see the sketch). The amplitude of the incident electric field is E_o, that of the reflected electric field is E_R, and the transmitted electric field is E_T. The material for z > 0 is characterized by a relative dielectric constant, ε_r, which is real (no losses in the medium). The material is characterized by the magnetic permeability of free space.

(a) Write expressions for the components of **E** and **H** in the incident wave e.g.

$$\mathrm{E_y} = \mathrm{E_0 e}^{\mathrm{i}[(\mathrm{k}\sin\theta)\mathrm{x} + (\mathrm{k}\cos\theta)\mathrm{z} - \omega \mathrm{t}]}$$

$$\text{etc. where } \mathrm{k} = \omega/\mathrm{c}.$$

(b) Write expressions for the components of **E**, **H** in the reflected wave.

(c) Write expressions for the components of **E**, **H** in the transmitted wave.

(d) Show that $\frac{E_R}{E_0} = \left[\frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi}\right]$

where n = $\sqrt{\varepsilon_r}$ and $\sin\phi = \frac{\sin\theta}{n}$

and $\frac{E_T}{E_0} = \left[\frac{2\cos\theta}{\cos\theta + n\cos\phi}\right]$.

(e) Show that the normal component of **B**, B_z, is continuous across the boundary at z = 0.

(f) Construct a graph of $\left(\frac{E_R}{E_0}\right)$ vs the angle of incidence, θ, for $\varepsilon_r = 4$.

Answer (10.7).

(a) Incident Wave:

$$\mathrm{E_Y = E_0 e^{i[(k\sin\theta)x + (k\cos\theta)z - \omega t]}}$$

$$\mathrm{H_x = \frac{-E_0}{Z_0}\cos\theta e^{i[(k\sin\theta)x + (k\cos\theta)z - \omega t]}}$$

$$\mathrm{H_z = \frac{E_O}{Z_O}\sin\theta e^{i[(k\sin\theta)x + (k\cos\theta)z - \omega t]}}$$

where Z_o = 377 Ω = $c\mu_o$.

(b) Reflected Wave:

$$E_y = E_R e^{i[(k\sin\theta)x - (k\cos\theta)z - \omega t]}$$

$$\mathrm{H_X = \frac{E_R}{Z_0}\cos\theta e^{i[(k\sin\theta)x - (k\cos\theta)z - \omega t]}}$$

$$\mathrm{H_z = \frac{E_R}{Z_0}\sin\theta e^{i[(k\sin\theta)x - (k\cos\theta)z - \omega t]}}$$

(c) Transmitted Wave:

$$E_y = E_T e^{i[(k\sin\theta)x + (k_m\cos\phi)z - \omega t]}$$

$$\mathrm{H_X \frac{-n\cos\phi}{Z_0} E_T e^{i[(k\sin\theta)x + (k_m\cos\phi)z - \omega t]}}$$

$$\mathrm{H_z = \frac{\sin\theta}{Z_0} E_T e^{i[(k\sin\theta)x + (k_m\cos\phi)z - \omega t]}}$$

Since curl $\mathbf{E} = i\omega\mu_0\mathbf{H}$ or $\frac{\partial E_y}{\partial z} = -i\omega\mu_0 H_x$

and $\mathrm{\frac{\partial E_y}{\partial x} = i\omega\mu_0 H_z}$

and $\mathrm{Curl}\ \mathbf{H} = -\mathrm{i}\omega\varepsilon_r\varepsilon\mathbf{E}$ or $\mathrm{\frac{\partial H_X}{\partial z} - \frac{\partial H_z}{\partial x} = i\omega\varepsilon_r\varepsilon_0 E_y}$

$\therefore \mathrm{\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} = -\varepsilon_r\left(\frac{\omega}{c}\right)^2 E_Y}$

or $\mathrm{k^2\sin^2\theta + k_m^2 = E_r\left(\frac{w}{c}\right)^2}$

or $k_m^2 = \varepsilon_r\left(\frac{\omega}{c}\right)^2 \qquad \therefore \qquad k_m = \sqrt{\varepsilon_r}\left(\frac{\omega}{c}\right) = n\left(\frac{\omega}{c}\right)$

$$\mathrm{k_m\sin\phi = k\sin\theta = \left(\frac{\omega}{c}\right)\sin\theta}$$

$$\therefore \sin\phi = \sin\theta/n$$

At z = 0 $E_o + E_R = E_T$ (1)

$$\mathrm{-\frac{E_0\cos\theta}{Z_O} + \frac{E_R\cos\theta}{Z_O} = -\frac{n\cos\phi}{Z_O}E_T}$$

or $-E_0+E_R=-\frac{n\cos\phi}{\cos\theta}E_T$ (2)

$\therefore \frac{2E_R}{E_T}=\left(1-\frac{n\cos\phi}{\cos\theta}\right)$

$2E_O=\left(1+\frac{n\cos\phi}{\cos\theta}\right)E_T$

$\therefore \frac{\mathbf{E_R}}{\mathbf{E_0}}=\left[\frac{\cos\theta-n\cos\phi}{\cos\theta+n\cos\phi}\right]$ where $n=\sqrt{\varepsilon_r}$

(d) $\frac{E_T}{E_O}\left[\frac{2\cos\theta}{\cos\theta+n\cos\phi}\right]$, where $\cos\phi=\sqrt{1-\frac{\sin^2\theta}{\varepsilon_r}}$

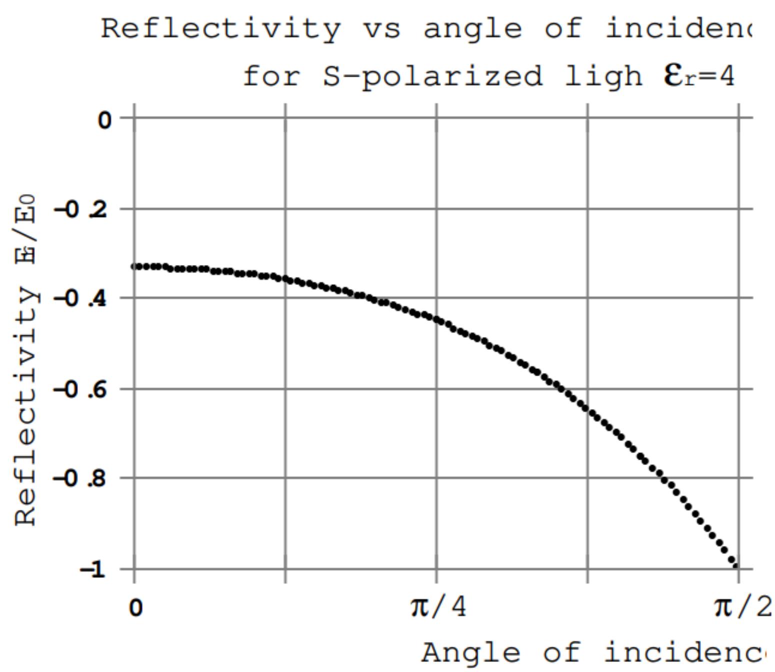
(e) At z = 0

on the left: $H_z=(E_O+E_R)\frac{\sin\theta}{Z_O}$

on the right: $H_z=E_T\frac{\sin\theta}{Z_O}$

Therefore, because of eqn (1), the normal component of $B_z = \mu_o H_z$ is continuous across the interface.

(f)



The ratio $\frac{E_R}{E_0}$ is plotted in the figure. Notice that

(1) The phase of the electric field is reversed in the reflected wave i.e. the total electric field at the interface is smaller than the incident electric field amplitude;

(2) The reflectivity approaches 1 at large angles of incidence i.e. as the beam becomes parallel with the interface plane. It is a common experience that surfaces appear more reflecting at shallow angles.

Problem (10.8).

Let p-polarized radiation, λ= 0.50 µm, be incident from vacuum on glass at an angle of incidence of 45°. The index of refraction of the glass is 1.5 and the glass is lossless. Let the plane of incidence be the x-z plane, and let the surface of the glass be parallel with the x-y plane and located at z=0.

(a) Write expressions for the incident fields (**E**,**H**) assuming a time dependence $e^{-i\omega t}$. Let the incident electric field amplitude be E_0= 1 V/m.

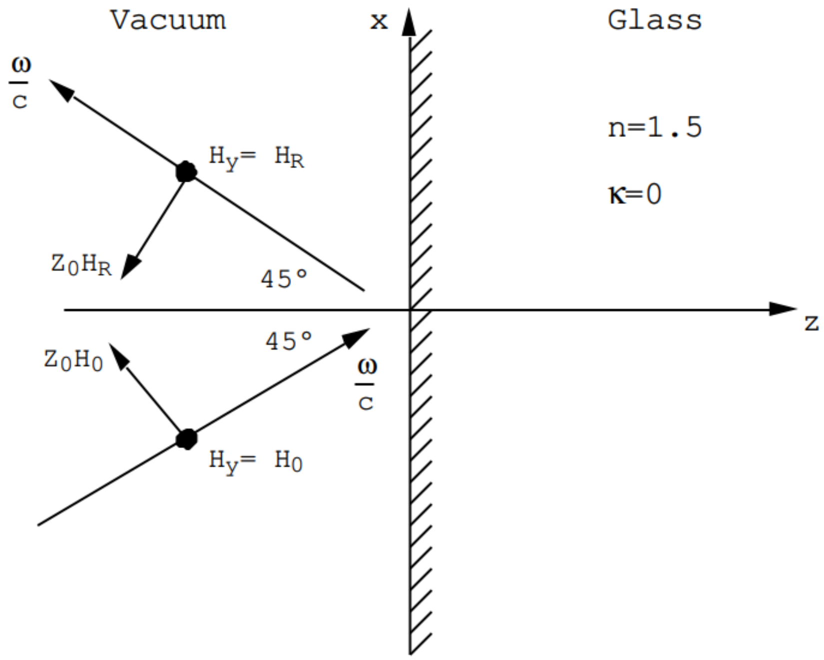
(b) Write expressions for the reflected fields. Let the reflected electric field amplitude be E_R.

(c) Write expressions for the transmitted fields. Let the transmitted electric field amplitude be E_T.

(d) Solve the appropriate boundary value problem to obtain the complex ratios E_R/E_0 and E_T/E_0.

(e) Calculate all of the components of the time averaged Poynting vectors for each of the incident, reflected, and transmitted waves.

Answer (10.8).



$\frac{\omega}{c}=\frac{2\pi}{\lambda}=4\pi\times10^{6}\ \mathrm{rad/sec}\ =1.2566\times10^{7}\mathrm{m}^{-1}$;

the component along x is $q=\frac{1}{\sqrt{2}}\frac{\omega}{c}=0.889\times10^{7}\mathrm{m}^{-1}$; the component along z is k= q= $0.889\text{x}10^{7}\ \mathrm{m}^{-1}$.

(a) Incident Wave:

$$H_0=\frac{E_0}{Z_0}$$

$$E_x=\frac{E_0}{\sqrt{2}}e^{iqx}e^{iqz}e^{-i\omega t}$$

$$E_z=-\frac{E_0}{\sqrt{2}}e^{iqx}e^{iqz}e^{-i\omega t}$$

$$H_y=\frac{E_0}{Z_0}e^{iqx}e^{iqz}e^{-i\omega t}$$

(b) Reflected Wave:

$$H_R=\frac{H_R}{Z_0}$$

$$E_x=-\frac{E_R}{\sqrt{2}}e^{iqx}e^{-iqz}e^{-i\omega t}$$

$$E_z=-\frac{E_R}{\sqrt{2}}e^{iqx}e^{-iqz}e^{-i\omega t}$$

$$H_y=\frac{E_R}{Z_0}e^{iqx}e^{-iqz}e^{-i\omega t}.$$

(c) In the glass $q^2 + k_m^2 = n^2\left(\frac{\omega}{c}\right)^2$,

therefore $\mathrm{k_m^2} = \left(\mathrm{n^2} - \frac{1}{2}\right)\left(\frac{\omega}{\mathrm{c}}\right)^2$, since $\mathrm{q^2} = \frac{1}{2}(\omega/\mathrm{c})^2$,

and $\mathrm{k_m} = 1.3229\left(\frac{\omega}{\mathrm{c}}\right) = 1.6624\times 10^7\ \mathrm{m^{-1}}$.

The angle of refraction is such that $\tan\phi = \frac{q}{k_m} = 0.534$, ϕ= 28.13°.

In the glass $\mathrm{E_T} = \frac{\mathrm{Z_0H_T}}{\mathrm{n}}$:

$$\mathrm{E_X} = \left(\frac{\mathrm{k_m}}{\mathrm{n}\frac{\omega}{\mathrm{c}}}\right)\mathrm{E_T}e^{\mathrm{i}q\mathrm{x}}e^{\mathrm{ik_m z}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_z} = \left(\frac{-\mathrm{q}}{\mathrm{n}\frac{\omega}{\mathrm{c}}}\right)\mathrm{E_T}e^{\mathrm{i}q\mathrm{x}}e^{\mathrm{ik_m z}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{H_Y} = \frac{\mathrm{nE_T}}{\mathrm{Z_0}}e^{\mathrm{iqx}}\mathrm{e^{ik_m z}}e^{-\mathrm{i}\omega\mathrm{t}},$$

where $\frac{\mathrm{k_m}}{\mathrm{n}\omega/\mathrm{c}} = 0.882$ and $\frac{q}{n\omega/c} = 0.4714$.

(d) Boundary Value Problem.

(i) Continuity of $\mathrm{H_y}$:

$$\frac{\mathrm{E_0}}{\mathrm{Z_0}} + \frac{\mathrm{E_R}}{\mathrm{Z_0}} = \frac{\mathrm{nE_T}}{\mathrm{Z_0}}$$

(ii) Continuity of $\mathrm{E_x}$:

$$\frac{E_0}{\sqrt{2}} - \frac{E_R}{\sqrt{2}} = (0.882)E_T.$$

Therefore $\mathrm{E_0} + \mathrm{E_R} = 1.5\ \mathrm{E_T}$

$\mathrm{E_0} - \mathrm{E_R} = 1.247\ \mathrm{E_T}$

from which $\frac{\mathbf{E_R}}{\mathbf{E_0}} = \mathbf{0.0920}$ and $\frac{\mathbf{E_T}}{\mathbf{E_0}} = \mathbf{0.7280}$.

(e) Time averaged Poynting Vectors.

(i) Incident Wave.

$$<\mathrm{S_X}> = -\frac{1}{2}\mathrm{Real}(\mathrm{E_Z H_Y^*})$$

$$<\mathbf{S_x}> = \frac{\mathbf{E_0^2}}{\mathbf{z_0 2\sqrt{2}}} = \mathbf{9.38\times 10^{-4}\ Watts\ /m^2}.$$

$$<\mathbf{S_z}> = \frac{\mathbf{E_0^2}}{\mathbf{z_0 2\sqrt{2}}} = \mathbf{9.38\times 10^{-4}\ Watts\ /m^2}.$$

(ii) Reflected Wave.

$$<\mathbf{S_x}> = \frac{\mathbf{E_R^2}}{\mathbf{z_0 2\sqrt{2}}} = \mathbf{7.94\times 10^{-6}\ Watts\ /m^2}$$

$$<\mathbf{S_z}> = -\frac{\mathbf{E_R^2}}{\mathbf{z_0 2\sqrt{2}}} = \mathbf{7.94\times 10^{-6}\ Watts\ /m^2}.$$

(iii) Transmitted Wave.

$$<\mathrm{S_x}> = \frac{1}{2\mathrm{n}}\frac{\mathrm{q}}{\mathrm{w/c}}\frac{\mathrm{n}}{\mathrm{Z_0}}\mathrm{E_T^2} = \frac{\mathrm{E_T^2}}{\mathrm{Z_0 2\sqrt{2}}}$$

$$\mathbf{< S_x >= 4.97 \times 10^{-4} Watts / m^2.}$$

$$< S_z >= \frac{1}{2n} \frac{k_m}{w/c} \frac{n}{Z_0} E_T^2 = 1.323 \frac{E_T^2}{2Z_0}$$

$$\mathbf{< S_z >= 9.30 \times 10^{-4}\ Watts / m^2.}$$

Problem (10.9).

Reverse the configuration of Problem (10.8); i.e. let p-polarized radiation be incident on a glass-vacuum interface from inside the glass. The interface is parallel with the x-y plane and it is located at z=0: the glass is on the left in the half-space z<0. Let the index of the glass be n=1.5 (the imaginary part of the index may be set equal to zero, κ=0). The vacuum wavelength of the light is λ= 0.50 µm, and the angle of incidence is 45°. The magnetic vector of the incident light is polarized along the y-direction.

(a) Calculate the z-component of the Poynting vector in the vacuum at z=0.

(b) Calculate the amplitude of the vacuum wave at z=0 if the incident wave electric field amplitude is E_0= 1 V/m

Answer (10.9).

(a) The wave-vector in the glass is given by

$$k^2 = n^2 \left(\frac{\omega}{c}\right)^2,$$

or

$$k = n\left(\frac{\omega}{c}\right).$$

For this problem $\left(\frac{\omega}{c}\right) = 1.2566 \times 10^7\ m^{-1}$ and k= 1.8849x10^7 m^{-1}.

The wave-vector component along the interface (along x) is

$$q = k \sin 45^\circ = \frac{k}{\sqrt{2}} = 1.3328 \times 10^7\ m^{-1}.$$

On the vacuum side of the interface the fields are proportional to

$$e^{iqx} e^{ik_v z} e^{-i\omega t}$$

where $q^2 + k_v^2 = \left(\frac{\omega}{c}\right)^2$,

therefore

$$k_V^2 = \left(\frac{\omega}{c}\right)^2 - q^2 = -0.1974 \times 10^{14}\ m^{-2}.$$

Notice that k_v^2 is negative. This means that the square root is pure imaginary.

$$k_v = \left(4.443 \times 10^6\right) i\ m^{-1} = i\alpha = i\frac{\omega}{c}\sqrt{\left(n^2/2\right) - 1}.$$

The wave in the vacuum is a pure exponential, it does not oscillate in space. The fields are confined to a distance of the order of $1/\alpha$ near the interface, i.e ~ 1 λ. In the vacuum

$$H_Y = H_T e^{iqx} e^{-\alpha z} e^{-i\omega t}$$

where α= 4.443x10^6 m^{-1}. In the vacuum curl**H** = -iωε$_0$ **E**, therefore

$$-i\omega\varepsilon_0 E_x = -\frac{\partial H_y}{\partial z} = \alpha H_y$$

$$-i\omega\varepsilon_0 E_z = -\frac{\partial H_y}{\partial x} = iqH_y,$$

or

$$E_x = \left(\frac{i\alpha}{\omega\varepsilon_0}\right) H_Y$$

$$\mathrm{E_z} = -\left(\frac{\mathrm{q}}{\omega\varepsilon_0}\right)\mathrm{H_Y}.$$

The time averaged Poynting vector at the interface is

$$< \mathrm{S_z} >= \frac{1}{2}\mathrm{Real}(\mathrm{E_X H_Y^*})$$

so

$$< S_z >= \frac{1}{2}\mathrm{Real}\left(\frac{i\alpha}{\omega\varepsilon_0}|H_T|^2\right) \equiv 0.$$

There is no energy flow from the glass to the vacuum. The light is totally reflected.

(b) From the continuity of the tangential components of **E** and **H** one finds

$$\mathrm{H_0 + H_R = H_T}$$

$$\frac{\mathrm{nZ_0H_0}}{\sqrt{2}} - \frac{\mathrm{nZ_0H_R}}{\sqrt{2}} = \left(\frac{\mathrm{i}\alpha}{\omega\varepsilon_0}\right)\mathrm{H_T} = \mathrm{iZ_0}\left(\frac{\alpha}{\omega/\mathrm{c}}\right)\mathrm{H_T}$$

or $\mathrm{H_0 - H_R = \frac{i}{2n}H_T}$, since $\frac{\alpha}{\omega/c} = \frac{1}{2\sqrt{2}}$.

Consequently, $\mathrm{H_0 + H_R = H_T}$

$\mathrm{H_0 - H_R = \frac{i}{3}H_T}$,

from which

$$\frac{\mathrm{H_T}}{\mathrm{H_O}} = \frac{6}{(3+\mathrm{i})} = (1.80 - 0.60\mathrm{i}) = 1.897\mathrm{e}^{-\mathrm{i}\phi}.$$

where ϕ= 18.43°,

and

$$\frac{\mathrm{H_R}}{\mathrm{H_0}} = \frac{1}{6}(3-\mathrm{i})\frac{\mathrm{H_T}}{\mathrm{H_0}} = (0.8 - 0.6\mathrm{i}) = \mathrm{e}^{-\mathrm{i}\theta},$$

where θ= 36.87°.

NB. $\left|\frac{\mathrm{H_R}}{\mathrm{H_0}}\right|^2 \equiv 1$ as expected.

The electric field amplitude in the glass is given by

$$\mathrm{E_0} = \left(\frac{\mathrm{Z_0}}{\mathrm{n}}\right)\mathrm{H_0},$$

so if E_0= 1 V/m, then H_0= 3.98x10^{-3} Amps/m. The vacuum wave amplitude is given by

$$\mathrm{H_T} = \left(1.897\mathrm{e}^{-\mathrm{i}\phi}\right)\mathrm{H_0} = 7.548\times10^{-3}\mathrm{e}^{-\mathrm{i}\phi} \quad \mathrm{Amps/m},$$

and

$$\mathrm{E_T = Z_0H_T} = 2.846\mathrm{e}^{-\mathrm{i}\phi} \quad \mathrm{V/m}, \qquad \text{where } \phi = 18.43^\circ.$$

Problem (10.10).

Light of wavelength λ= 0.50 μm falls from vacuum on a plane glass interface; the angle of incidence is 60°. Let the plane of incidence be the x-z plane, and let z be directed into the glass; the interface is located at z=0. The complex index of refraction of the glass, n+iκ, has components n=1.5, κ=0. The incident light is plane polarized but the electric vector has equal amplitudes, E_0, for the component perpendicular to the plane of incidence (the s-polarized component), and for the component parallel with the plane of incidence (the p-polarized component). Calculate the reflected electric field amplitudes and show that the electric field in the reflected light is plane polarized, but that the plane of polarization has been rotated relative to that of the incident light.

Answer (10.10).

From Snell's law

$$\sin\theta = n\sin\phi$$

where θ= 60°.Thus

$$\begin{aligned}&\sin\phi = 0.5774\\&\phi = 35.26^{\circ}\\&\cos\phi = 0.8165\\&\cos\theta = 1/2.\end{aligned}$$

For the s-polarized component

$$\mathrm{R_S} = \frac{\mathrm{E_R}}{\mathrm{E_0}} = \frac{\cos\theta - \mathrm{n}\cos\phi}{\cos\theta + \mathrm{n}\cos\phi} = -0.4202.$$

For the p-polarized component

$$\mathrm{R_p} = \frac{\mathrm{H_R}}{\mathrm{H_0}} = \frac{\mathrm{E_R}}{\mathrm{E_0}} = \frac{\mathrm{n}\cos\theta - \cos\phi}{\mathrm{n}\cos\theta + \cos\phi} = -0.0425.$$

The reflected light is polarized almost perpendicular to the plane of incidence. The angle which the electric vector makes with the plane of incidence is α, where

$\tan\alpha = \frac{0.4202}{0.0425}$, so that α = 84.2°.

The amplitude of the electric vector is 0.422 E_0.

Problem (10.11).

Light of wavelength λ= 0.5145 μm falls on a plane copper interface; the complex index of refraction for copper, $\sqrt{\varepsilon_r} = (n + i\kappa)$, has components n=1.19, and κ= 2.60, for a time dependence $e^{-i\omega t}$. Let the copper-vacuum interface lie in the x-y plane at z=0. The plane of incidence is the x-z plane, and the angle of incidence is 60°. The incident wave is plane polarized and its electric vector is oriented at 45° with respect to the plane of incidence. Take the amplitudes of the s-polarized and p-polarized components to be equal to E_0. Calculate the reflected wave electric field amplitudes and show that the reflected light is elliptically polarized.

Answer (10.11).

In the copper one has a spatial variation of the form

$$e^{iqx}e^{ikz}$$

where $\mathrm{q}^2 + \mathrm{k}^2 = \varepsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2$

and $q = \left(\frac{\omega}{c}\right)\sin 60^{\circ} = 0.8660\left(\frac{\omega}{c}\right)$.

Therefore $\mathrm{k}^2 = (\varepsilon_\mathrm{r} - 0.75)\left(\frac{\omega}{\mathrm{c}}\right)^2$.

For copper $\varepsilon_r = (n + i\kappa)^2 = \left(n^2 - \kappa^2\right) + 2ni\kappa$,

or $\varepsilon_r = -5.34 + i6.19$,

and $\mathrm{k}^2 = (-6.09 + \mathrm{i}6.19)\left(\frac{\omega}{\mathrm{c}}\right)^2$

$\mathrm{k}^2 = 8.686\mathrm{e}^{\mathrm{i}134.5^{\circ}}\left(\frac{\omega}{\mathrm{c}}\right)^2$,

so that $\mathrm{k} = 2.947\mathrm{e}^{\mathrm{i}\phi}\left(\frac{\omega}{\mathrm{c}}\right)$ where $\phi = 67.27^{\circ}$.

This can be written $\mathrm{k} = (\mathrm{n}_\theta + \mathrm{i}\kappa_\theta)\left(\frac{\omega}{\mathrm{c}}\right)$

where n_θ= 1.139 κ_θ= 2.718.

For the s-polarized wave

$$\frac{E_R}{E_0}=\frac{\cos\theta-(n_\theta+i\kappa_\theta)}{\cos\theta+(n_\theta+i\kappa_\theta)}=0.880e^{-i162.14^\circ}.$$

For the p-polarized wave

$$\frac{\mathrm{H_R}}{\mathrm{H_0}}=\frac{\varepsilon_\mathrm{r}\cos\theta-(\mathrm{n}_\theta+\mathrm{iK}_\theta)}{\varepsilon_\mathrm{r}\cos\theta+(\mathrm{n}_\theta+\mathrm{i}\kappa_\theta)}=0.637\mathrm{e}^{\mathrm{i}69.59^\circ}.$$

The reflected waves can be described at z=0 by

$$E_y=0.880E_0e^{-i(\omega t+162.14^\circ)}$$

and $E_{x'}=-0.637E_0e^{-i(\omega t-69.59^\circ)}$,

or $E_{x'}=0.637\text{ E }_0e^{-i(\omega t+110.41^\circ)}$

where the x' refers to a co-ordinate system in which the x'-axis lies in the plane perpendicular to the reflected wave wave-vector.

These expressions mean

$$\mathrm{E_{x'}}=0.637\mathrm{E_0}\cos(\omega\mathrm{t}+110.41^\circ)$$

$$\mathrm{E_Y}=0.880\mathrm{E_0}\cos(\omega\mathrm{t}+162.14^\circ).$$

The phase shift between these two components is

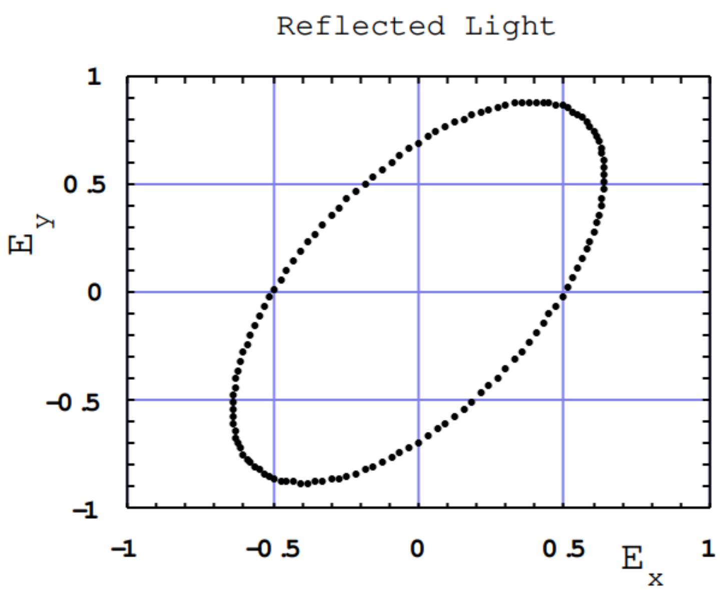
ϕ= 162.14 - 110.41 = 51.73° .

Shift the zero of time so as to make the component E_x vary as Cosωt:

$$E_{x'}=a\cos\omega t$$

$$E_y=b\cos(\omega t+51.73^\circ),$$

where a= 0.637E_0 and b= 0.880E_0. These relations are plotted below for E_0= 1 V/m.



This ellipse can be put in standard form by a co-ordinate rotation through the angle θ:

$$\mathrm{E_\xi}=\mathrm{E_X}\cos\theta+\mathrm{E_Y}\sin\theta$$

$$E_\eta = -E_X \, Sin\,\theta + E_Y \cos\theta.$$

Using these relations the electric field components in the rotated frame can be written:

$$E_\xi = 0.637\ \cos\theta\ \cos\omega t + 0.5451\ \sin\theta\ \cos\omega t - 0.6909\ \sin\theta\ \sin\omega t$$

$$E_\eta = -0.637\ \sin\theta\ \cos\omega t + 0.5451\ \cos\theta\ \cos\omega t - 0.6909\ \cos\theta\ \sin\omega t$$

These have the form

$$E_\xi = A\cos(\omega t + \alpha) = A\ \cos\alpha\ \cos\omega t - A\ \sin\alpha\ \sin\omega t$$

where

$$A\cos\alpha = 0.637\cos\theta + 0.5451\sin\theta$$
$$A\sin\alpha = 0.6909\sin\theta$$

and

$$E_\eta = B\sin(\omega t + \alpha) = \cos\alpha\sin\omega t + \sin\alpha\cos\omega t$$

where

$$B\cos\alpha = -0.6909\cos\theta$$
$$B\sin\alpha = 0.5451\cos\theta - 0.637\sin\theta.$$

These give two expressions for Tan α which when equated provide an equation for the angle of rotation θ.

$$\frac{0.637\sin\theta - 0.5451\cos\theta}{0.6909\cos\theta} = \frac{0.6909\sin\theta}{0.637\cos\theta + 0.5451\sin\theta}.$$

Solutions are θ= -31.01° and θ= 58.98°. Use θ= -31.01°

so that Cosθ= 0.857, Sinθ= - 0.515.

These can be used to write

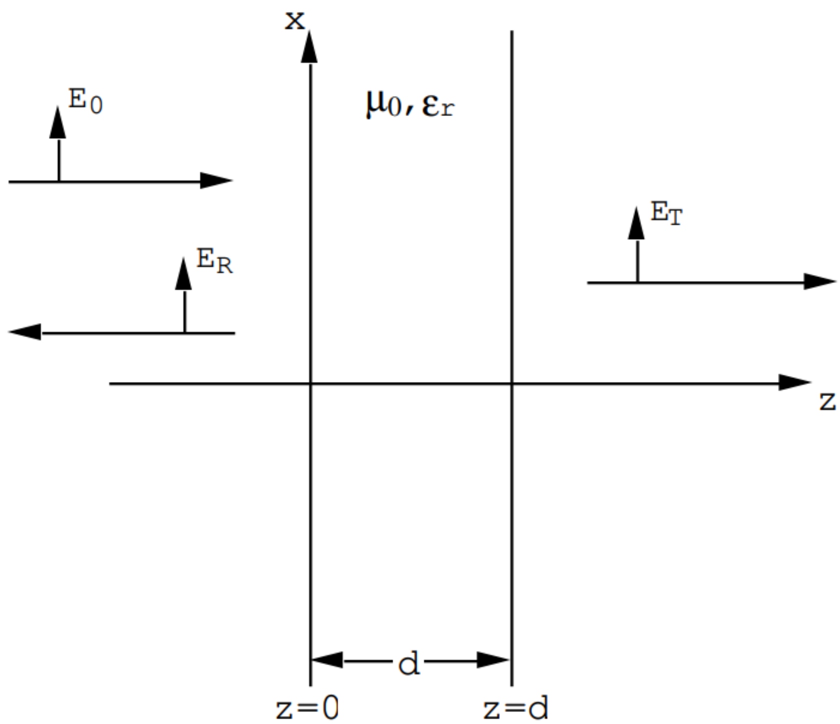
$E\xi = 0.265\cos\omega t + 0.356\sin\omega t = \mathbf{0.444}\ \cos(\omega \mathbf{t} - \mathbf{53.3}^\circ)$

$E_\eta = 0.796\cos\omega t - 0.592\sin\omega t = \mathbf{-0.992}\ \sin(\omega t - \mathbf{53.30})$.

The light is elliptically polarized. The ratio of the major to the minor axes of the ellipse is 2.23, and one of the principle axes of the ellipse is rotated 31° from the plane of incidence of the light. The electric vector is rotating counter-clockwise when viewed looking into the reflected beam along the +z direction.

Problem (10.12).

Consider a block of dielectric material of thickness d immersed in vacuum. A wave having an amplitude E_o is incident on the block as shown: the angle of incidence is θ = 0.



Calculate the amplitudes of the reflected and transmitted waves E_R, E_T.

HINT: Inside the dielectric block there is both a forward and a backward moving wave: ie. in the block

$$E_x = ae^{i[kmz-\omega t]} + be^{-i[kmz+\omega t]}$$

One must satisfy boundary conditions at both z = 0 and at z = d.

Answer (10.12).

In the dielectric block $k_m^2 = \varepsilon_r \left(\frac{\omega}{c}\right)^2$

$\therefore k_m = (n + ik)\left(\frac{\omega}{c}\right)$ if ε_r is complex.

We require curl **E** = $i\omega\mu_o$**H**

$\therefore$ since there is only an x-component of **E**

$$i\omega\mu_\circ H_Y = \frac{\partial E_X}{\partial z} = ik_m\left[ae^{i[kmz-\omega t]} - be^{-i[kmz+\omega t]}\right]$$

Incident Wave:

$$E_x = E_0 e^{i[\omega/c\, z-\omega t]}$$

$$H_y = \frac{E_0}{Z_0} e^{i[\omega/c\, z-\omega t]}$$

Reflected Wave:

$$E_x = E_R e^{-i[\omega/c\, z+\omega t]}$$

$$H_Y = -\frac{E_R}{Z_0} e^{-i[\omega/c\, z+\omega t]}$$

Boundary Conditions at z = 0

(1) Continuity of E_x $\mathbf{E_0} + \mathbf{E_R} = \mathbf{a} + \mathbf{b}$ (1)

(2) Continuity of H_y $\frac{E_0}{Z_0}-\frac{E_R}{Z_0}=\frac{ck_m}{\omega Z_0}[a-b]$

or $\mathbf{E}_0-\mathbf{E}_R=\frac{\mathbf{c}\mathbf{k}_m}{\omega}[\mathbf{a}-\mathbf{b}]$ (2) (2)

Now at z = d one can write the transmitted fields as

$$E_x=E_T e^{i[\omega/c\,(z-d)}-\omega t\Big]$$

$$\mathrm{H_y}=\frac{\mathrm{E_T}}{\mathrm{Z_0}}e^{\mathrm{i}[\omega/\mathrm{c}\,(\mathrm{z}-\mathrm{d})-\omega\mathrm{t}]}$$

∴ at z = d $E_x = E_T$ and $H_y = E_T/Z_o$

But in the dielectric at z = d one has

$$E_x=\left(ae^{ik_m d}+be^{-ik_m d}\right)e^{-i\omega t}$$

$$\mathrm{H_Y}=\frac{\mathrm{ck_m}}{\omega\mathrm{Z_0}}\left(a_{\mathrm{b}}e^{\mathrm{ik_m d}}-\mathrm{be}^{-\mathrm{ik_m d}}\right)\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}$$

Therefore from continuity of E_x one obtains

$\mathbf{ae^{ik_m d}+be^{-ik_m d}=E_T}$ (1)

and from continuity of H_y

$$\frac{ck_m}{\omega Z_0}\left(ae^{ik_m d}-be^{-ik_m d}\right)=\frac{E_T}{Z_0}$$

or

$\mathbf{a}e^{\mathbf{ik_m d}}-\mathbf{b}e^{-\mathbf{ik_m d}}=\left(\frac{\omega}{\mathbf{ck_m}}\right)\mathbf{E_T}$ (4)

From (3) and (4) one has

$\mathrm{ae^{ik_m d}-be^{-ik_m d}}=\left(\frac{\omega}{\mathrm{ck_m}}\right)\left[\mathrm{ae^{ik_m d}+be^{-ik_m d}}\right]$

$\therefore\quad \frac{\mathrm{b}}{\mathrm{a}}=\mathrm{e}^{2\mathrm{ik_m d}}\left[\frac{1-\left(\frac{\omega}{\mathrm{ck_m}}\right)}{1+\left(\frac{\omega}{\mathrm{ck_m}}\right)}\right]$

and from (1) and (2)

$2\mathrm{E_0}=\mathrm{a}\left\{\left[1+\left(\frac{\mathrm{ck_m}}{\omega}\right)\right]+\left[1-\left(\frac{\mathrm{ck_m}}{\omega}\right)\right]\left(\frac{\mathrm{b}}{\mathrm{a}}\right)\right\}$

or

$$\frac{\mathrm{a}}{\mathrm{E_o}}=\frac{2\left(1+\frac{\omega}{\mathrm{ck_m}}\right)}{\left[\left(2+\frac{\omega}{\mathrm{ck_m}}+\frac{\mathrm{ck_m}}{\omega}\right)+\left(2-\frac{\omega}{\mathrm{ck_m}}-\frac{\mathrm{ck_m}}{\omega}\right)\mathrm{e}^{2\mathrm{ik_m d}}\right]}$$

$$\frac{\mathbf{b}}{\mathbf{E_O}}=\frac{2\left(1-\frac{\omega}{\mathbf{ck_m}}\right)}{\left[\left(2+\frac{\omega}{\mathbf{ck_m}}+\frac{\mathbf{ck_m}}{\omega}\right)+\left(2-\frac{\omega}{\mathbf{ck_m}}-\frac{\mathbf{ck_m}}{\omega}\right)\mathrm{e}^{\mathbf{2ik_m}\mathrm{d}}\right]}$$

$$\frac{E_R}{E_0}=\frac{\left[\left(\frac{\omega}{ck_m}\right)-\left(\frac{ck_m}{\omega}\right)\right]\left[1-e^{2ik_m d}\right]}{\left[\left(2+\frac{\omega}{ck_m}+\frac{ck_m}{\omega}\right)+\left(2-\frac{\omega}{ck_m}-\frac{ck_m}{\omega}\right)e^{2ik_m d}\right]}$$

$$\frac{E_T}{E_0}=\frac{4e^{ik_m d}}{\left[\left(2+\frac{\omega}{ck_m}+\frac{ck_m}{\omega}\right)+\left(2-\frac{\omega}{ck_m}-\frac{ck_m}{\omega}\right)e^{2ik_m d}\right]}$$

If ε_r is real $k_m = n\omega/c$ and

$$\left(\frac{E_R}{E_0}\right) = \frac{(1-n^2)\left[1-e^{2ik_md}\right]}{\left[(n+1)^2-(n-1)2e^{2ik_md}\right]}$$

$$\left(\frac{E_T}{E_0}\right) = \frac{4ne^{ik_md}}{[(n+1)2-(n-1)2e^{2ik_md}]}$$

The above two equations are oscillatory functions of the wavelength.

if $2k_md = 2\pi, 4\pi, 6\pi$, etc.

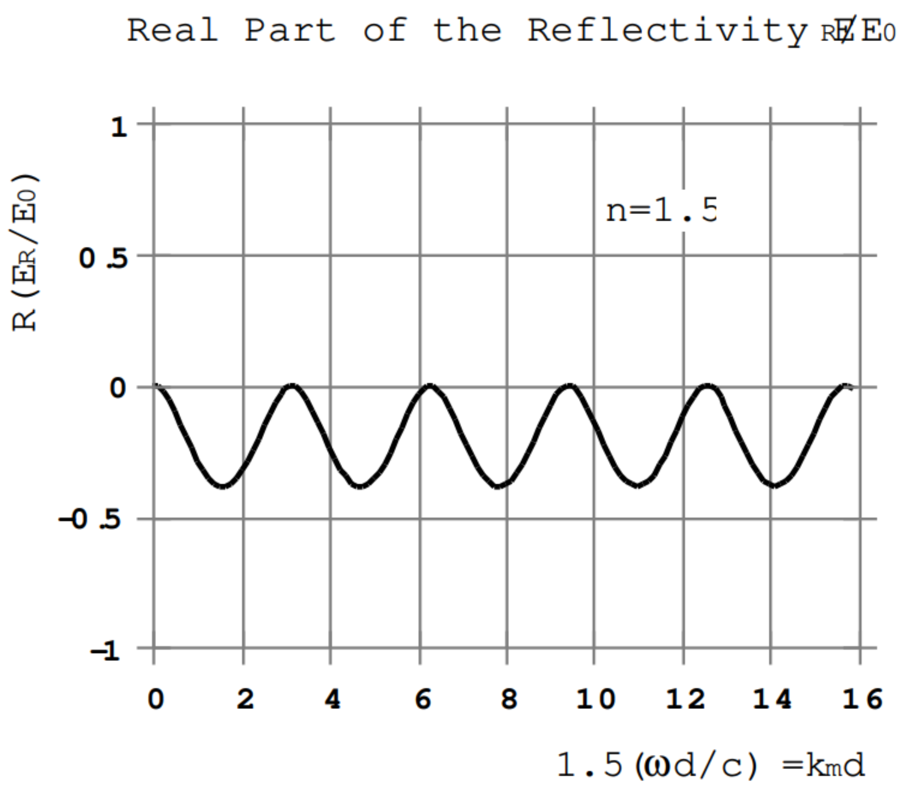
then $\frac{E_R}{E_0} = 0 \ \frac{E_T}{E_0} = \pm 1$

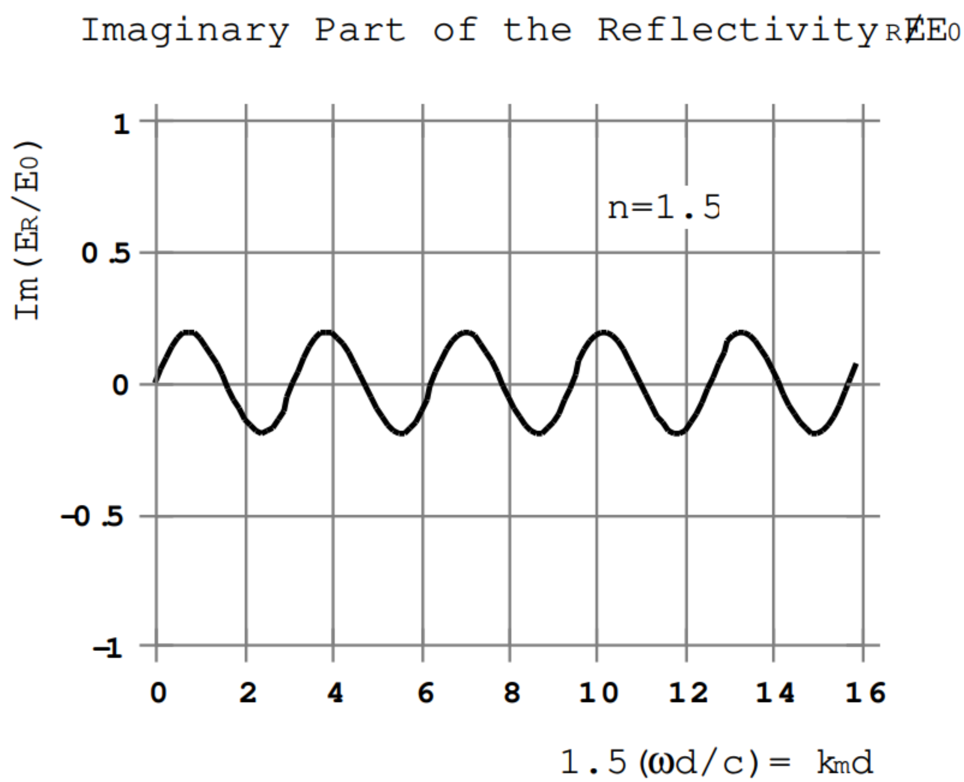
If $2k_md = \pi, 3\pi, 5\pi$, etc.

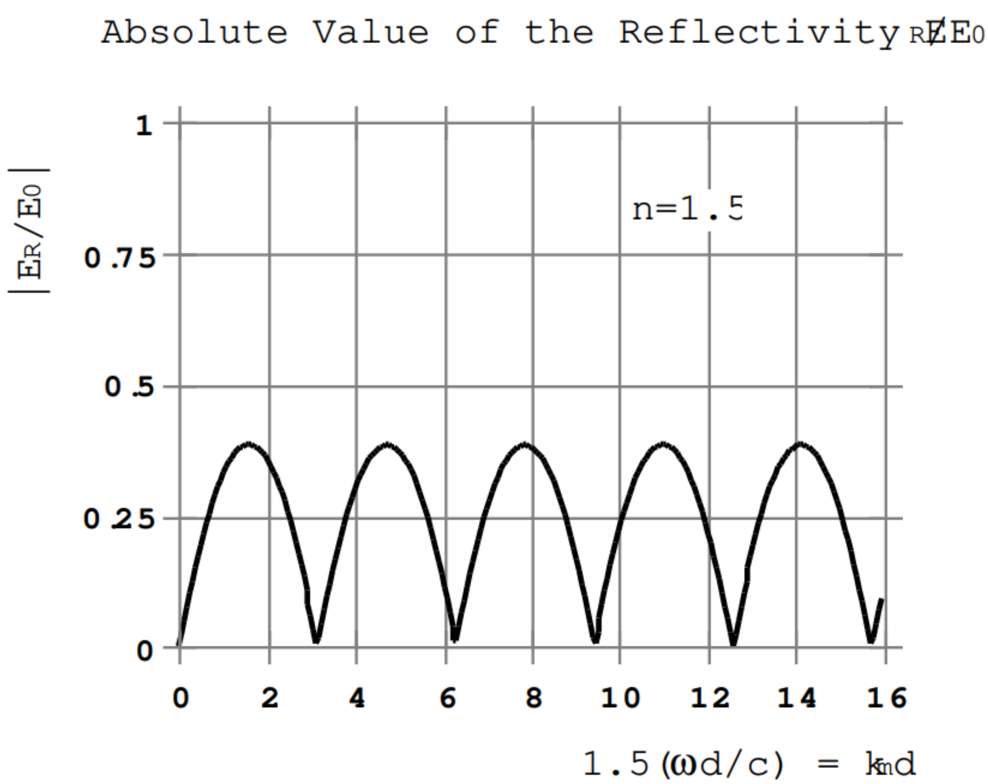
then $\frac{E_R}{E_0} = \frac{(1-n^2)}{(1+n^2)}$ i.e. a maximum

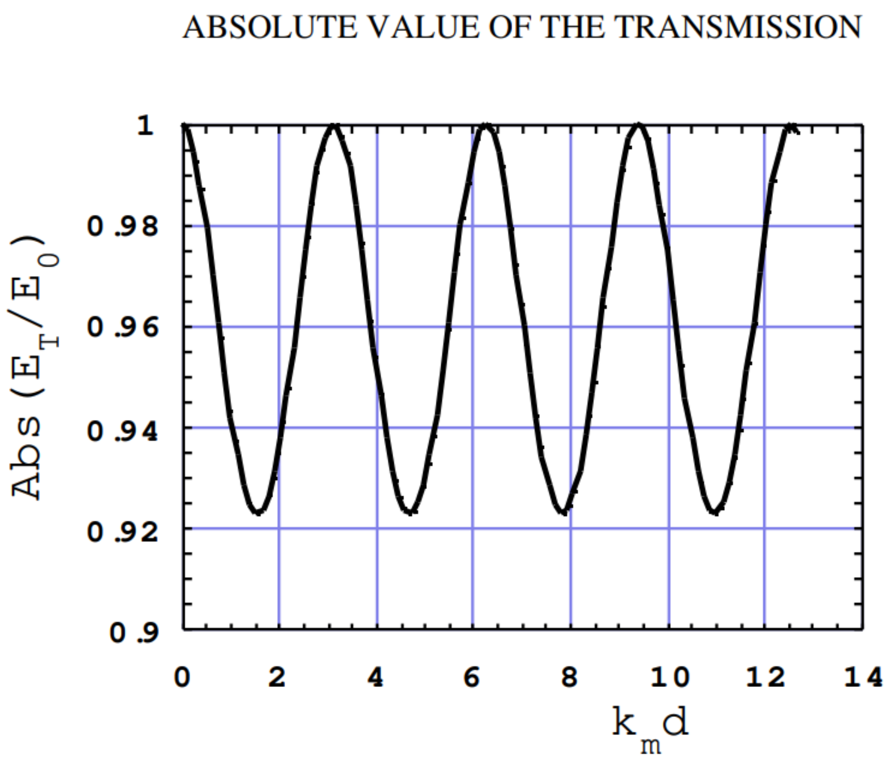
$\frac{E_T}{E_0} = \frac{\pm 2ni}{(n^2+1)}$

The variation with frequency of the reflectivity and the transmission coefficient are plotted below for a real dielectric constant ε_r= 2.25 (n= 1.5).




Imaginary Part of the Reflectivity E_R/E_0
$Im(E_R/E_0)$
n=1.5
1
0.5
0
-0.5
-1
0
2
4
6
8
10
12
14
16
$1.5(\omega d/c) = k_m d$


Absolute Value of the Reflectivity E_R/E_0
$|E_R/E_0|$
n=1.5
1
0.75
0.5
0.25
0
0
2
4
6
8
10
12
14
16
$1.5(\omega d/c) = k_m d$



Problem (10.13).

Let a material be described by electric and magnetic linear response: i.e.

$$\mathbf{D}=\varepsilon(\omega)\mathbf{E},$$

and

$$\mathbf{B}=\boldsymbol{\mu}(\boldsymbol{\omega})\mathbf{H},$$

where both ε(ω) and μ(ω) are complex numbers. These are usually written

$$\varepsilon(\omega)=\varepsilon_0\varepsilon_r=\varepsilon_1+i\varepsilon_2$$

and

$$\mu(\omega)=\mu_0\mu_r=\mu_1+\mu_2.$$

For a time dependence $e^{-i\omega t}$ the imaginary parts of the response functions, $\varepsilon_2(\omega)$ and $\mu_2(\omega)$, are greater than zero.

(a) According to Poynting's theorem the rate of increase of energy stored in the fields is given by

$$\frac{dW}{dt}=\mathbf{E}\cdot\frac{d\mathbf{D}}{dt}+\mathbf{H}\cdot\frac{d\mathbf{B}}{dt}.$$

Show that for a time dependence $e^{-i\omega t}$ the imaginary parts of ε and μ must be greater than zero for any finite frequency. This conclusion follows from the restriction that the time average of $\frac{dW}{dt}$ must be greater than or equal to zero according to the second law of thermodynamics.

(b) Show that for a time dependence $e^{-i\omega t}$ a plane wave solution of Maxwell's equations can be found in the form

$$E_x=E_0e^{i(k_mz-\omega t)} \qquad (1)$$

$$\mathrm{H_Y}=\frac{\mathrm{k_m}}{\omega\mu}\mathrm{E_0e^{i(k_mz-\omega t)}}, \qquad (2)$$

where $k_m^2 = \varepsilon_r \mu_r \left(\frac{\omega}{c}\right)^2$,

and $k_m = \sqrt{\varepsilon_r \mu_r} \left(\frac{\omega}{c}\right) \equiv (N + i\kappa) \left(\frac{\omega}{c}\right)$, where K>0

for a wave damped towards the interior of a semi-infinite slab.

(c) Calculate the time averaged value of the Poynting vector corresponding to the fields of eqns.(1) and (2). Show that

$\langle S_z \rangle = \frac{1}{2C} \frac{(N\mu_1 + K\mu_2)}{(\mu_1^2 + \mu_2^2)} |E_0|^2 e^{-2\kappa(\frac{\omega}{c})z}$. (3)

Notice that for a passive medium $<S_z>$ must be greater than, or equal, to zero; this means that $(N\mu_1 + K\mu_2) \geq 0$. For a nonmagnetic material $\mu_1 = \mu_0$ and $\mu_2 = 0$; thus for a non-magnetic material eqn.(3) states that n≥ 0 (for this case N=n).

(d) Calculate the time averaged energy densities corresponding to the waves of eqns.(1) and (2). Show that

$< W_E >= \frac{1}{2}\text{Real}\left(\frac{\varepsilon E E^*}{2}\right) = \frac{\varepsilon_1 |E_0|^2}{4} e^{-2K(\frac{\omega}{c})z}$, (4)

and

$< W_B >= \frac{1}{2}\text{Real}\left(\frac{\mu H H^*}{2}\right) = \frac{\varepsilon_0 \mu_0}{4} \left(\frac{\mu_1}{\mu_1^2 + \mu_2^2}\right) (N^2 + K^2) |E_0|^2 e^{-2K(\frac{\omega}{c})z}$ (5)

Expressions (4) and (5) do not appear to have much in common except the factor $|E_0|^2 e^{-2K(\frac{\omega}{c})z}$. However, from the definition

$$(\varepsilon_1 + i\varepsilon_2)(\mu_1 + i\mu_2) \equiv (N + iK)^2 \varepsilon_0 \mu_0$$

plus some tedious algebra, it can be shown that

$\varepsilon_1 = \left(\frac{(N^2 - K^2)\mu_1 + 2NK\mu_2}{(\mu_1^2 + \mu_2^2)}\right) \varepsilon_0 \mu_0$, (6)

and

$\varepsilon_2 = \left(\frac{2NK\mu_1 - (N^2 - K^2)\mu_2}{(\mu_1^2 + \mu_2^2)}\right) \varepsilon_0 \mu_0$. (7)

These can be used to write

$< W_E >= \frac{\varepsilon_0 \mu_0}{4} \left(\frac{(N^2 - K^2)\mu_1 + 2NK\mu_2}{(\mu_1^2 + \mu_2^2)}\right) |E_0|^2 e^{-2K(\frac{\omega}{c})z}$. (8)

(e) Calculate the total time averaged energy density associated with the electric and magnetic fields of eqns.(1) and (2). Show that since $< W >= \langle W_E \rangle + \langle W_B \rangle$ it follows that

$< W >= \frac{\varepsilon_0 \mu_0}{2} \frac{N(N\mu_1 + K\mu_2)}{(\mu_1^2 + \mu_2^2)} |E_0|^2 e^{-2K(\frac{\omega}{c})z}$. (9)

If this energy density is to be non-negative, it follows from eqn.(3) for $<S_z>$ which must be greater than or equal to zero, that N≥0. By comparison of eqns.(3) and (9) one finds also that

$$< S_z >= \left(\frac{c}{N}\right) < W >.$$

I know of no fundamental microscopic reason why the real part of the index of refraction should be confined to positive values. It is true, however, that for the metals that I have checked, Fe,Co,Ni,Cu,Ag,Au, and Al, the real part of the index of refraction, n, is greater than zero over the energy range 0.1 to 100 eV. For example,

(i) Cu: n is a minimum at 1.80 eV where n=0.21 and κ=4.25; the index then increases with energy but becomes less than 1 for energies greater than 9.0 eV.

(ii) Ag: n is a minimum at 3.5 eV where n=0.21 and κ=1.42; the index then increases with energy and becomes again less than 1 for energies greater than 25 eV.

(iii) Au: n is a minimum at 1.40 eV where n=0.08 and κ=5.44; the index then increases with energy but becomes less than 1 for energies greater than 22 eV.

(iv) Al: n is a minimum at 12.0 eV where n=0.033 and κ=5.44; the index then increases with energy but drops below 1 for energies greater than 95 eV.

Answer (10.13).

(a) Let $E_x = E_0 e^{-i\omega t} = E_0 \cos \omega t$

then $D_{\mathrm{X}} = (\varepsilon_1 + i\varepsilon_2) E_0 e^{-i\omega t}$

or

$$\mathrm{D_x} = \varepsilon_1 \mathrm{E_0 Cos\omega t} + \varepsilon_2 \mathrm{E_0}\, \mathrm{Sin}\, \omega \mathrm{t}.$$

$$\frac{dW_E}{dt} = E_x \frac{dD_x}{dt} = E_0 \cos \omega t \left(-\varepsilon_1 \omega E_0 \,\mathrm{Sin}\, \omega t + \varepsilon_2 \omega E_0 \cos \omega t\right),$$

$$\frac{\mathrm{d}W_{\mathrm{E}}}{\mathrm{dt}} = -\omega\varepsilon_1 \mathrm{E}_0^2 \sin \omega \mathrm{t} \cos \omega \mathrm{t} + \omega\varepsilon_2 \mathrm{E}_0^2 \cos^2 \omega \mathrm{t}.$$

Therefore $< \frac{dW_E}{dt} >= \omega\varepsilon_2 \frac{E_0^2}{2}$.

It follows that if $< \frac{dW_E}{dt} > \ \geq 0$ then $\varepsilon_2 \geq 0$ for any finite frequency.

Similarly, $\mathrm{H_y} = \mathrm{H_0 e^{-i\omega t}} = \mathrm{H_0} \cos \omega \mathrm{t}$,

and $\mathrm{B_y} = \mu \mathrm{H_y} = \mu_1 \mathrm{H_0} \cos \omega \mathrm{t} + \mu_2 \mathrm{H_0} \sin \omega t$.

$$\frac{\mathrm{dB_Y}}{\mathrm{dt}} = \omega \mathrm{H_0} \left(-\mu_1 \sin \omega \mathrm{t} + \mu_2 \cos \omega \mathrm{t}\right),$$

therefore

$$\frac{dW_B}{dt} = \mathbf{H} \cdot \frac{d\mathbf{B}}{dt}$$

$$\frac{\mathrm{d}W_B}{dt} = \omega H_0^2 \left(-\mu_1 \sin \omega t \cos \omega t + \mu_2 \cos^2 \omega t\right),$$

and

$$< \frac{dW_B}{dt} >= \omega\mu_2 \frac{H_0^2}{2}.$$

It follows that if $< \frac{dW_B}{dt} > \ \geq 0$ then $\mu_2 \geq 0$ for any finite frequency.

(b) Maxwell's equations for a time dependence $e^{-i\omega t}$ can be written

$$\mathrm{curl}\, \mathbf{E} = \mathrm{i}\, \omega\mu \mathbf{H} = i\omega\mu_r \mu_0 \mathbf{H} \qquad (i)$$

$$\mathrm{curl}\, \mathbf{H} = -\mathrm{i}\omega\varepsilon \mathbf{E} = -\mathrm{i}\omega\varepsilon_\mathrm{r}\varepsilon_0 \mathbf{E} \qquad (ii)$$

where from (i) div**H**=0 and from (ii) div**E**=0 because the divergence of any curl must vanish. The fields **E**,**H** therefore satisfy

$$\nabla^2 \mathbf{E} = -\varepsilon_\mathrm{r}\mu_\mathrm{r} \left(\frac{\omega}{\mathrm{c}}\right)^2 \mathbf{E},$$

$$\nabla^2 \mathbf{H} = -\varepsilon_\mathrm{r}\mu_\mathrm{r} \left(\frac{\omega}{\mathrm{c}}\right)^2 \mathrm{H}.$$

Let **E** be polarized along x and **H** be polarized along y. Then plane wave solutions of the above equations are

$$E_x = E_0 e^{i(k_m z - \omega t)}$$

and

$$\mathrm{H_y} = \mathrm{H_0 e^{i(k_m z - \omega t)}}$$

or

$$\mathrm{H_Y}=\frac{\mathrm{k_m}}{\omega\mu}\mathrm{E_0e^{i(k_mz-\omega t)}},\ \text{from eqn. (i)},$$

where $\mathrm{k_m^2}=\varepsilon_\mathrm{r}\mu_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2$

or $\mathrm{k_m}=(\mathrm{N}+\mathrm{iK})\left(\frac{\omega}{\mathrm{c}}\right)$,

where $\mathrm{N}+\mathrm{iK}=\sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}}$.

It is necessary to use the branch of the square root for which K≥0, since this branch corresponds to a disturbance which dies away with increasing z.

$$<\mathrm{S_z}>=\frac{1}{2}\mathrm{Real}\left(\mathrm{E_xH_y^*}\right),$$

$$<\mathrm{S_z}>=\frac{1}{2}\mathrm{Real}\left(\mathrm{E_0e^{ik_mz}}\frac{\mathrm{k_m^*}}{\omega\mu^*}\mathrm{E_0^*e^{-ik_m^*z}}\right)$$

$$<\mathrm{S_z}>=\frac{1}{2\mathrm{c}}\mathrm{Real}\left(\frac{(\mathrm{N}-\mathrm{iK})(\mu_1+\mathrm{i}\mu_2)}{\mu\mu^*}|\mathrm{E_0}|^2e^{-2\mathrm{K}\left(\frac{\omega}{\mathrm{c}}\right)\mathrm{z}}\right),$$

$$\langle\mathrm{S_z}\rangle=\frac{1}{2\mathrm{c}}\frac{(\mathrm{N}\mu_1+\mathrm{K}\mu_2)}{\left(\mu_1^2+\mu_2^2\right)}|\mathrm{E_0}|^2\mathrm{e}^{-2\mathrm{K}\left(\frac{\omega}{\mathrm{c}}\right)\mathrm{z}}.$$

(d) $\langle W_E\rangle=\left\langle\frac{\mathbf{E}\cdot\mathbf{D}}{2}\right\rangle=\frac{1}{4}\mathrm{Real}(E_xD_x^*)$, or

$$\langle W_E\rangle=\frac{1}{4}\mathrm{Real}\left(E_0e^{i(k_mz-\omega t)}(\varepsilon_1-i\varepsilon_2)E_0^*e^{-i(k_m^*z-\omega t)}\right),$$

$$<W_E>=\frac{1}{4}\mathrm{Real}\left((\varepsilon_1-i\varepsilon_2)|E_0|^2e^{i(k_m-k_m^*)z}\right).$$

But $(\mathrm{k_m}-\mathrm{k_m^*})=2\mathrm{iK}\left(\frac{\omega}{\mathrm{c}}\right)$, therefore

$$<W_E>=\frac{\varepsilon_1}{4}|E_0|^2e^{-2K\left(\frac{\omega}{c}\right)z}.$$

$$<W_B>=<\frac{\mu H^2}{2}>,$$

$$<W_B>=\frac{1}{4}\mathrm{Real}\left(\frac{(\mu_1+i\mu_2)}{\omega^2}\frac{k_mk_m^*}{\mu\mu^*}|E_0|^2e^{-2K\left(\frac{\omega}{c}\right)z}\right).$$

But $\mathrm{k_mk_m^*}=\left(\mathrm{N^2}+\mathrm{K^2}\right)\left(\frac{\omega}{\mathrm{c}}\right)^2$

and $\mu\mu^*=\left(\mu_1^2+\mu_2^2\right)$,

so that

$$<W_B>=\frac{1}{4c^2}\frac{\mu_1\left(N^2+K^2\right)}{\left(\mu_1^2+\mu_2^2\right)}|E_0|^2e^{-2K\left(\frac{\omega}{c}\right)z},$$

where $c^2=1/\varepsilon_0\mu_0$.

(e) Just add together $<W_E>$ and $<W_B>$ and use eqn.(6) above to get

$$<\mathrm{W}>=\frac{\varepsilon_0\mu_0}{2}\frac{\mathrm{N}(\mathrm{N}\mu_1+\mathrm{K}\mu_2)}{\left(\mu_1^2+\mu_2^2\right)}|\mathrm{E_0}|^2\mathrm{e}^{-2\mathrm{K}\left(\frac{\omega}{\mathrm{c}}\right)\mathrm{z}}.\qquad(9)$$

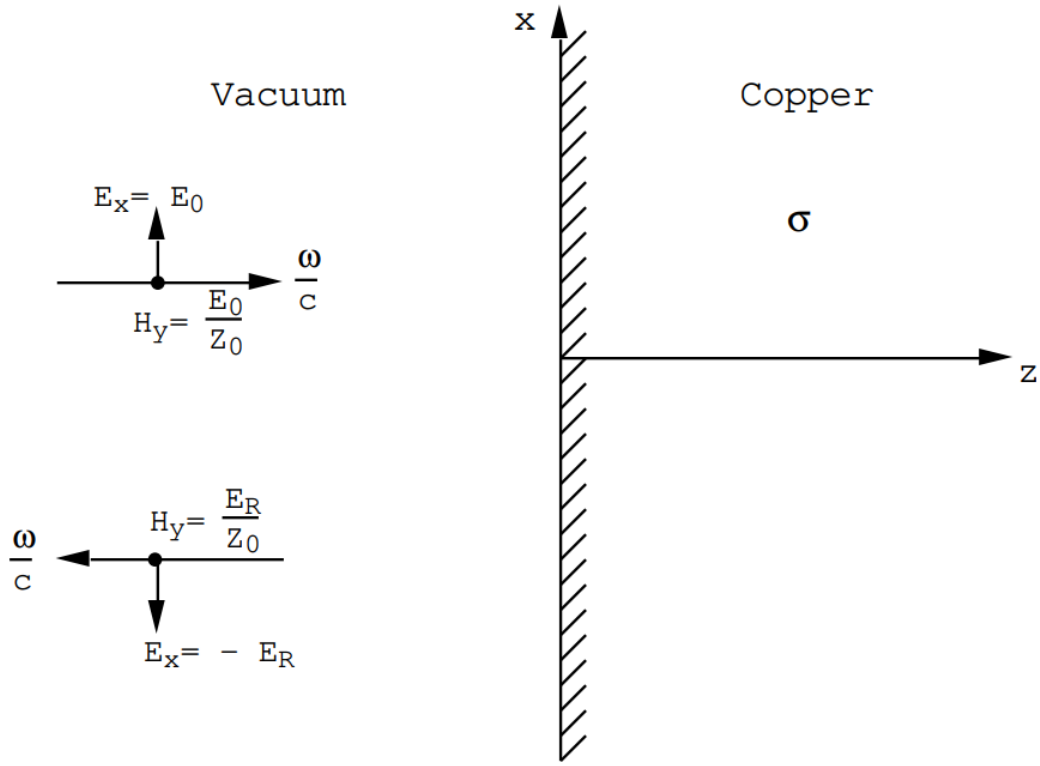
Problem (10.14).

Radiation having a frequency of 1 MHz falls at normal incidence from vacuum upon a thick copper sheet. The copper sheet is parallel with the x-y plane and the surface of the sheet lies at z=0. The resistivity of copper is $\rho=2.0\times10^{-8}$ Ohm-meters at room temperature.

(a) How much energy is absorbed per square meter by the copper sheet if the electric field strength in the incident wave is 1 V/m?

(b) What will be the energy absorbed per m^2 if the incident radiation falls on the surface at an angle of incidence of 45°? Let the incident radiation be p-polarized.

Answer (10.14).



In the metal

$$\begin{aligned} \operatorname{curl} \mathbf{E} &= i\omega\mu_0 \mathbf{H} \\ \operatorname{curl} \mathbf{H} &= \sigma \mathbf{E} \\ \operatorname{div} \mathbf{E} &= 0 \\ \operatorname{div} \mathbf{H} &= 0 \end{aligned}$$

therefore

$$\operatorname{curl} \operatorname{curl} \mathbf{H} = \mathbf{i}\omega\sigma\mu_0 \mathbf{H},$$

and $\operatorname{curl} \operatorname{curl} \mathbf{E} = \mathbf{i}\omega\sigma\mu_0 \mathbf{E},$

where

$$\mathrm{k}_{\mathrm{m}}^2 = \mathrm{i}\omega\sigma\mu_0,$$

or

$$\mathrm{k}_{\mathrm{m}} = \sqrt{\frac{\omega\mu_0\sigma}{2}}(1+\mathrm{i}) = \frac{(1+\mathrm{i})}{\delta}.$$

Also $\frac{\partial \mathrm{E}_{\mathrm{x}}}{\partial z} = \mathrm{i}\omega\mu_0 \mathrm{H}_{\mathrm{Y}}$

or

$$\mathrm{E}_{\mathrm{x}} = \left(\frac{\omega\mu_0}{\mathrm{k}_{\mathrm{m}}}\right) \mathrm{H}_{\mathrm{y}} = \left(\frac{\delta\omega}{\mathrm{c}}\right) \mu_0 \mathrm{c} \frac{(1-\mathrm{i})}{2} \mathrm{H}_{\mathrm{Y}}.$$

For this problem $\omega = 2\pi x 10^6$ radians/sec

$\frac{\omega}{c}=0.0209/\mathrm{m}$

$\delta=\sqrt{\frac{2}{\omega\mu_0\sigma}}=0.71\times10^{-4}\ \mathrm{m}=71\mu\mathrm{m}$

and $\frac{\delta\omega}{\mathrm{c}}=1.49\times10^{-6}$.

In the incident wave E_0= 1 V/m, and $\mathrm{H}_0=\frac{\mathrm{E}_0}{\mathrm{Z}_0}=2.65\times10^{-3}$ Amps/m.

Just inside the metal surface $\mathrm{H}_\mathrm{T}\cong2\mathrm{H}_0=5.31\times10^{-3}$ Amps/m.

Therefore $E_x=\left(1.49\times10^{-6}\right)(377)\frac{(1-i)}{2}\left(5.31\times10^{-3}\right)$

$E_x=\left(1.49\times10^{-6}\right)(1-i)\quad V/m$.

$\langle S_x\rangle=\frac{1}{2}\mathrm{Real}\left(H_yE_x^*\right)$

$\langle S_x\rangle=\mathrm{Real}\left(\left(2\cdot65\times10^{-3}\right)\left(1\cdot49\times10^{-6}\right)(1+i)\right)$

$\langle S_x\rangle=3.95\times10^{-9}\mathrm{Watts}/\mathrm{m}^2$.

(b) The incident wave is given by

$$\mathrm{H_Y}=\mathrm{H_0e^{iqx}e^{ikz}}e^{-\mathrm{i}\omega \mathrm{t}}$$

$$\mathrm{E_x}=\frac{\mathrm{Z_0H_0}}{\sqrt{2}}\mathrm{e^{iqx}e^{ikz}e^{-i\omega t}}$$

$$\mathrm{E_z}=-\frac{\mathrm{Z_0H_0}}{\sqrt{2}}e^{\mathrm{iqx}}e^{\mathrm{ikz}}e^{-\mathrm{i}\omega\mathrm{t}}.$$

The reflected wave is given by

$$\mathrm{H_y}=\mathrm{H_Re^{-ikz}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_x}=-\frac{\mathrm{Z_0H_R}}{\sqrt{2}}e^{iqx}e^{-\mathrm{ik}z}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_z}=-\frac{\mathrm{Z_0H_R}}{\sqrt{2}}e^{iqx}e^{-\mathrm{ik}z}e^{-\mathrm{i}\omega t}.$$

In the metal $-\nabla^2\mathbf{H}=i\omega\sigma\mu_0\mathbf{H}$

therefore

$$q^2+k_m^2=i\omega\sigma\mu_0$$

$$q^2+k_m^2=\frac{i\left(4\pi\times10^{-7}\right)\left(2\pi\times10^6\right)}{2\times10^{-8}}=3.95i\times10^8/\mathrm{m}^2$$

and

$$q=\frac{\omega}{\mathrm{c}\sqrt{2}}=0.0148/\mathrm{m},\quad \mathrm{i.e}\cdot\mathrm{q}^2=2.18\times10^{-4}/\mathrm{m}^2.$$

In other words, q^2 is completely negligible compared with k_m^2. This is, for all intents and purposes, the same problem as part (a). The energy absorbed from the incident wave will be **$3.95\mathrm{x}10^{-9}$ Watts/m^2**. For completeness, if

$\mathrm{H_Y}=\mathrm{H_T}e^{iqx}e^{ik_mz}$, then $\mathrm{E_x}=-\frac{\mathrm{ik_m}}{\sigma}\mathrm{H_Te^{iqx}e^{ik_mz}}$, where $\mathrm{H_T}\cong2/\mathrm{Z}_0$, and Z_0= $c\mu_0$.

13.11: Chapter- 11

Problem (11.1).

A strip-line is constructed from a metal strip 1 mm wide (W= 1 mm) separated from a ground plane by an oxide layer whose thickness, D, is 20 μm. The relative dielectric constant of the oxide layer is ε_r= 8.00, and its relative permeability is μ_r= 1.00.

(a) What is the velocity of an electromagnetic wave on this line?

(b) What is the characteristic impedance of the strip-line?

(c) A pulse on the line is 10 meters long and corresponds to a constant potential difference of 10 Volts. How much energy is stored in the pulse?

Answer (11.1).

(a) $v^2 = \frac{1}{\varepsilon\mu} = \frac{c^2}{\varepsilon_r} = \frac{c^2}{8}$ **; v= 1.06 x 10^8 m/sec.**

(b) In the dielectric material one finds curl**E**= iωμ$_0$ **H** for a wave having a time dependence $e^{-i\omega t}$. Therefore

$$\frac{\partial \mathrm{E_X}}{\partial \mathrm{z}} = \mathrm{i}\omega\mu_0\mathrm{Hy},$$

and

$$\mathrm{E_x} = \left(\frac{\omega\mu_0}{\mathrm{k}}\right)\mathrm{H_y} \qquad \text{where kv} = \omega.$$

Thus $\frac{\mathrm{E_x}}{\mathrm{H_y}} = \mathrm{v}\mu_0 = 133.2$ Ohms.

In the strip line the potential is V= E_xD, and the current is given by I= WH_y. It follows that the characteristic impedance is given by

$$\mathrm{Z_0} = \frac{\mathrm{V}}{\mathrm{I}} = \frac{\mathrm{E_x D}}{\mathrm{H_y W}} = \left(\frac{\mathrm{D}}{\mathrm{W}}\right)(133.2) = \mathbf{2.66\ ohms.}$$

(c) The electric field in the insulator is $\mathrm{E_X} = \frac{\mathrm{V}}{\mathrm{D}}$, so

$$E_x = \frac{10}{20\times10^{-6}} = 5\times10^5 \text{ Volts /m.}$$

$$\mathrm{H_y} = \frac{\mathrm{E_X}}{133\cdot2} = 3.754\times10^3 \text{ Amps/m.}$$

The energy density stored in the electric field is given by

$$\mathrm{W_E} = \frac{\varepsilon\mathrm{E}^2}{2} = \frac{\varepsilon_\mathrm{r}}{2}\varepsilon_0\mathrm{E_x^2} = 4\varepsilon_0\mathrm{E_x^2} \text{ Joules /m}^3.$$

The energy density stored in the magnetic field is given by

$$W_B = \frac{\mu H^2}{2} = \frac{\mu_0 H_Y^2}{2} \text{ Joules }/m^3.$$

But $\frac{E_X}{H_Y} = \frac{C}{\sqrt{8}}\mu_0$ or $\mathrm{H_y} = \frac{\sqrt{8}}{\mathrm{c}\mu_0}\mathrm{E_x}$

so that $\mathrm{W_B} = 4\varepsilon_0\mathrm{E_x^2}$ Joules /m^3.

The total energy density is $\mathrm{W} = \mathrm{W_E} + \mathrm{W_B} = 8\varepsilon_0\mathrm{E_X^2}$.

So W= (8)(8.84 x 10^{-12})(25x10^{10}) = 17.86 Joules/m^3

The volume which contains this energy density is given by

$$\text{Vol.} = (10)\left(10^{-3}\right)\left(20\times10^{-6}\right) = 2\times10^{-7}\ \mathrm{m}^3.$$

The total energy stored in the pulse is **3.54 x 10^{-6} Joules.**

Problem (11.2).

The space between the conductors in a co-axial cable is filled with polyethylene which has a relative dielectric constant ε_r= 2.25. The characteristic impedance of the cable is 50 Ohms. A 10 meter length of cable is used to connect a pulse generator to a load of R Ohms. The incident pulse amplitude is V_0.

(a) What is the amplitude of the reflected pulse if the cable is terminated by 50 Ohms?

(b) What is the amplitude of the reflected pulse if the cable is terminated by zero Ohms?

(c) What is the amplitude of the reflected pulse if the cable is terminated by an open circuit?

(d) What is the amplitude of the reflected pulse if the cable is terminated by 100 Ohms?

(e) What is the inductance per meter of cable?

(f) What is the capacitance per meter of cable?

Answer (11.2).

The velocity of a pulse on the cable is $v=\frac{c}{\sqrt{\varepsilon_r}}=\frac{2c}{3}=2.0\times 10^8\ m/sec$, and the characteristic impedance is Z_0= 50 Ohms. The reflection coefficient is given by $\frac{V_R}{V_0}=\frac{r-1}{r+1}$, where $r=\frac{R}{Z_0}$.

(a) R= 50 Ohms, therefore r=1 and $\mathbf{V_R= 0}$.

(b) R= 0 Ohms, therefore r= 0 and $\mathbf{V_R= - V_0}$.

(c) R= ∞ Ohms, therefore r= ∞ and $\mathbf{V_R= +V_0}$.

(d) R= 100 Ohms, therefore r= 2 and $\frac{\mathbf{V_R}}{\mathbf{V_0}}=\frac{\mathbf{1}}{\mathbf{3}}$.

(e) $V^2=\frac{1}{LC}$ and $\mathrm{Z_0}=\sqrt{\mathrm{L/C}}$ so that

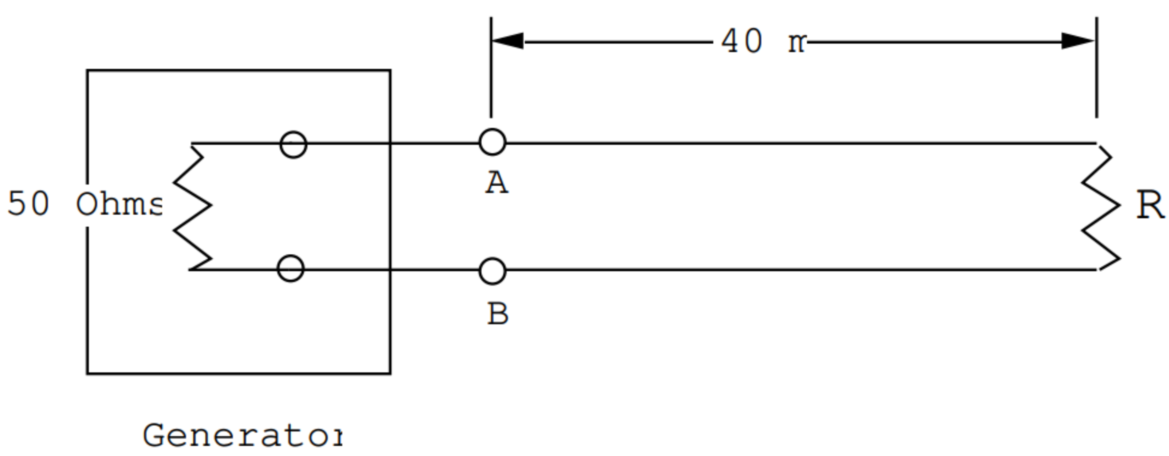
$$\mathrm{L/C}=2500 \quad \text{and} \quad \mathrm{LC}=\frac{1}{4\times 10^{16}}.$$

Consequently, $\mathrm{L}^2=\frac{2500}{4\times 10^{16}}$ and $\mathbf{L}=\frac{\mathbf{1}}{\mathbf{4}}\ \boldsymbol{\mu}\mathbf{Henry/m}$.

$$C=\frac{L}{2500}=\mathbf{100\ pF/m}.$$

Problem (11.3).

A typical co-axial cable has a characteristic impedance of 50 Ohms (Z_o = 50 Ohms). The dielectric material can be regarded as lossless and ε_r = 2.25. The cable is connected to a 50 Ohm pulse generator and is terminated by a resistance R Ohms (see the figure).



An oscilloscope is connected across AB: its impedance is effectively infinite so that it does not disturb the propagation of pulses on the line. The distance between AB and the end of the line is 40 meters. The generator emits a rectangular pulse whose amplitude is 5 Volts and whose length in time is 10^{-7} seconds.

(a) What is the velocity of pulses on this cable?

(b) Let R = 0. Make a sketch of the signal measured using the oscilloscope across AB.

(c) Let R = 0. Make a sketch of the signal measured using the oscilloscope connected across the resistor, R.

(d) Let R = 50 Ohms. Make a sketch of the signal measured across AB.

(e) Let R = 50 Ohms. Make a sketch of the signal measured across the resistor, R.

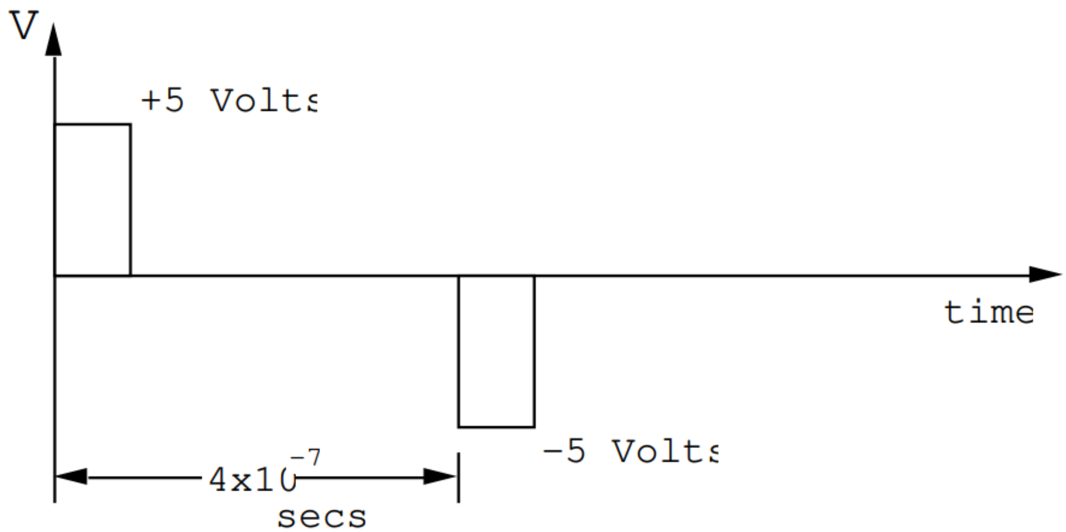
(f) Let R → ∞ (an open circuit). Make a sketch of the signal measured across AB.

(g) Let R → ∞. Make a sketch of the signal measured across the open end of the cable.

Answer (11.3).

(a) For this cable $\varepsilon_r = n^2 = 9/4$ therefore n = 3/2. The velocity of propagation $v = \frac{c}{n} = 2 \times 10^8 \text{ m/sec}$.

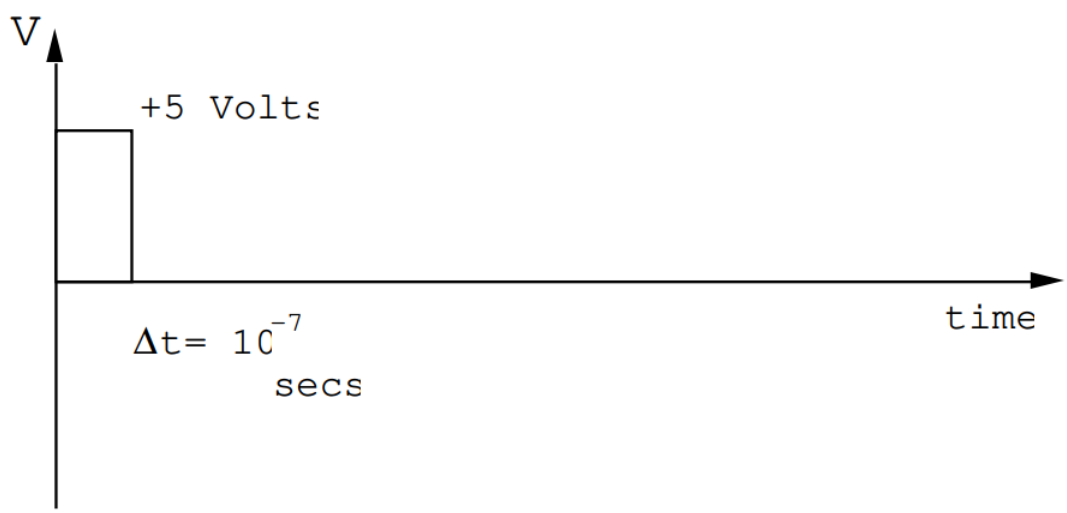
(b) Shorted Cable. At AB one sees the original pulse followed by the reflected pulse after a time delay of 80/v = 4×10^{-7} seconds (40 m out and 40 m back). The reflected pulse is inverted.



The reflected pulse is absorbed in the generator because the generator impedance is Z_o = 50 Ohms.

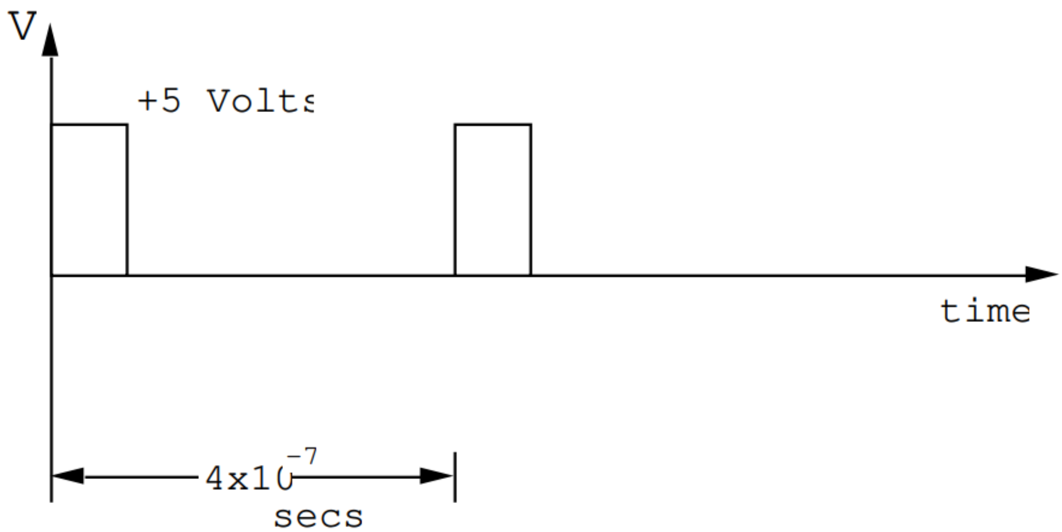
(c)Shorted Cable. Nothing will be seen across the short at the end of the cable (R = 0).

(d) Cable terminated by Z_o = 50 Ohms. One will measure only the initial pulse. There is no reflected pulse.



(e) Cable terminated by 50 Ohms. The voltage across the 50 Ohms will just look like the incident pulse but delayed by 40/v = 2×10^{-7} secs.

(f) Open circuit. At AB one will see the original pulse followed $80/(2 \times 10^8) = 4 \times 10^{-7}$ secs. later by a similar pulse. The reflected pulse will then be absorbed in the generator.



This is a standard technique for generating a delayed pulse.

(g) At the open end of the cable one will measure a single pulse whose amplitude is twice that of the original pulse. (One measures $V_o + V_R$). There will be a time delay of 2×10^{-7} secs.

Problem (11.4).

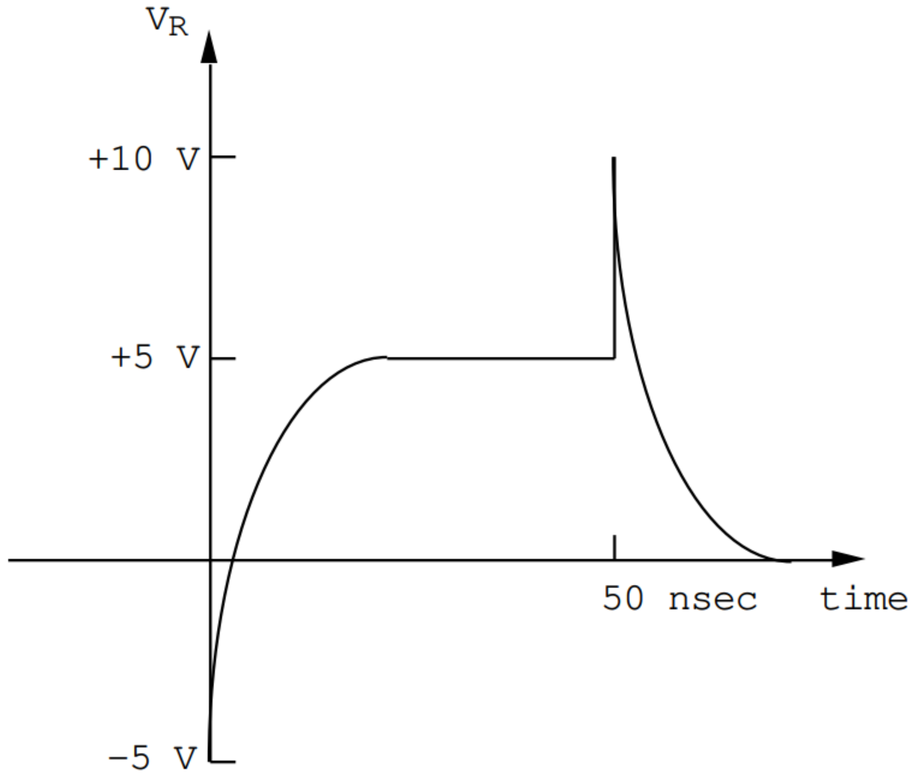
A certain co-axial cable is characterized by a velocity of v= 2.00×10^8 meters/sec., and it has a characteristic impedance of 50 Ohms. The cable is terminated by a capacitor C= 100 pF. A 10 Meter long rectangular pulse whose amplitude is 5 Volts is launched along the cable. Make a sketch of the reflected pulse. Carefully indicate the voltage and time scales; let the reflected pulse reach the observer at t=0. What is the maximum voltage in the reflected pulse?

Answer (11.4).

A 10 m pulse has a time duration of 5×10^{-8} seconds. The time constant associated with the capacitor is $CZ_0 = 5 \times 10^{-9}$ secs., therefore the capacitor will become fully charged during the time that the pulse is applied to it.

(i) Initially the capacitor behaves like a short circuit; the reflected pulse will have an amplitude of -5 Volts. This amplitude decays to +5 Volts as the capacitor becomes fully charged and looks like an open circuit. Note that when fully charged the potential across the capacitor is V_0+V_R= 10 Volts.

(ii) At the end of the incident pulse the capacitor, which has been charged to +10 Volts, deposits its charge back into the line at a rate determined by C and the characteristic impedance, Z_0.



Problem (11.5).

A certain co-axial cable is characterized by a velocity of V= 2.00 x 10^8 meters/sec., and it has a characteristic impedance of 50 Ohms. The cable is terminated by an inductor L= 0.25 µH. A 10 Meter long rectangular pulse whose amplitude is 5 Volts is launched along the cable. Make a sketch of the reflected pulse. Carefully indicate the voltage and time scales; let the reflected pulse reach the observer at t=0. What is the maximum voltage in the reflected pulse?

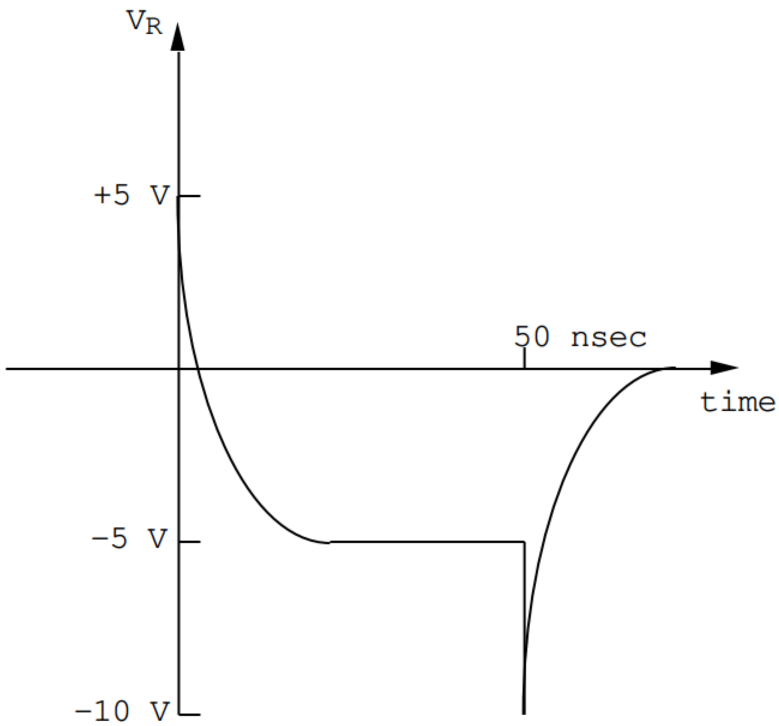
Answer (11.5).

The time duration of the pulse is 5 x 10^{-8} secs.= 50 nsecs., whereas the time constant associated with the inductor is $\tau=\frac{L}{Z_0}=0.5\times 10^{-8}$ secs. $=5$ nsecs ; thus the inductor will become fully charged with magnetic energy during the course of the pulse.

(i) At t=0 the inductor looks like an open circuit because it resists a change in the current flowing through it. The reflected pulse will therefore have an amplitude of +5 Volts, equal to the amplitude of the incident pulse. The reflected amplitude will decay with a time constant $\tau=L/Z_0$ as the current through the inductor reaches a steady state value. When the current has become constant, the inductor looks like a short circuit and the reflected pulse amplitude is -5 Volts.

(ii) The steady state value of the current through the inductor is just twice the current in the incident pulse, i.e. $I_0=\frac{2V_0}{Z_0}$ Amps, corresponding to a short circuit. Upon termination of the pulse, this current collapses to give an initial voltage

$$V=L\frac{dI}{dt}=-\frac{L}{\tau}I_0=-Z_0\left(\frac{2V_0}{Z_0}\right)=-2V_0.$$



Problem (11.6).

A certain co-axial cable is characterized by a velocity of V= 2.00 x 10^8 meters/sec., and it has a characteristic impedance of 50 Ohms. A piece of this cable 21 m long is used to connect a 250 MHz oscillator to a load impedance Z_L.

(a) What load impedance will be presented to the generator if Z_L is a 50 Ohm resistor?

(b) What load impedance will be presented to the generator if Z_L is a 1.00 μH inductor?

(c) What load impedance will be presented to the generator if Z_L is a 100 pF capacitor?

(d) What impedance will be presented to the generator in the above three cases if the co-axial cable has a length of 20.0 meters?

Answer (11.6).

At 250 MHz and for v= 2.00 x 10^8 m/sec. the wavelength on the cable is $\lambda = \frac{2\times10^8}{2.5\times10^8} = \frac{4}{5}$ meters .

(a) Terminated by the characteristic impedance. The generator looks into 50 Ohms.

(b) At 250 MHz. the impedance of a 1.0 μH inductor is given by Z_L= iLω = 1571i Ohms, since ω= 1.57 x 10^9 radians/sec. $\frac{Z_L}{Z_0} = i31.42$

$$2ikl = i\frac{4\pi(21)}{\lambda} = 105i\pi$$

which is equivalent to a phase shift of π. Since the impedance seen by the generator is

$$\frac{Z_G}{Z_0} = \frac{1+b/a}{1-b/a},$$

where

$$\frac{b}{a} = \left(\frac{z-1}{z+1}\right)e^{-2ikl},$$

and

$$z=\frac{Z_L}{Z_0},$$

one finds $\frac{b}{a}=\frac{1-i31\cdot 42}{1+i31\cdot 42}=-0.998-i0.0636.$

$$\frac{Z_G}{Z_0}=\frac{1+b/a}{1-b/a}=-i0.0319,$$

or **Z_G= -1.59i Ohms**. The load appears to the generator like a capacitor with C= 400 pF!

(c) The load impedance is a 100 pF capacitor.

For the capacitor $Z_c=\frac{-i}{c\omega}=-i6.366\text{ Ohms}$.

$$z=\frac{z_L}{z_0}=-i0.1273$$

As before e^{-2ikl} = -1 so that

$$b/a=\frac{1-z}{1+z}=0.9681+i0.2505.$$

$$\frac{Z_G}{Z_0}=\frac{1+b/a}{1-b/a}=i7.855,$$

from which $\boldsymbol{Z_G=i392.8}$ Ohms. .

The load appears to the generator like a 0.25 μH inductor.

(d) A 20 m cable contains an integer number of wavelengths, therefore the generator will look into the load impedance exactly as if the cable had zero length.

(a) Z_G = 50 Ohms.

(b) Z_G = i1571 Ohms (Inductive).

(c) Z_G = -i6.37 Ohms.

Problem (11.7).

A co-axial cable is characterized by a characteristic impedance of Z_o = 50 Ohms and a velocity of propagation of 2 x 10^8 m/sec. It is used to connect a 10 Ohm load to a generator. Calculate the impedance as seen from the generator for a cable having the following lengths, L:

(a) L = λ/8

(b) L = λ/4

(c) L = 3λ/8

(d) L = λ/2

(e) Calculate the Voltage Standing Wave Ratio, VSWR.

Answer (11.7).

For a load of Z_L = 10 one has a normalized impedance $Z_L=\frac{10}{50}=0.20$

$$\therefore\Gamma=\frac{z_L-1}{z_L+1}=-\frac{0.80}{1.2}=-\frac{2}{3}=\frac{2}{3}e^{i\pi}$$

(a) $L=\frac{\lambda}{8}$ $\quad\therefore e^{-2ikL}=e^{-i4\pi L/\lambda}=e^{-i\pi/2}=-i$

$$\therefore\Gamma e^{-2ikL}=\frac{2i}{3}$$

$$\therefore z_G=\frac{1+\Gamma e^{-2ikL}}{1-\Gamma e^{-2ikL}}=\frac{1+2i/3}{1-2i/3}=\frac{3+2i}{3-2i}=\frac{(3+2i)(3+2i)}{9+4}$$

$$\therefore \quad Z_G = \frac{5+12i}{13} = 0.385 + 0.923i$$

$$\therefore Z_G = 50 Z_G = \mathbf{19.23 + 46.15}\mathrm{i \ Ohms}$$

i.e. a large inductive component

(b) L = λ/4 $e^{-2ikL} = e^{-i4\pi L/\lambda} = e^{-i\pi} = -1$

$\therefore \Gamma \mathrm{e}^{-2\mathrm{ikL}} = 2/3$ (real)

$\therefore Z_G = \frac{1+2/3}{1-2/3} = \frac{5/3}{1/3} = 5$

∴ **Z_G = 250 Ohms** (purely real and relatively large!).

(c) $\mathrm{L} = \frac{3\lambda}{8} \quad \mathrm{e}^{-2\mathrm{ikL}} = \mathrm{e}^{-\mathrm{i}4\pi\mathrm{L}/\lambda} = \mathrm{e}^{\mathrm{i}\pi/2} = +\mathrm{i}$

$\therefore \Gamma e^{-2ikL} = -\frac{2i}{3}$

$$Z_G = \frac{1-2i/3}{1+2i/3} = \frac{3-2i}{3+2i} = \frac{(3-2i)(3-2i)}{9+4} = \frac{5-12i}{13}$$

∴ Z_G = **19.23 - 46.15i** Ohms

i.e. there is a large capacitive component.

(d) $\mathrm{L} = \frac{\lambda}{2} \ \mathrm{e}^{-2\mathrm{ikL}} = \mathrm{e}^{-\mathrm{i}4\pi\mathrm{L}/\lambda} = \mathrm{e}^{-\mathrm{i}2\pi} = +1$

$\therefore \Gamma e^{-2ikL} = \Gamma = -2/3$

$Z_G = \frac{1-2/3}{1+2/3} = \frac{1}{5} = 0.2$

∴ Z_G = **10 Ohms**.

i.e. The generator looks directly into the load.

(e) The standing wave ratio is given by

$$\mathrm{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+2/3}{1-2/3} = 5$$

In a slotted line there would be no change in the position of $|V_{min}|$ when the load was exchanged for a short.

Problem (11.8).

Given a co-axial cable for which Z_o = 50 Ohms and v = 2 x 10^8 m/sec. A piece of this cable of length L meters is used to connect a load impedance Z_L to the generator: Z_L = (10 + 20i) Ohms.

Calculate the impedance seen by the generator for

(a) $L = \frac{\lambda}{16}$

(b) $L = \frac{3\lambda}{16}$

(c) $\mathrm{L} = \frac{5\lambda}{16}$

(d) $\mathrm{L} = \frac{\lambda}{2}$

(e) Calculate the Voltage Standing Wave Ratio, VSWR.

Answer (11.8).

Z_L = (10 + 20i) Ohms

$Z_L = \left(\frac{1+2i}{5}\right)$

$$\Gamma = \frac{z_L - 1}{z_L + 1} = \frac{1+2i-5}{1+2i+5} = \frac{-4+2i}{6+2i} = \frac{-2+i}{3+i}$$

$$\therefore \Gamma=\frac{(-2+i)(3-i)}{9+1}=\frac{-5+5i}{10}=\frac{(-1+i)}{2}$$

$$\therefore \Gamma=\frac{1}{\sqrt{2}}e^{3\pi i/4}$$

Now $e^{-2ikL} = e^{-i4\pi L/\lambda}$ ∴ (a) $\frac{L}{\lambda}=\frac{1}{16}\ e^{-2ikL}=e^{-\pi i/4}$

(b) $\frac{L}{\lambda}=\frac{3}{16}\ \mathrm{e}^{-2\mathrm{ikL}}=\mathrm{e}^{-3\pi\mathrm{i}/4}$

(c) $\frac{L}{\lambda}=\frac{5}{16}\ \mathrm{e}^{-2\mathrm{ikL}}=\mathrm{e}^{-5\pi\mathrm{i}/4}$

(d) $\frac{L}{\lambda}=\frac{1}{2}\ \mathrm{e}^{-2\mathrm{ikL}}=\mathrm{e}^{-2\pi\mathrm{i}}\equiv+1$

So

(a) $\Gamma e^{-2ikL}=\frac{1}{\sqrt{2}}e^{i\pi/2}=\frac{i}{\sqrt{2}}$

$$z_G=\frac{1+\Gamma e^{-2ikL}}{1-\Gamma e^{-2ikL}}=\frac{1+i/\sqrt{2}}{1-i/\sqrt{2}}=\frac{\sqrt{2}+i}{\sqrt{2}-i}$$

$$\therefore \quad z_G=\frac{(\sqrt{2}+i)(\sqrt{2}+i)}{3}=\frac{1+i2\sqrt{2}}{3}$$

$$\therefore Z_G=50z_G=(16.67+47.14\ \mathrm{i})\ \text{Ohms.}$$

(b) $\Gamma\mathrm{e}^{-2\mathrm{ikL}}=\frac{1}{\sqrt{2}}\quad$ (purely real)

$$\therefore \quad z_G=\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}=5.83$$

∴ Z_G = (5.83)(50) = **291.4 Ohms** (Purely real!).

(c) $\Gamma e^{-2ikL}=\frac{1}{\sqrt{2}}e^{-i\pi/2}=-\frac{i}{\sqrt{2}}$

$$z_G=\frac{1-i/\sqrt{2}}{1+i/\sqrt{2}}=\frac{\sqrt{2}-i}{\sqrt{2}+i}$$

(The reciprocal of case (a))

$\therefore \quad z_G=\frac{1-i2\sqrt{2}}{3}$, and Z_G = **(16.67 - 47.14 i)** Ohms,

and now the generator load has a capacitive component.

(d) $\Gamma\mathrm{e}^{-2\mathrm{ikL}}=\Gamma\ \ z_G=\frac{1+\Gamma}{1-\Gamma}\equiv z_L$

∴ $Z_G \equiv Z_L$ = **(10 + 20i)** Ohms.

(e) $\mathrm{VSWR}=\frac{1+|\Gamma|}{1-|\Gamma|}=\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}=\mathbf{5.83}$

Problem (11.9).

A 50 Ohm piece of co-axial cable of length L meters is used to connect a load to a generator. The load impedance is given by

Z_L = (10 + 100 i) Ohms

Calculate the impedance seen by the generator for

(a) L/λ = 0.0732

(b) L/λ = 0.250

(c) L/λ = 0.3232

(d) L/λ = 0.5000

(e) Calculate the Voltage Standing Wave Ratio, VSWR.

Answer (11.9).

$z_L = \frac{1}{5} + 2i = \frac{1+10i}{5}$

$\therefore \Gamma = \frac{z_L - 1}{z_L + 1} = \frac{-4/5+2i}{6/5+2i} = \frac{-2+5i}{3+5i} = \mathbf{0.924e^{+0.921i}}$.

(a) $\mathrm{e}^{-2\mathrm{ikL}} = \mathrm{e}^{-0.92\mathrm{i}}$ $\therefore$ $\Gamma e^{-2ikL} = 0.923$ purely real

$$\therefore z_G = \frac{1+\Gamma e^{-2ikL}}{1-\Gamma e^{-2ikL}} = \frac{1+.9235+i0.00095}{1-.9235-i0.00095} = (25.16 - i0.34) \text{ Ohms.}$$

Z_G = **(1257.8 + i 16.9) Ohms.** Almost resistive!

(b) $e^{-2ikL} = e^{-i\pi} = -1$ $\therefore \Gamma e^{-2ikL} = \frac{2-5i}{3+5i} = -0.559 - 0.735i$

$$\therefore z_G = \frac{(1-0.559)-0.735i}{[1.559+0.735i]} = \frac{(.441-.735i)(1.559-.735i)}{2.971}$$

$\therefore$ $z_G = \frac{0.148-1.47i}{2.971}$ and Z_G = **2.48 - 24.75i** Ohms

Capacitive Loading.

(c) $\frac{\mathrm{L}}{\lambda} = 0.3232$ $\mathrm{e}^{-2\mathrm{ikL}} = \mathrm{e}^{-\mathrm{i}4\pi\mathrm{L}/\lambda = \mathrm{e}^{-4.06i}}$

$$\therefore \Gamma e^{-2ikL} = 0.923e^{-3.141i} = 0.923e^{-i\pi} = -0.923 \quad (\text{ real })$$

$$\therefore z_G = \frac{1-.9235-i0.00099}{1+.9235+i0.00099}$$

and Z_G = **(1.987 - i 0.027) Ohms.**

A small, nearly purely real, load.

(d) When $\mathrm{L}/\lambda = \frac{1}{2}$ one gets the same effect as connecting the load directly across the generator.

$\therefore Z_G = Z_L$ = **(10 + 100i) Ohms.**

(e) VSWR $= \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+.923}{1-.923} = \mathbf{25.16}$.

Problem (11.10).

A certain co-axial cable is characterized by a velocity of V= 2.00 x 10^8 meters/sec., and it has a characteristic impedance of 50 Ohms. The attenuation parameter for the cable is α= 0.02 per meter. A piece of this cable 21 m long is used to connect a 250 MHz oscillator to a load consisting of 100 pF shunted by a resistance of 5.0 Ohms. Calculate the load on the generator.

Answer (11.10).

The impedance of the capacitor is $z_c = \frac{-i}{c\omega} = -i6.366$ Ohms. The load impedance is ZC in parallel with a 5 Ohm resistor;

$$\frac{1}{Z_L} = \frac{1}{5} + \frac{1}{Z_C} = 0.20 + i\frac{\pi}{20},$$

so that Z_L= 3.092 -i 2.429 Ohms, and

$$z = \frac{z_L}{z_0} = 0.0618 - i0.0486 = 0.0786 \lfloor -38.15^\circ .$$

We have

$$z = \frac{Z_L}{Z_0} = \frac{1+\frac{b}{a}e^{2\alpha 1}e^{2ik1}}{1-\frac{b}{a}e^{2\alpha 1}e^{2ik1}},$$

where $e^{2ik1} = -1$ and where $e^{2\alpha 1} = e^{21(.04)} = 2.316$.

Let $\Gamma = \frac{z-1}{z+1} = (-0.880 - i0.086)$

and

$$b/a = \Gamma \exp(-2\alpha l - 2ikl) = (0.380 + i0.037)$$

The impedance seen by the generator is $\frac{Z_G}{Z_0} = \frac{1+b/a}{1-b/a}$.

$$\frac{Z_G}{Z_0} = (2.213 + i0.192),$$

therefore $\mathbf{Z_G}$ **= (110.7 + i 9.62) Ohms**.

This can be compared with an impedance Z_G = 500 + i 393 Ohms for the same length of lossless cable. In the limit of a very long cable the impedance seen by the generator must, of course, approach the characteristic impedance of 50 Ohms.

Problem (11.11).

A slotted line is terminated by a load impedance ZL = (10 + 10i) Ohms. The characteristic impedance is Z_o = 50 Ohms. The position of the voltage minimum is found to be at z_1. The load is then replaced by a short and the voltage minimum is found to be at z_2.

(a) How large is the shift $\frac{(z_1 - z_2)}{\lambda}$?

Is this shift positive (i.e. $z_1 > z_2$) corresponding to the shorted line minimum closer to the generator, or is it negative (i.e. $z_2 > z_1$) corresponding to the shorted line minimum closer to the load?

(b) Calculate the Voltage Standing Wave Ratio, VSWR.

Answer (11.11).

(a) Z_o = 50 Ohms Z_L = (10 + 10i) Ohms

$\therefore z_L = \frac{Z_L}{Z_O} = 0.2(1 + i)$

$$\Gamma = \frac{z_L - 1}{z_L + 1} = \frac{-0.8 + 0.2i}{1.2 + 0.2i} = \frac{-0.92 + 0.40i}{(1.2)^2 + 0.04}$$

= - 0.622 + 0.27i

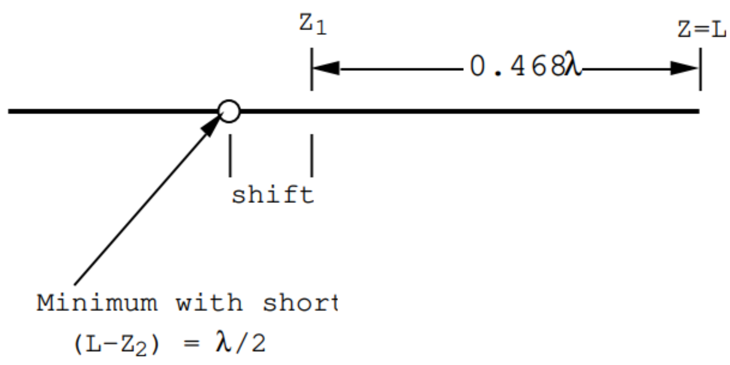
$$\therefore \Gamma = 0.678e^{2.731i} = 0.678e^{i(0.869)\pi}.$$

We have at a voltage minimum

$$\cos\left[\frac{4\pi}{\lambda}(L - z_1) - \theta\right] = -1$$

$$\therefore \frac{4\pi}{\lambda}(L - z_1) - .87\pi = \pi$$

$$\therefore L - z_1 = \frac{1.87\lambda}{4} = .468\lambda$$



So upon shorting the line, the minimum moves 0.0327 λ towards the generator.

(b) $\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+.678}{1-.678} = \mathbf{5.21}$

Problem (11.12).

A slotted line is terminated by a load impedance Z_L = (10 - 10i) Ohms. The characteristic impedance of the slotted line is 50 Ohms. The voltage minimum is found to be at z_1 on the line. When the load is replaced by a short the voltage minimum moves to z_2.

(a) Calculate the shift $\frac{(z_1 - z_2)}{\lambda}$. When the line is shorted does the minimum move towards the generator or towards the load?

(b) Calculate the Voltage Standing Wave Ratio, VSWR.

Answer (11.12).

(a) $\mathrm{Z_L} = 10(1-\mathrm{i})$ Ohms

$\frac{Z_L}{Z_0} = \frac{1}{5} - \frac{i}{5} = z_L$

$$\Gamma = \frac{z_L - 1}{z_L + 1} = \frac{-4/5 - i/5}{6/5 - i/5} = \frac{-4 - i}{6 - i} = \frac{-23 - 10i}{37}.$$

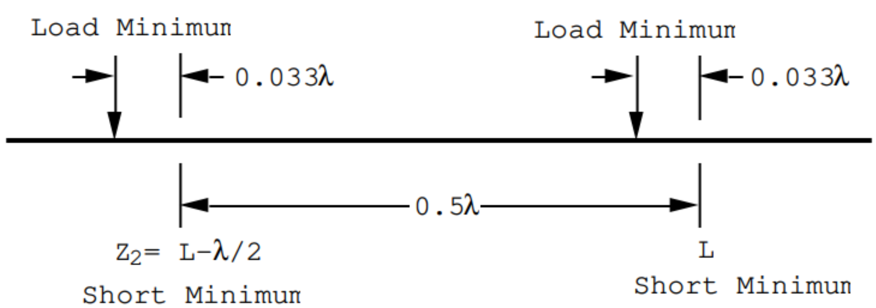
$$\therefore \Gamma = -0.622 - 0.27\mathrm{i} = 0.678\mathrm{e}^{\mathrm{i}1.13\pi}.$$

$\theta = 203.5^\circ = 1.131\pi$ radians.

Minimum when cos[2k(L-z) - θ] = -1

or $\frac{4\pi}{\lambda}(L-z) - 1.13\pi = \pi$

$\therefore L - z = 0.533\lambda$



When the load is replaced by a short, the minimum moves .0327 λ towards the load.

(b) $\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1.678}{1-.678} = \mathbf{5.21}$.

The same VSWR as for the impedance of problem (9.11).

Problem (11.13).

The Voltage Standing Wave Ratio is found to be S = 2.0 on a lossless 300 Ohm transmission line terminated by an unknown load impedance, Z_L. The nearest voltage minimum is $\frac{3\lambda}{10}$ from the load i.e. z_1 = (L - 0.3λ).

(a) When the above line is shorted where will the voltage minimum be located which is nearest the load, but not right at the load?

(b) Calculate the real and imaginary parts of the unknown load impedance, Z_L.

Answer (11.13).

(a) When the load is replaced by a short the minimum will be located $\frac{\lambda}{2}$ from the short.

$\therefore \mathrm{z_2} = \mathrm{L} - \frac{\lambda}{2}$

(b) $S = VSWR = 2.0 = \frac{1+|\Gamma|}{1-|\Gamma|}$

$\therefore |\Gamma| = 1/3 \; \Gamma = |\Gamma| e^{i\theta}$

Minimum at z_1 where

cos [2k(L-z) - θ] = -1

k $=\frac{2\pi}{\lambda}$ and $\frac{4\pi}{\lambda}(L-z_1)-\theta=\pi$

So $\pi+\theta=\left(\frac{4\pi}{\lambda}\right)\left(\frac{3\lambda}{10}\right)=1.2\pi$

∴ θ = 0.2 π

So $\Gamma=\frac{1}{3}e^{0.2\pi i}$

$\mathbf{\Gamma=0.270+.196i}$

But $z_L=\frac{Z_L}{Z_0}=\frac{1+\Gamma}{1-\Gamma}$

∴ z_L = 1.555 + .685 i

∴ Z_L = (z_L)(300) = **466.5 + 205.6i Ohms**

Problem (11.14). A slotted line is characterized by a velocity V= 3.00 x 10^8 m/sec, and by a characteristic impedance of 50 Ohms. The slotted line is connected to an oscillator on one end and to an unknown load on the other end. The voltage standing wave ratio is found to be VWSR= 2.0. Moreover, when the load is replaced by a short circuit the position of the voltage minimum shifts 5 cm towards the load. The position of the first minimum from the shorted end occurs 40.0 cm from the short. Calculate the impedance of the load.

Answer (11.14).

The voltage minimum on a shorted line occurs at a distance λ/2 from the short; therefore for this problem the generator frequency corresponds to a wavelength of λ= 80 cm= 0.80 meters. The velocity on the slotted line is c= 3 x 10^8 m/sec, so that the frequency is f= c/λ= 375 MHz. The corresponding circular frequency is ω= 2πf = 2.356 x 10^9 radians/sec. The wavevector on the line is k= 2π/λ = 7.854 m^{-1}. Let the load be at z=L, with the generator somewhere to the left (at z=0). For a time dependence $e^{i\omega t}$

$$V=ae^{-ikz}+be^{ikz}$$

and

$$\mathrm{z_0I=ae^{-ikz}-be^{ikz}}.$$

At z=L $V=ae^{-ikL}+be^{ikL}$

and

$$\mathrm{z_0I=ae^{-ikL}-be^{ikL}}$$

Thus

$$\frac{z_L}{z_0}=\frac{1+(b/a)e^{2ikL}}{1-(b/a)e^{2ikL}}=\frac{1+\Gamma e^{i\theta}}{1-\Gamma e^{i\theta}},$$

where

$$\left(\frac{\mathrm{b}}{\mathrm{a}}\right)\mathrm{e^{2ikL}}=\Gamma\mathrm{e}^{i\theta}=\frac{(\mathrm{Z_L/Z_0})-1}{(\mathrm{Z_L/Z_0})+1}.$$

One can write

$$V(z)=ae^{-ikz}\left(1+\Gamma e^{i\theta}e^{2ik(z-L)}\right);$$

clearly $|\mathrm{V_{max}}|=|\mathrm{a}|(1+\Gamma)$,

whereas $|\mathrm{V_{min}}|=|\mathrm{a}|(1-\Gamma)$,

so that $\frac{|V_{\max}|}{|V_{min}|}=\frac{1+\Gamma}{1-\Gamma}=2.0$

Therefore $\Gamma=1/3$. With the load connected the minimum occurs at z_1. At the minimum $e^{i(2k(z_1-L)+\theta)}=-1$, or $2\mathrm{k}\left(\mathrm{z}_1-\mathrm{L}\right)+\theta=\pm\pi\backslash.$

When the line is shorted the minimum occurs 40 cm from the load. With the load in place the minimum shifts 5 cm towards the generator. That means that z_1 is such that L-z_1= 45 cm = 0.5625λ. Thus

$$\theta=\pm\pi+2k\left(L-z_1\right),$$

or θ = ±π + 7.0686. The appropriate value is less than 2π so that θ = 3.9270 radians

$$\frac{Z_L}{Z_0}=\frac{1+\frac{1}{3}e^{3.927i}}{1-\frac{1}{3}e^{3.927i}}=0.562-i0.298$$

and the load impedance is **Z_L = 28.08 - i14.89 Ohms**. This is equivalent to a resistance of 28.08 Ohms in series with a $28.5\text{x}10^{-12}$ Farad capacitor.

13.12: Chapter- 12

Problem (12.1).

Microwave power of 1 Watt at a frequency of 24 GHz is transmitted through a piece of rectangular waveguide whose inside dimensions are 1 cm x 0.5 cm. Let the z-axis lie parallel with the waveguide axis, and let the microwaves be propagating in the +z direction. Use $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$.

(a) Write expressions for the electric and magnetic fields in the waveguide if the time variation is $e^{-i\omega t}$.

(b) Calculate the amplitudes of the electric and magnetic field components.

(c) Calculate the time-averaged energy density contained in the fields.

(d) With what velocity is the above energy density transported along the waveguide?

(e) Show that the magnetic field vector rotates with time at points which are part way across the width of the waveguide. Show that for points near x=a/4 the rotation is clockwise when viewed from a point on the plus y-axis and looking towards the x-z plane, whereas the rortation is counter-clockwise near x=3a/4.

Answer (12.1).

(a) For a frequency F= 24 GHz, $\omega = 2\pi F = 1.508 \times 10^{11}$ radians/sec. For the TE_{10} mode (all other modes are cut-off)

$$\mathbf{E_y} = \mathbf{E}_0 \, \mathbf{sin}\left(\frac{\pi \mathbf{x}}{\mathbf{a}}\right) \mathbf{e}^{\mathbf{i}(\mathbf{k_g z} - \omega \mathbf{t})},$$

where the waveguide walls are at x=0,a and at y=0,b: there is no spatial variation along the narrow dimension of the guide. The field components must satisfy the wave equation: in particular,

$$\nabla^2 E_y = \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2},$$

from which

$$\left(\frac{\pi}{a}\right)^2 + k_g^2 = \left(\frac{\omega}{c}\right)^2.$$

For the present case, $\frac{\pi}{a} = 314.2 \text{ m}^{-1}$

$$\frac{\omega}{c} = 502.7 \text{ m}^{-1}$$

so that

$$k_g = 392.4 \text{ m}^{-1}.$$

From curl**E**= $i\omega\mu_0$**H**, using the fact that **E** has only a y component, one finds

$$\mathbf{H_x} = -\left(\frac{\mathbf{k_g}}{\omega \mu_0}\right) \mathbf{sin}\left(\frac{\pi \mathbf{x}}{\mathbf{a}}\right) \mathbf{E}_0 \mathbf{e}^{\mathbf{i}(\mathbf{k_g z} - \omega \mathbf{t})},$$

and $i\omega\mu_0 \mathrm{H_z} = \frac{\partial \mathrm{E_y}}{\partial \mathrm{x}} = \left(\frac{\pi}{\mathrm{a}}\right) \mathrm{E}_0 \cos\left(\frac{\pi \mathrm{x}}{\mathrm{a}}\right) \mathrm{e}^{\mathrm{i}(\mathrm{k_g z} - \omega \mathrm{t})}$,

or

$$\mathbf{H_z} = \frac{-\mathbf{i}\pi}{\mu_0 \mathbf{a}} \mathbf{E}_0 \, \mathbf{cos}\left(\frac{\pi \mathbf{x}}{\mathbf{a}}\right) \mathbf{e}^{\mathbf{i}(\mathbf{k_g z} - \omega \mathbf{t})}.$$

Note that $E_y = -Z_g H_x$ where $\mathrm{Z_g} = -\left(\frac{\omega}{\mathrm{ck_g}}\right) \mathrm{Z}_0$, and $Z_0 = \mu_0 c =$ 377 Ohms.

(b)

$$\mathrm{S_z} = -\mathrm{E_y H_x} \text{ Watts }/\mathrm{m}^2.$$

$$< S_z >= -\frac{1}{2}\text{Real}(E_y H_x^\star) = \frac{1}{2}\frac{|E_0|^2}{|Z_g|}\sin^2\left(\frac{\pi x}{a}\right).$$

The average across the guide is given by

$$<< S_z >>= \frac{1}{4}\frac{|E_0|^2}{\left(\frac{\omega}{ck_g}\right) Z_0},$$

where E_0 is the electric field amplitude. Now $Z_g = \mathrm{Z}\left(\frac{\omega}{\mathrm{ck_g}}\right) = 482.9 \text{ Ohms}$, and $<<S_z>>ab=$ 1 Watt, therefore $<< S_z >>= 2\times 10^4 \text{ Watts }/m^2$,

so that **E_0 = 6216 Volts/meter**, or 31.1 Volts across the narrow dimension of the waveguide. The x-component of the magnetic field amplitude is $|\mathbf{H_x}| = \mathbf{12.87\ Amps\ /m}$. The amplitude of the longitudinal magnetic field component is **$|H_0|$= 10.31 Amps/m**.

(c) The time-averaged energy density contained in the fields is given by

$$< W >=< \varepsilon_0 E_Y^2/2 > + < \mu_0 H_x^2/2 > + < \mu_0 H_z^2/2 >,$$

or

$$< W >= \frac{\varepsilon_0 E_0^2 \sin^2(\pi \times /a)}{4} + \frac{1}{4\mu_0}\left(\frac{\mathrm{k_g^2}}{\omega^2}\mathrm{E_0^2}\sin^2(\pi\mathrm{x/a}) + \frac{\pi^2}{\mathrm{a^2}\omega^2}\mathrm{E_0^2}\cos^2(\pi\mathrm{x/a})\right).$$

Averaged over the guide cross-section, this expression gives

$$<< \mathrm{W} >>= \varepsilon_0\frac{\mathrm{E_0^2}}{4}\ \text{ Joules }/\mathrm{m}^3 = 85.4\times 10^{-6}\ \mathrm{J/m^3}.$$

(d) The group velocity is the rate of energy transport down the guide;

$$<< S_z >>= V_g << W >>.$$

It follows from this that

$$V_g = c\frac{k_q}{(\omega/c)} = 0.781c = \mathbf{2.34\times 10^8\ m/\ sec.}$$

The group velocity is also given by $\mathrm{V_g} = \frac{\partial\omega}{\partial \mathrm{k_g}}$.

(e) Near x=a/4 $\mathrm{H_x} = \frac{-\mathrm{k_g}}{\mu_0\omega}\frac{\mathrm{E_0}}{\sqrt{2}}e^{-\mathrm{i}\omega\mathrm{t}}$

$$\mathrm{H_z} = \frac{\pi}{\mathrm{a}\mu_0\omega}\frac{\mathrm{E_0}}{\sqrt{2}}e^{-(\mathrm{i}\omega\mathrm{t}-\pi/2)},$$

therefore if $\mathrm{H_x} = \frac{-\mathrm{k_q}}{\mu_0\omega}\frac{\mathrm{E_0}}{\sqrt{2}}\cos\omega t,$

then

$$\mathrm{H_z} = -\frac{\pi}{\mathrm{a}\mu_0\omega}\frac{\mathrm{E_0}}{\sqrt{2}}\sin\omega t.$$

These expressions describe an elliptically polarized wave (nearly circularly polarized because $\frac{\mathrm{k_q}}{(\pi/\mathrm{a})} = 1.25$ rotating in the direction from z to -x, i.e. clockwise looking from +y towards the x-z plane.

Similarly, near x=3a/4 $\mathrm{H_x} = -\frac{\mathrm{k_g}}{\mu_0\omega}\frac{\mathrm{E_0}}{\sqrt{2}}\cos\omega\mathrm{t}$, and

$\mathrm{H_z} = \frac{\pi}{\mathrm{a}\mu_0\omega}\frac{\mathrm{E_0}}{\sqrt{2}}\quad \sin\omega\mathrm{t}$,

corresponding to a counter-clockwise rotation looking from +y towards the xz plane.

Problem (12.2).

An attempt is made to propagate a 10 GHz microwave signal along a rectangular air-filled waveguide whose internal dimensions are 1 cm x 0.50 cm. Use ε_0 and μ_0 for the dielectric constant and the permeability.

(a) Write expressions for the electric and magnetic fields associated with the non-propagating TE_{10} mode.

(b) Over what distance is the amplitude of the microwave fields attenuated by 1/e?

(c) Calculate the z-component of the Poynting vector and show that it corresponds to a periodic flow of energy across the waveguide section whose time average is zero.

Answer (12.2).

(a) F= 10 GHz ω= 6.28 x 10^{10} rad./sec. $\frac{\omega}{c}=2.094\times 10^2\mathrm{m}^{-1}$.

$\frac{\pi}{a}=3.141\times 10^2\mathrm{m}^{-1}$.

For the TE_{10} mode $\mathrm{k}_\mathrm{g}^2+\left(\frac{\pi}{\mathrm{a}}\right)^2=\left(\frac{\omega}{\mathrm{c}}\right)^2$,

from which $k_g^2=-5.4831\times 10^4$, and $k_g = \pm i\ 2.342 \times 10^2\ \mathrm{m}^{-1}$,

a pure imaginary number. Let $k_g = i\alpha$.

$$E_y=E_0\sin\left(\frac{\pi x}{a}\right)e^{-\alpha z}e^{-i\omega t}$$

$$\mathrm{H}_\mathrm{x}=-\frac{\mathrm{i}\alpha}{\omega\mu_0}\mathrm{E}_0\sin\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)e^{-\alpha z}e^{-\mathrm{i}\omega t}$$

$$\mathrm{H}_\mathrm{z}=-\mathrm{i}\left(\frac{\pi}{\mathrm{a}\omega\mu_0}\right)\mathrm{E}_0\cos\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)e^{-\alpha z}e^{-\mathrm{i}\omega t}.$$

(b) The attenuation length is $\frac{1}{\alpha}=\frac{10^{-2}}{2.34}=4.27\times 10^{-3}$ meters , or

$$1/\alpha=4.27\ \mathrm{mm}.$$

(c) $S_z = - E_y H_x$, where for this problem

$$E_y=E_0\sin\left(\frac{\pi x}{a}\right)e^{-\alpha z}\cos\omega t,$$

and

$$\mathrm{H}_\mathrm{x}=-\left(\frac{\alpha}{\omega\mu_0}\right)\mathrm{E}_0\sin\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)e^{-\alpha z}\sin\omega\mathrm{t}.$$

Therefore, $\mathrm{S}_\mathrm{z}=-\mathrm{E}_\mathrm{y}\mathrm{H}_\mathrm{x}=\frac{\alpha}{\omega\mu_0}\mathrm{E}_0^2\sin^2\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)\mathrm{e}^{-2\alpha z}\sin\omega\mathrm{t}\cos\omega\mathrm{t}$

or $\boldsymbol{S_z=1.483\times 10^{-3}\mathrm{E}_0^2\sin^2\left(\frac{\pi\mathrm{x}}{\mathrm{a}}\right)\mathrm{e}^{-2\alpha z}\sin 2\omega t}$

since $\sin\omega t\cos\omega t=\frac{1}{2}\sin 2\omega t$.

Problem (12.3).

(a) Design a rectangular air-filled cavity to operate at 24 GHz in the TE_{103} mode. The cavity is to be constructed from a length of rectangular waveguide whose internal dimensions are 1 x 0.50 cm. Use ε_0 and μ_0 for the dielectric constant and the permeability.

(b) Write expressions for the fields in the cavity at resonance.

Answer (12.3).

(a) At 24 GHz ω= 1.508 x 10^{11} rad./sec $\frac{\omega}{c}=502.7\ \mathrm{m}^{-1}$.

For the TE_{10} mode the guide wave-number can be calculated from

$$k_g^2=\left(\frac{\omega}{c}\right)^2-\left(\frac{\pi}{a}\right)^2$$

where a= 0.01 m is the broad dimension of the guide:

$$k_g = 3.925 \times 10^2 \ m^{-1}.$$

The guide wavelength is $\lambda_g = 2\pi/k_g = 1.60 \times 10^{-2}$ m $= 1.60$ cm . The length of the cavity should be $L = \frac{3\lambda_g}{2}$ for the TE_{103} mode;

$$\mathbf{L = 2.40 \times 10^{-2} \ m = 2.40 \ cm.}$$

(b) For the forward propagating wave and a TE_{10} mode

$$E_y = E_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z} e^{-i\omega t},$$

For the backward propagating wave

$$E_y = E_0 \sin\left(\frac{\pi x}{a}\right) e^{-ik_g z} e^{-i\omega t}.$$

In the cavity one must set up a standing wave along z which has nodes at z=0 and at $z = L = \frac{3\lambda_q}{2}$; i.e.

$$E_y = E_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{n\pi z}{L}\right) \cos \omega t.$$

From this electric field one can calculate the other field components using curl $\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$. For the TE_{10} mode the electric field has only one component, E_y, and

$$\frac{\partial E_y}{\partial z} = \mu_0 \frac{\partial H_x}{\partial t} \qquad (1)$$

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (2)$$

From (1)

$$H_x = \left(\frac{1}{\mu_0 \omega}\right) \left(\frac{n\pi}{L}\right) E_0 \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{n\pi z}{L}\right) \sin \omega t$$

From (2)

$$H_z = -\left(\frac{1}{\mu_0 \omega}\right) \left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{n\pi z}{L}\right) \text{sinwt} .$$

For resonance k_g= 3λ/2 and therefore L= 2.40 cm.

Problem (12.4).

A rectangular waveguide is filled with material characterized by a relative dielectric constant ε_r= 9.00. The inside dimensions of the waveguide are a= 1 cm, b= 0.50 cm.

(a) Over what frequency interval would this guide support only the TE_{10} mode?

(b) Calculate the time-averaged energy density for the TE_{10} mode, and average the resulting expression over the guide cross section. Let the amplitude of the electric field be $E_y = E_0$.

(c) Calculate the time-averaged value of the Poynting vector, and average the resulting expression over the guide cross section. Let the amplitude of the electric field be $E_y = E_0$.

(d) A signal having an average power of 1 Watt is transmitted down the guide at a frequency of 7.5 GHz. Calculate (i) the wavelength along the guide, λ_g; (ii) the ratio of the guide wavelength to the free space wavelength for a 7.5 GHz plane wave; (iii) the group velocity, i.e. the velocity with which information can be transmitted down the guide; (iv) the amplitude of the electric field.

Answer (12.4).

(a) For the TE_{10} mode the fields have the form

$$E_y = E_0 \sin\left(\frac{\pi x}{a}\right) e^{i(k_g z - \omega t)},$$

$$H_x = -\left(\frac{k_g}{\omega\mu_0}\right) E_0 \sin\left(\frac{\pi x}{a}\right) e^{i(k_g z-\omega t)},$$

$$H_z = -\left(\frac{i}{\omega\mu_0}\right)\left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi x}{a}\right) e^{i(k_g z-\omega t)},$$

where $\omega^2\varepsilon\mu_0 = k_g^2 + \left(\frac{\pi}{a}\right)^2$

or $\varepsilon_r\left(\frac{\omega}{c}\right)^2 = k_g^2 + \left(\frac{\pi}{a}\right)^2$.

If a= 1 cm = 0.01 m $\left(\frac{\pi}{a}\right)^2 = 9.870 \times 10^4 \text{ m}^{-2}$.

The cut-off frequency corresponds to k_g = 0; i.e. $\sqrt{\varepsilon_r}\left(\frac{\omega}{c}\right) = \frac{\pi}{a}$. At cut-off $\frac{\omega}{c} = \frac{314.2}{\sqrt{\varepsilon_r}} = 104.7 \text{ m}^{-1}$,

or **F = 5.00 GHz.**

For the higher order modes, cut-off corresponds to the condition k_g= 0, so that

$$\varepsilon_r\left(\frac{\omega}{c}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2,$$

where $\frac{\pi}{a} = 314.2 \text{ m}^{-1}$, and $\frac{\pi}{b} = 628.4 \text{ m}^{-1}$.

For m=0 n=1 F_{01} = 10.00 GHz

m=1 n=1 F_{11} = 11.18 Ghz

m=1 n=2 F_{12} = 20.62 GHz

m=2 n=0 F_{20} = 10.00 GHz.

This waveguide will support only the TE_{10} mode for frequencies in the interval 5.00 to 10.00 GHz.

(b) The time-averaged energy density is given by

$$< W >=< \varepsilon E_y^2/2 > + < \mu_0 H_x^2/2 > + < \mu_0 H_z^2/2 >,$$

$$< W >= \frac{\varepsilon_r\varepsilon_0}{4} E_0^2 \sin^2\left(\frac{\pi_x}{a}\right) + + \frac{1}{4\mu_0\omega^2} k_g^2 E_0^2 \sin^2\left(\frac{\pi_x}{a}\right) + \frac{1}{4\mu_0\omega^2}\left(\frac{\pi}{a}\right)^2 E_0^2 \cos^2\left(\frac{\pi_x}{a}\right).$$

Take the spatial average over the cross-section of the waveguide:

$$<< W >>= \left(\varepsilon_r + \frac{1}{\omega^2\varepsilon_0\mu_0}\left(k_g^2 + \left(\frac{\pi}{a}\right)^2\right)\right)\frac{\varepsilon_0 E_0^2}{8},$$

$$\mathbf{<< W >>= \frac{\varepsilon_r\varepsilon_0}{4} E_0^2 \text{ Joules }/m^3}..$$

(c) $S_z = - E_y H_x$,

$$\langle \mathbf{S_z}\rangle = \frac{\mathbf{k_g}}{\mathbf{2\omega\mu_0}} \mathbf{E}_0^2 \sin^2\left(\frac{\pi \mathbf{x}}{\mathbf{a}}\right).$$

The average over the x co-ordinate gives

$$\mathbf{<< S_z >>= \frac{k_g}{4\omega\mu_0} E_0^2 \quad Watts\ /m^2}.$$

(d) The group velocity is such that $<<S_z>>= V_g<<W>>$, therefore

$$V_g = \frac{c}{\varepsilon_r}\left(\frac{k_q}{(\omega/c)}\right).$$

At 7.5 GHz $k_0 = \frac{\omega}{c} = 157.1 \text{ m}^{-1}$ and the free space wavelength is λ_0= 4.00 cm. The waveguide wave-vector is given by

$$k_g^2 = 9k_0^2 - \left(\frac{\pi}{a}\right)^2 = 12.337 \times 10^4,$$

and

$$\mathrm{k_g} = 3.513 \times 10^2\ \mathrm{m}^{-1}.$$

From this, the guide wavelength is

(i) $\boldsymbol{\lambda_g = \frac{2\pi}{k_g} = 1.788\ cm,}$, and

(ii) $\frac{\lambda_g}{\lambda_0} = 0.447$

(iii) $V_g = \frac{c}{9}\left(\frac{3.512}{1.571}\right) = \mathbf{0.745 \times 10^8\ meters/sec\ .}$

(iv) $\langle\langle S_z\rangle\rangle = \frac{k_q}{4\omega\mu_0} E_0^2 = \frac{1}{ab} = 2 \times 10^4\ \ \text{Watts}\ /m^2$.

From this $\mathrm{E_0^2} = 4\frac{(\omega/\mathrm{c})}{\mathrm{k_g}}(377)\left(2 \times 10^4\right) = 1.349 \times 10^7$,

so that $\mathbf{E_0}$ **= 3673 Volts/m.**

Problem (12.5).

It is desired to construct a cylindrical air-filled cavity which will resonate at 10 GHz in the TE_{01} doughnut mode (this is a very low loss mode which is often used to construct frequency meters). If the radius of the cavity is chosen to be R= 2.50 cm how long should the cavity be made?

Answer (12.5).

For the TE_{01} mode the tangential component of the electric field, E_θ, is proportional to the Bessel function $J_0'(\mathrm{k_c r}) = -\mathrm{J_1}(\mathrm{k_c r})$ where

$$\mathrm{k_c^2} = \varepsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 - \mathrm{k_g^2},$$

see eqn.(10.90b).

The component E_θ must be zero at the waveguide wall in order that the tangential component of the electric field be zero:

$$J_1(k_c R) = 0$$

or $\mathrm{k_c R} = 3.8317$ for the lowest mode.

Thus $k_c = \frac{3.832}{0.025} = 153.3\ \mathrm{m}^{-1}$.

For an air-filled waveguide ε_r= 1, so

$k_g^2 = 2.0373 \times 10^4\ \mathrm{m}^{-2}$ since $\frac{\omega}{\mathrm{c}} = 209.44\ \mathrm{m}^{-1}$ at 10 GHz. Consequently, k_g = 142.7 m^{-1} and the guide wavelength is $\lambda_\mathrm{g} = \frac{2\pi}{\mathrm{k_g}} = 4.40\ \mathrm{cm}$. But E_θ must vanish at the cavity end walls and therefore E_θ must be proportional to $\sin\left(\frac{\mathrm{n}\pi\mathrm{z}}{\mathrm{L}}\right)$. Thus $\mathrm{k_g} = \frac{\mathrm{n}\pi}{\mathrm{L}}$ and the cavity length must be an integral number of half wavelengths long. A convenient choice would be L= 4.40 cm.

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    - 9.1: Introduction to Plane Waves - *CC BY 4.0*
    - 9.2: Phasors - *CC BY 4.0*
    - 9.3: Elliptically Polarized Plane Waves - *CC BY 4.0*
    - 9.4: Gaussian Light Beams - *CC BY 4.0*
  - 10: Plane Waves II - *CC BY 4.0*
    - 10.1: Normal Incidence - *CC BY 4.0*
    - 10.2: Boundary Conditions - *CC BY 4.0*
    - 10.3: Application of the Boundary Conditions to a Plane Interface - *CC BY 4.0*